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# Some Remarks on Compact Strongly Pseudoconvex CR Manifolds

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***Dedicated to Professor M. Kuranishi on his 70th birthday***

## §0. Introduction.

In this note, we make some remarks on compact strongly pseudoconvex CR manifolds. These remarks are related to the problem of minimal embedding dimension of a compact strongly pseudoconvex CR manifold in complex Euclidean space and the classification problem of compact strongly pseudoconvex CR manifolds. Most are contained in [LuY1-3] and in our joint paper with Yung Yu [LuYY]. We hope that this expository note would be of interest for the study of the relationship between the geometry of a compact strongly pseudoconvex CR manifold and the singularities that it may bound, much in the spirit of Kuranishi's application of $\overline{\partial}_b$ to the deformation of isolated singularities [Ku1]. The first author most gratefully recalls the years at Columbia when he studied with Professor Kuranishi and was brought into the fascinating field of CR geometry, being inspired by Professor Kuranishi's mathematical power and depth.

## §1 Preliminary

**Definition 1.1.** *Let $X$ be a connected real manifold of dimension $2n-1$, $n \geq 2$. A CR structure on $X$ is an $(n-1)$-dimensional subbundle $S$ of the complexified tangent bundle $\mathbb{C}TX$ such that*

(a) $S \cap \overline{S} = \{0\}$,

(b) *If $L$, $L'$ are local sections of $S$, then so is $[L, L']$.*

**Definition 1.2.** *Let $L_1, \ldots, L_{n-1}$ be a local frame of $S$. Then $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section $N$*

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*of $TX$ such that $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}TX$. The matrix $(c_{ij})$ defined by*

$$[L_i, \overline{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

*is Hermitian and is called the Levi form. The CR manifold $X$ is called strongly pseudoconvex if its Levi form is definite at each point of $X$. This condition is independent of the choice of local frame and it implies the orientability of $X$.*

A fundamental invariant in CR geometry is the $\overline{\partial}_b$ cohomology introduced by Kohn-Rossi [KR]. The formulation below follows Tanaka [T].

**Definition 1.3.** *Let $A^k(X) = \wedge^k \mathbb{C}TX^*$ and $\mathcal{A}^k(X) = \Gamma(A^k(X))$. Then $\{\mathcal{A}^k(X), d\}$ is the deRham complex. With the notation in Definition 1.1, let*

$$A^{p,q}(X) = \{\varphi \in A^{p+q}(X) : \varphi(Y_1, \ldots, Y_{p-1}, \overline{Z}_1, \ldots, \overline{Z}_{q+1}) = 0 \ , \\ \text{for all } Y\text{'s in } \mathbb{C}TX \text{ and } Z\text{'s in } S\}$$

*and $\mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X))$. Then*

$$A^{p+1,q-1}(X) \subset A^{p,q}(X) \quad \text{and} \quad d\mathcal{A}^{p,q}(X) \subset \mathcal{A}^{p,q+1}(X) \ .$$

*Hence let $C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X)$ and $\mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X))$. Then*

$$\begin{array}{ccc} \mathcal{A}^{p,q}(X) & \xrightarrow{d} & \mathcal{A}^{p,q+1}(X) \\ \cup & & \cup \\ \mathcal{A}^{p+1,q-1}(X) & \xrightarrow{d} & \mathcal{A}^{p+1,q}(X) \end{array} \quad \text{induces} \quad \mathcal{C}^{p,q}(X) \xrightarrow{\overline{\partial}_b} \mathcal{C}^{p,q+1}(X).$$

*The cohomology groups of the resulting complex $\{\mathcal{C}^{p,q}(X), \overline{\partial}_b\}$ will be denoted by $H^{p,q}(X)$.*

The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex CR manifolds was developed by Kohn [Ko]. The theory of harmonic integrals on strongly pseudoconvex CR structures over small balls was due to Kuranishi [Ku2]. Using the former theory, Boutet de Monvel [B] proved that if $X$ is a compact strongly pseudoconvex CR manifold of dimension $2n-1$ and $n \geq 3$, then there exist $C^\infty$ functions $f_1, \ldots, f_N$ on $X$ such that each $\overline{\partial}_b h_j = 0$ and $f = (f_1, \ldots, f_N)$ defines an embedding of $X$ in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex CR manifold of dimension $\geq 5$ can be CR embedded in some complex

Euclidean space. Using the theory of harmonic integrals over small balls of special type, Kuranishi [Ku2] proved that any strongly pseudoconvex CR manifold of dimension $2n-1$ with $n \geq 5$ can be locally CR embedded as a real hypersurface in $\mathbb{C}^n$. For $n = 4$, Akahori [Ak] proved that Kuranishi's local embedding theorem is also true.

## §2 Concerning the minimal embedding dimension in complex Euclidean space

Let us first consider a compact strongly pseudoconvex manifold $X$ of dimension $2n-1$ where $n \geq 3$. As mentioned above, $X$ can be CR embedded in some $\mathbb{C}^N$. It is therefore of interest to study the minimal dimensional complex Euclidean space in which $X$ CR embeds. Our starting point is the following two theorems.

**Theorem 2.1.** *(Harvey-Lawson [HL], see also Yau [Y]) Let $X$ be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$, in $\mathbb{C}^N$. Then there exists a unique bounded complex analytic subvariety $V$ of dimension $n$ in $\mathbb{C}^N \setminus X$ such that $X$ is the boundary of $V$ in the $C^\infty$ sense. Further, $V$ is smooth except at finitely many isolated singular points.*

**Theorem 2.2.** *(Yau [Y]) Let $X$ be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$, which is the boundary of a Stein space $V$ with isolated singularities $p_1, \ldots, p_m$. Then for $1 \leq q \leq n-2$,*

$$H^{p,q}(X) \cong \bigoplus_{i=1}^{m} H_{p_i}^{q+1}(V, \Omega_V^p)$$

*where $\Omega_V^p$ is the sheaf of germs of holomorphic $p$-forms on $V$. If $p_1, \ldots, p_m$ are hypersurface singularities, then*

$$\dim H^{p,q}(X) = \begin{cases} 0 & p+q \leq n-2 \quad 1 \leq q \leq n-2 \\ \displaystyle\sum_{i=1}^{m} \tau_i & p+q = n-1, n \quad 1 \leq q \leq n-2 \\ 0 & p+q \geq n+1 \quad 1 \leq q \leq n-2 \end{cases}$$

*where $\tau_i$ is the number of moduli of $V$ at $p_i$.*

Theorem 2.2 provides a solution to the Kohn-Rossi conjecture [KR] that "in general, either there is no boundary cohomology (in degree

$(p, q)$, $q \neq 0$, $n-1$) or it must result from the interior singularities". Moreover it provides us with obstructions to CR embedding:

**Theorem 2.3.** *Let $X$ be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$. Then $X$ cannot be CR embedded in $\mathbb{C}^n$ unless all $H^{p,q}(X) = 0$, $1 \leq q \leq n-2$. Further, $X$ cannot be CR embedded in $\mathbb{C}^{n+1}$ if one of the following does not hold:*

(1) $H^{p,q}(X) = 0$ *for* $p+q \leq n-2$ *and* $1 \leq q \leq n-2$

(2) $\dim H^{p,q}(X) = \dim H^{p',q'}(X)$ *for* $\left.\begin{matrix} p+q \\ p'+q' \end{matrix}\right\} = n-1, n$ *and* $1 \leq q, q' \leq n-2$

(3) $H^{p,q}(X) = 0$ *for* $p+q \geq n+1$ *and* $\leq q \leq n-2$

We next consider an interesting class of CR manifolds.

**Definition 2.4.** *Let $X$ be a CR manifold with structure bundle $S$. Let $\alpha$ be a smooth $S^1$-action on $X$ and $\mathbf{v}$ be its generating vector field. The $S^1$-action $\alpha$ is called holomorphic if $\mathcal{L}_{\mathbf{v}}\Gamma(S) \subset \Gamma(S)$. It is called transversal if $\mathbf{v}$ is transversal to $S \oplus \overline{S}$ in $\mathbb{C}TX$ at every point of $X$.*

For a CR manifold $X$ which admits a transversal holomorphic $S^1$-action, the invariant Kohn-Rossi cohomology is defined as follows:

**Definition 2.5.** *With the notation in Definition 2.4, consider first the differential operator on $k$ forms $N : \mathcal{A}^k(X) \to A^k(X)$ defined by $N\varphi = \sqrt{-1}\mathcal{L}_{\mathbf{v}}\varphi$, $\varphi \in \mathcal{A}^k(X)$. Observe that $N$ leaves invariant the spaces $\mathcal{A}^{p,q}(X)$ and $C^{p,q}(X)$, and commutes with the operators $d$ and $\overline{\partial}_b$. Hence $N$ acts on the cohomology groups $H^{p,q}(X)$. Now define the invariant Kohn-Rossi cohomology by $\tilde{H}^{p,q}(X) = \{c \in H^{p,q}(X) : Nc = 0\}$.*

For a compact strongly pseudoconvex CR manifold $X$ of dimension $2n-1$, $n \geq 3$, which admits a transversal holomorphic $S^1$-action, the invariant Kohn-Rossi cohomology $\tilde{H}^{p,q}(X)$, for $1 \leq p+q \leq 2n-N-1$, are obstructions to CR embedding in $\mathbb{C}^N$. This is implied by the following theorem:

**Theorem 2.6.** *Let $X$ be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$, which admits a transversal holomorphic $S^1$-action. Suppose that $X$ is CR embeddable in $\mathbb{C}^N$. Then $\tilde{H}^{p,q}(X) = 0$ for all $1 \leq p+q \leq 2n-N-1$.*

The proof of Theorem 2.6 contains two main parts. The first part depends heavily on the work of Lawson-Yau [LY], which provides us with topological restrictions on $X$. In particular it can be shown that the de Rham cohomology groups $H^k(X) = 0$ for $1 \leq k \leq 2n - N - 1$. The second part follows Tanaka's differential geometric study on the $\overline{\partial}_b$ cohomology groups [T]. The existence of the vector field $\mathbf{v}$ with $[\mathbf{v}, \Gamma(S)] \subset \Gamma(S)$ entails a formalism analogous to Kähler geometry linking the various cohomology groups via harmonic forms. The details of the proof of Theorem 2.6 are contained in [LuY2].

For 3 dimensional compact strongly pseudoconvex CR manifolds, global CR embedding in complex Euclidean space may fail and much work has been done recently on this phenomenon. See for example [BlE], [BuE], [L]. We only remark that as a consequence of the global invariants to be discussed in the next section, we find obstructions to CR embedding in $\mathbb{C}^3$, assuming that the 3 dimensional strongly pseudoconvex CR manifold is CR embeddable in some $\mathbb{C}^N$ to begin with. These obstructions provide us with numerous examples of such 3 dimensional CR manifolds not CR embeddable in $\mathbb{C}^3$.

## §3 Concerning invariants of compact strongly pseudoconvex CR manifolds

As a first step towards the difficult classification problem of compact strongly pseudoconvex CR manifolds, it would be useful to understand the following notion of equivalence which is weaker than CR equivalence.

**Definition 3.1.** *Assume that $X_1$, $X_2$ are compact strongly pseudoconvex CR manifolds of dimension $2n-1$, $n \geq 2$, which are CR embeddable in some $\mathbb{C}^{N_1}$, $\mathbb{C}^{N_2}$ respectively. $X_1$, $X_2$ are called algebraically equivalent if the corresponding varieties $V_1$, $V_2$, which are bounded by $X_1$, $X_2$ in $\mathbb{C}^{N_1}$, $\mathbb{C}^{N_2}$ according to Theorem 2.1, have isomorphic singularities $Y_1$, $Y_2$, i.e., $(V_1, Y_1) \cong (V_2, Y_2)$ as germs of varieties.*

Thus, for $n = 2$, we are restricting ourselves to embeddable compact strongly pseudoconvex CR manifolds. It is not difficult to show that CR equivalence implies algebraic equivalence.

In case a compact strongly pseudoconvex CR manifold $X$ of dimension $2n-1$ embeds in $\mathbb{C}^{n+1}$, $n \geq 2$, it is the boundary of a complex hypersurface $V$ with isolated singularities $p_1, \ldots, p_m$. In this case, an Artinian algebra can be associated to $X$ as follows.

**Definition 3.2.** *With the above notation, let $f_i$ be a defining function of the germ $(V, p_i)$, $1 \leq i \leq m$. Then the $\mathbb{C}$-algebra $A_i =$*

$\mathcal{O}_{n+1}\Big/\left(f_i, \frac{\partial f_i}{\partial z_0}, \dots, \frac{\partial f_i}{\partial z_n}\right)$ *is a commutative local Artinian algebra called the moduli algebra of* $(V, p_i)$. *The moduli algebra is independent of the choice of defining function. We associate to the CR manifold* $X$ *the Artinian algebra* $A(X) = \bigoplus_{i=1}^{m} A_i$.

By the work of Mather-Yau [MY] on isolated hypersurface singularities, it can be shown that the associated Artinian algebras are complete algebraic CR invariants in the following sense.

**Theorem 3.3.** *[LuY3] Two compact strongly pseudoconvex real codimension 3 CR manifolds* $X_1$, $X_2$ *are algebraically equivalent if and only if the associated Artinian algebras* $A(X_1)$, $A(X_2)$ *are isomorphic* $\mathbb{C}$ *algebras.*

We remark that there are Torelli type examples in which the Artinian algebras $A(X_t)$ associated to a family of compact strongly pseudoconvex real codimension 3 CR manifolds $X_t$ suffice to distinguish CR equivalence. For example, in the family $X_t = \{(x, y, z) \in \mathbb{C}^3 : x^6 + y^3 + z^2 + tx^4y = 0$ and $|x|^2 + |y|^2 + |z|^2 = \varepsilon^2\}$ where $\varepsilon > 0$ is a small fixed number and $t \in \mathbb{C}$ with $4t^2 + 27 \neq 0$, $X_{t_1}$, $X_{t_2}$ are CR equivalent if and only if $A(X_{t_1})$, $A(X_{t_2})$ are isomorphic $\mathbb{C}$ algebras.

For the rest of this section, we consider embeddable 3 dimensional compact strongly pseudoconvex CR manifolds. By taking resolutions of the singularities of the subvariety $V$ bounded by such a CR manifold $X$ in complex Euclidean space, numerical invariants under algebraic equivalence may be defined, as follows.

**Definition 3.4.** *Let* $\pi : M \to V$ *be a resolution of the singularities* $Y$ *of* $V$ *such that the exceptional set* $A = \pi^{-1}(Y)$ *has normal crossing, i.e., the irreducible components* $A_i$ *of* $A$ *are nonsingular, they intersect transversally and no three meet at a point. According to Artin [Ar], there exists a unique minimal positive divisor* $Z$, *called the fundamental cycle, with support on* $A$, *such that* $Z \cdot A_i \leq 0$ *for all* $i$. *For any positive divisor* $D = \sum d_i A_i$, *let* $\mathcal{O}_M(-D)$ *be the sheaf of germs of holomorphic functions on* $M$ *vanishing to order* $d_i$ *on* $A_i$, *let* $\mathcal{O}_D = \mathcal{O}_M/\mathcal{O}_M(-D)$ *and let* $\chi(\mathcal{O}_D) = \sum_{i=0}^{2}(-1)^i \dim H^i(M, \mathcal{O}_D)$. *It can be proved that* $p_f(X) \overset{\text{def}}{=} 1 - \chi(\mathcal{O}_Z)$, $p_a(X) \overset{\text{def}}{=} \sup(1 - \chi(\mathcal{O}_D))$ *where* $D$ *ranges over all positive divisors with support on* $A$ *and* $p_g(X) \overset{\text{def}}{=} \dim H^1(M, \mathcal{O})$ *are defined independent of the resolution* $\pi$ *and are invariants of* $X$ *under algebraic equivalence. The detailed proofs are contained in [LuYY]. We refer to*

*$p_f(X)$, $p_a(X)$ and $p_g(X)$ as the fundamental genus, arithmetic genus and geometric genus of $X$ respectively.*

The following facts are known

- $0 \le p_f(X) \le p_a(X) \le p_g(X)$
- $p_f(X) = 0 \Leftrightarrow p_a(X) = 0 \Leftrightarrow p_g(X) = 0$.

Further numerical invariants under algebraic equivalence are given by $m_Z(X) \stackrel{\text{def}}{=} Z \cdot Z$, $q(X) \stackrel{\text{def}}{=} \dim H^0(M - A, \Omega^1)/H^0(M, \Omega^1)$, $\chi(X) \stackrel{\text{def}}{=} K \cdot K + \chi_T(A)$ and $\omega(X) \stackrel{\text{def}}{=} K \cdot K + \dim H^1(M, \Omega^1)$, where $\Omega^1$ is the sheaf of germs of holomorphic 1-form on $M$, $\chi_T(A)$ is the topological Euler characteristic of $A$ and $K$ is the canonical divisor on $M$. These invariants are defined independent of the choice of the resolution $\pi$. Since $K$ is a divisor with rational coefficient, $\chi(X)$ and $\omega(X)$ are in general rational numbers.

Using the above invariants, one may attempt a rough algebraic classification of embeddable 3 dimensional compact strongly pseudoconvex CR manifolds. In particular,

**Definition 3.5.** *An embeddable 3 dimensional compact strongly pseudoconvex CR manifold $X$ is called a rational (resp. elliptic) CR manifold if $p_a(X) = 0$ (resp. $p_a(X) = 1$).*

If $X$ is a rational or an elliptic CR manifold embeddable in $\mathbb{C}^3$ and $M_0$ is the minimal good resolution of the subvariety $V$ bounded by $X$ in $\mathbb{C}^3$, then the weighted dual graph for the exceptional set of $M_0$ is completely classified. The same also holds for those $X$ embeddable in $\mathbb{C}^3$ and has $p_g(X) = 1$. With the weighted dual graphs classified, the topology of the embedding of the exceptional set in $M_0$ is well understood.

As an application, one obtains obstructions to embedding in $\mathbb{C}^3$ for the above three classes of CR manifolds when their weighted dual graphs fail to have the required forms. For example, a rational CR manifold whose weighted dual graph is not a direct sum of the graphs $A_k$, $D_k$, $E_6$, $E_7$, $E_8$ is not embeddable in $\mathbb{C}^3$.

Similarly in view of the following theorem, one obtains numerical obstructions to embedding in $\mathbb{C}^3$ for those CR manifolds failing the conditions in the theorem.

**Theorem 3.6.** *Let $X$ be a compact strongly pseudoconvex 3 dimensional CR manifold embeddable in $\mathbb{C}^3$. Then*

(1) *$\chi(X)$ and $\omega(X)$ are integers.*

(2) $10p_g(X) + \omega(X) \geq 0$.

(3) *If* $p_a(X) = 1$, *then* $\chi(X) \geq -3$.

(4) *If* $X$ *admits a transversal holomorphic* $S^1$*-action, then* $6p_g(X) + \chi(X) > 0$.

The proof of Theorem 3.6 is contained in [LuY1], [LuYY]. We remark that (4) depends on the Durfee conjecture which is solved by Xu and the second author in [XY].

It is interesting to note that there are compact strongly pseudoconvex 3 dimensional CR manifolds with arbitrarily large minimal embedding dimensions. For any positive integer $N$, take any 2 dimensional strongly pseudoconvex complex manifold with maximal compact analytic set $A$ which is a smooth rational curve having self intersection number $-N$. The corresponding weighted dual graph is hence $\overset{\bullet}{-N}$. On blowing down $A$, one gets a 2 dimensional rational singularity $(V, p)$. The minimal embedding dimension of $(V, p)$ is $-A \cdot A + 1 = N + 1$. Let $X$ be the intersection of $V$ with a small sphere centered at $p$. Then the minimal embedding dimension of $X$ is $N + 1$.

Work is in progress on determining the weighted dual graph associated to $X$ as above, in terms of the CR manifold $X$ intrinsically. This however is a difficult problem.

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