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On Scattering by Two Degenerate Convex Bodies

Mitsuru Ikawa

§1. Introduction

Let n be an odd integer ≥ 3, and let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^n$ such that

$$\Omega = \mathbb{R}^n - \overline{\mathcal{O}} \quad \text{is connected.} \tag{1.1}$$

We assume that

$$\Gamma = \partial\mathcal{O} \quad \text{is smooth.}$$

Denote by $\mathcal{S}(z)$ the scattering matrix for $\mathcal{O}$. The scattering matrix $\mathcal{S}(z)$ is an $\mathcal{L}(L^2(S^{n-1}))$-valued holomorphic function defined in $\{z \in \mathbb{C};\ \operatorname{Re} z < 0\}$, where we denote by $\mathcal{L}(E)$ the space of all the bounded operators from E into itself. As a fundamental property of the scattering matrix, it is shown in Lax-Phillips [7]:

Theorem 5.1 of Chapter V. *The scattering matrix $\mathcal{S}(z)$ is holomorphic on the real axis and meromorphic in the whole plane, having a pole at exactly those points z for which there is a nontrivial z-outgoing local solution of*

$$\begin{cases} (\triangle + z^2)\,u = 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \Gamma. \end{cases}$$

In the study of scattering by obstacles, the problem to know relationships between the geometry of obstacles and the distribution of poles of scattering matrices is one of the most interesting and important problems. It is conjectured that the more rays of geometric optics are trapped by $\mathcal{O}$ the more solutions of the wave equation are trapped by $\mathcal{O}$, and that the more solutions of the wave equation are trapped, the nearer to the real axis it appears the poles of the scattering matrix.

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Concerning this problem, Melrose [9] proved that, if $\mathcal{O}$ is nontrapping in the sense of geometric optics, for any $a > 0$ the logarithmic domain

$\{z;\ \mathrm{Im}\, z \le a \log(|z|+1)\}$ has at most a finite number of poles of $\mathcal{S}(z)$.

For trapping obstacles, Bardos-Guillot-Ralston [1] considered the following example:

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$$

where

$$\mathcal{O}_l,\quad l = 1, 2 \ \text{ are strictly convex and } \overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_1} = \phi.$$

They showed that, for any $\varepsilon > 0$, the logarithmic domain

$$\{z;\ \mathrm{Im}\, z \le \varepsilon \log(|z|+1)\} \quad \text{has an infinite number of poles of } \mathcal{S}(z).$$

Next Ikawa [3] considered the same example and showed the following result: Set $d = \mathrm{distance}(\mathcal{O}_1, \mathcal{O}_2)$, and let A_l, $l = 1, 2$, be the point on $\Gamma_l = \partial \mathcal{O}_l$ such that

$$\mathrm{distance}(\mathcal{O}_1, \mathcal{O}_2) = |A_1 - A_2|.$$

Then, there is a positive constant c_0 determined by d and the geometry of Γ_l near A_l $(l = 1, 2)$ such that, in the strip $\{z;\ 0 < \mathrm{Im}\, z < \frac{2}{3} c_0\}$ the poles of $\mathcal{S}(z)$ distribute asymptotically at the points $\frac{\pi}{d} j + \sqrt{-1} c_0$, $j = 0, \pm 1, \pm 2, \cdots$.

After that, Gérard [2] proved that, for any $a > 0$, the poles of $\mathcal{S}(z)$ in the strip $\{z;\ 0 < \mathrm{Im}\, z < a\}$ distribute asymptotically on the points

$$\frac{\pi}{d} j + \sqrt{-1} c_m, \quad j = 0, \pm 1, \pm 2, \cdots, \quad m = 0, 2, \cdots, m_0$$

where

$$0 < c_0 \le c_1 \le c_2 \le \cdots \le c_{m_0} < a.$$

The constants c_m, $m \ge 1$ are also determined by d and the geometry of Γ_l near A_l, $l = 1, 2$.

The formula which gives c_m indicates that, when all the principal curvatures of Γ_l at A_l, $l = 1, 2$, become small, the constants c_m become also small, and when all the principal curvatures vanish at A_l $(l = 1, 2)$, all the c_m determined by the formula are equal to 0.

This fact indicates us that, if all the principal curvatures vanish at A_l, $\mathcal{S}(z)$ may have a sequence of of poles converging to the real axis.

But the methods used in [3] and [2] are no more valid in the case where all the principal curvatures vanish. We considered in [4] an example of $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ in $\mathbb{R}^3$ such that the principal curvatures of Γ_l vanish only at A_l of finite order, and showed that there exist an infinite number of poles in a domain $\{z;\ \operatorname{Im} z \le |\operatorname{Re} z|^{-\gamma}\}$ for some positive constant γ. The proof of this result is based on the trace formula due to Bardos-Guillot-Ralston [1]. On the other hand, as to the position of poles near to the real axis, by taking account of the results of [3] and [2], it seems very likely that the poles of $\mathcal{S}(z)$ in the domain $\{z;\ \operatorname{Im} z \le |\operatorname{Re} z|^{-\gamma}\}$ exist only near the points $\frac{\pi}{d}j$, $j = \pm 1, \pm 2, \cdots$. But it seems very difficult to get more information on the distribution of poles by the means of the trace formula.

In this paper we shall consider an example of obstacle in $\mathbb{R}^2$ consisting of two convex bodies, whose curvature vanishes of finite order at A_l. Precisely, let $\mathcal{O}_1$ be a bounded open set in $\mathbb{R}^2$ with smooth boundary Γ_1 such that

(1) $\mathcal{O}_1 \subset \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 < 0\}$,
(2) $A_1 = (0,0) \in \Gamma_1$,
(3) Γ_1 is represented near A_1 as

$$x_2 = -{x_1}^{2m}$$

where m is a positive integer ≥ 2,
(4) the curvature of Γ_1 does not vanish on $\Gamma_1 - \{A_1\}$.

Let $\mathcal{O}_2$ be a bounded open set in $\mathbb{R}^2$ with smooth boundary Γ_2 such that

(1) $\mathcal{O}_2 \subset \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 > d\}$ where d is a positive constant,
(2) $A_2 = (0,d) \in \Gamma_2$,
(3) Γ_2 is represented near A_2 as

$$x_2 = d + {x_1}^{2m},$$

(4) the curvature of Γ_2 does not vanish on $\Gamma_2 - \{A_2\}$.

We set

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2, \quad \Gamma = \Gamma_1 \cup \Gamma_2$$

and

$$\Omega = \mathbb{R}^2 - \overline{\mathcal{O}}.$$

Consider the following boundary value problem with parameter $\mu \in \mathbb{C}$

$$(1.2) \qquad \begin{cases} (\triangle + \mu^2)u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{in } \Gamma \end{cases}$$

for $g(x) \in C^\infty(\Gamma)$. For $\operatorname{Im} \mu < 0$, (1.1) has a unique solution in $L^2(\Omega)$. Denote the solution $u(x)$ as

$$u(x) = (U(\mu)g)(x).$$

Then by using the regularity theorem for elliptic operators, $U(\mu)$ can be regarded as a continuous operator from $C^\infty(\Gamma)$ into $C^\infty(\overline{\Omega})$ for each μ such that $\operatorname{Im} \mu < 0$. Thus, $U(\mu)$ becomes an $\mathcal{L}(C^\infty(\Gamma), C^\infty(\overline{\Omega}))$-valued holomorphic function in $\{\mu; \operatorname{Im} \mu < 0\}$, where $\mathcal{L}(E, F)$ denotes the set of all the continuous operators from E into F.

We would like to consider the analytic continuation of $U(\mu)$ into $\{\mu; \operatorname{Im} \mu \geq 0\}$. The result that I will show is the following theorem:

Theorem 1. *Assume that*

$$(1.3) \qquad m \geq 4,$$

and set

$$\alpha = \frac{1}{m-1}.$$

Then, for any $\varepsilon_1,\ \varepsilon_2 > 0$, *there exists a positive constant* $C_{\varepsilon_1 . \varepsilon_2}$ *such that* $U(\mu)$ *can be continued analytically into*

(1.4)

$$\{\mu;\ \operatorname{Im} \mu \leq |\operatorname{Re} \mu|^{-(1+2\alpha)^{-1} - \varepsilon_1},\ |\operatorname{Re} \mu| \geq C_{\varepsilon_1, \varepsilon_2}\} \\ - \cup_{r=-\infty}^{\infty} \{\mu;\ \operatorname{Im} \mu \geq 0 \ \textit{and}\ |\frac{\pi}{d} r - \operatorname{Re} \mu| < \varepsilon_2\}.$$

Recall that the poles of $\mathcal{S}(z)$ coincide with those of $U(\mu)$. Therefore, even though Theorem 1 is of the analytic continuation of $U(\mu)$ for an obstacle in $\mathbb{R}^2$, it gives us a partial answer to the above question.

§2. Geometric optics near the periodic ray $a_1 a_2$

In order to consider the solution of the reduced wave equation (1.2) for high frequency, that is, for $|\operatorname{Re} \mu|$ large, the geometric optics in Ω plays an important role. Especially, it is essential to know the asymptotic behavior of rays trapped by $\mathcal{O}$, which are the ones approaching to the periodic rays when $t \to \infty$. In our case, the periodic ray in Ω is only the one going and returning between A_1 and A_2. Thus we consider the behavior of rays in the domain $\Omega(\delta)$ $(\delta > 0)$ surrounded by the following four curves

$$x_1 = \delta, \quad x_1 = -\delta, \quad x_2 = -{x_1}^{2m}, \quad x_2 = d + {x_1}^{2m}$$

and set

$$S_l(\delta) = \overline{\Omega(\delta)} \cap \Gamma_l, \quad l = 1, 2.$$

From now on, in this section we shall denote the point in $\mathbb{R}^2$ as $Q = (x, y),\ x, y \in \mathbb{R}$.

Let

$$Q = (x, -x^{2m}) \in S_1(\delta) \text{ and } \Xi = (\xi, \sqrt{1-\xi^2}) \in S^1,$$

and denote by $X(Q, \Xi)$ the ray starting from Q in the direction Ξ, that is,

$$X(Q, \Xi) = \{Q + s\Xi; s \geq 0\}.$$

Denote by Q' and Ξ' the first fitting point of $X(Q, \Xi)$ at Γ_2 and the direction of the reflected ray respectively. Setting $Q' = (x', d + {x'}^{2m})$, we have

$$\Xi' = \Xi - 2(\Xi, N(Q'))N(Q')$$

where $N(Q')$ denotes the unit outer (with respect to $\mathcal{O}_2$) normal of Γ_2, that is,

$$N(Q') = (1 + (2m{x'}^{2m-1})^2)^{-1/2}\,(2m{x'}^{2m-1}, -1).$$

Set $\Xi' = (\xi', -\sqrt{1-\xi'^2})$. Then we have a mapping

$$T : (x, \xi) \to (x', \xi').$$

It is obvious that, when the both x and ξ tend to zero, x' and ξ' also tend to zero. As an approximation of the mapping T we shall consider the following mapping $\tilde{T}$, which maps (x, ξ) to $\tilde{T}(x, \xi) = (x', \xi')$ given by

$$\text{(2.1)} \qquad \begin{cases} x' = x + \xi \\ \xi' = \xi + 4m{x'}^{2m-1} = \xi + 4m(x+\xi)^{2m-1}. \end{cases}$$

Let $f(s)$ be a smooth function defined for s near to 0, and let $\{m_j\}_{j=0}^{\infty}$ be an increasing sequence such that $m_j \to \infty$ as $j \to \infty$. We say that $f(s)$ has an asymptotic expansion for $s \to 0$

$$f(s) \sim a_0\, s^{m_0} + a_1\, s^{m_1} + \cdots + a_j\, s^{m_j} + \cdots$$

when, for any $M > 0$, there are j_0 and C_M such that

$$|f(s) - \sum_{j=0}^{j_0} a_j\, s^{m_j}| \leq C_M\, |s|^M.$$

Lemma 2.1. *Suppose that $m \geq 2$. Then, there is a one parameter family of a pair of functions $g(s)$ and $h(s)$ defined for small s having asymptitic expansions*

$$(2.2) \qquad \begin{cases} g(s) \sim a_0\, s^{\alpha} + a_1\, s^{\alpha+1} + \cdots + a_j\, s^{\alpha+j} + \cdots \\ h(s) \sim b_0\, s^{\alpha+1} + b_1\, s^{\alpha+2} + \cdots + b_j\, s^{\alpha+j+1} + \cdots \end{cases}$$

and satisfying

$$(2.3) \qquad \tilde{T}(g(s), h(s)) - (g(\frac{s}{s+1}), h(\frac{s}{s+1})) \sim 0.$$

In the asymptotic expansion (2.2), *a_1 is a free parameter, $a_0 = \pm(\alpha/2)^{\alpha}$ and $b_0 = \mp\alpha a_0$ are independent of a_1, and b_1 is given by*

$$b_1 = a_0\, \frac{\alpha(\alpha+1)}{2} - (\alpha+1)a_1,$$

and a_j and b_j $(j \geq 2)$ depend on a_1.

Proof. We look for formal series

$$(2.4) \qquad \begin{cases} g(s) \sim a_0 s^{\gamma} + a_1 s^{\gamma+1} + \cdots \\ h(s) \sim b_0 s^{\beta} + b_1 s^{\beta+1} + \cdots \end{cases}$$

as they satisfy (2.3), which can be written as

$$(2.5) \qquad g(\frac{s}{1+s}) - g(s) \sim h(s),$$

$$(2.6) \qquad h(\frac{s}{1+s}) - h(s) \sim 4m(g(s) + h(s))^{2m-1}.$$

We choose a_j, b_j $(j = 0, 1, \cdots)$ so that (2.5) and (2.6) hold. Note that for $p \in \mathbb{R}$ we have

$$\left(\frac{s}{s+1}\right)^p \sim s^p - ps^{p+1} + \frac{1}{2}p(p+1)s^{p+2} - \cdots.$$

Substitute (2.4) and the above expansion into (2.5) and (2.6). Equating the leading terms of the both sides of (2.5) we have

$$-a_0 \gamma s^{\gamma+1} = b_0 s^{\beta},$$

which implies that $\beta = \gamma + 1$ and $b_0 = -\gamma a_0$. Substituting the just obtained relations into (2.6) and equating the leading terms of the both sides of (2.6) we have

$$-\beta b_0 s^{\beta+1} = 4m \left(a_0 s^{\gamma}\right)^{2m-1}.$$

Therefore it follows that

$$\begin{aligned} \gamma(2m-1) &= \beta + 1 = \gamma + 2, \\ 4m{a_0}^{2m-1} &= -b_0\beta = a_0\gamma\beta = a_0\gamma(\gamma+1). \end{aligned}$$

Thus we have

$$\begin{aligned} \gamma &= (m-1)^{-1} = \alpha, \\ 4m{a_0}^{2m-2} &= \frac{1}{m-1}\frac{m}{m-1} = m\alpha^2. \end{aligned}$$

Now, substitute these γ, β, a_0 and b_0 and equate the second terms of the both sides of (2.5). The we have

$$\left\{ a_0 \frac{\alpha(\alpha+1)}{2} - a_1(\alpha+1) \right\} s^{\alpha+2} = b_1 s^{\alpha+2}.$$

Choose arbitrary a_1 and take b_1 as

$$b_1 = a_0 \frac{\alpha(\alpha+1)}{2} - a_1(\alpha+1).$$

Then the second term of the left hand side of (2.6) is

$$\left\{ b_0 \frac{\beta(\beta+1)}{2} - b_1(\beta+1) \right\} s^{\beta+2}.$$

On the other hand, the second term of the right hand side of (2.6) is

$$4m(2m-1){a_0}^{2m-1}(a_1+b_0)s^{(2m-2)\alpha+\alpha+1}.$$

Evidently it holds that $(2m-2)\alpha + \alpha + 1 = \beta + 2$, and we can check easily by a direct calculus that

$$b_0 \frac{\beta(\beta+1)}{2} - b_1(\beta+1) = 4m(2m-1){a_0}^{2m-1}(a_1+b_0).$$

For $j \geq 2$, the $(j+1)$-th term of the left hand side of (2.5) is

$$\{-(\alpha+j)a_j + \text{ linear combination of } a_0, a_1, \cdots, a_{j-1}\}\, s^{\alpha+j+1}.$$

Thus, b_j should satisfy

$$-(\alpha+j)a_j + \text{ linear combination of } a_0,\ a_1, \cdots, a_{j-1} = b_j.$$

Similarly, $(j+1)$-th term of the left hand side of (2.6) is

$$\{-(\beta+j)b_j + \text{ linear combination of } b_0,\ b_1, \cdots, b_{j-1}\}\, s^{\beta+j+1}.$$

The $(j+1)$-th term of the right hand side of (2.6) is

$$4m\left\{(2m-1){a_0}^{2m-2}a_j + \text{terms determined by } b_0,\ b_1, \cdots, b_{j-1} \text{ and } a_0,\ a_1, \cdots, a_{j-1},\ a_j\right\} s^{\beta+j+1}.$$

Now consider a linear equation in unknown $(a_j,\ b_j)$:

$$\begin{cases} (\alpha+j)a_j + b_j = F_j \\ (\alpha+1)(\alpha+2)a_j + (\beta+j)b_j = G_j. \end{cases}$$

Since $(\alpha+j)(\beta+j) - (\alpha+1)(\alpha+2) \neq 0$ for all $j \geq 2$, the above equation has a unique solution for any given $(F_j,\ G_j)$. Thus for $j \geq 2$, we can choose the coefficients a_j, b_j successively in such a way that the asymptotic expansions of the both sides of (2.3) are equal. Q.E.D.

Lemma 2.1 gives us an asymptotic behavior of broken rays in Ω. Choose j_0 and set

$$g^{(j_0)} = a_0\, s^{\alpha} + a_1\, s^{\alpha+1} + \cdots + a_{j_0}\, s^{\alpha+j_0},$$
$$h^{(j_0)} = b_0\, s^{\alpha+1} + b_1\, s^{\alpha+2} + \cdots + b_{j_0}\, s^{\alpha+1+j_0}$$

and

$$x_n^0 = g^{(j_0)}(n^{-1}),\quad \xi_n^0 = h^{(j_0)}(n^{-1}).$$

Since $\dfrac{s}{1+s} = (n+1)^{-1}$ for $s = n^{-1}$, we have

$$x_{n+1}^0 = g^{(j_0)}\Big(\frac{n^{-1}}{1+n^{-1}}\Big),\quad \xi_{n+1}^0 = h^{(j_0)}\Big(\frac{n^{-1}}{1+n^{-1}}\Big).$$

Then, for any M fixed, if we choose j_0 sufficiently large, we have the following estimate

$$|\tilde{T}(x_n^0, \xi_n^0) - (x_{n+1}^0, \xi_{n+1}^0)| \leq C_M\, n^{-M} \quad \text{for all } n.$$

This gives us an approximate behavior of broken ray in Ω which converges to the periodic ray a_1a_2.

Let us denote as

$$\tilde{T}(x_n^0+s,\xi_n^0+t)=(\tilde{x}_{n+1}^0+s',\tilde{\xi}_{n+1}^0+t'),$$

where we set $(\tilde{x}_{n+1}^0,\tilde{\xi}_{n+1}^0)=\tilde{T}(x_n^0,\,\xi_n^0)$. Define mapping $\tilde{T}_n$ by

$$\tilde{T}_n:(s,t)\to(s',t'),$$

which maps a neighborhood of $(0,0)\in\mathbb{R}^2$ in to a neighborhood of $(0,0)\in\mathbb{R}^2$. Then we have

$$\begin{aligned}A_n&=\left.\frac{\partial(s',t')}{\partial(s,t)}\right|_{s=t=0}\\&=\begin{pmatrix}1&1\\2m(2m-1)\,(x_{n+1}^0)^{2m-2}&1+2m(2m-1)\,(x_{n+1}^0)^{2m-2}\end{pmatrix}.\end{aligned}$$

Substituting the expansion of x_n^0, we have

$$A_n\sim\begin{pmatrix}1&1\\d_0n^{-2}+d_1n^{-3}+\cdots&1+d_0n^{-2}+d_1n^{-3}+\cdots\end{pmatrix},$$

where $d_0=(\alpha+1)(\alpha+2)$. Set

$$S_j^n(s,t)=\tilde{T}_n\circ\tilde{T}_{n-1}\circ\cdots\circ\tilde{T}_j(s,t)=(X_j^n(s,t),\Xi_j^n(s,t)),$$

and

$$D_j^n(s,t)=\frac{\partial S_j^n(s,t)}{\partial(s,t)}=\begin{pmatrix}g_{j,11}^n(s,t)&g_{j,12}^n(s,t)\\g_{j,21}^n(s,t)&g_{j,22}^n(s,t)\end{pmatrix}.$$

Evidently we have

$$D_1^n(0,0)=A_n\circ A_{n-1}\circ\cdots\circ A_1.$$

Lemma 2.2. *Suppose that $m\geq 4$. Then, we have an asymptotic expansion of $D_j^n(0,0)$ in $n^{-\alpha}$ of the form*

$$D_j^n(0,0)\,D_j\sim$$
$$\begin{pmatrix}n^{\alpha+2}+a_{11,1}n^{\alpha+1}+\cdots&n^{-\alpha-1}+a_{12,1}n^{-\alpha-2}+\cdots\\(\alpha+2)n^{\alpha+1}+a_{21,1}n^{\alpha}+\cdots&-(\alpha+1)n^{-\alpha-2}+a_{22,1}n^{-\alpha-3}+\cdots\end{pmatrix},$$

where D_j is a nonsingular 2×2-matrix.

Proof. In this proof, we write $D_j^n(0,0)$ as D^n for the simplicity. Suppose that D^n has an asymptotic expansion of the form

$$D^n \sim \left(g_{ij}(n^{-1})\right)_{i,j=1,2} = G(n^{-1})$$

where $g_{ij}(s)$ are functions with asymptotic expansion for $s \to 0$

$$g_{ij}(s) \sim a_{ij,0}s^{\gamma_{ij}} + a_{ij,1}s^{\gamma_{ij}+\alpha} + a_{ij,2}s^{\gamma_{ij}+2\alpha} + \cdots.$$

Since

$$\begin{aligned} D^{n+1} &= A_{n+1}D^n \\ &= \left\{E + \begin{pmatrix} 0 & 1 \\ d_0n^{-2} + d_1n^{-3} + \cdots & d_0n^{-2} + d_1n^{-3} + \cdots \end{pmatrix}\right\} D^n, \end{aligned}$$

we have

$$D^{n+1} - D^n = \begin{pmatrix} 0 & 1 \\ d_0n^{-2} + d_1n^{-3} + \cdots & d_0n^{-2} + d_1n^{-3} + \cdots \end{pmatrix} D^n.$$

Thus, it suffices to look for 2×2-valued function $G(s)$ satisfying

$$G(\frac{s}{1+s}) - G(s) = \begin{pmatrix} 0 & 1 \\ d_0s^2 + d_1s^3 + \cdots & d_0s^2 + d_1s^3 + \cdots \end{pmatrix} G(s).$$

By the same argument as of Lemma 2.1 we get an aymptotic expansion of $G(s)$ for $s \to 0$, and $D^n = G(n^{-1})$ satisfies the required properties. Here we use essentially the assumption $m \geq 4$ for the purpose of the possibility of successive determination of the coefficients of $G(s)$.

Lemma 2.3. *For any multi-index γ we have*

$$\left|\partial_{s,t}^{\gamma} X_1^n(s,t)\big|_{s=t=0}\right| \leq C^{|\gamma|}\, n^{-\alpha}(n^{2+2\alpha})^{|\gamma|},$$
$$\left|\partial_{s,t}^{\gamma} \Xi_1^n(s,t)\big|_{s=t=0}\right| \leq C^{|\gamma|}\, n^{-\alpha-1}(n^{2+2\alpha})^{|\gamma|},$$

where $C > 0$ is a constant independent of γ.

Take other functions

$$\begin{aligned} \tilde{g}(s) &\sim \tilde{a}_0s^{\alpha} + \tilde{a}_1s^{\alpha+1} + \cdots, \\ \tilde{h}(s) &\sim \tilde{b}_0s^{\alpha} + \tilde{b}_1s^{\alpha+1} + \cdots \end{aligned}$$

with the properties of Lemma 2.1 of the type (2.3), that is,

$$\tilde{T}(\tilde{g}(s), \tilde{h}(s)) - (\tilde{g}(\frac{s}{1+s}), \tilde{h}(\frac{s}{1+s})) \sim 0.$$

Set

$$y_n^0 = \tilde{a}_0 n^{-\alpha} + \tilde{a}_1 n^{-(\alpha+1)} + \cdots$$
$$\eta_n^0 = \tilde{b}_0 n^{-(\alpha+1)} + \tilde{b}_1 n^{-(\alpha+2)} + \cdots$$

and define $\tilde{S}_n$ by

$$\tilde{S}_n(s,t) = (s', t'),$$

where (s,t) and (s',t') are combined by the relation $\tilde{T}(y_n^0 + s, \eta_n^0 + t) = (\tilde{y}_{n+1}^0 + s', \tilde{\eta}_{n+1}^0 + t')$. We set similarly

$$\tilde{S}_j^n = \tilde{S}_n \circ \tilde{S}_{n-1} \circ \cdots \circ \tilde{S}_j.$$

Now by using Lemmas 2.2 and 2.3 we have

Proposition 2.4. *Let $j_{0,l}$ $(l = 1, 2)$ be fixed. Then there are functions $k_l(s)$, $(l = 1, 2)$ with asymptotic expansion for $s \to 0$*

$$k_l(s) \sim c_{l,0} s^{\alpha+2} + c_{l,1} s^{\alpha+3} + \cdots$$

satisfying

$$S_{j_{0,1}}^n(0, k_1(\frac{1}{n})) \sim \tilde{S}_{j_{0,2}}^n(0, k_2(\frac{1}{n})) \quad \textit{for} \;\; n \to \infty.$$

§3. Construction of asymptotic solutions

From now on we shall use again the notation $x = (x_1, x_2)$ to denote a point of $\mathbb{R}^2$. Let us construct an asymptotic solution of (1.2) for an oscillatory data. Since the curvature of the boundary Γ_l is positive except at A_l, the behavior of asymptotic solutions going out from $\Omega(\delta)$ is same as in the case that the bodies are strictly convex. Therefore it is essential to consider asymptotic solutions in $\Omega(\delta)$ for oscillatory data given on $S_1(\delta)$. Let $\omega \in S^1 = \{\omega \in \mathbb{R}^2; |\omega| = 1\}$, and let $f(x) \in C_0^\infty(S_1(\delta))$, and set

$$g(x, \mu) = e^{-i\mu x \cdot \omega} f(x). \tag{3.1}$$

We shall use a standard method for construction, but as remarked in Section 2 it is crucial to know the behavior of the phase functions when the number of reflections increases to the infinity.

With aid of Proposition 2.4 we have the following

Proposition 3.1. *Let ω be an element of S^1 near $(0,1)$, and set*

$$\varphi_1(x) = x \cdot \omega.$$

For any positive integer N, there is a sequence of real valued smooth functions defined in a neighborhood of $\Omega(\delta)$ with the following expansions in $n^{-\alpha}$:

$$\begin{aligned}\frac{\partial \varphi_n}{\partial x_2}(x) = b_0(x)n^{-1-\alpha} &+ b_1(x)n^{-1-2\alpha} \\ &+ \cdots + b_M(x)n^{-1-(M+1)\alpha}, \\ \varphi_{2n}(x) = c_0(x) + 2nd &+ c_1(x)n^{-1-2\alpha} \\ &+ \cdots + c_M(x)n^{-1-(M+1)\alpha},\end{aligned}$$

$$\begin{aligned}\varphi_{2n+1}(x) = \tilde{c}_0(x) + (2n+1)d + \tilde{c}_1(x)n^{-1-2\alpha} &+ \tilde{c}_2(x)n^{-1-3\alpha} \\ &+ \cdots + \tilde{c}_M(x)n^{-1-(M+1)\alpha},\end{aligned}$$

where M is a positive integer and $b_j(x)$, $c_j(x)$, $\tilde{c}_j(x)$, $j = 1, 2, \cdots, M$, are smooth functions.

Moreover, $\varphi_j(x)$, $j = 1, 2, \cdots$, satisfy the eikonal equation

$$|\nabla \varphi_j(x)| = 1 \quad \text{in } \Omega(\delta)$$

and the difference $\varphi_{j+1} - \varphi_j$ on the boundary satisfies

$$\begin{aligned}(\varphi_{2n} - \varphi_{2n-1})(x) = e_0(x) + e_{N-1}(x)n^{-1-N\alpha} &+ e_N(x)n^{-1-(N+1)\alpha} \\ + \cdots + e_M(x)n^{-1-(M+1)\alpha} &\quad \text{for all } x \in S_1(\delta),\end{aligned}$$

$$\begin{aligned}(\varphi_{2n+1} - \varphi_{2n})(x) = \tilde{e}_0(x) + \tilde{e}_{N-1}(x)n^{-1-N\alpha} &+ \tilde{e}_N(x)n^{-1-(N+1)\alpha} \\ + \cdots + \tilde{e}_M(x)n^{-1-(M+1)\alpha} &\quad \text{for all } x \in S_2(\delta),\end{aligned}$$

where $e_0(x)$ and $\tilde{e}_0(x)$ satisfy the following estimate

$$|e_0(x)|, \ |\tilde{e}_0(x)| \le C_N \, |x_1|^N.$$

Now we construct a sequence of asymptotic solutions by using the sequence $\{\varphi_j\}_{j=1}^{\infty}$ of phase functions in the above proposition. First let

$\mu = k + i\sigma$ with $\sigma < 0$ and set

$$u_j(x,\mu) = \exp(-i\mu\varphi_j(x))\, v_j(x,\mu),$$
$$v_j(x,\mu) = \sum_{p=0}^{P} v_{jp}(x)(i\mu)^{-p},$$

and we shall construct v_{jp} successively by the following procedure:

Set

$$T_j = 2\,\nabla\varphi_j\cdot\nabla + \triangle\varphi_j.$$

Let $v_{00}(x)$ be solution of

$$\begin{cases} T_0\, v_{00} = 0 & \text{in } \Omega(\delta), \\ v_{00}(x) = f(x) & \text{on } S_1(\delta) \end{cases}$$

and $v_{0p}(x),\ p = 1, 2, \cdots, P$ be the successive solutions of

$$\begin{cases} T_0\, v_{0p} = -\triangle\, v_{0,p-1} & \text{in } \Omega(\delta), \\ v_{0p}(x) = 0 & \text{on } S_1(\delta). \end{cases}$$

Let $j \geq 1$ and suppose that $v_{j-1,p}(x)$ are defined. Define v_{jp} as the solutions of

$$\begin{cases} T_j\, v_{jp} = \triangle\, v_{j,p-1} & \text{in } \Omega(\delta), \\ v_{jp}(x) = v_{j-1,p} & \text{on } S_{\epsilon(j)}(\delta) \end{cases}$$

where we take $v_{j,-1} \equiv 0$ and

$$\epsilon(j) = \begin{cases} 1 & \text{for } j \text{ even}, \\ 2 & \text{for } j \text{ odd}. \end{cases}$$

About the asymptotic behavior of v_{np} for $n \to \infty$, we have the following lemma which is a direct consequence of the properties of $\varphi_n(x)$ in Proposition 3.1.

Lemma 3.2. *For each p fixed, we get the following asymptotic expansion of $v_{np}(x)$ in $n^{-\alpha}$:*

$$v_{2n,p}(x) \sim w_{p0}(x)n^p + w_{p1}(x)n^{p-\alpha} + w_{p2}(x)n^{p-2\alpha} + \cdots + w_{pK}(x)n^{p-K\alpha},$$

and

$$v_{2n+1,p}(x) \sim \tilde{w}_{p0}(x)n^p + \tilde{w}_{p1}(x)n^{p-\alpha} + \tilde{w}_{p2}(x)n^{p-2\alpha} + \cdots + \tilde{w}_{pK}(x)n^{p-K\alpha},$$

where $w_{pj}(x)$ and $\tilde{w}_{pj}(x)$ are smooth.

Now define $u(x,\mu)$ for $\operatorname{Im}\mu = \sigma < 0$ by

$$u(x,\mu) = \sum_{n=0}^{\infty} (-1)^n\, u_n(x,\mu). \tag{3.2}$$

It is evident that $u(x,\mu)$ converges absolutely, and we see from the construction of u_j that the following relations hold:

$$(\triangle + \mu^2)\, u(x,\mu) = (i\mu)^{-P} \sum_{n=0}^{\infty} \exp(-i\mu\varphi_n(x))\, \triangle v_{nP}(x), \tag{3.3}$$

$$\begin{aligned}&[u(x,\mu) - \exp(-i\mu\varphi_0(x))\, f(x)]_{S_1(\delta)}\\ &= \sum_{n=1}^{\infty} \{\exp(-i\mu\varphi_{2n}(x)) - \exp(-i\mu\varphi_{2n-1}(x))\}\, v_{2n}(x,\mu)\end{aligned} \tag{3.4}$$

and

(3.5)

$$u(x,\mu)|_{S_2(\delta)} = \sum_{n=0}^{\infty} \{\exp(-i\mu\varphi_{2n+1}(x)) - \exp(-i\mu\varphi_{2n}(x))\}\, v_{2n+1}(x,\mu).$$

Let η and ε_0 be an arbtrary positive constant. With the aid of Lemma 3.2 we have from (3.3)

$$|(\triangle + \mu^2)\, u(x,\mu)| \le C_{N,\eta,\varepsilon_0}\, |\mu|^{-P} \quad \text{for all } \operatorname{Im}\mu \le -\varepsilon_0,\quad x \in \Omega(\delta). \tag{3.6}$$

Similarly we have from (3.4)

$$|u(x,\mu) - g(x,\mu)| \le C_{N,\eta,\varepsilon_0}\, |\mu|^{-\eta N} \quad \text{for all } x \in S_1(|\mu|^{-\eta}) \text{ and } \operatorname{Im}\mu \le -\varepsilon_0 \tag{3.7}$$

and from (3.5)

$$|u(x,\mu)| \le C_{N,\eta,\varepsilon_0}\, |\mu|^{-\eta N} \quad \text{for all } x \in S_2(|\mu|^{-\eta}) \text{ and } \operatorname{Im}\mu \le -\varepsilon_0. \tag{3.8}$$

Now, note that for any broken ray starting from a point in $\Omega(\delta)$ and for any $a > 0$ it holds the either of the following two cases:

(i) the broken ray fits $S_1(a)$ within $[a^{-2(m-1)}]$-times reflections.

(ii) the broken ray goes out from $\Omega(\delta)$ within $[a^{-2(m-1)}]$-times reflections.

Then, by using the techniques in Ikawa [4] and that of Vainberg [10] jointly, we can easily construct $\tilde{u}(x,\mu)$ by an explicit procedure from $u(x,\mu)$ satisfying the following estimates, which show that $u(x,\mu)$ an good approximate solution to (1.2) for an oscillatry data $g(x,\mu)$ defined (3.1):

For any $N > 0$ and $\varepsilon_0 > 0$ we have for all $\operatorname{Im}\mu \leq -\varepsilon_0$

(i) $\quad \tilde{u}(\cdot,\mu)$ is $C^\infty(\overline{\Omega})$-valued holomorphic function,

(ii) $\quad (\triangle + \mu^2)\tilde{u}(x,\mu) = 0 \quad \text{in } \Omega,$

(iii) $\quad |\tilde{u}(x,\mu) - g| \leq C_{N,\varepsilon_0} |\mu|^{-N} \quad \text{for all } x \in \Gamma_1,$

(iv) $\quad |\tilde{u}(x,\mu)| \leq C_{N,\varepsilon_0} |\mu|^{-N} \quad \text{for all } x \in \Gamma_2.$

When we want to extend the above results beyond the real axis, there is a difficulty that the convergence of $u_n(x,\mu)$ is not exponential with respect to $n \to \infty$. But the summation (3.2) is of a similar form to the zeta functions. Thus, we shall use the technique of analytic continuation of the zeta functions. We shall consider in the next section the analytic continuation and estimates of the zeta function so that we may use it for the analytic extension of $u(x,\mu)$ beyond the real axis.

§4. Analytic continuation of the zeta function and its generalization

In order to consider the analytic continuation of $u(x,\mu)$ defined by (3.2), we express $u(x,\mu)$ as a sum of zeta functions.

Even though the analytic continuation of the zeta function is well known (see for example Veech [11]), we shall give a proof because the one used here is modified a little and we need estimates of the dependency of the functions on parameters.

In this section, several notations will be used in different meanings from the ones in the previous sections, except α.

Let m be a positive integer and let z and s be complex numbers. For $|z| < 1$ we define the function $F(z,s:m)$ by

$$F(z,s:m) = \sum_{n \geq m} z^n n^{-s}. \tag{4.1}$$

Obviously, the right hand side of (4.1) converges absolutely for $|z| < 1$, which implies that the function $F(z,s:m)$ is holomorphic in $\{z; |z| < 1\}$ for any $s \in \mathbb{C}$.

We consider the analytic continuation of F. First assume $\operatorname{Re} s > 0$, and set

$$I(z, s : m) = \int_0^\infty \frac{z^m e^{-mx} x^{s-1}}{1 - ze^{-x}} \, dx.$$

We see that, for each $\operatorname{Re} s > 0$, $I(z, s : m)$ is holomorphic in $z \in D = \mathbb{C} - [1, \infty)$. As it is well known, $F(z, s : m)$ has the following integral representation:

$$F(z, s : m) = \frac{1}{\Gamma(s)} I(z, s : m) \qquad \text{for } |z| < 1. \tag{4.2}$$

On the other hand, the definition (4.1) gives us

$$z \frac{\partial F}{\partial z}(z, s : m) = F(z, s - 1 : m) \qquad \text{for all} \quad |z| < 1.$$

Let a be a positive integer. Then we have for $\operatorname{Re} s > 0$ and $|z| < 1$ the expression

$$F(z, s - a : m) = \frac{1}{\Gamma(s)} \left(z \frac{\partial}{\partial z} \right)^a I(z, s : m). \tag{4.3}$$

By means of the above integral representation we shall show the following lemma:

Lemma 4.1. *For any $s \in \mathbb{C}$ and m positive integer, $F(z, s : m)$ as a function in z variable can be continued holomorphically into the domain $D = \mathbb{C} - [1, \infty)$. Moreover, we have the following estimate:*

$$|F(z, s : m)| \le C_{K,a} \frac{\Gamma(\operatorname{Re} s + a)}{|\Gamma(s + a)|} m^{-\operatorname{Re} s} |z|^m (1 + |z|)^a \tag{4.4}$$
$$\textit{for all } \operatorname{Re} s > -a \textit{ and } z \in K,$$

where K is an arbitrary compact subset of D, a is an arbitrary positive integer and $C_{K,a}$ is a constant independent of m.

Proof. By using the fact that $I(z, s : m)$ is holomorphic in $z \in D$ for any $\operatorname{Re} s > 0$, the expression (4.3) proves Lemma 4.1 except the estimate (4.4). It is easy to show by the induction that

$$\left(z \frac{\partial}{\partial z} \right)^a \frac{z^m}{1 - ze^{-x}} = \frac{m^a z^m}{(1 - ze^{-x})^{a+1}} \{1 + c_{a,1}(m) ze^{-x} +$$
$$+ c_{a,2}(m)(ze^{-x})^2 + \cdots + c_{a,a}(m)(ze^{-x})^a\},$$

where the coefficients $c_{a,l}(m)$, $l = 1, 2, \dots, a$ are polynomials of m^{-1} of order less than a, and they satisfy

$$|c_{a,l}(m)| \le C_a \qquad \text{for all } m.$$

Thus, if we set

$$\max_{\substack{x \ge 0 \\ z \in K}} |1 - ze^{-x}| = c_K,$$

we have for all $\operatorname{Re} s > 0$

$$\begin{aligned}
&\left| \left(z \frac{\partial}{\partial z} \right)^a I(z, s : m) \right| \\
\le& m^a \, |z|^m \, (c_K)^{-(a+1)} \, C_a (1 + |z|)^a \int_0^\infty e^{-mx} |x^{s-1}| \, dx \\
\le& m^a \, |z|^m \, (c_K)^{-(a+1)} \, C_a (1 + |z|)^a \, m^{-\operatorname{Re} s} \, \Gamma(\operatorname{Re} s).
\end{aligned}$$

Substituting this estimate into (4.3) we get immediately for all $\operatorname{Re} s > 0$

$$|F(z, s - a : m)| \le (c_K)^{-(a+1)} \, C_a \, \frac{\Gamma(\operatorname{Re} s)}{|\Gamma(s)|} \, m^{a - \operatorname{Re} s} \, |z|^m \, (1 + |z|)^a.$$

Denoting $s - a$ in the above inequality by s anew, we get (4.4). Q.E.D.

In order to consider the analytic continuation of $u(x, \mu)$ beyond the real axis, we have to consider the analytic continuation of the following function originally defined for $\operatorname{Im} \mu < 0$:

$$\begin{aligned}
\text{(4.5)} \qquad R_\beta(\mu : q) = \sum_{n \ge |k|^\beta} \exp\big(&- i\mu(n + c_0 n^{-1-2\alpha} \\
&+ c_1 n^{-1-3\alpha} + \cdots + c_M n^{-1-(M+2)\alpha}) \big) \, n^q.
\end{aligned}$$

Let us set

$$D_{r,\beta,\varepsilon} = \{\mu = ik + \sigma; 2r\pi + \varepsilon \le |k| \le 2(r+1)\pi - \varepsilon, \ \sigma \le r^{-\beta}\}.$$

For $\sigma < 0$, as remarked in the above, the right hand side converges absolutely. Now consider the holomorphic extension of $R_\beta(\mu : q)$ into $\sigma > 0$.

Lemma 4.2. *Let $\beta > (1 + 2\alpha)^{-1}$ and let $\varepsilon > 0$. For any positive integer r, $R_\beta(\mu : q)$ can be prolonged analytically into $D_{r,\beta,\varepsilon}$. Moreover, we have the following estimates:*

$$\text{(4.6)} \qquad |R_\beta(\mu : q)| \le C_{\beta,\varepsilon} \, r^{q\beta} \qquad \textit{for all } \mu \in D_{r,\beta,\varepsilon}$$

and

$$\left|R_\beta(\mu:q) - F(e^{-i\mu}, -q:[r^\beta])\right| \le C_{\beta,\varepsilon}\, c_1\, r^{q\beta-\gamma} \quad \text{for all } \mu \in D_{r,\beta,\varepsilon},$$

where $\gamma = (1+\alpha)\beta - 1 > 0$.

Proof. First suppose that $c_j = 0$ for all $j \ge 1$. For each $n \ge 0$ we have

$$\exp\big(-i\mu(n + c_0 n^{-1-2\alpha})\big)\, n^q = z^n \sum_{l=0}^{\infty} \frac{(-i\mu)^l}{l!}\, c_0{}^l\, n^{-(1+2\alpha)l}\, n^q, \tag{4.8}$$

where we set $z = \exp(-i\mu)$. Suppose that $\mu \in D_{r,\beta,\varepsilon}$ $(r > 0)$ and set $m = [r^\beta]$. Note that

$$\sum_{n\ge m} z^n\, n^{-(1+2\alpha)l}\, n^q = F(z, (1+2\alpha)l - q : m).$$

Let $|z| < 1$ and take the summation in $n \ge m$ of the both sides of (4.8). Since the both summations converges absolutely we have a relation

$$R_\beta(\mu:q) = \sum_{l=0}^{\infty} \frac{(-i\mu)^l}{l!}\, c_0{}^l\, F(z, (1+2\alpha)l - q : m),$$

which implies

$$R_\beta(\mu:q) - F(e^{-i\mu}, -q:m) = \sum_{l=1}^{\infty} \frac{(-i\mu)^l}{l!}\, c_0{}^l\, F(e^{-i\mu}, (1+2\alpha)l - q : m). \tag{4.9}$$

We see easily that $\{z = \exp(-i\mu);\ \mu \in D_{r,\beta,\varepsilon}\}$ is contained in a compact subset K of $D = \mathbb{C} - [1,\infty)$ for all r. Then by Lemma 4.1, each term of the right hand side of (4.9) can be extended holomorphically into $D_{r,\beta,\varepsilon}$. Therefore, if we show that the right hand side of (4.9) converges absolutely in $D_{r,\beta,\varepsilon}$, it follows that $R_\beta(\mu:q)$ can be extended analytically into $D_{r,\beta,\varepsilon}$.

Thus, by applying the previous lemma we have for all $\mu \in D_{r,\beta,\varepsilon}$

$$\left|\frac{(-i\mu)^l}{l!}\, c_0{}^l\, F(z, (1+2\alpha)l - q, m)\right| \le C_{K,q} |z|^m (1+|z|)^q \frac{|k|^l}{l!}\, |c_0|^l\, m^{-(1+2\alpha)l+q}.$$

Here we applied Lemma 4.1 by taking $s = (1+2\alpha)l - q$, and used the fact that $\Gamma(\operatorname{Re} s + a) = |\Gamma(s+a)|$. Note that

$$|z^m| = |e^{-ikm+m\sigma}| = e^{m\sigma} \le Ce^{r^{-\beta}\cdot r^{\beta}} = C,$$
$$m^{-(1+2\alpha)l}|k|^l \le (|k|^{-(1+2\alpha)\beta+1})^l = C|k|^{-\gamma l}$$

where we set $\gamma = (1+2\alpha)\beta - 1 > 0$, and

$$|z| \le \exp(r^{-\beta}) \le C \quad \text{for all} \quad z = e^{-i\mu},\ \mu \in D_{r,\beta,\varepsilon}.$$

Then we have

$$|R_\beta(\mu) - F(e^{-i\mu}, -q : m)|$$
$$\le C_{K,q}|k|^{q\beta} \sum_{l=1}^{\infty} \frac{1}{l!}(c_0 k^{-\gamma})^l \le C_{K,q}\, c_0 |k|^{q\beta-\gamma}.$$

Thus the desired properties of $R_\beta(\mu : q)$ are proved for the special case.

Next consider the general case, that is, the case that c_j, $(j \ge 1)$ are not necessarily zero. We introduce some notations. Set

$$l = (l_0, l_1, \cdots, l_M) \in \{0, 1, \cdots\}^{M+1},$$
$$c = (c_0, c_1, \cdots, c_M) \quad \text{and} \quad A = (0, 1, \cdots, M),$$

and denote as

$$|l| = l_0 + l_1 + \cdots + l_M, \quad A\cdot l = l_1 + 2l_2 + \cdots + M\, l_M,$$
$$c^l = \prod_{j=0}^{M} c_j^{l_j}, \quad l! = \prod_{j=0}^{M} l_j!.$$

Now we have the following expansion

$$\exp\left(-\mu(n + c_0 n^{-1-2\alpha} + \cdots + c_M n^{-1-(2+M)\alpha})\right) n^q$$
$$= z^n \sum \frac{(-i\mu)^{|l|}}{l!} c^l\, n^{-1-(2|l|+A\cdot l)\alpha} n^q$$

Thus, by replacing the expansion (4.8) by the above one, we can achieve the same argument as the special case, and get Lemma 4.2. Q.E.D.

§5. Proof of Theorem 1

First we shall show that the function $u(x,\mu)$ defined by (3.2) can be extended analytically into the domain

$$D_{\beta,\varepsilon} = \cup_{|r|\geq C_{\beta,\varepsilon}} D_{r,\beta,\varepsilon}$$

where $C_{\beta,\varepsilon}$ is a positive integer depending on β and ε. Secondly, we shall show that $u(x,\mu)$ is a good approximation of the solution of (1.2) for all $\mu \in D_{\beta,\varepsilon}$

Let $\mu \in D_{r,\beta,\varepsilon}$ and set $m_r = |r|^{\beta}$. We express the function $u(x,\mu)$ defined by (3.2) as follows:

$$(5.1)\qquad \begin{aligned} u(x,\mu) =& \sum_{n=0}^{m_r} u_{2n}(x,\mu) + \sum_{n>m_r} u_{2n}(x,\mu) \\ & - \sum_{n=0}^{m_r} u_{2n+1}(x,\mu) - \sum_{n>m_r} u_{2n+1}(x,\mu) \\ =& u_{\mathrm{e}}^{(1)}(x,\mu) + u_{\mathrm{e}}^{(2)}(x,\mu) - u_{\mathrm{o}}^{(1)}(x,\mu) - u_{\mathrm{o}}^{(2)}(x,\mu). \end{aligned}$$

Recall Proposition 3.1 and Lemma 3.2. Then we have

$$\begin{aligned} u_{\mathrm{e}}^{(2)}(x,\mu) = \sum_{p=0}^{P} (-i\mu)^{-p} \sum_{n\geq m_r} & \exp(-i\mu(c_0(x) + 2nd + c_1(x)n^{-1-2\alpha} \\ & + \cdots + c_M(x)n^{-1-(M+1)\alpha}))\,\{w_{p0}(x)n^p + \cdots + w_{pK}n^{p-K\alpha}\}. \end{aligned}$$

Thus, for each $x \in \Omega(\delta)$ fixed, $u_{\mathrm{e}}^{(2)}(x,\mu)$ can be expressed by a summation of following terms:

$$(-i\mu)^{-p} R_\beta(\mu : p - j\alpha),\ p = 0,\cdots,P,\ j = 0,\cdots,K,$$

from which we see that $u_{\mathrm{e}}^{(2)}(x,\mu)$ can be extended analytically into $D_{r,\beta,\varepsilon}$ beyond the real axis. Moreover, applying the estimate in Lemma 4.2 we have

$$(5.2)\qquad |u_{\mathrm{e}}^{(2)}(x,\mu)| \leq C_{N,\beta,\varepsilon} \sum_{p=0}^{P} |\mu|^{-p}\,|r|^{\beta p} \leq C'_{N,\beta,\varepsilon}$$

for all $\mu \in D_{r,\beta,\varepsilon}$

(Recall that the constants P, K, M are determined by N through Proposition 3.1). Similarly we see that $u_{\mathrm{o}}^{(2)}(x,\mu)$ also can be extended analytically into $D_{r,\beta,\varepsilon}$ and

$$(5.3) \qquad |u_{\mathrm{o}}^{(2)}(x,\mu)| \le C_{N,\beta,\varepsilon} \sum_{p=0}^{P} |\mu|^{-p}\,|r|^{\beta p} \le C'_{N,\beta,\varepsilon} \qquad \text{for all } \mu \in D_{r,\beta,\varepsilon}$$

holds.

Consider $u_{\mathrm{e}}^{(1)}(x,\mu)$. Since it is a finite sum of entire functions, it is also an entire function. But it is important to get an estimate for $\mu \in D_{r,\beta,\varepsilon}$. For all $n \le m_r$ we have

$$\operatorname{Re}\varphi_{2n} \le 2n\sigma \le m_r\,\sigma \le s_0,$$

where s_0 is independent of r. Therefore we have for all $\mu \in D_{r,\beta,\varepsilon}$

$$|u_{2n}(x,\mu)| \le e^{s_0} \sum_{p=0}^{P} m_r^p\,|\mu|^{-p} \le C_\beta,$$

from which it follows that

$$(5.4) \qquad |u_{\mathrm{e}}^{(1)}(x,\mu)| \le C_\beta\, m_r.$$

Evidently the same estimate holds for $u_{\mathrm{o}}^{(1)}(x,\mu)$. Thus we have the following

Lemma 5.1. *The function $u(x,\mu)$ defined by (3.1) can be extended analytically into $D_{\beta,\varepsilon}$ and the following estimate holds:*

$$(5.5) \qquad |u(x,\mu)| \le C_{N,\beta,\varepsilon}\,|\mu|^\beta \quad \textit{for all } x \in \Omega(\delta),\ \mu \in D_{\beta,\varepsilon}.$$

Next consider $(\triangle + \mu^2)u(x,\mu)$. By applying the above argument to the expression (3.3), we get easily

$$(5.6) \qquad |(\triangle + \mu^2)u(x,\mu)| \le C_{N,\beta,\varepsilon}\,|\mu|^{-P}\,|r|^{\beta P} \qquad \text{for all } x \in \Omega(\delta), \quad \mu \in D_{r,\beta,\varepsilon}.$$

For the estimate of boundary value, we can use the same argument as above.

An in Section 3, by using the techniques in [4] and that of [10] jointly, we can easily construct by an explicit procedure from $u(x,\mu)$ a function $\tilde{u}(x,\mu)$ with the following properties:

Proposition 5.2. *Let $N > 0$, $\varepsilon_0 > 0$ and $\beta > (1+2\alpha)^{-1}$ be fixed. For the oscillatory data $g(x,\mu)$ of the form (3.1) we can construct a function $\tilde{u}(x,\mu)$, which is $C^\infty(\overline{\Omega})$-valued holomorphic function in $D_{\beta,\varepsilon}$, satisfying for all $\mu \in D_{\beta,\varepsilon}$*

(i) $(\triangle + \mu^2)\tilde{u}(x,\mu) = 0$ *in* Ω,

(ii) $|\tilde{u}(x,\mu) - g| \leq C_{N,\beta,\varepsilon_0} |\mu|^{-N}$ *for all* $x \in \Gamma_1$,

(iii) $|\tilde{u}(x,\mu)| \leq C_{N,\beta,\varepsilon_0} |\mu|^{-N}$ *for all* $x \in \Gamma_2$.

Theorem 1 in Introduction can be derived from the above proposition by a standard arugument.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan