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Super Lie Groups

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In recent years the theory of super Lie groups has been studied by many authors in different formulations. See [1] for general references. We have developed the theory of super manifolds in previous notes [2] and [3]. With the same formulation used in the latter, here we shall consider super Lie groups and prove some fundamental existence theorems.

§1. Preliminary

In this note we shall basically follow the arguments and notations in [3]. However, we shall make some change in notations so that our arguments will be more coherent with the theory of ordinary Lie groups.

Let M be a super manifold and $\mathcal{O}_z$ the set of all germs of super smooth functions at a point $z \in M$. A super tangent vector at $z \in M$ was defined in [3]. But in this note we define a super tangent vector as follows.

A mapping v of $\mathcal{O}_z$ into Λ whose image of $f \in \mathcal{O}_z$ is written by $v \cdot f \in \Lambda$ is called a *super tangent vector* at $z \in M$ if v satisfies the following conditions: for $f, g \in \mathcal{O}_z$ and $a \in \Lambda$,

$$\begin{aligned} &1) \quad v \cdot (f+g) = v \cdot f + v \cdot g, \\ &2) \quad v \cdot (fa) = (v \cdot f)a, \\ &3) \quad v \cdot (fg) = (v \cdot f)g(z) + (-1)^{fg}(v \cdot g)f(z), \end{aligned}$$

where f, g in $(-1)^{fg}$ denote their parities of f, g. Then the set of all super tangent vectors at $z \in M$ forms a super vector space called the *super tangent space* at $z \in M$, denoted by $T_z(M)$. This change is not at all essential. Actually, this $T_z(M)$ can be identified with the old $T_z(M)$ in [3] in a natural way. See [1] for the details of super linear algebra. When (z^i) is a local coordinate around $z \in M$, $\left\{\left(\frac{\vec{\partial}}{\partial z^i}\right)_z\right\}$ forms a base

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of the super vector space $T_z(M)$. Then *super vector fields* on a domain in M are defined as usual. The *bracket* $[X,Y]$ of super vector fields X,Y is defined as follows:

$$[X,Y]\cdot f = X\cdot(Y\cdot f) - (-1)^{XY}Y\cdot(X\cdot f),$$

for a super smooth function f on a domain M. On the underlying non-super manifold of a super manifold M, we have defined a non-super tangent space $\mathcal{T}_z(M)$ at a point $z\in M$, which can be identified with the even part $T_z(M)_{[0]}$ of the super tangent space $T_z(M)$ as [3]. Thus each even super vector field X on a super manifold M can be regarded as a vector field $\tilde{X}$ on the underlying non-super manifold of M. Since both operate on functions on the left hand side, we have the following:

$$\widetilde{[X,Y]} = [\tilde{X},\tilde{Y}],$$

for even super vector fields X,Y on M. This formula is different from the previous one in [3].

The almost super structures on a super manifold M have been defined in [3]. Even after our change, the almost super structures are defined as before, since they are defined in terms of multiplications of even super numbers. That is, for an even super tangent vector $v\in T_z(M)_{[0]}$ and $K\in\Gamma_{[0]}$, *the almost super structure* J^K is defined by

$$J^K(\tilde{v}) = \widetilde{(\zeta^K v)}.$$

Then the almost super structure J^K is a linear endomorphism of the non-super tangent space $\mathcal{T}_z(M)$ of M.

After our change of the signature as above, all the previous theorems obtained in [3] hold without any changes.

§2. Super Lie groups and super Lie algebras

A *super Lie algebra* $\mathfrak{g}$ is a super vector space provided with a bracket operation $[\,,\,]$ which satisfies the following: for $u,v,w\in\mathfrak{g}$ and $a,b\in\Lambda$,

(1) $[au,vb] = a[u,v]b,$

(2) $[u+v,w] = [u,w]+[v,w],$

(3) $[u,v+w] = [u,v]+[u,w],$

(4) $[u,v]+(-1)^{uv}[v,u] = 0,$

(5) $[u,[v,w]]+(-1)^{u(v+w)}[v,[w,u]]+(-1)^{w(u+v)}[w,[u,v]] = 0.$

The formula (5) is called *the super Jacobi identity*, which is equivalent to the following (5′).

$$(5') \qquad [u,[v,w]] = [[u,v],w] + (-1)^{uv}[v,[u,w]].$$

The even part of $\mathfrak{g}$ will be denoted by $\mathfrak{g}_{[0]}$. When $\mathfrak{g}$ is $(m|n)$ dimensional as a super vector space, the space $\mathfrak{g}_{[0]}$ can be regarded as an $(m|n)$-dimensional super manifold $\mathbf{R}^{(m|n)}$ in a natural way.

A group G is called a *super Lie group* if G satisfies the following conditions.

(1) G is also a super manifold and
(2) the multiplication in G is super smooth. That is, the mapping $G \times G \ni (a,b) \longrightarrow ab^{-1} \in G$ is super smooth.

When G is a super Lie group, the N-th skeleton G_N $(N \geq 0)$ of the super Lie group G is an ordinary Lie group, whose Lie algebra will be denoted by $\mathfrak{g}_N$. For $N = 0$, the 0-th skeleton G_0 is usually called the *body* of a super Lie group G and is denoted by G_B whose Lie algebra will be denoted by $\mathfrak{g}_B$.

A left-invariant super vector field on a super Lie group is super smooth. We denote by $\mathfrak{g}$ the set of all *left-invariant* super vectors on G. When the super Lie group G is $(m|n)$-dimensional as a super manifold, the super tangent space $T_e(M)$ at the identity $e \in G$ is an $(m|n)$-dimensional super vector space and hence, so is $\mathfrak{g}$. Moreover $\mathfrak{g}$ is a super Lie algebra in a natural way and is called *the super Lie algebra* of the super Lie group G. The set of all even elements of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{[0]}$, which is an infinite-dimensional Lie algebra over $\mathbf{R}$, and is $\mathbf{R}$-linearly isomorphic with the tangent space $\mathcal{T}_e(G)$ of the underlying non-super manifold of G. Then the above argument implies that the Lie algebra of the projective limit of the family $\{\mathfrak{g}_N\}$ is canonically isomorphic with the Lie algebra $\mathfrak{g}_{[0]}$ as a Lie algebra over $\mathbf{R}$. Thus from this point, we shall identify these Lie algebras as follows: $\mathfrak{g}_{[0]} = \varprojlim \mathfrak{g}_N$.

For $N \geq 1$, we denote by p^N_{N-1} the projection of G_N onto G_{N-1} ($\mathfrak{g}_N$ onto $\mathfrak{g}_{N-1}$) and moreover by A_N ($\mathfrak{a}_N$) the kernel of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ ($p^N_{N-1} : \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1}$, respectively). Then the Lie algebra $\mathfrak{a}_N$ is abelian and A_N is homeomorphic to a Euclidean space. We obtain the following exact sequences of Lie algebras and Lie groups.

$$\{0\} \longrightarrow \mathfrak{a}_N \longrightarrow \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1} \longrightarrow \{0\},$$

$$\{e\} \longrightarrow A_N \longrightarrow G_N \longrightarrow G_{N-1} \longrightarrow \{e\},$$

where $N \geq 1$ and the 1st exact sequence is splitting.

§3. Fundamental Existence Theorems

We shall prove two fundamental theorems on super Lie groups.

Theorem 1. *Let G be a super Lie group with the super Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ a super Lie subalgebra of $\mathfrak{g}$. Then there exists uniquely a connected super Lie subgroup H of G whose super Lie algebra is $\mathfrak{h}$. Furthermore if the body H_B of H is a closed subgroup of G_B, then H itself is a closed super Lie subgroup of the super Lie group G.*

Proof. The super Lie subalgebra $\mathfrak{h}$ defines a super differential system D on G. That is, $D_a = (L_a)_{*e}(\mathfrak{h}) \subset T_a(G)$ for $a \in G$ where L_a denotes the left-translation by $a \in G$ and $\mathfrak{h}$ is regarded as a subspace of $T_e(G)$. Then D is an involutive super differential system on G. By Frobenius' Theorem on a super manifold obtained in [3], there exists uniquely a maximum connected integral super submanifold H of the super differential system D through the identity $e \in G$. In fact this super submanifold H is a unique connected super Lie subgroup with the super Lie algebra $\mathfrak{h}$. The second part of the theorem can be proved inductively.

The following lemma gives a sufficient condition for a projective limit of a projective family of Lie groups to be a super Lie group.

Lemma. *Let $\mathfrak{g}$ be a finite dimensional super Lie algebra and $\{G_N\}_{N\geq 0}$ a projective family of connected Lie groups such that the Lie algebra of G_N is the N-th skeleton $\mathfrak{g}_N$ of the even part $\mathfrak{g}_{[0]}$ of $\mathfrak{g}$. If each kernel A_N of the projection p^N_{N-1} of G_N onto G_{N-1} is homeomorphic to a Euclidean space, then the projective limit G of the family $\{G_N\}_{N\geq 0}$ is a super Lie group with the super Lie algebra $\mathfrak{g}$.*

Proof. Let V_B an open set in $\mathfrak{g}_B = \mathfrak{g}_0$ which is diffeomorphic with $U_B = \exp(V_B) \subset G_B = G_0$ through the exponential mapping exp of $\mathfrak{g}_B$ into G_B. Let $U = (p_B)^{-1}(U_B)$ be a domain of G, whose N-th skeleton is denoted by U_N ($N \geq 0$). Similarly we define $V = (p_B)^{-1}(V_B) \subset \mathfrak{g}_{[0]}$ and $V_N = p_N(V) \subset \mathfrak{g}_N$. Then by induction on $N \geq 0$, we can prove that $\exp : \mathfrak{g}_N \longrightarrow G_N$ is a diffeomorphism of V_N onto U_N for each $N \geq 0$ since the kernel A_N of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ is abelian and homeomorphic to a Euclidean space. Thus $\exp : \mathfrak{g}_{[0]} \longrightarrow G$ defines a diffeomorphism of $V \subset \mathfrak{g}$ onto $U \subset G$. Through this diffeomorphism, we introduce a super manifold structure on $U \subset G$ regarding the domain $V \subset \mathfrak{g}_{[0]}$ a super manifold. We shall prove that for any $a \in U$ the left-translation L_a is a super diffeomorphism around the identity $e \in G$. It is sufficient to show that $(L_a)_* \circ J^K = J^K \circ (L_a)_*$ on $\mathcal{T}_e(G) = \mathfrak{g}_{[0]}$

for each almost super structure J^K ($K \in \Gamma_{[0]}$). Let $a = \exp(X) \in U$ ($X \in V \subset \mathfrak{g}_{[0]} = T_e(G)$). Then we have the following.

$$(\exp)_{*X} = (L_a)_{*e} \circ (\frac{1 - e^{-ad(X)}}{ad(X)}) \quad \text{on} \quad \mathcal{T}_X(\mathfrak{g}_{[0]}) = \mathfrak{g}_{[0]}$$

where $\mathfrak{g}_{[0]}$ is regarded as a non-super regular manifold. Since we introduce a super structure on $U \subset G$ through the exponential mapping and $ad(X) \circ J^K = J^K \circ ad(X)$ on $\mathfrak{g}_{[0]}$, the above formula shows that $(L_a)_{*e} \circ J^K = J^K \circ (L_a)_{*e}$ on $\mathcal{T}_e(G)$. Thus it follows that $(L_a)_{*g} \circ J^K = J^K \circ (L_a)_{*g}$ on $\mathcal{T}_g(G)$ if $g \in U$ is sufficiently close to $e \in G$. And hence L_a is super smooth around the identity e. Now we take $\{(L_a \circ \exp)^{-1}, L_a(U)\}_{a \in G}$ as an atlas defining the super structure on G. The above shows that this is a well-defined super structure on G. Let $a = \exp(X) \in G$. Then $ad(a) = e^{ad(X)}$ on $\mathfrak{g}_{[0]}$ which is commutative with J^K ($K \in \Gamma_{[0]}$). This implies that the right-translation R_a is also super smooth. Therefore the mapping $G \times G \ni (a, b) \longmapsto ab \in G$ is super smooth. Let ψ be the mapping of G onto G which maps $a \in G$ to $a^{-1} \in G$. Then on $\mathcal{T}_e(G)$, we have $(\psi)_{*e} = -id$ and hence $\psi_{*e} \circ J^K = J^K \circ \psi_{*e}$. On the other hand we have $\psi_{*g} = (R_{g^{-1}})_{*e} \circ \psi_{*e} \circ (L_{g^{-1}})_{*g}$ on $\mathcal{T}_g(G)$. Therefore ψ is also super smooth and hence G is a super Lie group with the super Lie algebra $\mathfrak{g}$.

Theorem2. *Let $\mathfrak{g}$ be a finite dimensional super Lie algebra. Then there exists a super Lie group whose super Lie algebra is $\mathfrak{g}$.*

Proof. For $N \geq 0$, let G_N be the connected and simply connected Lie group whose Lie algebra is the N-th skeleton $\mathfrak{g}_N$ of $\mathfrak{g}_{[0]}$. Then the projection p^N_{N-1} of $\mathfrak{g}_N$ onto $\mathfrak{g}_{N-1}$ induces the projection p^N_{N-1} of G_N onto G_{N-1} for each $N \geq 1$. Since the kernel of $p^N_{N-1} : \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1}$ is an abelian Lie algebra, the kernel of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ is an abelian Lie group, denoted by A_N. Then we have the exact sequence of the Lie groups:

$$\{e\} \longrightarrow A_N \longrightarrow G_N \longrightarrow G_{N-1} \longrightarrow \{e\}.$$

Therefore we have the following long exact sequence of homotopy groups:

$$\begin{aligned} &\cdots \longrightarrow \pi_2(G_{N-1}) \longrightarrow \\ \longrightarrow \pi_1(A_N) \longrightarrow \pi_1(G_N) &\longrightarrow \pi_1(G_{N-1}) \longrightarrow \\ \longrightarrow \pi_0(A_N) \longrightarrow \pi_0(G_N) &\longrightarrow \pi_0(G_{N-1}) \longrightarrow \{0\}. \end{aligned}$$

By the assumption, both G_N and G_{N-1} are connected and simply connected and then we have $\pi_1(G_N) = \pi_1(G_{N-1}) = \{e\}$ and $\pi_0(G_N) = \{e\}$.

On the other hand, it is well known that $\pi_2(G)=\{e\}$ for any Lie group G. Thus the kernel A_N is abelian and homeomorphic to a Euclidean space. Therefore the family $\{G_N\}_{N\geq 0}$ satisfies the conditions in the above lemma, and hence the projective limit G of the family $\{G_N\}_{N\geq 0}$ is a super Lie group with the super Lie algebra $\mathfrak{g}$.

References

[1] B. DeWitt, Supermanifolds, Cambridge University Press, 1984.
[2] K. Yagi, Super differential calculus, Osaka J. Math., **25** (1988), 243–257.
[3] ———, Super manifolds, Osaka J. Math., **25** (1988), 900–932.

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