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# Boundary Conditions in Conformal Field Theory*

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**Abstract.**

We consider conformal field theories on manifolds with a boundary, and the constraints placed by modular invariance on their partition functions. In particular, the partition functions on an annulus with particular boundary conditions are given by the fusion rules. This leads to a simple derivation of the Verlinde formula. We note the remarkable fact that, for some integrable models, these partition functions have the same form away from criticality, with the modular parameter $q$ of the annulus replaced by a temperature-like variable, and give a partial explanation of this in the case of the Ising model.

## §1. Introduction

Most studies of conformal field theory have focussed on properties of correlation functions on manifolds without boundaries, considering first the sphere, and then later manifolds of higher genus. An important property of correlation functions in such geometries is that they decompose into a sum of terms which factorize into holomorphic and antiholomorphic parts:

$$G(\{z_i,\bar{z}_i\},\{\tau_j,\bar{\tau}_j\}) = \sum_{I,J} h_{IJ}\mathcal{F}_I(\{z_i\},\{\tau_j\})\overline{\mathcal{F}}_J(\{\bar{z}_i\},\{\bar{\tau}_j\}) \tag{1}$$

where the arguments of the correlation function are labelled by $(z_i, \bar{z}_i)$ and the moduli of the surface are labelled by $(\tau_j, \bar{\tau}_j)$. This factorization has its roots in the fact that the local conformal symmetry is described by two commuting Virasoro algebras. The conformal blocks $\mathcal{F}_I$ and $\overline{\mathcal{F}}_J$ are in principle calculable for a given central charge and a given set of scaling dimensions, and the fact that, when sewn together as in eq.(1), they yield physically sensible correlation functions, gives strong

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constraints on the $h_{IJ}$ and the other parameters specifying the theory. One such constraint comes from duality, or crossing symmetry of the four-point function. When points $z_1$ and $z_2$ are close, the operator product expansion allows us to express $\phi_i(z_1)\phi_j(z_2)$ as a sum of operators $\phi_k(z_1)$, thus allowing the four-point function to be calculated in terms of two-point functions. Such a construction is possible not just for primary operators, corresponding to highest weight states of the Virasoro (or extended Virasoro) algebra, but also for all the descendant operators corresponding to the other states in the representations. For a rational conformal field theory, a feature of these representations is the existence of null states, which decouple from the theory. The fact that these null states should not be generated in the operator product expansion leads to the so-called fusion rules: if $\phi_i$ and $\phi_j$ correspond to highest weight representations labelled by $i$ and $j$ respectively, then the number of times that the representation $k$ occurs in the fusion of $\phi_i$ and $\phi_j$ is denoted by $N_{ij}^k$. It should be emphasized that these non-negative integers are properties of the algebra and its representations, and do not depend on the particular use to which they being put in this context.

Another important constraint comes from the requirement of modular invariance of the partition function (zero-point function) on a torus. The torus may be thought of as being constructed by identifying opposite edges of the parallelogram with vertices at $(0, \omega_1, \omega_2, \omega_1 + \omega_2)$. The partition function depends only on the modular parameter $q \equiv e^{2\pi i\tau}$ where $\tau = w_2/w_1$. The generators of translations in $z$ and $\bar{z}$ on the torus are $L_0 - c/24$ and $\overline{L}_0 - c/24$, where $L_0$ and $\overline{L}_0$ are Virasoro generators. Thus

$$Z(q, \bar{q}) = \operatorname{Tr} q^{L_0 - c/24} \bar{q}^{\overline{L}_0 - c/24}. \tag{2}$$

Now the states over which the trace is being taken are organized into highest weight representations, and $L_0$ is diagonal in each representation. Thus eq.(2) may be written as

$$Z = \sum_{i,j} n_{ij} \chi_i(q) \chi_j(\bar{q}) \tag{3}$$

where $n_{ij}$ counts the number of times the representation $(i, j)$ occurs in the theory, and

$$\chi_i(q) = q^{h_i - c/24} \sum_n d_i(n) q^n \tag{4}$$

is the character of the representation $i$, in other words the generating function for the degeneracy $d_i(n)$ of the states at level $n$ in the representation. For a rational conformal field theory, the sum over $i$ and $j$ in eq.(3) is finite.

Although $Z$ is parametrized by $q$, different $q$'s connected by the modular transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ label the same torus, and therefore give the same $Z$. Since in eq.(3) $Z$ is expressed as a finite sesquilinear combination of the characters, it is not hard to show that the set of characters must transform according to a finite-dimensional representation of the modular group. Under $T$ the character $\chi_i$ is simply multiplied by a phase $e^{2\pi i h_i}$, but under $S$ the transformation law is non-trivial:

$$\chi_i(q) = \sum_j S_i^j \chi_j(\tilde{q}). \tag{5}$$

Once one knows the matrix $\mathbf{S}$, it becomes a finite problem to enumerate all the possible sets of integers $n_{ij}$ for which $Z$ is modular invariant. Since $\mathbf{S}$ may always be chosen unitary, one solution is always the diagonal invariant with $n_{ij} = \delta_{ij}$.

From the point of view of eq.(1), the two constraints discussed above are part of a larger set of constraints which come from the invariance under reparametrizations of the Riemann surface. In fact, since the points $\{z_i\}$ at which correlation functions are to be evaluated may viewed as punctures in the surface, there is in fact no real distinction between the $\{z_i\}$ and the $\{\tau_i\}$. However, it turns out that most of these constraints are redundant from the point of view of consistency of conformal field theory. In fact, the two constraints discussed above, together with the modular covariance of the one-point functions on the torus, are sufficient to guarantee the consistency on an arbitrary Riemann surface with an arbitrary number of insertions of operators [14], [2].

Actually, even the conditions of crossing symmetry of the four-point function in the pane, and of modular invariance of the partition function on the torus are not really independent, for, as was first suggested by E. Verlinde [1], there exists a remarkable relation between the coefficients $N_{jk}^i$ of the fusion rules, and the elements of the matrix $\mathbf{S}$:

$$\sum_i S_i^j N_{kl}^i = \frac{S_k^j S_l^j}{S_o^j}. \tag{6}$$

The derivation of this formula is usually made in the context of what may be called holomorphic conformal field theory. Both sides of the identity relate to properties of only the holomorphic (or the antiholomorphic) half of a conformal field theory. It is necessary to think about the holomorphic part of a conformal field as having a separate definition of its own, with its correlation functions being essentially the holomorphic parts of the correlation functions of physical operators. By their nature, these correlation functions are multi-valued and therefore do not represent physical correlation functions. Manipulations with them are therefore ill-defined in terms of the path integral which (at least heuristically) defines the physical theory. In order to make the above procedure well-defined, it has proved necessary [2] to adopt an axiomatic approach to the study of these objects, abstracting from known examples such properties which will lead to sensible physical correlation functions when the holomorphic and antiholomorphic parts are sewn together appropriately. While this procedure is in the end justified by its results, it is desirable to have a more physical way of understanding such results as eq.(6).

When conformal field theory is defined on a manifold with a boundary, there is only one algebra, rather than two. This is because, as will be explained later, the holomorphic and antiholomorphic fields $W^{(r)}$ and $\overline{W}^{(r)}$ which generate the left and right algebras are not independent of each other on the boundary. As a result, a single algebra is realized directly on the physical fields of the theory. The partition function of the theory defined on an annulus turns out to be expressible as a linear combination of characters of the algebra (rather than a sesquilinear combination as in the case of the torus), and the modular transformation $\tau \to -1/\tau$ has a simple physical interpretation. When boundaries are present, we have some freedom in choosing the boundary condition consistent with the extended algebra. As we shall discuss, there is an isomorphism between the set of possible boundary conditions and the space of conformal blocks. By considering annuli with different types of boundary conditions it is then possible to relate their partition functions to the fusion rules. In fact we shall show that* for boundary conditions of type $j$ and $k$ on either boundary of the annulus, the partition function is simply

$$Z = \sum_i N^i_{jk} \chi_i(q) \tag{7}$$

where $q$ is the modular parameter of the annulus. Combining this result with the modular properties of $Z$ then leads to the Verlinde formula eq.(6).

*In fact, eq.(2) is slightly modified for non-selfconjugate representations, as discussed in Sec. 3.

There are other reasons for studying conformal field theory on manifolds with boundaries. One is the obvious connection with open string theory. Another relates to the connection with statistical mechanics. Given a conformal field theory, it is useful to know an underlying lattice model whose continuum limit at criticality the theory represents. There are indications that for every conformal field theory obtained by the GKO coset construction [4], there exists some lattice model which is integrable not only at the critical point, but also on some one-parameter curve in the coupling constant space away from criticality [5]. Thus in general one would like to solve the inverse problem: starting from some conformal field theory, construct the solution of some integrable lattice model away from criticality, and then perhaps even calculate the Boltzmann weights. The first step in this program would be to identify the appropriate microscopic degrees of freedom. One may then hope that fixing the boundary degrees of freedom into one of these microscopic states will lead, in the continuum limit, to a conformally invariant boundary condition of the type we consider in this paper. This has been already understood for simple models like the Ising model and the 3-state Potts model [6], and, more recently, Saleur and Bauer [7], and Saleur [8], have obtained results for the A-D-E series of models with $c < 1$, and for the SU($N$) vertex models. In such cases, we can show that the result eq.(7) is consistent with their findings. In general, then, one may hope to build appropriate models by choosing microscopic states labelled by highest weight representations of the algebra of the conformal field theory.

One of the remarkable results of the Kyoto group [5] has been that one-point functions, local height probabilities, in the solvable lattice theories away from criticality are given in terms of functions related to characters of affine Lie algebras. For the RSOS models of Andrews *et al.* [11] the dependence of these on the boundary conditions is simply proportional to a Virasoro character depending on a temperature-like variable $q$. Saleur and Bauer [7] further showed that the character which enters is the same one as occurs at criticality, in a finite system. Thus two remarkable sets of facts beg for an explanation. Although we shall not provide this in this paper, we shall indicate the lines along which a physical explanation may ultimately be given.

The layout of this paper is as follows. In Sec. 2 basic facts about conformal field theories on manifolds with boundaries, and establish our notation. In the next Section, we discuss conformal field theory on an annulus, analysing more carefully the idea of a boundary state, discussing modular properties, and finally deriving eq.(1). In Sec. 4, we compare eq.(7) with known results for lattice models. Then we relate this to properties of the corner transfer matrix at criticality. Finally we discuss how similar results arise away from criticality and show how the Virasoro characters arise in the case of the Ising model.

## §2. Boundary operators in conformal field theory

In this section we recall some basic ideas of conformal field theory on a manifold with a boundary [9], [6]. For this purpose, the prototype geometry is that of the upper half plane $\mathrm{Im} z > 0$. We suppose that some boundary condition, labelled by $(\alpha)$, is prescribed on the real axis. The action of the conformal field theory is locally invariant under infinitesimal conformal transformations $z \to z + a(z)$, where, in order to preserve the geometry, $a(z)$ must be real when $z$ lies on the real axis. Thus, if $a(z) = \sum_n a_n z^{n+1}$, the parameters $a_n$ should be real. As we shall see, this has the consequence that there is only one set of corresponding Virasoro generators $L_n$, rather than $(L_n, \overline{L}_n)$ as in the usual case.

In deriving the analog of the conformal Ward identity, the global variation of the action is written as usual as a contour integral

$$\frac{1}{2\pi i}\oint_C a(z)T(z)dz - \frac{1}{2\pi i}\oint_C a(\bar{z})\overline{T}(\bar{z})d\bar{z} \tag{8}$$

where $C$ is a sufficiently large contour, which surrounds all the points at which correlation functions are to be evaluated. It is convenient to take $C$ to be a large semicircle $C_+$ in the upper half plane, together with a portion of the real axis. In order that the conformal Ward identity continue to be valid as the arguments of the correlation functions approach the real axis, the contribution to eq.(8) from this portion of the contour should vanish. Thus we insist that $T = \overline{T}$ on the real axis. This is a necessary condition on the type of boundary condition $(\alpha)$ we consider. In Cartesian components, it is equivalent to $T_{xy} = 0$, expressing in field theory terms the fact that no momentum flows across the boundary.

The theory is now developed in analogy with that in the plane. The Virasoro generators are

$$L_n = \frac{1}{2\pi i}\int_{C_+} z^{n+1}T(z)dz - \frac{1}{2\pi i}\int_{C_+} \bar{z}^{n+1}\overline{T}(\bar{z})d\bar{z}. \tag{9}$$

From the boundary condition we see that $\overline{T}(\bar{z})$ may be taken as the analytic continuation of $T(z)$ into the lower half plane, so that the terms in eq.(9) may be combined as

$$L_n = \frac{1}{2\pi i}\oint_C z^{n+1}T(z)dz \tag{10}$$

where $C$ is a full circle. It is then clear, by analogy with the usual case, that the $L_n$ satisfy the Virasoro algebra.

In radial quantization, the fact that $T(z)$ is non-singular as $z \to 0$ implies that the vacuum state $|0\rangle$ satisfies $L_n|0\rangle = 0$ for $n \geq -1$. Highest weight states of irreducible representations of the Virasoro algebra are created by primary operators $\phi(0)$ acting on $|0\rangle$. Note that $\phi$ is a *boundary* operator. Its $L_0$-eigenvalue is a boundary, or surface, scaling dimension, which determines, for example, how its two-point function decays as a function of distance along the boundary. A boundary operator and its bulk counterpart have, in general, different scaling dimensions. For example, in the Ising model with free boundary conditions, the spin operator has a boundary scaling dimension of $\frac{1}{2}$, while in the bulk its dimensions are $(\frac{1}{16}, \frac{1}{16})$. An operator which is primary with respect to the usual bulk Virasoro algebras may have no primary counterpart in the set of boundary operators. For example, in the same model the energy operator is primary in the bulk, but appears as a descendant of the identity operator when classified according to the irreducible representations of the boundary Virasoro algebra.

Now consider a related geometry, that of an infinitely long strip, whose width we take to be $\pi$ in suitable units. This is related to the upper half plane by the conformal mapping $w = \ln z$. Writing $w = t + i\sigma$, the generator of $t$-translations, or Hamiltonian, $H$, is given in terms of the generator of scale transformations in the upper half plane, $L_0$, by [6], [10]

$$H = \int_0^\pi T_{tt} d\sigma = L_0 - \frac{c}{24} \tag{11}$$

where the second term comes from the Schwartzian derivative in the transformation law for $T$. Likewise, the other Virasoro generators may be identified with appropriate Fourier components of $T_{tt}$ and $T_{t\sigma}$. As in the upper half plane, they satisfy a Virasoro algebra. In the case we have described, the boundary conditions on either side of the strip will be of the same type ($\alpha$). However, the existence of the Virasoro algebra depends only on the fact that the boundary conditions are conformally invariant, that is $T_{t\sigma} = 0$ on $\sigma = 0, \pi$. Thus we are free to consider the more general case when the boundary conditions on either side of the strip are not necessarily the same. We shall label this pair of boundary conditions by $(\alpha\beta)$, and call the corresponding Hamiltonian $H_{\alpha\beta}$. The eigenstates of $H_{\alpha\beta}$ will fall into irreducible representations of the Virasoro algebra. Denote by $n^h_{\alpha\beta}$ the number of times that the irreducible representation with highest weight $h$ occurs in the spectrum of $H_{\alpha\beta}$.

Note that $n^0_{\alpha\alpha} = 1$.

If we transform this more general boundary condition $(\alpha\beta)$ back to the upper half plane, there is a discontinuity in the boundary condition at $z = 0$. In radial quantization, this corresponds to a 'vacuum' state which is no longer annihilated by $L_{-1}$. We may consider this state as equivalent to the action of a boundary operator $\phi_{\alpha\beta}(0)$ acting on the true vacuum $|0\rangle$. It is a highest weight state with weight $h_{\alpha\beta}$ equal to the lowest value of $h$ for which $n^h_{\alpha\beta} \neq 0$. If we act on this state with other local operators, we obtain the other representations with non-zero $n^h_{\alpha\beta}$.

Thus we arrive at one of the fundamental ideas underlying our arguments: juxtaposition of different (conformally invariant) boundary conditions is equivalent to the insertion of boundary operators. This is similar to the definition of disorder and twist operators in the bulk. It is not necessarily true that all boundary operators are of this form, but it will be seen later that all possible highest weights $h$ may be realized by an appropriate choice of $(\alpha\beta)$.

To use the Ising model once more as an example, there are three distinct conformally invariant boundary conditions: when the boundary spins are all in the $s = 1$ state, all in the $s = -1$ state, or are all free. These we denote by $(+)$, $(-)$ and $(f)$. The non-zero $n^h_{\alpha\beta} = n^h_{\beta\alpha}$ are then [6]

$$\begin{aligned} (++) \quad \text{or} \quad (--): &\quad h = 0 \\ (ff): &\quad h = 0, \tfrac{1}{2} \\ (+-): &\quad h = \tfrac{1}{2} \\ (+f) \quad \text{or} \quad (-f): &\quad h = \tfrac{1}{16}. \end{aligned} \tag{12}$$

In the case $(ff)$, the lowest representation has $h = 0$, but by inserting a spin operator at $z = 0$, we get the representation with $h = \frac{1}{2}$. With the other boundary conditions, insertion of a spin operator does not change the representation. This is because the boundary conditions already break the $Z_2$ spin symmetry of the bulk theory.

Note that since the two-point function $\langle\phi_{\alpha\beta}(x_1)\phi_{\beta\alpha}(x_2)\rangle$ is evidently non-zero, $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$ should be considered as conjugate operators. If the discontinuity in the boundary condition does not occur at the origin, but rather at $z = x_1$, then the corresponding state $\phi_{\alpha\beta}(x_1)|0\rangle = e^{x_1 L_{-1}}\phi_{\alpha\beta}|0\rangle$ is no longer an eigenstate of $L_0$, but nevertheless lies in the representation corresponding to the highest weight $h_{\alpha\beta}$. Similarly, other local operators acting at $z_1$ will give states lying in the other representations with $n^h_{\alpha\beta} \neq 0$, although these will not be highest weight states.

So far, we have discussed only the Virasoro algebra. In many interesting conformal field theories there also exists an extended algebra, generated by currents in addition to the stress tensor. The above considerations may be extended to this case, if we consider boundary conditions which are also invariant under this larger symmetry. In the upper half plane, this is equivalent to requiring that $W^{(r)} = \overline{W}^{(r)}$ on the real axis, where $\{W^{(r)}, \overline{W}^{(r)}\}$ represents the set of conserved currents. In this case, the eigenstates of $H_{\alpha\beta}$ will be organized into highest weight representations of the extended algebra. These representations will be labelled by an index $i$, whose specification includes the $L_0$-eigenvalue of the highest weight state. We then define the non-negative integer $n^i_{\alpha\beta}$ to be the number of times that the representation $i$ occurs in the spectrum of $H_{\alpha\beta}$. In the case of extended algebras, it is possible to have representations which are not self-conjugate. In this case, we denote the representation conjugate to $i$ by $\tilde{i}$. From the previous argument we see that $n^i_{\alpha\beta} = n^{\tilde{i}}_{\beta\alpha}$. It is, of course, also possible, and indeed interesting, to consider boundary conditions which break the extended symmetry, but which are nevertheless conformally invariant. We shall not consider such a situation in this paper.

## §3. Conformal field theory on an annulus

Suppose that the strip is now made periodic in the $t$-direction, so that different values of $t$ are identified modulo $2\pi\mathrm{Im}\tau$. Here $\tau$ is a complex number which, for the purposes of this paper, will always be purely imaginary. The manifold is now topologically an annulus. The modular parameter is $q \equiv e^{2\pi i\tau}$. The partition function in this geometry is then

$$Z_{\alpha\beta}(q) = \mathrm{Tr}\, e^{-(2\pi i\mathrm{Im}\tau)H_{\alpha\beta}} = \sum_i n^i_{\alpha\beta}\chi_i(q) \tag{13}$$

where $\chi_i(q) \equiv q^{-c/24}\mathrm{Tr}_i\, q^{L_0}$ is the Virasoro character of the representation $i$ defined in the introduction. From eq.(5) this may be written

$$Z_{\alpha\beta}(q) = \sum_{i,j} n^i_{\alpha\beta} S^j_i \chi_j(\tilde{q}) \tag{14}$$

where $\tilde{q} = e^{-2\pi i/\tau}$. On the other hand, we may calculate this partition function using the Hamiltonian acting in the $\sigma$-direction. This will be the

Hamiltonian $H^{(P)}$ for the cylinder, which is related by the exponential mapping $\zeta = e^{-i(t+i\sigma)}$ to the Virasoro generators in the whole $\zeta$-plane by

$$H^{(P)} = (\mathrm{Im}\tau)^{-1}\left(L_0^{(P)} + \overline{L}_0^{(P)} - \frac{c}{12}\right) \tag{15}$$

where we have used the superscript $(P)$ to stress that they are not the same as the generators of the boundary Virasoro algebra. The partition function is then

$$Z_{\alpha\beta}(q) = \langle\alpha|e^{-\pi H^{(P)}/\mathrm{Im}\tau}|\beta\rangle = \langle\alpha|(\tilde{q}^{1/2})^{L_0^{(P)}+\overline{L}_0^{(P)}-\frac{c}{12}}|\beta\rangle \tag{16}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are boundary states. Since we now have a well-defined Hilbert space, on which the $(L_n^{(P)}, \overline{L}_n^{(P)})$ act, we may describe these states more precisely. The boundary conditions are $e^{i\pi s/2}W^{(r)} = e^{-i\pi s/2}\overline{W}^{(r)}$ on $\sigma = 0, \pi$. Here $s$ is the spin of $W$. The phase factors arise from the factor of $i$ in the exponential mapping. The generators $(W_n^{(r)}, \overline{W}_n^{(r)})$ are Fourier components of these operators with respect to $t$. Then it is straightforward to show that the boundary condition is equivalent to

$$\left(W_n^{(r)} - (-1)^s\overline{W}_{-n}^{(r)}\right)|\alpha\rangle = 0 \tag{17}$$

together with a similar condition on $|\beta\rangle$. Note in particular that $(L_n - \overline{L}_{-n})|\alpha\rangle = 0$, reflecting the reparametrization invariance of the boundary state.

The solution of the conditions eq.(17) has been analysed by Ishibashi [12] and Onogi and Ishibashi [13]. Consider the representation $j$ of the antiholomorphic algebra $\{\overline{W}^{(r)}\}$. Its states are linear combinations of states of the form $\prod_I \overline{W}_{-n_I}^{(r_I)}\overline{|j;0\rangle}$, where the $\overline{W}_{-n_I}^{(r_I)}$ are lowering operators and $\overline{|j;0\rangle}$ is the highest weight state. Define the (antiunitary) operator $U$ acting on this space by

$$\begin{aligned} U\overline{|j;0\rangle} &= \overline{|j;0\rangle}^* \\ U\overline{W}_{-n_I}^{(r_I)} &= (-1)^{s_{r_I}}\,\overline{W}_{-n_I}^{(r_I)}U. \end{aligned} \tag{18}$$

Note that this definition is consistent with projecting out the null states to obtain the irreducible representation. Let denote an orthonormal basis of the representation $j$ of the holomorphic algebra by $|j;N\rangle$, and the corresponding basis of the isomorphic representation of the antiholomorphic algebra by $\overline{|j;N\rangle}$. Then the state

$$|j\rangle \equiv \sum_N |j;N\rangle \otimes U\overline{|j;N\rangle} \tag{19}$$

is a solution of eq.(17). To see that eq.(19) satisfies eq.(17), consider the vectors $\langle k;N_1| \otimes U\overline{\langle l;N_2|}$, which form a basis for the dual space. Then

$$\begin{aligned}
\langle k;N_1|\otimes U\overline{\langle l;N_2|} & \left(W_n^{(r)} - (-1)^s \overline{W}_{-n}^{(r)}\right)|j\rangle \\
&= \sum_N \langle k;N_1|W_n^{(r)}|j;N\rangle \overline{\langle l;N_2|j;N\rangle}^* \\
&\quad - (-1)^s \sum_N \langle k;N_1|j;N\rangle \overline{\langle l;N_2|}U^\dagger \overline{W}_{-n}^{(r)} U\overline{|j;N\rangle} \\
&= \delta_{kj}\delta_{lj}\left(\langle j;N_1|W_n^{(r)}|j;N_2\rangle - \langle j;N_2|W_{-n}^{(r)}|j;N_1\rangle^*\right) \\
&= 0
\end{aligned} \tag{20}$$

where we have used the orthonormality of the basis, and the hermiticity property $W_n^{(r)\dagger} = W_{-n}^{(r)}$.

So far, we have not specified which particular theory we are considering. For a given extended algebra, with a given value of the central charge $c$, there are in general several different theories corresponding to different modular invariant combinations of characters on the torus. Now we specialize to the 'diagonal' theories, that is, those whose partition function on the torus is the diagonal combination $Z_{\text{torus}} = \sum_i \chi_i(q)\chi_i(\bar{q})$. For these theories, each representation $j$ appears just once in the spectrum of $H^{(P)}$ in the sector in which $L_0 = \overline{L}_0$. Therefore the states $|j\rangle$ defined in eq.(19) form a complete set in the space of conformal blocks. Writing, therefore, $\langle\alpha| = \langle\alpha|j\rangle\langle j|$ and similarly for $|\beta\rangle$, and substituting into eq.(16), we see that

$$Z_{\alpha\beta}(q) = \sum_j \langle\alpha|j\rangle\langle j|\beta\rangle\chi_j(\tilde{q}). \tag{21}$$

In the case when the characters are linearly independent, which happens if no two representations have the same Virasoro character, we have immediately, comparing eq.(14) and eq.(21)

$$\sum_i S_i^j n_{\alpha\beta}^i = \langle\alpha|j\rangle\langle j|\beta\rangle. \tag{22}$$

This is the most important equation of this paper. Since, by reversing the arguments of this rest of this section, it may be deduced from the Verlinde formula, its validity must be more general that the case of independent Virasoro characters for which we have derived it. To understand this, however, it is necessary to define the matrix $\mathbf{S}$ not by eq.(5), but at a more fundamental level as representing a map from the space of conformal blocks (isomorphic to the space of allowed boundary conditions) to the space of conformal blocks (isomorphic to the space of highest weight operators corresponding to discontinuities in the boundary conditions.) While several reasonable arguments thus suggest themselves, we have not so far succeeded in constructing a convincing one without invoking further assumptions. Therefore, we shall continue with the argument with this proviso in mind.

Now we show that there exists a boundary state $|\tilde{0}\rangle$ such that $n^i_{\tilde{0}\tilde{0}} = \delta^i_0$, that is, the only representation which occurs in $H_{\tilde{0}\tilde{0}}$ is the identity representation, corresponding to the conformal block of the unit operator. From eq.(22), such a state must satisfy

$$|\langle\tilde{0}|j\rangle|^2 = S^j_0. \tag{23}$$

Taking the limit $\tilde{q} \to 0$ in eq.(5), we see that $\chi_i(q) \sim \tilde{q}^{-c/24} S^0_j$, and hence $S^0_j > 0$. Then, using the relation $\mathbf{S} = \mathbf{S}^2\mathbf{S}^\dagger$, it follows that $S^j_0 = (S^2)^k_0 (S^k_j)^* = S^0_j$, since only the identity operator corresponds to the character $\chi_0$. Hence the right hand side of eq.(23) is positive, which shows that if we define the state

$$|\tilde{0}\rangle = \sum_j (S^j_0)^{1/2} |j\rangle \tag{24}$$

it will have the required properties. Of course eq.(24) is not unique, for we could choose different relative phases of the terms in the sum.

In a similar way, we may define a boundary state $|\tilde{l}\rangle$, with the property that $n^i_{\tilde{0}\tilde{l}} = \delta^i_l$, that is, only the representation $l$ appears in the spectrum of $H_{\tilde{0}\tilde{l}}$. From eq.(22), such a state is

$$|\tilde{l}\rangle = \sum_j \frac{S^j_l}{(S^j_0)^{1/2}} |j\rangle. \tag{25}$$

Notice that $n^i_{\tilde{l}\tilde{0}} = \delta^i_{l^*}$. This corresponds to choosing a bra boundary state

$$\langle\tilde{l}| = \sum_j \frac{(S^j_l)^*}{S^{j\,1/2}_0} \langle j|. \tag{26}$$

In order to have the representation $l$ rather than $l^{\check{}}$ running in the $t$-direction, the correct bra state is

$$(27)\qquad \langle \tilde{l^{\check{}}}| = \sum_j \frac{S_l^j}{S_0^{j\,1/2}}\langle j|.$$

Now consider the boundary conditions $(\tilde{k^{\check{}}}\tilde{l})$. Applying eq.(22) once again,

$$(28)\qquad \sum_i S_i^j n^i_{\tilde{k^{\check{}}}\tilde{l}} = \langle\tilde{k^{\check{}}}|j\rangle\langle j|\tilde{l}\rangle = \frac{S_k^j S_l^j}{S_0^j}.$$

However, $n^i_{\tilde{k^{\check{}}}\tilde{l}}$ may be related to the fusion rules. To see this, consider the geometry consisting of a very long strip. For 'time' $t < t_1$, the boundary conditions on both sides of the strip are $(\tilde{0}\tilde{0})$, so that only states in the representation of the identity propagate. At time $t_1$, the boundary condition on the right hand side changes to $(\tilde{l})$, so that for $t_1 < t < t_2$, the boundary conditions are $(\tilde{0}\tilde{l})$. The only states which may propagate then belong to representation $l$. We may see this in another way: the change of boundary condition at $t = t_1$ corresponds to the insertion of a boundary operator $\phi_{\tilde{0}\tilde{l}}$. This transforms according to the representation $l$, by the arguments of Sec. 2. Thus, when it acts on a state in the representation 0, it will give only states in the representation $l$. In terms of the fusion rules, this is just the statement that $N^i_{0l} = \delta^i_l$. Now the boundary condition on the left hand side changes to $(\tilde{k^{\check{}}})$ at time $t_2$, corresponding to the insertion of an operator $\phi_{\tilde{k^{\check{}}}\tilde{0}} = \phi_{\tilde{0}\tilde{k}}$, transforming according to the representation $k$. The number of times the representation $i$ can occur when this operator acts on the space of states in the representation $k$ is precisely the same number which arises in the computation by Belavin, Polyakov and Zamolodchikov [3] of the four-point function in the plane using the operator product expansion, which they identify as the fusion rule coefficient $N^i_{kl}$. Thus $n^i_{\tilde{k^{\check{}}}\tilde{l}} = N^i_{kl}$. Substituting this into eq.(28) then gives the Verlinde formula

$$(29)\qquad \sum_i S_i^j N^i_{kl} = \frac{S_k^j S_l^j}{S_0^j}.$$

In deriving eq.(29) we have been careful to consider only states and operators transforming according to a single irreducible representation. It is interesting to analyse the above situation when we consider 'time'

$t$ as running in the opposite direction. For $-t < -t_2$ the representation content is given by $n^i_{\tilde{l}\tilde{k}^*} = N^i_{l^* k^*}$. At time $-t_2$ the operator $\phi_{\tilde{k}^*\tilde{0}}$, transforming according to the representation $k$, acts. Since $N^i_{l^* k^*}$ may be non-zero for more than one value of $i$, if we try to apply the same argument as above cancellations may occur, and we may conclude only that the representation content for $-t_2 < -t < -t_1$ satisfies

$$n^j_{\tilde{l}\tilde{0}} \leq N^j_{ik} N^i_{l^* k^*} = N^j_{il^*} N^i_{kk^*} \tag{30}$$

where we have used the associativity property of the fusion algebra to obtain the last expression. Now $N^0_{kk^*} = 1$ and $N^j_{0l^*} = \delta^j_{l^*}$, so that eq.(30) is consistent with the result $n^j_{\tilde{l}\tilde{0}} = \delta^j_{l^*}$, but we may draw no stronger conclusion by running the argument in this direction.

## §4. Examples

In comparing our results with calculations on specific lattice models, we do not know *a priori* which boundary conditions on the microscopic model will lead to boundary states satisfying eq.(17). In general, this can only be settled by a detailed analysis of the spectrum of $H_{\alpha\beta}$ to see whether its states fall into irreducible representations of the extended algebra. In certain simple cases, however, it is clear how to proceed. We shall discuss two of these below.

### 4.1. Ising model

The matrix $\mathbf{S}$ for the $c = \frac{1}{2}$ representations of the Virasoro algebra is

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{31}$$

where the rows and columns are labelled by the highest weights $(0, \frac{1}{2}, \frac{1}{16})$. Thus the boundary states are

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle + \frac{1}{\sqrt[4]{2}}|\sigma\rangle \\ |\tilde{\tfrac{1}{2}}\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle \\ |\tilde{\tfrac{1}{16}}\rangle &= |0\rangle - |\epsilon\rangle \end{aligned} \tag{32}$$

where we have labelled the states $|j\rangle$ by the corresponding physical scalar bulk operators. Note that $|\tilde{0}\rangle$ and $|\tilde{\tfrac{1}{2}}\rangle$ differ in the sign of the coefficient of the $Z_2$-odd state $|\sigma\rangle$. It is then natural to identify them with $|+\rangle$ and $|-\rangle$ respectively. In fact $|-\rangle$, since it only differs from $|+\rangle$ in the assignment of a phase factor, is an equally good candidate for $|\tilde{0}\rangle$. There are two other possibilities for $|\tilde{0}\rangle$, obtained from the above two by changing the sign of the coefficient of $|\epsilon\rangle$. These correspond to fixing the *dual* spins to $\pm 1$ respectively on the boundary. Under duality, the energy-like state $|\epsilon\rangle$ changes sign. Also, the free boundary state $|f\rangle$ goes into an equal superposition of the $|+\rangle$ and $|-\rangle$ states. Thus we may identify $|f\rangle$ with $|\tilde{\tfrac{1}{16}}\rangle$.

Note that, although the states in eq.(32) should be orthogonal, this is not obvious from their definition. However, it should be remembered that the states $|j\rangle$ are not normalizable. If the inner product is defined *e.g.* as

$$(+|-) \equiv \lim_{q\to 1} \frac{\langle +|q^{L_0+\overline{L}_0}|-\rangle}{\Big(\langle +|q^{L_0+\overline{L}_0}|+\rangle\langle -|q^{L_0+\overline{L}_0}|-\rangle\Big)^{1/2}}, \tag{33}$$

we obtain zero.*

We have therefore reproduced the cases $(++)$, $(+-)$, and $(+f)$ of eq.(12). The remaining case $(ff)$ then follows from the fusion rules

$$\left[\tfrac{1}{16}\right] \times \left[\tfrac{1}{16}\right] = [0] + \left[\tfrac{1}{2}\right]. \tag{34}$$

This verifies the results stated in eq.(12), which were first derived in ref. [6] by a slightly different argument, and confirmed by direct calculation by Saleur and Bauer [7].

### 4.2. $A_m$ RSOS models

These lattice models, originally formulated and solved by Andrews, Baxter and Forrester [11], correspond, in the continuum limit at criticality, to the diagonal modular invariants with $c = 1 - 6/m(m+1)$. The Virasoro characters $\chi_{r,s}$ of these theories are most easily labelled by the coordinates $(r, s)$ of the corresponding highest weight $h_{r,s}$ in the Kac table, with $\chi_{r,s} = \chi_{m-r,m+1-s}$, where $1 \le r \le m-1$ and $1 \le s \le m$. The site variables of the lattice model take values on the vertices of the Dynkin diagram of $A_m$. Thus it is natural also to label them by $s$ also, where $1 \le s \le m$. In fact, this correspondence is very natural, because,

*In general, states $|\tilde{k}\rangle$ and $|\tilde{l}\rangle$ are orthogonal in this sense if $N^0_{k^* l} = 0$.

as was shown by explicit calculation by Saleur and Bauer [7], on the annulus we have

$$Z_{1s} = \chi_{1,s}. \tag{35}$$

Since $r = s = 1$ corresponds to the representation with $h = 0$, this result means that we should identify $|1\rangle$, the state where all the boundary spins are in state $s = 1$, with $|\tilde{0}\rangle$. In the same way, $|s\rangle$, where all the boundary spins are in state $s$, should be identified with $|\widetilde{h_{1,s}}\rangle$. The partition functions when the spins are in states $s$ and $s'$ on either side of the strip are then, by the analysis of Sec. 3, given by the fusion rules [3]:

$$Z_{ss'} = \sum_{p=0}^{p_{\max}} \chi_{1,|s-s'|+2p+1} \tag{36}$$

where $p_{\max} = \min(s-1, s'-1, m-s, m-s')$. This may be shown to be in agreement with the results of Saleur and Bauer [7]. They also identified another type of boundary condition, corresponding to varying $r$ in the Kac table. In the RSOS models, neighboring sites must be in states corresponding to adjacent vertices on the Dynkin diagram. Thus it is appropriate to consider a boundary condition where the boundary spins are in state $r$, and their neighbors, one layer in from the boundary, in state $r+1$. In this case $1 \le r \le m-1$. Saleur and Bauer [7] find that

$$Z_{r1} = \chi_{r,1}. \tag{37}$$

Note that $r = 1$ corresponds to $s = 1$, since the spins on the layer next to the boundary must then be in state 2. From the fusion rule

$$[h_{r,1}] \times [h_{1,s}] = [h_{r,s}] \tag{38}$$

we then see that

$$Z_{rs} = \chi_{r,s} \tag{39}$$

consistent with a further result of Saleur and Bauer [7].

## §5. Relation to the corner transfer matrix

In the above we have pictured the annulus as a band formed by the identification of two opposite edges of a rectangle, of dimensions $\pi \times 2\pi \mathrm{Im}\tau$. Instead, we may choose a metric corresponding to identifying the annulus with the region $a \leq |\zeta| \leq R$ in the $\zeta$-plane with a Euclidean metric [15], [16]. This corresponds to the conformal mapping

$$\zeta = Re^{iw/\mathrm{Im}\tau}. \tag{40}$$

The ratio of inner to outer radii of the annulus is then $a/R = \tilde{q}^{1/2}$. Since the two geometries are related by a conformal mapping, the partition functions in the two cases are identical up to a universal factor involving the conformal anomaly [16]. This drops out of any expectation values, so we shall ignore it.

Let us consider for definiteness the $A_m$ RSOS models discussed in the last section. The ground states of these models are labelled by two adjacent vertices on the Dynkin diagram $r, r+1$. Let us fix the degrees of freedom on the outer boundary $|\zeta| = R$ into this state. Suppose that the interactions along the inner edge are infinitely strong so that the sites along this edge are all in the same state $s$. Then, as discussed in the last section, the partition function for this geometry is equal to $\chi_{rs}(q)$. If we now sum over all possible values for the state on the inner boundary, keeping the state on the outer boundary fixed, the probability of finding the sites on the inner boundary in the state $s$ is

$$P(s; r, r+1) = \frac{\chi_{rs}(q)}{\sum_s \chi_{rs}(q)}. \tag{41}$$

It is interesting to evaluate the behavior of this quantity as $R/a \to \infty$, that is $\tilde{q} = (a/R)^2 \to 0$. In that case, we find

$$P(s; r, r+1) \sim \frac{\sum_{r',s'} S_{rs}^{r's'} \tilde{q}^{h_{r's'}}}{\sum_s \sum_{r',s'} S_{rs}^{r's'} \chi_{r's'}(\tilde{q})}. \tag{42}$$

The denominator is dominated by the term with $r' = s' = 1$, which goes to a constant as $\tilde{q} \to 0$, with a coefficient which is independent of $s$. Also, for the minimal models the matrix $S_{rs}^{r's'}$ essentially factorizes in its $r$ and $s$ dependence into the outer product of two S-matrices for the SU(2) WZW model [17]. This implies that if we form the combinations of single site probabilities whose expectation value with the given boundary condition on the outer edge is

$$\langle \Phi_{\bar{s}} \rangle_{r,r+1} \equiv \sum_s S_{1s}^{1\bar{s}} P(s; r, r+1), \tag{43}$$

then its behavior as $R/a \to \infty$ is dominated by the term in the sum over $r'$ in eq.(42) which has the smallest value of $h_{r'\bar{s}}$, that is $r' = \bar{s}$. Thus the expectation value in eq.(23) behaves like $R^{-2h_{\bar{s}\bar{s}}}$, with corrections coming only from irrelevant powers of $R$. This finite size scaling behavior is that expected of a local scaling operator with scaling dimension $2h_{\bar{s}\bar{s}}$. In fact, it was shown by Huse [18] how the local relevant scaling operators in the RSOS models are formed from the local height probabilities. The coefficients are precisely the elements of $\mathbf{S}$ which appear in eq.(23). They are also the eigenvectors of the adjacency matrix for the $A_m$ Dynkin diagram.

Scaling theory also predicts how the expectation value in eq.(23) should behave in the *infinite* system $(R \to \infty)$ *away* from criticality. All that happens is that $R$ is replaced by the correlation length $\xi$. In discussing the continuum limit at criticality, we expect the finite-size scaling behavior to be universal. The same is true way from criticality as long as $\xi$ is large. However, for finite $\xi$, different microscopic models will lead to different behavior for the one-point functions in eq.(23). By adjusting the couplings to the irrelevant operators, we may obtain different correction terms. We may imagine fine-tuning these irrelevant couplings so that, even at finite $\xi$, we obtain expressions for the one-point functions similar to those in eq.(41), with, however, now $\tilde{q} = (a/\xi)^2$. In that case, these one-point functions will be expressible in terms of Virasoro characters.

This is what seems to be happening in the integrable case, as was first shown by the Kyoto group [5]. In fact, as shown by Saleur and Bauer [7], if we consider an infinite system away from criticality with a boundary condition at infinity of the type $(r, r+1)$, and with the site at the origin in the state $s$ (note that in this case the inner circle has to be shrunk down to be a single lattice spacing), then the partition function is proportional to $\chi_{rs}(q)$, where $q$ is related to $\tilde{q} = (a/\xi)^2$. This remarkable fact implies that the spectrum of the CTM away from criticality, with given boundary conditions, is identical to that of the Hamiltonian $H_{rs}$ for a strip at criticality. Since the latter is proportional to $L_0$, one suspects the existence of a spectrum-generating Virasoro algebra away from criticality. The physical significance, if any, of this symmetry (which is not conformal symmetry since we are away from the critical point) has so far been elusive. In the next section we give a partial explanation of how it arises in the Ising model.

## §6. Virasoro algebra in the non-critical Ising model

The zero-field Ising model shares with many other solvable models the property of commuting transfer matrices. It was shown by Baxter [19], that, together with some reasonable analyticity assumptions, this implies a simple structure for the eigenvalue spectrum of the corner transfer matrix on the lattice. For a simplified account of how this works for the Ising model, see ref. [21]. The result is that the computation of the partition function with a given ground state at infinity, and a given value of the spin at the origin, is reduced to the evaluation of a one-dimensional partition sum. This sum is essentially the partition function for the original model defined on a half-line, except that the interactions grow linearly with the distance from the origin.

For example, for the $A_m$ RSOS models, the site variables $s_j$ take the values $1, 2, \ldots, m$, with the constraint that $|s_j - s_{j+1}| = 1$. A given configuration of the model on the half-line $j = 0, 1, 2, \ldots$ thus may be thought of as a random walk on the Dynkin diagram of $A_m$. The 'world-line' of this walk corresponds to a 'path'. The partition function in this case is simply

$$Z = \mathrm{Tr}\, q^{\frac{1}{4}\sum_j j|s_{j-1}-s_{j+1}|} \tag{44}$$

where $q < 1$ is a parameter appearing in the Boltzmann weights. Low temperature corresponds to $q \to 0$, and the model approaches criticality as $q \to 1$. The remarkable thing noticed by the Kyoto group [5] is that if eq.(44) is evaluated with the boundary condition that, for sufficiently large $j$, neighboring sites all take the values $(r, r+1)$, and that $s_0 = s$, then the result is proportional to the Virasoro character $\chi_{rs}(q)$. This implies that the space of allowed paths forms a highest weight representation of the Virasoro algebra.

In order to understand this result, one would like to describe, and preferably get a physical picture of, the action of the Virasoro generators $L_n$ on the space of paths. Clearly, the exponent in eq.(44) should be identified with $L_0$, up to an additive constant. For the case $m = 3$ of the Ising model, the other generators may also be identified, as follows. It is useful to relabel the sites so that $j$ runs over half-integer values, and to define the quantity

$$n_j \equiv \frac{1}{4}|s_{j-\frac{1}{2}} - s_{j+\frac{1}{2}}|. \tag{45}$$

Note that $n_j = 1$ if there is a 'domain wall' at site $j$. Otherwise $n_j = 0$. Note that in the Ising case, domain walls only occur for half-integer (resp. integer) values of $j$ depending on whether $s_0$ is odd (resp. even), and that they only occur when $s_j = 2$. It is straightforward to show that there is a 1-1 correspondence between paths and the configurations of domain walls. The appearance of a variable taking the values 0 and 1 leads the physicist to think of fermions. Let us therefore introduce operators $b_j$ with canonical anticommutation relations $\{b_j^\dagger, b_{j'}\} = \frac{1}{2}\delta_{jj'}$, and let $n_j = b_j^\dagger b_j$, so that

$$L_0 = \sum_j j b_j^\dagger b_j + \text{const.} \tag{46}$$

where $j$ is summed over either non-negative half-integers, or non-negative integers, depending on whether $s_0$ is odd or even. The expression in eq.(46) is precisely the standard expression for $L_0$ for a free Majorana fermion. It is thus easy to write down the other $L_n$ which generate a Virasoro algebra with $c = \frac{1}{2}$.

Let us summarize this calculation. Starting from a fermionic field $b(z)$ one defines the mode expansion

$$b(z) = -i \sum_{j=-\infty}^{\infty} b_j z^{-j-\frac{1}{2}} \tag{47}$$

where one may choose the boundary conditions on $b(z)$ so that $j$ is either half-integrally or integrally moded. Hermiticity implies that $b_j^\dagger = b_{-j}$. The 'stress tensor' is

$$T(z) = -\frac{1}{2} :b\partial_z b:= \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (n + \frac{1}{2}) :b_m b_n: z^{-m-n-2} \tag{48}$$

from which we may identify $L_n$ as the coefficient of $z^{-n-2}$. For the half-integrally moded case, we have, carrying out the normal ordering,

$$\begin{aligned} L_0 &= \frac{1}{2}\left(b_{\frac{1}{2}}^\dagger b_{\frac{1}{2}} + 2b_{\frac{3}{2}}^\dagger b_{\frac{3}{2}} + 3b_{\frac{5}{2}}^\dagger b_{\frac{5}{2}} + \cdots + b_{\frac{3}{2}}^\dagger b_{\frac{3}{2}} + 2b_{\frac{5}{2}}^\dagger b_{\frac{5}{2}} + \cdots\right) \\ &= \frac{1}{2}b_{\frac{1}{2}}^\dagger b_{\frac{1}{2}} + \frac{3}{2}b_{\frac{3}{2}}^\dagger b_{\frac{3}{2}} + \cdots \end{aligned} \tag{49}$$

which is of the form eq.(46). Some of the other $L_n$ are

$$L_{-1} = \sum_{j>0} (2j-1) b_{j+1}^\dagger b_j \tag{50}$$

and

$$L_{-2} = \sum_{j>0} (j+1) b_{j+2}^\dagger b_j + \frac{1}{2} b_{\frac{3}{2}}^\dagger b_{\frac{1}{2}}^\dagger . \tag{51}$$

These two examples illustrate the general result quite clearly: the $L_n$ with $n < 0$ either move domain walls to the right by $(-n)$ units, or create a pair of domain walls. The $L_n$ with $n > 0$ do the opposite. Thus the space of paths, with given boundary conditions, forms a representation of the Virasoro algebra with $c = \frac{1}{2}$. The highest weight state is the one with the minimum number of domain walls dictated by the boundary conditions as far to the left as possible. It is therefore annihilated by the $L_n$ with $n > 0$. It is also not difficult to show that this gives an irreducible representation.

A similar analysis goes through for the case when $s_0 = 2$, so that $j$ is integrally moded. It should be pointed out that the stress tensor $T(z)$ introduced above is solely a mathematical construct used to obtain the other $L_n$. It would be very interesting to understand its possible physical significance. Also, we have shown how the Virasoro algebra acts on the one-dimensional model which is the extreme anisotropic limit of the original two-dimensional model. What rôle do the $L_n$ play in the two-dimensional case, when domain walls are allowed to wander? It should also be pointed out that the Virasoro algebra we have described above seems to be different from that proposed by Thacker and Itoyama [20]. In their case, the $L_n$ with $n \neq 0$ act on the whole line rather than the half-line. Also, their explanation of why only one Virasoro algebra occurs (rather than the two which occur on manifolds without boundaries at criticality) is different from ours. From our point of view, there is only one Virasoro algebra even at criticality because the definition of the corner transfer matrix requires the imposition of boundary conditions.

Although the above domain wall picture may be applied to the unrestricted SOS model (with $c = 1$), by simultaneously including half-integrally and integrally moded fermions, the extension to general RSOS models with $c < 1$ seems problematic. It is conceivable that some kind of lattice Feigin-Fuchs construction may lead to further progress. Recent work by Pasquier and Saleur [22] also suggests that quantum groups may hold the key to this interesting problem.

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