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Null Field Construction in Conformal and Superconformal Algebras

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Abstract

We present the vertex operator construction of null fields (singular vertex operators) based on the oscillator representation of conformal and superconformal algebras with central charge extention. As a consequence of the null field construction we determine the degenerate highest weights and prove the corresponding Kac determinant formulae for the conformal ($N=0$) and superconformal ($N=1, 2$) algebras with N supercharges. We also present an explicit construction of spin fields in the $N=1$ oscillator representations of the Neveu-Schwarz and Ramond algebras with extended central charges.

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§ I. Introduction

Recently conformally invariant quantum field theory has been a subject for intensive study in mathematics and theoretical physics[1]. The mathematical structure of conformal invariance is represented by infinite dimensional Lie algebras[2] which are often referred to as the Virasoro algebras[3] or the affine Kac-Moody algebras[4]. Superconformal invariance can be incorporated into the game by considering the supersymmetric generalizations[5] of conformal field theories.

A first concrete finding in physics which points to the importance of the investigation on two dimensional conformal symmetry with an infinite number of parameters came from the studies of dual resonance models[6] in late 60's. The gauge operators[7] of generalized Ward identities for the dual resonance models were discovered to form the infinite dimensional Lie algebras[6] (the Virasoro algebras) containing a subgroup of projective SL(2, C) symmetry[8]. In the string picture of the dual resonance models (the dual string theories)[9] the gauge symmetry is just the invariance under diffeomorphism of the string coordinates.

It should also be mentioned that current algebras[10] in two dimensions are nowadays recognized to be just the representations of affine Kac-Moody algebras. The Sugawara construction[11] of the energy-momentum tensor provides an interrelation between the Virasoro and the Kac-Moody algebras.

Another breakthrough for conformal field theory put forth by Polyakov[12] in 1970 was the application of local scale invariance implemented by conformal symmetry to the study of critical behavior of various lattice models in statistical mechanics. The basic idea is that the physics of two dimensional lattice systems is local because of nearest neighbor interactions and is scale invariant at the critical temperature, where the experimentally accessible parameters are the critical exponents determining the power law behavior of correlation functions, which could then be studied in a conformally invariant field theory formulation.

In mathematics two dimensional conformal field theory has been a fruitful laboratory for the study of the representation theory of the infinite dimensional Lie algebras[1,2]. As Frenkel and Kac[13] remarked, the discovery[6,7,8] of the Virasoro algebras and vertex operators[1,6] in the dual resonance models has put a great impetus to the remarkable development of the subject by mathematicians. Among others the Kac determinant formulae[14] were obtained which play a crucial role in discussing the unitarity[15] and the reducibility of the representations of the Virasoro algebra and the super-Virasoro algebras (i.e. the Neveu-Schwarz[16] and Ramond algebras[17]). The Frenkel-Kac construction[13] based on the vertex operator representations provides the basic representations of affine Kac-Moody algebras. It will also be an inevitable clue to dealing with the symmetry properties[18] of the compactified space of superstring theories[19].

A paper by Belavin, Polyakov and Zamolodchikov (BPZ)[20] triggered a feedback of the representation theory of the infinite dimensional Lie algebras into theoretical physics enriching the apparently diverse areas ranging from critical pheonomena in statistical mechanics to superstring theories of all the interactions in particle physics. In particular, BPZ presented a new approach to two dimensional conformal field theory. They showed that there is a special class of such theories, so-called degenerate conformal theories, in which all the correlation functions satisfy linear differential equations. Particularly there are sepcial conformal theories where there are only a finite number of conformal fields. BPZ suggested that there is a close connection between these degenerate conformal field theories and lattice models in statistical mechanics. The representation theory of the Virasoro algebras provides a list of possible rational values for the critical exponents[15] of the statistical models.

Bershadsky, Knizhnik and Teitelman (BKT)[21], and Friendan, Qiu and Shenker (FQS)[22] studied superconformal field theories in two dimensions. In superconformal theories superfields correspond to irreducible representations of the Neveu-Schwarz algebra. The superconformal field theories also contain spin fields[22] which correspond to the irreducible representations of the Ramond algebra. The spin fields are nonlocal with respect to the fermionic part of the superfields. Projecting on the sector of even fermion number by selecting the bosonic parts of the superconformal fields and a subset of the spin fields gives a new local bosonic field theory, called the spin model[23]. The construction of the spin model is a direct generalization of the construction of the superstring from the Neveu-Schwarz-Ramond model, and is of particular interest in the superstring theories as is discussed by Friedan, Martinec and Shenker (FMS)[23].

Many recent works on string theory have been developed in the framework of two dimensional conformal field theory[24]. The bosonic string represents one of the unitary representations of the $N=0$ conformal algebra, while the ferminonc string that of the $N=1$ superconformal algebra. N stands for the number of supercharges in the corresponding algebra. When we consider the compactification of the extra-dimensional space in which the strings are living, we need to be equipped with the representation theory of a suitable higher N superconformal algebra depending on the manifold structure of the compactified space.

In view of these remarkable developments in theoretical physics and mathematics over the past several years and also of presumably growing interests over the years to come in the study of the algebraic structures of two dimensional conformal field theories, we shall present in this paper some significant results of our recent investigation[25,26,27] on the representation theory of the affine Lie algebras.

In this paper we shall study the oscillator representations of the conformal and superconformal algebras with central charge extentions. Our aim is to achieve a null state construction*), thereby proving the Kac determinant formulae for higher N superconformal algebras. We have completed our study for the $N=0$, 1, and 2 algebras, and we shall report here the corresponding results[25,26,27]. Our method is systematic and quite powerful for obtaining degenerate highest weight formulae. It is likely that the $N=4$ case can be studied in a similar way[28].

For the $N=1$ representation we shall also present the explicit construction of the spin fields which create the Ramond ground states with level degeneracy from the Neveu-Schwarz vacuum. With this result a complete formulation for the full correlation functions with $N=1$ supersymmetry could in principle be given a la BPZ approach.

We have some remarks in order. The group of conformal transformations in two dimensions is generally described by two complex coordinates $z=x_1+ix_2$ and $\bar{z}=x_1-ix_2$. In conformal field theory the central object to study is the energy-momentum tensor T_{ab} $(a, b=1, 2)$, which generates local coordinate transformations over x_1 and x_2. In two dimensions T_{ab} has generally four components. In scale invariant systems T_{ab} is symmetric and traceless, and thus has two independent components. From the conservation law $\partial_a T_{ab}=0$, the two independent combinations $T(z)=T_{11}-iT_{12}$ and $\bar{T}(\bar{z})=T_{11}+iT_{12}$ only depend on z and $\bar{z}$ respectively. For conformal treatment of the systems it will be convient to consider z and $\bar{z}$ as two independent complex variables and regard $T(z)$ and $\bar{T}(\bar{z})$ as the operators which generate the coordinate transformations in the complex space C^2. The Euclidean plane and Minkowski space-time can be obtained as appropriate real sections of this complex space[20]. Correspondingly we have a doubling of the Virasoro algebras which are generated by $T(z)$ and $\bar{T}(\bar{z})$ respectively. In the following, however, we shall omit the discussion on the $\bar{z}$ dependence unless it is required for precise presentation, since the discussion is equivalent to the z dependence.

§ II. Virasoro Algebra and Null Fields

2.1. Introductory Review

The *Virasoro algebra* is the infinite dimensional Lie algebra with

*) Gervais and Neveu[37] also attempted to analyze the similar problem from somewhat different viewpoint motivated by the study of the Liouville theory.

generators L_n, $n \in Z$, satisfying the commutation relations

$$[L_m, L_n]=(m-n)L_{m+n}+\frac{1}{12}c(m^3-m)\delta_{m+n,0} \tag{2.1}$$

where c is a central element, i.e. $[L_n, c]=0$, so that c is assigned a numerical value in any irreducible representation. The number c is also called the *central charge.* The *Verma module* $V(c, h)$ is a representation of the Virasoro algebra generated by a *ground-state* vector $|h\rangle$ satisfying

$$L_0|h\rangle=h|h\rangle, \qquad L_n|h\rangle=0, \qquad n>0 \tag{2.2}$$

and spanned by the linearly independent vectors $|h\rangle$ and

$$L_{-k_n}L_{-k_{n-1}}\cdots L_{-k_1}|h\rangle, \qquad k_n\geq k_{n-1}\geq\cdots\geq k_1\geq 1 \tag{2.3}$$

with *level* $N=\sum_{i=1}^n k_i$. The eigenvalue $h+N$ of L_0 is the *energy* of that state. The condition (2.2) implies that $|h\rangle$ is a *highest-weight state.* We assume that both c and h are real. In the representation theory of the Virasoro algebra, the *ket vacuum* $|0\rangle\rangle$ is the ground state of lowest energy which statisfies the conditions*):

$$L_n|0\rangle\rangle=0, \qquad n\geq -1. \tag{2.4a}$$

Similarly the *bra vacuum* $\langle\langle 0|$ satisfies

$$\langle\langle 0|L_n=0, \qquad n\leq 1 \tag{2.4b}$$

The "in" state $|h\rangle$ is created from the vacuum $|0\rangle\rangle$ by operating the *primary* conformal field $\phi_n(z)$ on it:

$$|h\rangle=\lim_{z\to 0}\phi_h(z)|0\rangle\rangle \tag{2.5a}$$

while the "out" primary state is defined by the formula**).

*) This definition of the vacuum itself already implies that the conformal dimension h as the eigenvalue of L_0 must be non-negative, i.e. $h\geq 0$, in the unitary representation. As will be discussed later, the condition $h\geq 0$ is also obtained from the physical imposition that two-point correlation functions do not diverge at large distances (see eq. (2.50)).

**) The precise definition of the "out" primary state with the $\bar{z}$ dependence for the conformal dimension $\bar{h}$ included is given in terms of the corresponding Virasoro operators $\bar{L}_n$ as

$$\langle h, \bar{h}|=\lim_{z,\bar{z}\to 0}\langle\langle 0|\phi_{h,\bar{h}}(z, \bar{z})z^{2L_0}\bar{z}^{2\bar{L}_0}$$

while that of the "in" state as

$$|h, \bar{h}\rangle=\lim_{z,\bar{z}\to 0}\phi_{h,\bar{h}}(z, \bar{z})|0\rangle\rangle.$$

(2.5b) $$\langle h| = \lim_{z\to\infty} \langle\langle 0|\phi(z)z^{2L_0}$$

The primary field $\phi_h(z)$ with *conformal dimension* (or *weight*) h transforms in the simplest way

(2.6) $$\phi_h(z) \longrightarrow \left(\frac{dw}{dz}\right)^h \phi_h(w)$$

under the coordinate transformation $z \to w(z)$, and satisfies the commutation relations

(2.7) $$[L_n, \phi_h(z)] = z^n\{z\partial_z + h(n+1)\}\phi_h(z)$$

Now there are specific values $h = h_{(r,s)}$ (r, s = positive integers) of conformal dimension h which make the corresponding Verma modules $V(c, h_{(r,s)})$ *reducible* (or *degenerate*) and consequently let *null states* $|\chi_{h_{(r,s)}, N=rs}\rangle$ with conformal dimension $h_{(r,s)}+N$ appear at level $N=rs$. The norm of a null state $|\chi_{h,N}\rangle$ vanishes:

(2.8a) $$\langle\chi_{h,N}|\chi_{h,N}\rangle = 0$$

and it satisfies the primary condition*)

(2.8b) $$L_n|\chi_{h,N}\rangle = \delta_{n,0}(h+N)|\chi_{h,N}\rangle, \qquad n = 0, 1, 2, \cdots$$

So there must be a corresponding *null field* $\chi_{h,N}(x)$ which is constructed from the *degenerate* primary field $\phi_h(z)$ and satisfies

(2.9) $$[L_n, \chi_{h,N}(z)] = z^n\{z\partial_z + (h+N)(n+1)\}\chi_{h,N}(z)$$

If $F^{(N)}(c, h)$ denotes the inner product matrix of the states at level N in the Verma module $V(c, h)$, then the zeros of its determinant $K^{(N)}(c, h) \equiv \det[F^{(N)}(c, h)]$ indicate the existence of the corresponding null fields at the level N. In particular, the conformal dimension $h_{(r,s)}$ of a degenerate primary field $\phi_{h_{(r,s)}}(z)$ is generally the multiple roots of $K^{(N)}(c, h)$ for $N \geq rs$. $K^{(N)}(c, h)$ is called the *Kac determinant*[14] and plays the essential role in the determination of the unitary representations of conformal algebras[15].

We present the two simplest examples of the null states.

Example 1. When $h = h_{(1,1)} = 0$, a null state $|\chi_{0,1}\rangle$ appears at level $N=1$ and is given by

*) Consequently $|\chi_{h,N}\rangle$ is orthogonal to any state, thus one may consistently put $|\chi_{h,N}\rangle = 0$.

$$|\chi_{0,1}\rangle = L_{-1}|h\rangle, \qquad h = h_{(1,1)} = 0. \tag{2.10}$$

$h_{(1,1)}$ is a solution of $K^{(1)}(c, h) = 0$ where

$$K^{(1)}(c, h) = 2h. \tag{2.11}$$

Example 2. When $h = h_{(1,2)}$ or $h = h_{(2,1)}$, a null state $|\chi_{h(1,2),2}\rangle$ or $|\chi_{h(2,1),2}\rangle$ appear at level $N = 2$ and is given by

$$\left\{L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right\}|h\rangle, \qquad h = h_{(1,2)} \text{ or } h_{(2,1)} \tag{2.12}$$

where $h_{(1,2)}$ and $h_{(2,1)}$ are the two roots of the quadratic equation in

$$\begin{aligned} K^{(2)}(c, h) &= 4h\left(8h^2 + h(c-5) + \frac{1}{2}c\right) \\ &= 32h(h - h_{(1,2)})(h - h_{(2,1)}) \end{aligned} \tag{2.13}$$

The physical significance of the null states was stressed in the paper by BPZ. They used a null state $|\chi_{h,N}\rangle$ to derive differential equations for the correlation functions

$$\langle \phi_h(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle \equiv \langle\langle 0|[\phi_h(z)\phi_1(z_1)\cdots\phi_n(z_n)]_R|0\rangle\rangle \tag{2.14}$$

where $[\cdots]_R$ denotes radial ordering, i.e. $\phi_i(z_i)$ is to be put to the left of $\phi_j(z_j)$ when $|z_i| > |z_j|$. The correlation functions are invariant under *global conformal transformations* generated by $L_{\pm 1}$, and L_0. Particularly we have translation invariance under $z_i \to z'_i = z_i - z$, which gives

$$\langle \phi_h(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle = \langle \phi_1(z'_1)\cdots\phi_n(z'_n)\phi_h(0)\rangle. \tag{2.15}$$

Suppose we take $h = h_{(1,2)}$, then the state $|\chi_{h(1,2),2}\rangle$ is null and orthogonal to all the states. So we have

$$\begin{aligned} &\langle \phi_1(z'_1)\cdots\phi_n(z'_n)\chi_{h(1,2),2}(0)\rangle \\ &\quad = \left\langle \phi_1(z'_1)\cdots\phi_n(z'_n)\left(L_{-2} - \frac{3}{2(2h_{(1,2)}+1)} L_{-1}^2\right)\phi_{h(1,2)}(0)\right\rangle = 0 \end{aligned} \tag{2.16}$$

Using the commutation relations (2.7) and the fact that L_{-2} and L_{-1} annihlate the bra vaccuum $\langle\langle 0|$ due to (2.4b) we arrive at the following differential equation

$$\begin{aligned} &\left\{\sum_{i=1}^{n}\left(\frac{1}{z'_i}\frac{\partial}{\partial z'_i} - \frac{h_i}{z'^2_i}\right) + \frac{3}{2(2h_{(1,2)}+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial z'_i}\frac{\partial}{\partial z'_j}\right\} \\ &\quad \times \langle \phi_1(z'_1)\cdots\phi_n(z'_n)\phi_{h(1,2)}(0)\rangle = 0 \end{aligned} \tag{2.17}$$

This equation can be rewritten, using global conformal invariance of the correlation functions, as

$$\left\{\frac{3}{2(2h_{(1,2)}+1)}\frac{\partial^2}{\partial z^2}+\sum_{i=1}^{n}\frac{1}{z_i-z}\frac{\partial}{\partial z_i}-\frac{h_i}{(z_i-z)^2}\right\} \times\langle\phi_{h_{(1,2)}}(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle=0 \tag{2.18}$$

BPZ showed that the differential equations thus obtained from the null states and satisfied by the degenerate fields impose hard constraints on the operator algebra[20]. In the case of "minimal" theories[20] it may even be the case that the whole set of differential equations uniquely determine any correlation function, thereby solving the theories completely.

We shall proceed to explain our generic construction of null fields $\chi_{h,N}(z)$ in terms of vertex operators.

2.2. Oscillator representation

The oscillator representation with central charge extention of the Virasoro operators $L_{\pm n}$ ($n=0, 1, 2, \cdots$) is given by[29,30]

$$L_n(\lambda)=(a_0+\lambda n)a_n+\sum_{j=1}^{\infty}a_{-j}a_{n+j}+\frac{1}{2}\sum_{j=1}^{n-1}a_ja_{n-j} \tag{2.19a}$$

$$L_{-n}(\lambda)=(a_0-\lambda n)a_{-n}+\sum_{j=1}^{\infty}a_{-n-j}a_j+\frac{1}{2}\sum_{j=1}^{n-1}a_{-n+j}a_{-j} \tag{2.19b}$$

$$L_0(\lambda)=\frac{1}{2}(a_0^2-\lambda^2)+R=\frac{1}{2}(a_0^2-\lambda^2)+\sum_{j=1}^{\infty}a_{-j}a_j \tag{2.19c}$$

where R is the level operator of the harmonic oscillator space, and a_n ($n=0, \pm1, \pm2, \cdots$) satisfy the hermiticity

$$a_n^\dagger=a_{-n} \tag{2.20}$$

and the commutation relations

$$[a_m, a_n]=m\delta_{m+n,0} \tag{2.21}$$

The central charge c is related to the c-number parameter λ by

$$c=1-12\lambda^2 \tag{2.22}$$

For real λ we have $c\leq 1$.

We also write

(2.23) $$a_0 = p_0$$

and introduce a hermitian coordinate q_0 as conjugate to the momentum p_0:

(2.24) $$[q_0, a_n] = i\delta_{n,0}$$

Then it is useful to introduce operators $\varphi(z)$, $A(z)$ and $T(z)$ written in terms of a complex variable z as

(2.25a) $$\varphi(z) = q_0 - i(p^0 - \lambda) \ln z - i \sum_{n=1}^{\infty} [(a_{-n}/n)z^n - (a_n/n)z^{-n}]$$

(2.25b) $$A(z) = i\partial_z \varphi(z) = \sum_{n=-\infty}^{\infty} (a_n - \lambda\delta_{n,0})/z^{n+1}$$

(2.25c) $$T(z) = \frac{1}{2} : A(z)A(z): - \lambda\partial_z A(z) = \sum_{n=-\infty}^{\infty} L_n(\lambda)/z^{n+2}$$

$T(z)$ is nothing but the energy-momentum tensor of the free scalar field $\varphi(z)$ with modified boundary condition at $z=0$ and ∞. It describes a Coulomb-like system with a fixed charge $\lambda(-\lambda)$ at $z=0$ (∞)[31].

The *vertex operator* of momentum t is defined by

(2.26) $$\begin{aligned} \mathscr{V}(t, z) &= : \exp\{it\varphi(z)\}: \\ &= \exp\left(t \sum_{n=1}^{\infty} (a_{-n}/n)z^n\right) \exp\left(-t \sum_{n=1}^{\infty} (a_n/n)z^{-n}\right) \exp(itq_0) z^{t(p_0-\lambda)} \end{aligned}$$

Note that in the normal-ordered expression a_n is moved to the right of $a_{-n}(n>0)$ and p_0 to the right of q_0. $\mathscr{V}(t, z)$ satisfies the commutation relation

(2.27) $$[L_n(\lambda), \mathscr{V}(t, z)] = z^n\{z\partial_z + h_0(t)(n+1)\}\mathscr{V}(t, z),$$

where

(2.28) $$h_0(t) \equiv \frac{1}{2}(t+\lambda)^2 - \frac{1}{2}\lambda^2 = \frac{t^2}{2} + \lambda t$$

Eq. (2.27) implies by definition that $\mathscr{V}(t, z)$ transforms as a primary field $\phi_h(z)$ with conformal dimension $h = h_0(t)$, i.e. under the conformal transformation $z \to z + \varepsilon(z)$ $\mathscr{V}(t, z)$ transforms as

(2.29) $$\mathscr{V}(t, z) \to \left(1 + \frac{d\varepsilon}{dz}\right)^{h_0(t)} \mathscr{V}(t, z + \varepsilon(z))$$

Choosing t as the two solutions $t_\pm$ of the equation

(2.30) $$h_0(t_\pm)=1$$

where

(2.31) $$t_\pm=-\lambda\pm\sqrt{\lambda^2+2}, \qquad t_+t_-=-2,$$

we get the vertex operators $\mathscr{V}(t_\pm, z)$ with conformal dimension $=1$, which will be crucial operators playing a central role in our construction of null fields.

In particular we have

(2.32) $$[L_n(\lambda), \mathscr{V}(t_\pm, z)]=\partial_z\{z^{n+1}\mathscr{V}(t_\pm, z)\}$$

Consequently, by defining $\oint dz\equiv(2\pi i)^{-1}\oint dz$, we have the two "charge screening" operators $S_\pm$ of conformal dimension $=0$[31]:

(2.33) $$S_\pm=\oint_C dz\mathscr{V}(t_\pm, z)$$

which commute with all $L_n(\lambda)$:

(2.34) $$[L_n(\lambda), S_\pm]=0$$

but satisfy

(2.35) $$[p_0, S_\pm]=t_\pm S_\pm.$$

Therefore $S_\pm$ carries momentum $t_\pm$ whereas it preserves conformal properties wherever inserted in the correlation functions. The contour C in (2.33) has to be taken properly as is discussed in detail in Ref. 31.

For later convenience we introduce the operators

(2.36) $$a_n(z)=\oint_{C_{0,z}} d\zeta(\zeta-z)^n A(\zeta), \qquad n=0, \pm 1, \pm 2, \cdots$$

where $C_{0,z}$ encloses the points 0 and z. By definition we have

(2.37) $$a_n(0)=a_n-\lambda\delta_{n,0}.$$

The ground state *ket vacuum* $|0;\lambda\rangle$ with zero conformal weight is defined by

(2.28) $$a_n(0)|0;\lambda\rangle=0, \qquad n\geq 0$$

Then it satisfies the following requirements of conformal invariance at $z=0$

$$L_n(\lambda)|0;\lambda\rangle=0, \qquad n\geq -1 \tag{2.39}$$

By use of the relation (2.20) we have the hermiticity relations

$$a_n(0)^\dagger = a_{-n}(0)|_{\lambda\to\lambda^*} \tag{2.40}$$

and

$$L_n(\lambda)^\dagger = L_{-n}(-\lambda^*). \tag{2.41}$$

Consequently, for real λ, we obtain the state $\langle 0;-\lambda|$ as the *bra vacuum* which satisfies the conditions for the conformal invariance at $z=\infty$:

$$\langle 0;-\lambda|L_{-n}(\lambda)=0, \qquad n\geq -1 \tag{2.42}$$

In the following we restrict our discussion to real λ. Hence our oscillator representation of the Virasoro operators becomes non-unitary. Because of this we could study the conformal properties of null fields in the harmonic space of hermitian oscillators. It is also to be remarked that, as eq. (2.41) shows, the hermitian conjugation operated on $L_n(\lambda)$ is equivalent to changing the sign of the mode number n and the λ parameter. This property of the hermitian conjugation is a universal manifestation of the operation in our representation of the conformal and superconformal algebras.

A vertex operator representation of an infinite set of the *secondary* fields (*descendants*)[20] $\phi_h^{(-k_1,-k_2,\cdots,-k_N)}(z)$ $(k_i\geq 1, N=1,2,\cdots)$ can be obtained as

$$\mathscr{V}^{(-k_1,-k_2,\cdots,-k_N)}(t,z)=L_{-k_1}(z)L_{-k_2}(z)\cdots L_{-k_N}(z)\mathscr{V}(t,z) \tag{2.43}$$

where the dimension h_N of the fields is $h_N(t)=h_0(t)+\sum_{i=1}^N k_i$ and the operators $L_{-k}(z)$ are given by the contour integrals

$$L_{-k}(z)=\oint_{C_{0,z}} d\zeta\, \frac{T(\zeta)}{(\zeta-z)^{k-1}} \tag{2.44}$$

In (2.43) the integration contour associated with each of $L_{-k_i}(z)$ enclose the points zero and z as well as the points $\zeta_{i+1}, \zeta_{i+2}, \cdots, \zeta_N$ which are the integration variable corresponding to the operators $L_{-k_j}(z)$ $(j=i+1, i+2,\cdots,N)$ sitting to the right of $L_{-k_i}(z)$.

The correlation functions for the product of primary or secondary vertex opertors

$$X_N=\mathscr{V}_1(t_1,z_1)\mathscr{V}_2(t_2,z_2)\cdots\mathscr{V}_N(t_N,z_N) \tag{2.45}$$

are represented by the vacuum expectation values of its *radial-ordered product*:*

(2.46) $$\langle X_N\rangle \equiv \langle 0; -\lambda|[X_N]_R|0; \lambda\rangle$$

which is invariant under the global conformal transformations generated by $L_n(\lambda)$ for $n=\pm 1$ and 0. The asymptotic condition

(2.47) $$\sum_{i=1}^{N} t_i + 2\lambda = 0$$

has to be satisfied in order to obtain nontrivial results for $\langle X_N\rangle$. The condition (2.47) can be seen by considering the relation

(2.48) $$\langle A(\zeta)X_N\rangle = \sum_{i=1}^{N} \frac{t_i}{\zeta - z_i}\langle X_N\rangle$$

and taking the limit $\zeta\to\infty$ as

(2.49) $$\lim_{\zeta\to\infty} \zeta\langle A(\zeta)X_N\rangle = -2\lambda\langle X_N\rangle$$

Now the ground state $|0; t+\lambda\rangle$ with conformal dimension $h_0(t)$ is obtained by

(2.50) $$|h_0(t)\rangle = |0; t+\lambda\rangle = \lim_{z\to 0} \mathscr{V}(t, z)|0; \lambda\rangle = e^{itq_0}|0; \lambda\rangle$$

The state $|0; t+\lambda\rangle$ is a primary state and satisfies**

(2.51) $$L_n(\lambda)|0; t+\lambda\rangle = h_0(t)\delta_{n,0}|0; t+\lambda\rangle, \qquad n\geq 0$$

Here we have some remarks in order. Each conformal vertex operator $\mathscr{V}(t, z)$ of dimension $h_0(t)$ is paired with another vertex operator $\mathscr{V}(-2\lambda-t, z)$ of the same conformal dimension $h_0(-2\lambda-t)=h_0(t)$ which can be regarded as a sort of conjugate to $\mathscr{V}(t, z)$. In the correlation functions (2.46) we should adopt appropriate vertex operators to represent each conformal field of a given dimension h so that we could obtain non-vanishing values for the correlators. For the two point correlator with conformal dimension $h_0(t)$, for example, we have

(2.52) $$\langle \mathscr{V}(-2\lambda-t, z)\mathscr{V}(t, z')\rangle = \frac{1}{(z-z')^{2h_0(t)}}, \qquad |z|>|z'|.$$

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*) The radial-ordering is defined to be the time-ordering operation with respect to τ_i where τ_i is given by $z_i=\exp(\tau_i+i\sigma_i)$ $0\leq\sigma_i<2\pi$.

**) This is obvious by noting that $a_n|0; t+\lambda\rangle = (t+\lambda)\delta_{n,0}|0; t+\lambda\rangle$.

From this expression we find for the norm of a primary state with $h=h_0(t)$

$$\begin{aligned}\langle h_0(t)|h_0(t)\rangle &= \lim_{z\to\infty}\langle 0;\ -\lambda|\mathscr{V}(-2\lambda-t, z)z^{2h_0(t)}\mathscr{V}(t, 0)|0;\ \lambda\rangle \\ &= \lim_{z\to\infty} z^{2h_0(t)}\langle\mathscr{V}(-2\lambda-t, z)\mathscr{V}(t, 0)\rangle \\ &= 1\end{aligned} \tag{2.53}$$

where we have used the definition (2.5b) for the "out" primary state. The result implies that in order to get a nonvanishing result for the norm of any primary state the conjugate vertex operator is to be used for defining the "out" primary state.

A local operator with conformal dimension $h=0$ is a conformally invariant *identity* operator of the algebra. In Coulomb theory it has two representatives: $\mathscr{V}(t=0, z)$ and $\mathscr{V}(t=-2\lambda, z)$. Actually, in conformity with the convention of eq. (2.53), $\mathscr{V}(0, z)$ is the identity operator to be applied on the "in" state, whereas $\mathscr{V}(-2\lambda, z)$ on the "out" state. Correspondingly we have the norm of the ground state vacuum as the nonvanishing matrix element of the vertex operator $\mathscr{V}(-2\lambda, \infty)$

$$\langle h=0|h=0\rangle=\langle 0;\ -\lambda|\mathscr{V}(-2\lambda, \infty)|0;\ \lambda\rangle=1. \tag{2.54}$$

On the other hand, if one interchanges the role of the vertex operators $\mathscr{V}(t, z)$ and $\mathscr{V}(-2\lambda-t, z)$, one obtains the two point correlator

$$\langle\mathscr{V}(t, z)\mathscr{V}(-2\lambda-t, z')\rangle=\frac{1}{(z-z')^{2h_0(t)}}, \qquad |z|>|z'| \tag{2.55}$$

and correspondingly the norm of the primary state with $h=h_0(t)$

$$\begin{aligned}\langle h_0(t)|h_0(t)\rangle &= \lim_{z\to\infty}\langle 0;\ -\lambda|\mathscr{V}(t, z)z^{2h_0(t)}\mathscr{V}(-2\lambda-t, 0)|0;\ \lambda\rangle \\ &= \lim_{z\to\infty} z^{2h_0(t)}\langle\mathscr{V}(t, z)\mathscr{V}(-2\lambda-t, 0)\rangle \\ &= 1\end{aligned} \tag{2.56}$$

instead of (2.53). Hence eq. (2.54) is to be replaced by

$$\langle h=0|h=0\rangle=\langle 0;\ -\lambda|\mathscr{V}(-2\lambda, 0)|0;\ \lambda\rangle. \tag{2.57}$$

However it should be alarmed that this convention is actually equivalent to another choice of the ket vacuum given by $\mathscr{V}(-2\lambda, 0)|0;\ \lambda\rangle=|0;\ -\lambda\rangle$, where the Virasoro operators are represented by $L_n(-\lambda)$ instead of $L_n(\lambda)$.

Now, coming back to the correlation functions, we mention that the charge screending operators $S_\pm$ defined by (2.33) will also be required to enter in the N-point correlator (2.46) in order to recover the balance of the "charges" $\{t_i\}$ without effecting its conformal properties. Due to the constraint (2.47) some of the vertex operators in (2.45) are necessarily the identity operator $\mathscr{V}(-2\lambda, z)$ or those coming from the charge screening operators $S_\pm$ inserted in the correlator.

2.3. Null fields

Singular vertex operators corresponding to the null fields $\chi_{h,N}(z)$ are given in three equivalent expressions[25]

$$\Phi_r^\pm(t, z)=\oint_{C_r} dz_r \int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 \mathscr{V}(t_\pm, z_r)\cdots\mathscr{V}(t_\pm, z_1)\mathscr{V}(t, z) \tag{2.58a}$$

$$\begin{aligned}&=\oint_{C_r} dz_r \int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 f_r^\pm(t; z, z_i)\\ &\times : \exp\left(-t_\pm \oint_{C_{z,z_i}} d\zeta A(\zeta) g_r(\zeta; z, z_i)\right)\cdot \mathscr{V}(rt_\pm+t, z):\end{aligned} \tag{2.58b}$$

$$\begin{aligned}&=\oint_{C_r} dz_r \int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 f_r^\pm(t; z, z_i)\\ &\times : \exp\left(t_\pm \sum_{i=1}^{r}\left(\sum_{l=0}^{\infty}\frac{(z_i-z)^{l+1}}{(l+1)!}A^{(l)}(z)\right)\right)\cdot \mathscr{V}(rt_\pm+t, z):\end{aligned} \tag{2.58c}$$

where the integration contours C_r and C_{z,z_i} are illustrated in Fig. 1. The complex functions $f_r^\pm$ and g_r are defined by

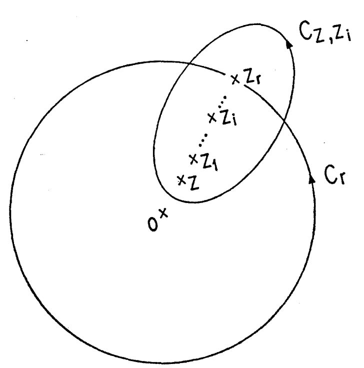

Fig. 1 The contours C_{z,z_i} and C_r for the complex integrations over ζ and z_r are shown. The contours may be freely modified as long as one keeps track of singularities at zero, z_i and z correctly.

(2.59a)
$$f_r^{\pm}(t; z, z_i)=\prod_{j>i}^{r}(z_j-z_i)^{t_\pm^2}\prod_{i=1}^{r}(z_i-z)^{t_\pm t}$$

and

(2.59b)
$$g_r(\zeta; z, z_i)=\ln\left(\prod_{i=1}^{r}(\zeta-z_i)/(\zeta-z)^r\right)$$

$A^{(l)}(z)$ stands for the lth derivative of $A(z)$ and is related to $a_{-n}(z)$ through the relation ($l\geq 0$):

(2.60)
$$\frac{1}{l!}A^{(l)}(z)=a_{-l-1}(z)-\oint_{C_0}\frac{A(\zeta)}{(\zeta-z)^{l+1}}d\zeta$$

The proof of getting form the first (2.58a) to the second expression (2.58b) goes as follows:
Define

(2.61)
$$\mathscr{V}(t, z)=\mathscr{V}_+(t, z)\mathscr{V}_-(t, z)\mathscr{V}_0(t, z)$$

where

(2.62a)
$$\mathscr{V}_+(t, z)=\exp\left(t\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right)$$

(2.62b)
$$\mathscr{V}_-(t, z)=\exp\left(-t\sum_{n=1}^{\infty}\frac{a_n}{n}z^{-n}\right)$$

(2.62c)
$$\mathscr{V}_0(t, z)=e^{itq_0}z^{t(p_0-\lambda)}$$

Then we have

(2.63a)
$$\mathscr{V}_-(t_1, z_1)\mathscr{V}_+(t_2, z_2)=\mathscr{V}_+(t_2, z_2)\mathscr{V}_-(t_1, z_2)(z_1)^{-t_1t_2}(z_1-z_2)^{t_1t_2}$$

(2.63b)
$$\begin{aligned}\mathscr{V}_-(t_\pm, z_i)\mathscr{V}_0(t_\pm, z_i)\mathscr{V}(t, z)=\mathscr{V}(t, z)\mathscr{V}_-(t_\pm, z_i)\mathscr{V}_0(t_\pm, z_i)&\\ \times z^{-t_\pm t}(z_i-z)^{t_\pm t}&\end{aligned}$$

(2.63c)
$$z^{t_1(p_0-\lambda)}e^{it_2q_0}=e^{it_2q_0}z^{t_1(p_0-\lambda)}z^{t_1t_2}$$

and also

(2.63d)
$$\begin{aligned}\mathscr{V}(t, z)=\mathscr{V}_+(-rt_\pm, z)\mathscr{V}(rt_\pm+t, z)\mathscr{V}_-(-rt_\pm, z)\mathscr{V}_0(-rt_\pm, z)&\\ \times z^{rt_\pm t+r^2t_\pm^2}.&\end{aligned}$$

From these relations we get

(2.64a)
$$\begin{aligned}(2.58\text{a})=\oint_{C_r}dz_r\int_z^{z_r}dz_{r-1}\cdots\int_z^{z_2}dz_1 f_r^{\pm}(t; z, z_i)&\\ \times\mathscr{V}_+(t_\pm, z_r)\cdots\mathscr{V}_+(t_\pm, z_1)\mathscr{V}_+(-rt_\pm, z)\mathscr{V}(rt_\pm+t, z)&\end{aligned}$$

$$\times \mathscr{V}_-(t_\pm, z_r)\cdots\mathscr{V}_-(t_\pm, z_1)\mathscr{V}_-(-rt_\pm, z)(z_1\cdots z_r z^{-r})^{t_\pm(p_0-\lambda)}$$

Using the relation

$$a_n-\lambda\delta_{n,0}=a_n(0)=\oint_{C_0} d\zeta\zeta^n A(\zeta), \tag{2.65}$$

we obtain

$$\begin{aligned}\mathscr{V}_+(t, z)&=\exp\left(t\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right)\\ &=\exp\left(t\sum_{n=1}^{\infty}\oint_{C_0} d\zeta A(\zeta)\frac{1}{n}\frac{z^n}{\zeta^n}\right)\\ &=\exp\left(-t\oint_{C_{0,z}} d\zeta A(\zeta)\ln\left(1-\frac{z}{\zeta}\right)\right)\end{aligned} \tag{2.66a}$$

where $|z|<|\zeta|$ and the contour $C_{0,z}$ encircles the points 0 and z. Similarly we have

$$\mathscr{V}_-(t, z)=\exp\left(t\oint_{C_0} d\zeta A(\zeta)\ln\left(1-\frac{\zeta}{z}\right)\right) \tag{2.66b}$$

and

$$z^{t(p_0-\lambda)}=\exp\left(t\oint_{C_0} d\zeta A(\zeta)\ln z\right) \tag{2.66c}$$

with $|\zeta|<|z|$ and the contour C_0 encircling the point zero. Then, from (2.64a) we find

$$\begin{aligned}(2.58\text{a})=&\oint_{C_r} dz_r\int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 f_r^{\pm}(t; z, z_i)\\ &\times\exp\left(-t_\pm\oint_{C_+} d\zeta A(\zeta)\sum_{i=1}^{r}\ln\left(\frac{\zeta-z_i}{\zeta-z}\right)\right)\mathscr{V}(rt_\pm+t, z)\\ &\times\exp\left(t_\pm\oint_{C_-} d\zeta A(\zeta)\sum_{i=1}^{r}\ln\left(\frac{z_i-\zeta}{z-\zeta}\right)\right)\end{aligned} \tag{2.64b}$$

where the contous C_+ and C_- are shown in Fig. 2. Finally, by use of the normal ordering and the contour relation $C_+-C_-=C_{z,z_i}$, we obtain

$$\begin{aligned}(2.58\text{a})=&\oint_{C_r} dz_r\int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 f_r^{\pm}(t; z, z_i)\\ &\times:\exp\left(-t_\pm\int_{C_{z,z_i}} d\zeta A(\zeta)\ln\left(\prod_{i=1}^{r}(\zeta-z_i)/(\zeta-z)^r\right)\mathscr{V}(rt_\pm+t, z):\\ &=(2, 58\text{b})\quad \text{q.e.d.}\end{aligned} \tag{2.64c}$$

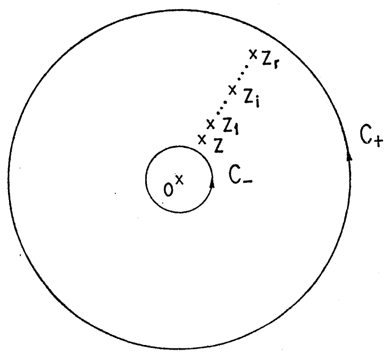

Fig. 2 The contours $C_\pm$ in eq. (2. 64b) are shown. The contour relation $C_+ - C_- = C_{z,z_i}$ can be easily seen from the figure.

The proof of arriving at the third expression (2.58c) is straightforward by noting $A(\zeta)=i\partial_\zeta\varphi(\zeta)$ and making a partial integration. Consider

$$\oint_{C_{z,z_i}} d\zeta A(\zeta) \sum_{i=1}^{r} \ln\left(\frac{\zeta-z_i}{\zeta-z}\right) = -\oint_{C_{z,z_i}} d\zeta i\varphi(\zeta)\partial_\zeta\left(\sum_{i=1}^{r}\ln\left(\frac{\zeta-z_i}{\zeta-z}\right)\right)$$

$$(2.67) \qquad = \sum_{i=1}^{r} (i\varphi(z)-i\varphi(z_i))$$

$$= -\sum_{i=1}^{r}\sum_{l=0}^{\infty} \frac{1}{(l+1)!}(z_i-z)^{l+1}A^{(l)}(z)$$

which gives the equality

$$(2.68) \qquad (2.58\text{b})=(2.58\text{c}) \qquad \text{q.e.d.}$$

Now we study the conformal properties of the null field $\Phi_r^\pm(t, z)$. First we shall recall that under a *finite* transformation

$$(2.69) \qquad z \to z' = z+\varepsilon(z) = z + \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1}$$

the vertex operator $\mathscr{V}(t, z)$ transforms as

$$(2.70\text{a}) \qquad \mathscr{V}(t, z) \to U(\lambda)\mathscr{V}(t, z)U(\lambda)^{-1} = \left(\frac{dz'}{dz}\right)^{h_0(t)} \mathscr{V}(t, z')$$

where

$$(2.70\text{b}) \qquad U(\lambda) = \exp\left(\sum_{n=-\infty}^{\infty} \varepsilon_n L_n(\lambda)\right)$$

For an infinitesimal transformation eq. (2.70) is expressed in the differential form (2.27). In particular, if we choose $t=t_{\pm}$ which are given by (2.31), we have from (2.70)

$$U(\lambda)\mathscr{V}(t_{\pm}, z)dzU(\lambda)^{-1}=\mathscr{V}(t_{\pm}, z')dz' \tag{2.71}$$

Hence for the first expression (2.58a) we obtain

$$\begin{aligned} &U(\lambda)\Phi_r^{\pm}(t, z)U(\lambda)^{-1} \\ &=\oint_C dz'_r \int_{z'}^{z'_r} dz'_{r-1}\cdots\int_{z'}^{z'_2} dz_1\mathscr{V}(t_{\pm}, z'_r)\cdots\mathscr{V}(t_{\pm}, z'_1)\left(\frac{dz'}{dz}\right)^{h_0(t)}\mathscr{V}(t_1, z') \\ &=\left(\frac{dz'}{dz}\right)^{h_0(t)}\Phi_r^{\pm}(t, z') \end{aligned} \tag{2.72}$$

From this we get for the infinitesimal transformation

$$[L_n(\lambda), \Phi_r^{\pm}(t, z)]=z^n\{z\partial_z+h_0(t)(n+1)\}\Phi_r^{\pm}(t, z) \tag{2.73}$$

which shows by definition that $\Phi_r^{\pm}(t, z)$ transforms as a primary field with conformal dimension $h_0(t)$. It is worth mentioning that the conformal transformation properties of $\Phi_r^{\pm}(t, z)$ is controlled by the rightmost vertex operator $\mathscr{V}(t, z)$ in (2.58a).

The last contour integral over z_r in (2.58) imposes the so-called "on-sell" constraint[8] on the allowed values of momentum t to obtain nonvanishing $\Phi_r^{\pm}(t, z)$'s. The contour integral over z_r is dragging the path integration's over z_i. Consequently, when all z_i's ($i=1, 2, \cdots, r$) turn around 0 and z^*, the integral acquires additional phases, the sum of which has to vanish in order to give a nonvanishing $\Phi_r^{\pm}(t, z)$. Consequently we obtain the "on-shell' condition:

$$rt_{\pm}t+\frac{r(r-1)}{2}t_{\pm}^2+r+N=0 \tag{2.74}$$

where

$$N=\sum_{l=1}^{N} ln_l \tag{2.75}$$

corresponding to a certain set $\{n_1, n_2, \cdots, n_N\}$ of integers.

*) Actually the contour C_r in (2.58) may be deformed in such a way as that z_r only encircles those points of z and $z_i (i=1, \cdots, r-1)$. This is obvious by noting from the third expression (2.58c) that the integrand of (2.58) is regular at $z_i=0$. In this deformed case z_i's turn around z only. But the similar argument to that in the text goes through in this case as well since it is the combination of the factors $z_j-z_i\,(j>i)$ and z_i-z that matters for the phase counting.

Let us define

$$t_{(r,s)}=\frac{1+r}{2}t_{+}+\frac{1+s}{2}t_{-}. \tag{2.76}$$

Then, solving (2.74) for t by use of the relation (see eq. (2.31))

$$t_{-}=-\frac{2}{t_{+}}, \tag{2.77}$$

we find for the case of $\Phi^{+}(t, z)$

$$t=t_{(-r,N/r)} \quad (r, N=\text{positive integers}) \tag{2.78}$$

or

$$rt_{+}+t=t_{(r,N/r)} \quad (r, N=\text{positive integers}) \tag{2.79}$$

The case for $\Phi_r^{-}(t, z)$ can be solved correspondingly. For the following consideration it is a matter of convenience which of the two choices, $\Phi_r^{+}(t, z)$ or $\Phi_r^{-}(t, z)$, is taken, so we restrict our discussion to $\Phi_r^{+}(t, z)$ without loss of generality.

Now the constraint (2.78) or (2.79) is only the *necessary* condition for the existence of a nontirival $\Phi_r^{+}(t, z)$. The *necessary* and *sufficient* condition for it is actually given by

$$t=t_{(-r,s)} \quad (r, s=\text{positive integers}) \tag{2.80}$$

or

$$rt_{+}+t=t_{(r,s)} \quad (r, s=\text{positives integers}) \tag{2.81}$$

which implies that N and r (r, N=postive integers) satisfy

$$\frac{N}{r}=s \quad (s=\text{positive integers}) \tag{2.82}$$

or N is given by

$$N=rs \quad (r, s=\text{positive integers}) \tag{2.83}$$

The proof is given in our later discussion on the Kac determinant.* For the moment we presume (2.82).

What N in eqs. (2.74) and (2.75) stands for becomes apparent by operating $\Phi_r^{+}(t=t_{(-r,s)}, z=0)$ on the ket vacuum $|0; \lambda\rangle$:

---

*) What is actually happening is that $\Phi_r^{+}(t_{(-r,s)}, z)$ vanishes unless the condition (2.83) is valid.

$$\begin{aligned}\Phi_r^+(t_{(-r,s)}, 0)|0; \lambda\rangle = \oint dz_r \int_0^{z_r} dz_{r-1} \cdots \int_0^{z_2} dz_1 f_r^+(t_r^+; 0, z_i) \\ \times \exp\left(t_+ \sum_{i=1}^{r} \sum_{l=1}^{\infty} \frac{z_i^l}{l} a_{-l}(0)\right) \mathscr{V}(t_{(r,s)}, 0)|0; \lambda\rangle\end{aligned} \tag{2.84}$$

where use has been made of the relation (2.60) and the normal ordering has been carefully performed. The operator $a_{-l}(0)$ increases the level by l. Thus we find that N stands for the level of the null state. With the notation of Section 2.1 we obtain

$$\begin{aligned}\Phi_r^+(t_{(-r,s)}, 0)|0; \lambda\rangle &= \chi_{h=h_{(r,s)}, N=rs}(0)|0; \lambda\rangle \\ &= \mathscr{L}_{h=h_{(r,s)}, N=rs} \mathscr{V}(t_{(r,s)}, 0)|0; \lambda\rangle\end{aligned} \tag{2.85}$$

where

$$h_{(r,s)} = h_0(t_{(r,s)}) \tag{2.86}$$

and

$$\begin{aligned}\mathscr{L}_{h_{(r,s)}, rs} = \oint_{C_r} dz_r \int_0^{z_r} dz_{r-1} \cdots \int_0^{z_2} dz_1 f_r^+(t_r; 0, z_i) \\ \times \exp\left(t_+ \sum_{i=1}^{r} \sum_{l=1}^{\infty} \frac{z_i^l}{l} a_{-l}(0)\right).\end{aligned} \tag{2.87}$$

$\Phi_r^+(t_{(-r,s)}, 0)$ transforms as a primary field with conformal dimension $h_0(t_{(-r,s)})$ whereas $\mathscr{V}(t_{(r,s)}, 0)$ does with conformal dimension $h_0(t_{(r,s)})$. Noting the relation

$$h_0(t_{(-r,s)}) = h_0(t_{(r,s)}) + rs \tag{2.88}$$

we find that the conformal dimension of $\Phi_r^+(t_{(-r,s)}, 0) = \chi_{h_{(s\ s)}, rs}(0)$ is exactly the sum of the level excitation $N = rs$ and the conformal dimension $h = h_{(r,s)}$ of the ground state

$$|h_{(r,s)}\rangle = \mathscr{V}(t_{(t,s)}, 0)|0; \lambda\rangle \tag{2.89}$$

Before proceeding to the usage of $\Phi_r^+(t_{(-r,s)}, z)$, we mention the conjugate singular vertex operator of it. As stated previously, we could consider another expression $\Phi_s^-(t_{(r,-s)}, z)$ which gives rise to the same null state $|\chi_{h_{(r,s)}, rs}\rangle$ as $\Phi_r^+(t_{(-r,s)}, z)$ does. This is simply due to the fact that, just as $\mathscr{V}(t, z)$ has the conjugate operator $\mathscr{V}(-2\lambda - t, z)$, $\Phi_r^+(t_{(-r,s)}, z)$ is paired with the conjugate operator $\Phi_s^-(-2\lambda - t_{(-r,s)}, z) = \Phi_s^-(t_{(r,-s)}, z)$ by noting the relation

$$-2\lambda - t_{(-r,s)} = t_{(r,-s)} \tag{2.90}$$

As discussed in section 2.2, the norm of the null state $|\chi_{h(r,s),rs}\rangle$ is given by

$$(2.91)\quad \langle\chi_{h(r,s),rs}|\chi_{h(r,s),rs}\rangle=\lim_{z\to\infty} z^{2(h(r,s)+rs)}\langle\Phi_s^-(t_{(r,-s)},z)\Phi_r^+(t_{(-r,s)},0)\rangle$$

which vanishes identically since there is a positive excess of momentum by $rt_+ + st_-$ ($r, s=$ positive integers) in the two point correlator. There is no way to restore the balance of "charges" by inserting the "charge screening" operators $S_\pm$ or the identity operator $\mathscr{V}(-2\lambda, \infty)$ in the null field correlation function. Thus, our state $\Phi_r^+(t_{(-r,s)}, 0)|0;\lambda\rangle$ is qualified as null state in all respects.

Here we present some examples of the null states obtained by use of our singular vertex operators $\Phi_r^+(t_{(-r,s)}, z)$. We remind the reader that a suitable choice of $\Phi_r^+(t_{(-r,s)}, z)$ or $\Phi_s^-(t_{(r,-s)}, z)$ may be taken for simplicity of the calculations.

(i) $r=1, s=1$ ($N=1$): From (2.86) we find $h_{(1,1)}=0$. Eqs. (2.84) and (2.85) give

$$(2.92)\quad \begin{aligned}|\chi_{h(1,1),1}\rangle&=\oint_{C_1} dz_1 z_1^{t_+ + t_{(-1,1)}}\\ &\times:\exp\left(t_+\sum_{l=1}^{\infty}\frac{z_1^l}{l}a_{-l}(0)\right)\mathscr{V}(t_{(1,1)},0):|0;\lambda\rangle\end{aligned}$$

By noting $t_+t_{(1,1)}=t_+t_-=-2$ we get

$$(2.93)\quad |\chi_{h(1,1),1}\rangle=t_+a_{-1}(0)\mathscr{V}(t_{(1,1)},0)|\,0;\lambda\rangle=\frac{t_+}{t_{(1,1)}}L_{-1}(\lambda)|h_{(1,1)}\rangle$$

which is just the null state expression of eq. (2.10).

Hence, by boosting back to the point z in terms of the translation operator $L_{-1}(\lambda)$, we find generically that the operator

$$(2.94)\quad \chi_{h(1,1),1}(z)=L_{-1}(z)\phi_{h(1,1)}(z)=\phi_{(1,1)}^{(-1)}(z)$$

is the null field.

(ii) $r=1, s=2$ ($N=2$): First note that

$$t_{(-1,2)}=\frac{3}{2}t_-,\qquad t_+t_{(-1,2)}=\frac{3}{2}t_+t_-=-3.$$

Therefore we get

$$|\chi_{h(1,2),2}\rangle=\oint_{C_1} dz_1 z_1^{-3}:\exp\left(t_+\sum_{l=0}^{\infty}\frac{z_1^l}{l}a_{-l}(0)\right)\mathscr{V}(t_{(1,2)},0):|0;\lambda\rangle$$

$$=\left(\frac{t_+}{2}a_{-2}(0)+\frac{t_+^2}{2!}a_{-1}(0)^2\right)\mathcal{V}(t_{(1,2)},0)|0;\lambda\rangle \tag{2.95}$$
$$=\frac{2(t_+-t_{(1,2)})}{1-2t_{(1,2)}(t_{(1,2)}-\lambda)}\left(L_{-2}(\lambda)-\frac{3}{2(2h_{(1,2)}+1)}L_{-1}(\lambda)^2\right)\cdot|h_{(1,2)}\rangle$$

which is again the expression (2.12) of the null state at level $N=2$.

Hence we see generically that the operator

$$\begin{aligned}\chi_{h(1,2),2}(z)&=\left\{L_{-2}(z)-\frac{3}{2(2h_{(1,2)}+1)}L_{-1}(z)^2\right\}\phi_{h(1,2)}(z)\\&=\phi^{(-2)}_{h(1,2)}(z)-\frac{3}{2(2h_{(1,2)}+1)}\phi^{(-1,-1)}_{h(1,2)}(z)\end{aligned} \tag{2.96}$$

is the null field.

In the above calculations it was necessary to rewrite $a_{-l}(0)$ in terms of $L_{-k}(\lambda)$. This transformation can always be done, but it is a highly nontrivial operation. Therefore we shall discuss the subject in the next section.

2.4. Kac determinant

In the previous two sections we studied the oscillator representation of the Virasoro algebra and gave the general construction of the null fields in terms of the singular vertex operators. There the null states are expressed in terms of the mode operators of the harmonic oscillators. On the other hand, the null states are expected to be constructed generically from the corresponding degenerate primary states by multiplying appropriate Virasoro operators to them. To achieve this construction, the null states given by the singular vertex operators must be reexpressed in terms of the Virasoro operators. It is the purpose of the present section to show explicitly the algebraic relation between the oscillator representation of level N states and the corresponding generic expression given in terms of the Virasoro operators. In addition, we will derive the Kac determinant[14] from the viewpoint of the osicillator representation. Our method provides an explicit proof of the Kac formula.

In the last section we learned that for the special value of $t=t_{(r,s)}$ where (r, $s=$ positive integers)

$$t_{(r,s)}=\frac{1+r}{2}t_++\frac{1+s}{2}t_- \tag{2.97}$$

$$t_\pm=-\lambda\pm\sqrt{\lambda^2+2} \tag{2.98}$$

there exists a null state

$$|\chi_{h(r,s),rs}\rangle = \mathscr{L}_{h(r,s),rs}|0;\ t_{(r,s)}+\lambda\rangle \tag{2.99}$$

at the level $N=rs$ with conformal weight

$$h_0(t_{(-r,s)}) = h_0(t_{(r,s)}) + rs \tag{2.100}$$

where $\mathscr{L}_{h(r,s),rs}$ is an operator consisting of the oscillators $a_{-n}(0)=a_{-n}$ $(n>0)$. The general construction of $\mathscr{L}_{h(r,s),rs}$ was given in Section 2.3. The null state satisfies by definition the primary state condition:

$$L_n(\lambda)|\chi_{h(r,s),rs}\rangle = 0 \qquad \text{for } n\geq 1 \tag{2.101}$$

Generally, a representation of the Virasoro algebra is specified by the value of the central charge c and of the conformal weight h. In the oscillator representation these are determined by the parameters λ and t. For the consideration that follows, we should notice that the correspondence between (c, h) and (λ, t) is four-fold degenerate; Both λ and $-\lambda$ give the same value of c, and even when λ is fixed, both t and $-2\lambda-t$ correspond to the weight (or dimension) $h=h_0(t)=h_0(-2\lambda-t)$.

Before proceeding to the main issue, it is convenient to define an operator P which we will call *parity*:

$$Pa_nP = -a_n \qquad \text{for all } n, \tag{2.102a}$$

$$Pq_0P = -q_0, \tag{2.102b}$$

or explicitly

$$P = (-)^{\sum_{n\neq 0}(1/n)a_{-n}a_n + (1/2)(p_0^2+q_0^2-1)}. \tag{2.103}$$

Under this operation, the Virasoro operators and the ground state are transformed as

$$PL_n(\lambda)P = L_n(-\lambda) \qquad \text{for all } n \tag{2.104}$$

and

$$P|0;\ t+\lambda\rangle = |0;\ -t-\lambda\rangle \tag{2.105}$$

Now, let us investigate the algebraic relations between the oscillator representations and the generic expressions in terms of the Virasoro operators for the level N states. There are $p(N)$ independent states at the level N in the Fock space of the oscillators a_n, where the degeneracy $p(N)$ is given by the number of ways of writing N as a sum of positive integers.* We can denote these states as

(2.106) $$a^{-J}|0;\, t+\lambda\rangle, \qquad J=1, 2, \cdots, p(N)$$

where

(2.107a) $$a^{J}=\text{const.}\times a_1^{n_1}a_2^{n_2}\cdots a_N^{n_N}$$

and

(2.107b) $$a^{-J}=\text{const.}\times a_{-N}^{n_N}a_{-N+1}^{n_{N-1}}\cdots a_{-1}^{n_1}$$

with

(2.108) $$\sum_{k=1}^{N} kn_n=N, \qquad n_k=0, 1, 2, \cdots, N$$

and the appropriate normalization constants are chosen so that the states be normalized as:

(2.109) $$\langle 0;\, t+\lambda|a^I a^{-J}|0;\, t+\lambda\rangle=\delta^{IJ}$$

In the same way we define the products of the Virasoro operators for the level N:

(2.110a) $$L^J(\lambda)=L_1(\lambda)^{n_1}L_2(\lambda)^{n_2}\cdots L_N(\lambda)^{n_N},$$

and

(2.110b) $$L^{-J}(\lambda)=L_{-N}(\lambda)^{n_N}L_{-N+1}(\lambda)^{n_{N-1}}\cdots L_{-1}(\lambda)^{n_1}$$

with a constraint on $\{n_1, \cdots, n_N\}$ similar to (2.108).

The relation between the states $a^{-J}|0;\, t+\lambda\rangle$ and the states $L^{-J}(\lambda)|0;\, t+\lambda\rangle$ is easily obtained; operate (2.110b) on the primary state $|0;\, t+\lambda\rangle$, translate it into the expression in terms of the oscillators using (2.19b), and carry all annihilation operators to the right by use of the commutators (2.21) so as to annihilate the ground state $|0;\, t+\lambda\rangle$, then we have

(2.111) $$L^{-I}(\lambda)|0;\, t+\lambda\rangle=\sum_J C_{IJ}(a_0, \lambda)a^{-J}|0;\, t+\lambda\rangle$$

where the coefficient matrix C_{IJ} depends on the parameter λ and the momentum operator a_0 which equals $t+\lambda$ when applied on the state

*) This is also given as the expansion coefficients of the infinite product:
$$\prod_{l=1}^{\infty}\frac{1}{1-x^l}=\sum_{N=0}^{\infty}p(N)x^N.$$

$|0; t+\lambda\rangle$. In the following we shall study the properties of this matrix $C_{IJ}(a_0, \lambda)$. Especially note that, unless the determinant of C_{IJ} vanishes, eq. (2.111) can be solved inversely. Consequently, the null state (2.99) can be reexpressed in terms of the Virasoro operators $L_{-n}(\lambda)$ $(n>0)$. This is the point which is most relevant to our problem in this section.

The answer for the inversion problem is summarized into the following proposition,

Proposition. *For the matrix $C_{IJ}(a_0, \lambda)$ at the level N, one has*

(2.112a) a) $\det[C(t+\lambda, \lambda)] = \text{const.} \times \prod_{\substack{r,s>0 \\ 1\le rs\le N}} (t-t_{(-r,-s)})^{p(N-rs)}$,

(2.112b) b) $\det[C(t+\lambda, -\lambda)] = \text{const.} \times \prod_{\substack{r,s>0 \\ 1\le rs\le N}} (t-t_{(r,s)})^{p(N-rs)}$,

These formulas exhibit all the zeros of the determinant of $C_{IJ}(a, \lambda)$. From (2.112a) we find that, indeed, eq. (2.111) can be solved inversely as

$$a^{-I}|0; t+\lambda\rangle = \sum_J [C(t+\lambda, \lambda)^{-1}]_{IJ} L^{-J}(\lambda)|0; t+\lambda\rangle, \tag{2.113}$$

unless t equals $t_{(-r,-s)}$ ($r, s=$ positive integers, $1\le rs\le N$).

Let us first prove eq. (2.112b), then coming to the proof of eq. (2.112a). The procedure for obtaining eq. (2.111) does not depend on the specific value $t+\lambda$ of the momentum of the state $|0; t+\lambda\rangle$. Therefore, with the same coeffiicient $C_{IJ}(a_0, \lambda)$, we can obtain the similar relation for the state $|0; -t-\lambda\rangle$ simply by making the substitution $t\to -2\lambda - t$ in eq. (2.111):

$$L^{-I}(\lambda)|0; -t-\lambda\rangle = \sum_J C_{IJ}(a_0, \lambda) a^{-J}|0; -t-\lambda\rangle \tag{2.114}$$

Operating the parity transformation P to eq. (2.114), we get*

$$L^{-I}(-\lambda)|0; t+\lambda\rangle = \sum_J C_{IJ}(a_0, -\lambda) a^{-J}|0; t+\lambda\rangle \tag{2.115}$$

From eqs. (2.109) and (2.115) the eoefficient matrix can be expressed in terms of the expectation value of the ground state:

$$\begin{aligned} C_{IJ}(t+\lambda, -\lambda) &= \langle 0; t+\lambda| a^J L^{-I}(-\lambda)|0; t+\lambda\rangle \\ &= \langle 0; t+\lambda| L^I(\lambda) a^{-J}|0; t+\lambda\rangle \end{aligned} \tag{2.116}$$

where the reality of $C_{IJ}(t+\lambda, -\lambda)$ and the relation (2.41), i.e. $[L^I(\lambda)]^\dagger=$

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*) Eq. (2.115) can also be obtained by making the consecutive substitutions $\lambda\to -\lambda$, then $t\to t+2\lambda$ in (2.111).

$L^{-I}(-\lambda)$ for real λ, are used.

As stated previously, there exists a null state

$$|\chi\rangle=\sum_J A_J a^{-J}|0;\, t+\lambda\rangle \tag{2.117}$$

with suitable coefficients A_J, when t is one of the special values $t_{(r,s)}$ ($rs=N$). Thereby for $t=t_{(r,s)}$

$$\sum_J C_{IJ}(t+\lambda,\, -\lambda)A_J=\langle 0;\, t+\lambda|L^I(\lambda)|\chi\rangle=0 \tag{2.118}$$

that is, $C_{IJ}(t_{(r,s)}+\lambda,\, -\lambda)$ has a zero eigenvector A_J. Thus the following relation holds for positive integers r, s ($rs=N$):

$$\det[C(t_{(r,s)}+\lambda,\, -\lambda)]=0 \tag{2.119}$$

Also for $rs<N$, this determinant has zeros, in other words, $C_{IJ}(t+\lambda,\, -\lambda)$ has zero eigenvectors; For $t=t_{(r,s)}$ ($rs<N$) ther exists a null state $|\chi\rangle$ at the level $rs=K(<N)$. Then there are $p(N-rs)$ degenerate independent states

$$L^{-j}(\lambda)|\chi\rangle=\sum_J B^{jJ}a^{-J}|0;\, t+\lambda\rangle \tag{2.120}$$

with coefficients B^{iJ} to become null states at the level N (where $L^{-j}(\lambda)$ belongs to the level $N-K$). From eq. (2.116) and (2.120)

$$\begin{aligned}\sum_J C_{IJ}(t+\lambda,\, -\lambda)B^{jJ}&=\langle 0;\, t+\lambda|L^I(\lambda)L^{-j}(\lambda)|\chi\rangle\\ &=\sum_i f^{Iji}\langle 0;\, t+\lambda|L^i(\lambda)|\chi\rangle=0\end{aligned} \tag{2.121}$$

where $L^i(\lambda)$ belongs to the level K and f^{Iji} is a suitable coefficient of the expansion

$$\langle 0;\, t+\lambda|L^I(\lambda)L^{-j}(\lambda)=\sum_i f^{Iji}\langle 0;\, t+\lambda|L^i(\lambda) \tag{2.122}$$

This expansion can be achieved by use of the commutators of the Virasoro algebra and the fact that $L_{-n}(\lambda)$ ($n\geq 0$) satisfies

$$\langle 0;\, t+\lambda|L_{-n}(\lambda)=\delta_{n,0}\langle 0;\, t+\lambda|h_0(t) \tag{2.123}$$

Therefore, $\det[C(t+\lambda,\, -\lambda)]$ has $p(N-rs)$-fold zeros at the point $t=t_{(r,s)}$ ($1\leq rs\leq N$), and is proportional to the right hand side of eq. (2.112b). Since the determinant is a polynomial in t, eq. (2.112b) will hold if and only if its zeros are saturated with $t_{(r,s)}$ ($1\leq rs\leq N$). Indeed we can prove this fact by showing that the order of the determinant with respect to t

equals that of the right hand side of eq. (2.112b), i.e. $\sum_{\substack{r,s>0 \\ 1\leq rs\leq N}} p(N-rs)$. The determinant depends on t only through a_0, so that it is sufficient to count the order in a_0. From the definition of C_{IJ} and the oscillator representation (2.19) of $L_n(\lambda)$, the maximal order of C_{IJ} with respect to a_0 for fixed I coincides with the number of L_{-n} of which L^{-I} consists. Thereby, the order of the determinant becomes

$$d_N = \sum_{\substack{\{n_i\} \\ \Sigma\, kn_k=N}} \sum_{k=1}^{N} n_k \tag{2.124}$$

Now, consider the quantity[32]

$$\begin{aligned}
\sum_{N=1}^{\infty} x^N d_N &= \sum_{N=1}^{\infty} \sum_{\substack{\{n_i\} \\ \Sigma\, kn_k=N}} \sum_{k=1}^{N} n_k x^{\Sigma\, l n_l} \\
&= \sum_{N=1}^{\infty} \sum_{\{n_i\}} \sum_{k=1}^{\infty} \delta_{\Sigma\, kn_k, N} n_k x^{\Sigma\, l n_l} \\
&= \sum_{\{n_i\}} \sum_{k=1}^{\infty} n_k x^{kn_k} \prod_{l\neq k} x^{l n_l} \\
&= \sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \prod_{l=1}^{\infty} \frac{1}{1-x^l} \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} x^{ks} \sum_{n=0}^{\infty} p(n) x^n \\
&= \sum_{N=1}^{\infty} x^N \sum_{\substack{r,s>0 \\ 1\leq rs\leq N}} p(N-rs)
\end{aligned} \tag{2.125}$$

Thus we have

$$d_N = \sum_{\substack{r,s>0 \\ 1\leq rs\leq N}} p(N-rs) \tag{2.126}$$

which completes the proof of eq. (2.112b).

Eq. (1.112a) can be derived easily from (2.112b). From (2.114) we have

$$\begin{aligned}
C_{IJ}(t+\lambda, \lambda) &= \langle 0; t+\lambda | a^J L^{-I}(\lambda) | 0; t+\lambda \rangle \\
&= \langle 0; t+\lambda | L^I(-\lambda) a^{-J} |\, 0; t+\lambda \rangle
\end{aligned} \tag{2.127}$$

Inserting the identity $P^2=1$ into the right hand side, we obtain

$$C_{IJ}(t+\lambda, \lambda) = \langle 0; -t-\lambda | L^I(\lambda) a^{-J} | 0; -t-\lambda \rangle \varepsilon_J, \tag{2.128}$$

where $\varepsilon_J = \pm 1$ is defined by

$$P a^{-J} P = \varepsilon_J a^{-J} \tag{2.129}$$

Comparing (2.128) with (2.116) under the substitution $t \to -t-2\lambda$, we get

$$C_{IJ}(-t-\lambda, -\lambda) = C_{IJ}(t+\lambda, \lambda)\varepsilon_J \tag{2.130}$$

especially for $t = t_{(-r,-s)} = -t_{(r,s)} - 2\lambda$

$$C_{IJ}(t_{(r,s)}+\lambda, -\lambda) = C_{IJ}(t_{(-r,-s)}+\lambda, \lambda)\varepsilon_J \tag{2.131}$$

Since the matrix on the left hand side has zero eigenvectors A_J, the one on the right hand side also has zero eigenvenctors $A'_J = \varepsilon_J A_J$. Thus we conclude that eq. (2.112a) holds.

Now we proceed to discuss the Kac determinant. In particular, we derive the determinant formula as a corollary of the proposition (2.112). In order to state the result we first define the Kac matrix for the level N as

$$F^{(N)}_{IJ} \equiv \langle 0; t+\lambda | L^I(\lambda) L^{-J}(\lambda) | 0; t+\lambda \rangle \tag{2.132}$$

Using eqs. (2.111) and (2.115) (Note again that $[L^I(\lambda)]^\dagger = L^{-I}(-\lambda)$),

$$\begin{aligned} F^{(N)}_{IJ} &= \sum_{K,L} \langle 0; t+\lambda | a^K C_{IK}(a_0, -\lambda) C_{JL}(a_0, \lambda) a^{-L} | 0; t+\lambda \rangle \\ &= \sum_K C_{IK}(t+\lambda, -\lambda) C_{JK}(t+\lambda, \lambda) \end{aligned} \tag{2.133}$$

Then we have

$$\begin{aligned} K^{(N)}(c, h) &= \det [F^{(N)}] = \det [C(t+\lambda, -\lambda) C(t+\lambda, \lambda)^t] \\ &= \det [C(t+\lambda, -\lambda)] \det [C(t+\lambda, \lambda)] \\ &= \text{const.} \times \prod_{\substack{r,s>0 \\ 1\le rs\le N}} [(t-t_{(r,s)})(t-t_{(-r,-s)})]^{p(N-rs)} \end{aligned} \tag{2.134}$$

Being aware that

$$\begin{aligned} (t-t_{(r,s)})(t-t_{(-r,-s)}) &= 2\left\{\left(\frac{t^2}{2}+\lambda t\right) - \left[\frac{1}{2}\left(\frac{r}{2}t_+ + \frac{s}{2}t_-\right)^2 - \frac{\lambda^2}{2}\right]\right\} \\ &= 2(h - h_{(r,s)}) \end{aligned} \tag{2.135}$$

with $h = h_0(t)$, we obtain the Kac formula:

$$K^{(N)}(c, h) = \text{const.} \times \prod_{\substack{r,s>0 \\ 1\le rs\le N}} (h - h_{(r,s)})^{p(N-rs)} \tag{2.136}$$

where we recall that

$$h_{(r,s)} = h_0(t_{(r,s)}) = \frac{1}{2}\left(\frac{r}{2}t_+ + \frac{s}{2}t_-\right)^2 - \frac{\lambda^2}{2} \tag{2.137}$$

with $t_{(r,s)}$ given by (2.97). The expression is exactly what Kac obtained for the zeros of the Kac determinant.

To conclude this section, the algebraic relations for the level N states to transform from the oscillator representation to the generic expression by the Virasoro operators are established, and, consequently, the Kac determinant is proved through the use of the obtained results.

§ III. Super-Virasoro Algebras and Null Fields

3.1. Neveu-Schwarz and Ramond algebras

In this section we discuss $N=1$ superconformal (super-Virasoro) algebras[21,22], where N is the number of two dimensional supersymmetries. In addition to the Virasoro operators L_n, there is another set of generators J_n in the superconformal case.

They are the components in the mode expansion of the super gauge current:

$$J(z)=\sum_n J_n z^{-n-3/2} \tag{3.1}$$

L_n and J_n satisfy the following algebra,

$$[L_n, L_m]=(n-m)L_{n+m}+\frac{c_1}{8}n(n^2-1)\delta_{n+m,0}, \tag{3.2a}$$

$$[L_n, J_m]=\left(\frac{n}{2}-m\right)J_{n+m}, \tag{3.2b}$$

$$\{J_n, J_m\}=2L_{n+m}+\frac{c_1}{2}\left(n^2-\frac{1}{4}\right)\delta_{n+m,0}. \tag{3.2c}$$

where c_1 is a central charge and its normalization is changed by a factor 3/2 from c of the previous section as is obviously seen in eq. (3.2a):

$$c=\frac{3}{2}c_1. \tag{3.3}$$

In the super Virasoro algebras we have two sectors depending on the boundary condition. The Neveu-Schwarz[16] (N-S) sector obeys the relation $J(e^{2\pi i}z)=J(z)$, whereas the Ramond[17] (R) sector $J(e^{2\pi i}z)=-J(z)$. Therefore, the mode suffix n of J_n runs over $\boldsymbol{Z}+\frac{1}{2}(\boldsymbol{Z})$ for the N-S (R) case. The suffix of L_n, of course, always runs over $\boldsymbol{Z}$.

The primary fields are paired into supermultiplets. Denoting them by $\phi(z)$, $\psi(z)$, they satisfy the following transformation law:

(3.4a) $$[L_n, \phi(z)] = z^n(z\partial_z + (n+1)h)\phi(z),$$

(3.4b) $$[L_n, \psi(z)] = z^n\left(z\partial_z + (n+1)\left(h + \frac{1}{2}\right)\right)\psi(z),$$

(3.4c) $$[J_n, \phi(z)] = z^{n+1/2}\psi(z),$$

(3.4d) $$\{J_n, \psi(z)\} = z^{n-1/2}\left(z\partial_n + \left(n + \frac{1}{2}\right)2h\right)\phi(z).$$

The primary states $|h\rangle$ are defined by

(3.5a) $$L_n|h\rangle = 0, \qquad n \geq 1$$

(3.5b) $$J_n|h\rangle = 0, \qquad n \geq \frac{1}{2}$$

(3.5c) $$L_0|h\rangle = h|h\rangle,$$

and they are generated from the $SL(2, C)$ invariant vacuum $|0\rangle\rangle$ by multiplying the primary fields.

(3.6) $$|h\rangle = \lim_{z\to 0} \phi(z)|0\rangle\rangle.$$

Here we should make two comments. The first is that there is not $SL(2, C)$ invariant state in the R-sector, so that the vacuum belongs to the N-S sector. The seconed is that the primary states of the R-sector, however, can be generated from the N-S vacuum by multiplying *spin fields*, which will be described later.

The Ramond primary states are doubly degenerate unless conformal dimension h equals $c_1/16$, because of the relation $J_0^2 = L_0 - c_1/16$. In that case, we can take such a degenerate pair $|h^+\rangle$ and $|h^+\rangle$ as satisfying

(3.7) $$|h^-\rangle = J_0|h^+\rangle.$$

The state $|h^-\rangle$ is not normalized to have unit norm, but $\langle h^-|h^-\rangle = h - c_1/16$ when $\langle h^+|h^+\rangle = 1$. For the case of $h = c_1/16$, $|h^-\rangle$ becomes null state, while $|h^+\rangle$ is a unique ground state (primary state) and two dimensional supersymmetry is maintained.

3.2. Null fields and Kac determinants

The oscillator representation of the superconformal algebra can be constructed as a supersymmetric extension of that of the Virasoro algebra. We need one boson and one fermion oscillator fields:

(3.8) $$\varphi(z)=q_0-i(p_0-\lambda)\ln z+i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n},$$

(3.9) $$\gamma(z)=\sum_n b_n z^{-n-1/2}.$$

where the commutation relations of the oscillators are

(3.10) $$\begin{aligned}&[a_n, a_m]=n\delta_{n+m,0},\\&[q_0, p_0]=i,\\&\{b_n, b_m\}=\delta_{n+m,0},\\&[\text{otherwise}]=0.\end{aligned}$$

The index n of the fermionic mode b_n runs over $Z+1/2(Z)$ for the N-S (R) sector. Using them we can write down the energy-momentum tensor and supergauge current as follows,

(3.11) $$T(z)=\frac{1}{2}:A(z)^2:-\lambda\partial_z A(z)-\frac{1}{2}:\gamma(z)\partial_z\gamma(z):-\frac{1}{8z}:\gamma(z)^2:,$$

(3.12) $$J(z)=\gamma(z)A(z)-2\lambda\partial_z\gamma(z),$$

where $A(z)=i\partial_z\varphi(z)$. The last term of (3.11) equals $-1/16z^2$ for the R-sector, whereas it vanishes for the N-S sector. The mode operators in the Laurent expansions of $T(z)$ and $J(z)$ satisfy the superconformal algebra (3.2) with the central charge

(3.13) $$c_1=1-8\lambda^2.$$

We are interested in the real λ case, i.e. $c_1<1$, as was in Section II.

In terms of the oscillator fields, the multiplet of primary fields is constructed as follows:

(3.14a) $$\mathscr{V}(t, z)=:\exp(it\varphi(z)):,$$

(3.14b) $$\mathscr{W}(t, z)=:t\gamma(z)\mathscr{V}(t, z):.$$

These operators represent the fields $\phi(z)$, $\psi(z)$ in eq. (3.4), respectively, with the conformal weight $h=h_0(t)=t^2/2+\lambda t$.

As was in the non-supersymmetric case, the ket vacuum $|0;\lambda\rangle$ which belongs to the N-S sector is defined by

(3.15) $$\begin{aligned}&a_n|0;\lambda\rangle=0 \qquad n\geq 1,\\&b_n|0;\lambda\rangle=0 \qquad n\geq\frac{1}{2},\\&a_0|0;\lambda\rangle=\lambda|0;\lambda\rangle,\end{aligned}$$

which satisfies

$$L_n(\lambda)|0;\lambda\rangle=0 \qquad n\geq -1, \tag{3.16}$$
$$J_n(\lambda)|0;\lambda\rangle=0 \qquad n\geq -\frac{1}{2},$$

The corresponding bra vacuum is given by the state $\langle 0;-\lambda|$ and the correlation functions of the fields can be calculated by sandwitching the corresponding suitable vertex operators between the two vacua as before (see the discussion in Section 2.2).

A primary state in the N-S sector is obtained from the vacuum according to eq. (3.6).

$$\lim_{z\to 0}\mathscr{V}(t,z)|0;\lambda\rangle=|0;t+\lambda\rangle \tag{3.17}$$

This state satisfies

$$L_n(\lambda)|0;t+\lambda\rangle=0 \qquad n\geq 1 \tag{3.18a}$$

$$L_0(\lambda)|0;t+\lambda\rangle=\left(\frac{t^2}{2}+\lambda t\right)|0;t+\lambda\rangle \tag{3.18b}$$

$$J_n(\lambda)|0;t+\lambda\rangle=0 \qquad n\geq\frac{1}{2} \tag{3.18c}$$

For the R-sector the fermionic zero mode has the propety $b_0^2=1/2$, hence we have degenerate ground states $|0;t+\lambda;\pm\rangle$ which satisfy

$$|0;t+\lambda;\mp\rangle=\sqrt{2}\,b_0|0;t+\lambda;\pm\rangle \tag{3.19}$$

These states can be obtained from the ground state with momentum λ, i.e.

$$|0;t+\lambda;\pm\rangle=\lim_{z\to 0}\mathscr{V}(t,z)|0;\lambda;\pm\rangle, \tag{3.20}$$

and satisfy the primary state condition,

$$L_n(\lambda)|0;t+\lambda;\pm\rangle=0 \qquad n\geq 1, \tag{3.21a}$$

$$L_0(\lambda)|0;t+\lambda;\pm\rangle=\left(\frac{t^2}{2}+\lambda t+\frac{1}{16}\right)|0;t+\lambda;\pm\rangle, \tag{3.21b}$$

$$J_n(\lambda)|0;t+\lambda;\pm\rangle=0 \qquad n\geq 1, \tag{3.21c}$$

and

$$J_0(\lambda)|0;t+\lambda;+\rangle=\frac{t+\lambda}{\sqrt{2}}|0;t+\lambda;-\rangle. \tag{3.22}$$

Comparing eqs. (3.7) and (3.22), one can conclude that the states $\alpha_\pm|0;t+\lambda;\pm\rangle$ represent the Ramond states $|h^\pm\rangle$ where $\alpha_+=1$, $\alpha_-=(t+\lambda)/\sqrt{2}=\sqrt{h-c_1/16}$. Note that the state $|0;\lambda;\pm\rangle$ is not an SL(2, C) invariant vacuum in spite of eq. (3.20), but a ground state with conformal dimension 1/16.

Now let us construct null fields. In order to do so we have to find the conformally invariant vertex operator. If we choose $t=t_+$ or t_- which are the solutions of the equation $h_0(t)=1/2$, i.e.

$$t_\pm=-\lambda\pm\sqrt{\lambda^2+1}, \tag{3.23}$$

then $\mathscr{W}(t_\pm, z)$ becomes conformally invariant:

$$[L_n, \mathscr{W}(t_\pm, z)]=\partial_z(z^{n+1}\mathscr{W}(t_\pm, z)), \tag{3.24a}$$

$$\{J_n, \mathscr{W}(t_\pm, z)\}=\partial_z(z^{n+1/2}\mathscr{V}(t_\pm, z)). \tag{3.24b}$$

Thereby, we can construct the singular vertex operator (SVO)

$$\Phi_r^\pm(t,z)=\oint_{C_r} dz_r\int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1\mathscr{W}(t_\pm,z_r)\cdots\mathscr{W}(t_\pm,z_1)\mathscr{V}(t,z) \tag{3.25a}$$

$$=\oint_{C_r} dz_r\int_z^{z_r} dz_{r-1}\cdots\int_z^{z_2} dz_1 f^\pm(t;z,z_i)$$

$$:\gamma(z_r)\cdots\gamma(z_1)\exp\left(t_\pm\sum_{i=1}^{r}\sum_{l=0}^{\infty}\frac{(z_i-z)^{l+1}}{(l+1)!}A^{(l)}(z)\right)\cdot\mathscr{V}(rt_\pm+t,z): \tag{3.25b}$$

where $f_r^\pm(t;z,z_i)$ is defined by (2.57) and the integration contour C_r is illustrated in Fig. 1.

This SVO has two important properties:

a) $\Phi_r^\pm(t,z)$ has the same commutation relations with the L_n, J_n's as $\mathscr{V}(t,z)$ does.

b) $\Phi_r^\pm(t,z)$ exists nontrivially (i.e. is nonvanishing) if and only if t takes one of the values

$$t=\frac{1-r}{2}t_\pm+\frac{1+s}{2}t_\pm \tag{3.26}$$

where r and s take positive integers which satisfy $r-s=$ even (odd) for the N-S (R) sector. Although the proof of the above goes in the same way as in section II, one should be careful that *the on shell constraint* gives only the necessary conditions, $r=$ integer and $s=2N/r$ where $N=$ integer or half integer, and that the sufficiency of the above condition b) is obtained from the argument about the number of zeros of the Kac determinant (see Section 2.4).

In accordance with the argument for the nonsupersymmetric case, we obtain the formulas of degenerate highest weights for the Neveu-Schwarz and the Ramond algebras, respectively

(3.27) $$h^{N-S}_{(r,s)}=\frac{1}{2}\left(\frac{r}{2}t_{+}+\frac{s}{2}t_{-}\right)^{2}-\frac{\lambda^{2}}{2}$$

where r, $s=$ positive integer and $r-s=$ even, and

(3.28) $$h^{R}_{(r,s)}=\frac{1}{2}\left(\frac{r}{2}t_{+}+\frac{s}{2}t_{-}\right)^{2}-\frac{\lambda^{2}}{2}+\frac{1}{16}$$

where r, $s=$ positive integer and $r-s=$ odd.

Counting the degeneracy of the null fields (see Appendix), we obtain the Kac formula[14,22] for each sector,

(3.29) $$\det[F^{(N)}_{N-S}]=\text{const.}\times\sum_{\substack{0>r,s\in Z\\ rs\leq 2N\\ r-s=\text{even}}}(h-h^{N-S}_{(r,s)})^{p_{NS}(N-rs/2)}$$

(3.30a) $$\det[F^{(N)}_{R+}]=\text{const.}\times\prod_{\substack{0<r,s\in Z\\ rs\leq 2N\\ r-s=\text{odd}}}(h-h^{R}_{(r,s)})^{p_{R}(N-rs/2)}$$

(3.30b) $$\det[F^{(N)}_{R-}]=\text{const.}\times\left(h-\frac{c_1}{16}\right)^{p_R(N)}\prod_{\substack{0<r,s\in Z\\ rs\leq 2N\\ r-s=\text{odd}}}(h-h^{R}_{(r,s)})^{p_{R}(N-rs/2)}$$

where $F^{(N)}_{R\pm}$ are the inner product matrix of the Verma module on the primary state $|h^{\pm}\rangle$ in the R-sector, and the number of partitions $p_{NS}(N)$ and $p_R(N)$ are defined by the equations

(3.31a) $$\sum_N p_{NS}(N)x^N=\frac{\prod_{0<l\in Z+1/2}(1+x^l)}{\prod_{0<n\in Z}(1-x^n)}$$

(3.31b) $$\sum_N p_{R}(N)x^N=\frac{\prod_{0<l\in Z}(1+x^l)}{\prod_{0<n\in Z}(1-x^n)}$$

3.3. Spin fields

As mentioned before, a Ramond primary state is obtained from the ground state $|0;\lambda;\pm\rangle$ according to eq. (3.20). If we have the operator which creates this state from the ket vacuum, say

(3.32) $$\lim_{z\to 0}W^{\pm}(z)|0;\lambda\rangle=|0;\lambda;\pm\rangle,$$

then combining the vertex operator $\mathscr{V}(t, z)$ we can construct the primary field which generate the Ramond primary state from the ket vacuum

$$\alpha_{\pm}|0; t+\lambda; \pm\rangle = \lim_{z\to 0} S^{\pm}(t, z)|0; \lambda\rangle, \tag{3.33}$$

$$S^{\pm}(t, z) = \alpha_{\pm}\mathscr{V}(t, z)W^{\pm}(z). \tag{3.34}$$

The field $S^{\pm}(t, z)$ is called the *spin field*[22] and can indeed be constructed by use of the fermion emission vertex[34,35] in dual string theory.

Let us describe it in the following. Let the suffix N-S and R explicitly denote the Neveu-Schwarz and Ramond sectors respectively to distinguish the fermionic fields in each sector, since we have to deal with both of them simultaneously. Following ref. 35 the operator $W^{\pm}(z)$ is constructed as

$$W^{\pm}(z) = e^{zL_{-1}^{(R)}}\tilde{W}^{\pm}(z) \tag{3.35}$$

$$\tilde{W}^{\pm}(z) = \langle 0; \lambda|\langle 0'|\exp(I_1(z) + iI_2(\varepsilon))|0'\rangle\, |0; \lambda; \pm\rangle, \tag{3.36}$$

where

$$I_1(z) = \oint_{C_1} dx\, \gamma_R(x-z)\gamma_{N-S}(x). \tag{3.37}$$

$$I_2(\varepsilon) = \oint_{C_2} dx\, \gamma'_{N-S}(x-\varepsilon)\gamma_{N-S}(x), \tag{3.38}$$

and $\gamma'_{N-S}(z)$ is a redundant N-S field of which the vacuum is denoted by $|0'\rangle$ and $\langle 0'|$. The Virasoro operator $L_{-1}^{(R)}$ consists only of the Ramond oscillators, not including the bosonic oscillators. The integration contours C_1 and C_2 are shown in Fig. 3. ε can be thought of as small but nonzero and $\tilde{W}^{\pm}(z)$ can be proved to be independent of it.

An important property of $W^{\pm}(z)$ is

$$W^{\pm}(z)\gamma_{N-S}(y) = \gamma_R(y)W^{\pm}(z). \tag{3.39}$$

Thereby we have

$$\begin{aligned} S^{\pm}(t, y)J^{N-S}(z) - J^{R}(z)S^{\pm}(t, y) &= \alpha_{\pm}\gamma_R(z)W^{\pm}(y)[\mathscr{V}(t, y), A(z)] \\ &= \alpha_{\pm}W^{\pm}(y)\gamma_{NS}(z)[\mathscr{V}(t, y), A(z)], \end{aligned} \tag{3.40}$$

of which the r.h.s. vanishes unless $y=z$ because of

$$[\mathscr{V}(t, y), A(z)] = 0 \qquad \text{for } y \neq z. \tag{3.41}$$

From the above relation we can obtain the supergauge condition for the

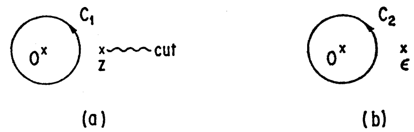

Fig. 3 The integration contours (a) C_1 and (b) C_2 respectively for eqs. (3.37) and (3.38) are shown.

spin field

$$K^{\dagger}_{-n-1/2}S^{\pm}(t,1)-iS^{\pm}(t,1)K_n=\alpha^2_{\pm}S^{\mp}(t,1), \tag{3.42}$$

where

$$K_n=\oint_{|z|<1}dz\,z^{n+1/2}(1-z)^{1/2}J(z), \tag{3.42}$$

and

$$\alpha^2_{+}=1, \qquad \alpha^2_{-}=\frac{(t+\lambda)^2}{2}=h-\frac{c_1}{16}.$$

The proof of (3.42) is the following. Owing to the eq. (3.40), we have

$$\begin{aligned} S^{\pm}(t,1)K_n&=\int_{|z|<1}dz\,z^{n+1/2}(1-z)^{1/2}S^{\pm}(t,1)J^{NS}(z)\\ &=\int_{|z|<1}dz\,z^{n+1/2}(1-z)^{1/2}J^{R}(z)S^{\pm}(t,1)\end{aligned} \tag{3.44}$$

Using the relation

$$(z^{3/2}J_{\lambda}(z))^{\dagger}=(1/z^*)^{3/2}J_{-\lambda}(1/z^*) \tag{3.45}$$

where we explicitly denote the λ dependence of $J(z)$, the r.h.s. of the above becomes

$$\left(\oint_{|1/z^*|>1}d\left(\frac{1}{z^*}\right)\left(\frac{1}{z^*}\right)^{(-n-1/2)+1/2}\left(\frac{1}{z^*}-1\right)^{1/2}J^{R}_{-\lambda}\left(\frac{1}{z^*}\right)\right)^{\dagger}S^{\pm}(t,1) \tag{3.46}$$

The integration contour of $1/z^*$ which encircles both the points 0 and 1 can be separated into two contours C_0 and C_1, where $C_0(C_1)$ circles around the point 0(1). Then, rewriting the contribution with C_0 in terms of the integration variable $w=1/z^*$ and putting that with C_1 back to the expression by z, we have

$$\left(i\oint_{|w|<1} dw\, w^{(-n-1/2)+1/2}(1-w)^{1/2}J^R_{-\lambda}(w)\right)^{\dagger} S^{\pm}(t,1) \qquad (3.47)$$
$$+\oint_{C_1} dz\, z^{n+1/2}(1-z)^{1/2}J^R_{\lambda}(z)S^{\pm}(t,1)$$

The operator $(J_{-\lambda})^{\dagger}$ acts on the space of λ (not $-\lambda$), so that the first term of (3.47) can be written as $-iK^{\dagger}_{-n-1/2}S^{\pm}(t,1)$. Hence we obtain

$$K^{\dagger}_{-n-1/2}S^{\pm}(t,1)-iS^{\pm}(t,1)K_n \qquad (3.48)$$
$$=-i\oint_{C_1} dz\, z^{n+1/2}(1-z)^{1/2}J^R_{\lambda}(z)S^{\pm}(t,1)$$

where the contour C_1 encircling the point 1 excluding the origin. Now the task is the evaluation of the r.h.s. Due to the property $[I_1(z)+iI_2(\varepsilon), \gamma_R(y-z)]=0$ which can easily be proved, we can write

$$\gamma_R(z)W^{\pm}(y)=\sum_{l=0}^{\infty}(z-y)^{l-1/2}W^{\pm}_l(y) \qquad (3.49)$$

where

$$W^{\pm}_l(y)=e^{yL^{(R)}_{-1}}\langle 0;\lambda|\langle 0'|\exp(I_1(y)+iI_2(\varepsilon))b_{-l}|0'\rangle|0;\lambda;\pm\rangle, \qquad (3.50)$$

so that

$$J^R_{\lambda}(z)S^{\pm}(t,1)=(\gamma_R(z)A(z)-2\lambda\partial_z\gamma_R(z))\alpha_{\pm}\mathscr{V}(t,1)W^{\pm}(1)$$
$$=\left(\frac{t}{z-1}\gamma_R(z)-2\lambda\partial_z\gamma_R(z)\right)\alpha_{\pm}W^{\pm}(1)\mathscr{V}(t,1)$$
$$+\alpha_{\pm}\gamma_R(z)W^{\pm}(1){:}\,A(z)\mathscr{V}(t,1){:} \qquad (3.51)$$
$$=(t+\lambda)(z-1)^{-3/2}\alpha_{\pm}W^{\pm}_0(1)\mathscr{V}(t,1)$$
$$+\sum_{l=0}^{\infty}(z-1)^{l-1/2}\alpha_{\pm}\{(t-2\lambda(l+1/2))W^{\pm}_{l+1}(1)\mathscr{V}(t,1)$$
$$+W^{\pm}_l(1){:}\,A(z)\mathscr{V}(t,1){:}\}$$

Substituting the above into the r.h.s. of eq. (3.48), only the first term survives through the contour integration and becomes

$$\frac{t+\lambda}{\sqrt{2}}(\alpha^{\pm}/\alpha^{\mp})S^{\mp}(t,1)=\alpha^2_{\pm}S^{\mp}(t,1),$$

where we have used the relation $W^{\pm}_0(1)=(1/\sqrt{2})W^{\mp}(1)$. Thus we have proved eq. (3.42).

Similarly we can show

$$(3.52)\qquad [L_n, S^{\pm}(t, z)]=z^n\left\{z\partial_z+(n+1)\left(\frac{t^2}{2}+\lambda t+\frac{1}{16}\right)\right\}S^{\pm}(t, z)$$

of which the proof will be omitted. This explicitly demonstrates that $S^{\pm}(t, z)$ tranforms as a primary field with conformal dimension $h=h_0(t)+1/16$.

§ IV. $N=2$ Superconformal Algebras and Null Fields

4.1. Algebraic relations

In this section we are to discuss on the $N=2$ superconformal algebras[5]. Comparing with the $N=1$ superconformal algebras which have been argued in the last section, the higher N algebras have not only extra supersymmetries but also some (bosonic) gauge symmetries[5]; in the $N=2$ case there is $U(1)$ or $O(2)$ gauge symmetry. Here the current and the mode operators of the $U(1)$ gauge symmetry are denoted by

$$(4.1)\qquad I(z)=\sum_n I_n z^{-n-1}.$$

Then the $N=2$ superconformal algebra is given as follows:

$$(4.2a)\qquad [L_n, L_m]=(n-m)L_{n+m}+\frac{c_2}{4}n(n^2-1)\delta_{n+m,0}$$

$$(4.2b)\qquad [L_n, J^i_m]=\left(\frac{n}{2}-m\right)J^i_{n+m}$$

$$(4.2c)\qquad \{J^i_n, J^j_m\}=2\delta^{ij}L_{n+m}+(n-m)i\varepsilon^{ij}I_{n+m}+c_2\left(n^2-\frac{1}{4}\right)\delta^{ij}\delta_{n+m,0}$$

$$(4.2d)\qquad [I_n, I_m]=c_2 n\delta_{n+m,0}$$

$$(4.2e)\qquad [I_n, J^i_m]=i\varepsilon^{ij}J^j_{n+m}$$

$$(4.2f)\qquad [I_n, L_m]=nI_{n+m}$$

where the constant c_2 is the central charge and is related to the c and c_1 in the normalization

$$(4.3)\qquad c=\frac{3}{2}c_1=3c_2.$$

In this case there are three sectors depending on the boundary conditions, which are summarized in Table 1. It is convenient to define the supercurrents $J^{\pm}_n=(1/\sqrt{2})(J^1_n\pm iJ^2_n)$. Then the corresponding commutation

Table 1. There are three sectors for $N=2$ superconformal algebra. In each sector the indices of the generators run over integers or half-integers as shown in the table.

<table>
<tr><th rowspan="2">Sectors</th><th colspan="4">Indices of generators</th></tr>
<tr><th>L_n</th><th>J^1_n</th><th>J^2_n</th><th>I_n</th></tr>
<tr><td>Neveu-Schwarz</td><td rowspan="4">Z</td><td>$Z+\frac{1}{2}$</td><td>$Z+\frac{1}{2}$</td><td rowspan="2">Z</td></tr>
<tr><td>Ramond</td><td>Z</td><td>Z</td></tr>
<tr><td rowspan="2">Twisted</td><td>Z</td><td>$Z+\frac{1}{2}$</td><td rowspan="2">$Z+\frac{1}{2}$</td></tr>
<tr><td>$Z+\frac{1}{2}$</td><td>Z</td></tr>
</table>

relations become

(4.4a) $$\{J^+_n, J^-_m\}=2L_{n+m}+(n-m)I_{n+m}+c_2\left(n^2-\frac{1}{4}\right)\delta_{n+m,0},$$

(4.4b) $$\{J^\pm_n, J^\pm_m\}=0,$$

(4.4c) $$[L_n, J^\pm_m]=\left(\frac{n}{2}-m\right)J^\pm_{n+m},$$

(4.4d) $$[I_n, J^\pm_m]=\pm J^\pm_{n+m}.$$

From (4.4d), we can see the current $J^\pm$ carries the $U(1)$ charge ± 1.

A primary multiplet consists of two bosonic and two fermonic fields. Denoting them by $F(z)$, $G^i(z)$ ($i=1$, 2 or $\pm$) and $H(z)$, the general transformation laws in the Neveu-Schwarz (N-S) and Ramond (R) sectors are written as follows,

(4.5a) $$[L_n, F(z)]=z^n(z\partial_z+(n+1)h)F(z),$$

(4.5b) $$[L_n, G^\pm(z)]=z^n(z\partial_z+(n+1)(h+1/2))G^\pm(z).$$

(4.5c) $$[L_n, H(z)]=z^n(z\partial_z+(n+1)(h+1))H(z),$$

(4.5d) $$[J^\pm_n, F(z)]=z^{n+1/2}G^\pm(z),$$

(4.5e) $$\{J^\pm_n, G^\pm(z)\}=0,$$

(4.5f) $$\{J^\pm_n, G^\mp(z)\}=z^{n-1/2}(z\partial_z+(n+1/2)(2h\pm\tau))F(z)\mp z^{n+1/2}H(z),$$

(4.5g) $[J_n^{\pm}, H(z)] = \pm z^{n-1/2}(z\partial_z + (n+1/2)(4h+1\pm 2\tau))G^{\pm}(z),$

(4.5h) $[I_n, F(z)] = z^n \tau F(z),$

(4.5i) $[I_n, G^{\pm}(z)] = z^n(\tau\pm 1)G^{\pm}(z),$

(4.5j) $[I_n, H(z)] = z^n \tau H(z) - z^{n-1}2nhF(z).$

The multiplet is characterized by two physical quantities, i.e. the conformal dimension h and the conformal $U(1)$ charge τ of the primary field $F(z)$. Those of the partner fields $G^{\pm}(z)$ and $H(z)$ are $(h+1/2, \tau\pm 1)$ and $(h+1, \tau)$ respectively, and the corresponding structure of the multiplet is summarized in Fig. 4.

In the twisted (T) sector, the global $U(1)$ charge I_0 or τ is absent, hence the multiplet in the T sector is characterized only by conformal dimension h.

The primary states are defined by

(4.6a) $L_n|h, \tau\rangle = 0 \qquad n\geq 1,$

(4.6b) $J_n^i|h, \tau\rangle = 0 \qquad n\geq 1/2,\ i=1, 2,$

(4.6c) $I_n|h, \tau\rangle = 0 \qquad n\geq 1,$

(4.6d) $L_0|h, \tau\rangle = h|h, \tau\rangle,$

(4.6e) $I_0|h, \tau\rangle = \tau|h, \tau\rangle,$

and are generated from the ket vacuum $|0\rangle\rangle$ as before, where the ket vacuum is defined by

(4.7a) $L_n|0\rangle\rangle = 0 \qquad n\geq -1,$

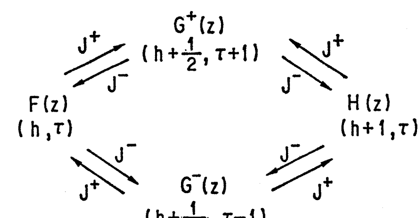

Fig. 4 The supersymmetric structure of the $N=2$ primary multiplet is shown. The arrows represent the transformation law by the supergauge generators $J^{\pm}$. The conformal dimension and the $U(1)$ charge are indicated in the parenthesis for each field.

(4.7b) $$J^i_n|0\rangle\rangle=0 \qquad n\geq -1/2,$$

(4.7c) $$I_n|0\rangle\rangle=0 \qquad n\geq 0.$$

4.2. Null field constructions and Kac determinants

Introducing two bosonic and two fermionic oscillators with real parameters λ and μ

(4.8a) $$\varphi(z)=q_0-i(p_0-\lambda)\ln z+i\sum_{n\neq 0}\frac{a_n}{n}z^{-n},$$

(4.8b) $$\phi(z)=Q_0-i(P_0-\mu)\ln z+i\sum_{n\neq 0}\frac{c_n}{n}z^{-n},$$

(4.8c) $$\gamma^i(z)=\sum_n b^i_n z^{-n-1/2} \qquad (i=1,2),$$

we can represent the currents in the following generic form:

(4.9a) $$\begin{aligned}T(z)=&\frac{1}{2}:A(z)^2:+\frac{1}{2}:C(z)^2:-\lambda\partial_z A(z)-\mu\partial_z C(z)\\&-\frac{1}{2}:\gamma^i(z)\partial_z\gamma^i(z):-\frac{1}{8z}:\gamma^i(z)\gamma^i(z):\end{aligned}$$

(4.9b) $$J^i(z)=\gamma^i(z)A(z)-2\lambda\partial_z\gamma^i(z)-\varepsilon^{ij}[\gamma^j(z)C(z)-2\mu\partial_z\gamma^j(z)]$$

(4.9c) $$I(z)=-\frac{i}{2}\varepsilon^{ij}:\gamma^i(z)\gamma^j(z):-2i\lambda C(z)+2i\mu A(z)$$

where $A(z)=i\partial_z\varphi(z)$, $C(z)=i\partial_z\phi(z)$.

In eq. (4.8) the suffix n of b^i_n runs over $Z+1/2$ or Z corresponding to those of J^i_n in each sector and the index n of c_n runs over $Z+1/2$ in the T sector. Commutation relations for the oscillators are

(4.10a) $$[a_n, a_m]=[c_n, c_m]=n\delta_{n+m},$$

(4.10b) $$[q_0, p_0]=[Q_0, P_0]=i,$$

(4.10c) $$\{b^i_n, b^j_m\}=\delta_{n+m,0}\delta^{ij},$$

(4.10d) $$[a_n, c_m]=[a_n, b^i_m]=[c_n, b^i_m]=0,$$

and we also use the notation $a_0\equiv p_0$ amd $c_0\equiv P_0$ for the zero modes. From eq. (4.10c) $(b^i_0)^2=1/2$ for the R case, then the last term of eq. (4.9a) becomes $-1/(8z^2)$ $(-1/(16z^2)$ for T case and it vanishes for the N-S case).

The components of the currents (4.9) satisfy the algebra (4.2) with central cahrge

(4.11) $$c_2=1-4(\lambda^2+\mu^2).$$

In the following we can take $\mu=0$ for simplicity, which does not spoil the generality of the discussion.

The vertex operators which correspond to a primary fields multiplet are defined as

(4.12a) $$V(t, u, z)=:\exp[it\varphi(z)+iu\phi(z)]:,$$

(4.12b) $$V^{\pm}(t, u, z)=:(t\pm iu)\gamma^{\pm}(z)V(t, u, z):,$$

(4.12c) $$V_{12}(t, u, z)=:i\left[\frac{t^2+u^2}{2}\varepsilon^{ij}\gamma^i(z)\gamma^j(z)-tC(z)+uA(z)\right]V(t, u, z):,$$

where

$$\gamma^{\pm}(z)=\frac{1}{\sqrt{2}}(\gamma^1(z)\pm i\gamma^2(z)).$$

They form an $N=2$ superconformal multiplet $(F, G^{\pm}, H)$ with the conformal weight h and $U(1)$ charge τ

(4.13a) $$h=\frac{1}{2}(t^2+u^2)+\lambda t,$$

(4.13b) $$\tau=-i2\lambda u,$$

where we take u pure imaginary for $c_2<1$ so as for τ to be real. The ket vacuum $|0; \lambda, 0\rangle$ is defined by

(4.14a) $$a_n|0; \lambda, 0\rangle=c_n|0; \lambda, 0\rangle=b^i_n|0; \lambda, 0\rangle=0 \quad (n>0, i=1, 2),$$

(4.14b) $$a_0|0: \lambda, 0\rangle=\lambda|0; \lambda, 0\rangle,$$

(4.14c) $$c_0|0; \lambda, 0\rangle=0.$$

This vacuum satisfies the requirement of superconformal invariance

(4.15a) $$L_n(\lambda)|0; \lambda, 0\rangle=0 \qquad n\geq -1$$

(4.15b) $$J^i_n(\lambda)|0; \lambda, 0\rangle=0 \qquad n\geq -1/2$$

(4.15c) $$I_n(\lambda)|0; \lambda, 0\rangle=0 \qquad n\geq 0$$

while the ground state $|0; t+\lambda, u\rangle$ with a_0 (c_0) momentum$=t+\lambda$ (u)

satisfies the primary state conditions

$$
\begin{aligned}
&L_n(\lambda)|0;t+\lambda,u\rangle=0 && n\geq 1\\
&J_n^i(\lambda)|0;t+\lambda,u\rangle=0 && n\geq 1/2\\
&I_n(\lambda)|0;t+\lambda,u\rangle=0 && n\geq 1\\
&L_0(\lambda)|0;t+\lambda,u\rangle=\left(\frac{t^2+u^2}{2}+\lambda t\right)|0;t+\lambda,u\rangle\\
&I_0(\lambda)|0;t+\lambda,u\rangle=-2i\lambda u|0;t+\lambda,u\rangle
\end{aligned}
\tag{4.16}
$$

Now consider three special vertex operators $V_{12}(-2\lambda, 0, z)$, $V^{\pm}(1/2, \mp i/2\lambda, z)$. They are conformally invariant vertex operators satisfying the following commutation relations,

$$
\begin{aligned}
&[L_n, V_{12}(-2\lambda,0,z)]=\partial_z(z^{n+1}V_{12}(-2\lambda,0,z)),\\
&[J_n^{\pm}, V_{12}(-2\lambda,0,z)]=\pm\partial_z(z^{n+1/2}V^{\pm}(-2\lambda,0,z)),\\
&[I_n, V_{12}(-2\lambda,0,z)]=0,
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
&[L_n, V^{\pm}(1/2\lambda, \mp i/2\lambda, z)]=\partial_z(z^{n+1}V^{\pm}(1/2\lambda, \mp i/2\lambda, z))\\
&\{J^{\pm}, V^{\pm}(1/2\lambda, \mp i/2\lambda, z)\}=0\\
&\{J^{\mp}, V^{\pm}(1/2\lambda, \mp i/2\lambda, z)\}=\partial_z\left(z^{n+1}V\left(\frac{1}{2\lambda}, \mp\frac{i}{2\lambda}, z\right)\right)\\
&[I_n, V^{\pm}(1/2\lambda, i/2\lambda, z)]=0
\end{aligned}
\tag{5.18}
$$

In calculating them we have used the relation $V_{12}(1/2\lambda, \mp i/2\lambda, z) = \pm\partial_z V(1/2\lambda, \mp i/2\lambda, z)$.

Utilizing these conformally invariant vertex operators we can construct the singular vertex operators (SVO) for the null fields*)

$$
\begin{aligned}
\Phi_r(t,u,z)=&\oint_{C_r} dz_z V_{12}(-2\lambda,0,z_z)\int_z^{z_r} dz_{r-1}V_{12}(-2\lambda,0,z_{r-1})\cdots\\
&\cdots\int_z^{z_2} dz_1 V_{12}(-2\lambda,0,z_1)V(t,u,z)
\end{aligned}
\tag{4.19a}
$$

$$
\Theta^{\pm}(t,u,z)=\oint_{C_r} dy V^{\pm}(1/2\lambda, \mp i/2\lambda, y)V(t,u,z)
\tag{4.19b}
$$

where the contour C_r is defined similarly to that shown in Fig. 1.

*) The repeated operation of $V^{\pm}$ in eq. (4.19b) is forbidden since it gives rize to an ill-defined expression for the short distance behaviour of the c_0 momentum sector.

These SVO's have important properties:

a) $\Phi_r(t, u, z)$ and $\Theta^{\pm}(t, u, z)$ have the same commutation relations with the L_n, J^i_n, I_n's as $V(t, u, z)$ does.

b) $\Phi_r(t, u, z)$ exists nontrivialy (i.e. is nonvanishing) if and only if t takes one of the values

$$t_{(-r,s)}=\frac{1-r}{2}t_- - \frac{s}{t_-} \tag{4.20}$$

where r and s take positive integers and $t_- = -2\lambda$.

c) $\Theta^{\pm}(t, u, z)$ exists nontrivially if and only if t and u satisfy the relation

$$-\frac{1}{t_-}(t \mp iu)+l+1/2=0 \tag{4.21}$$

where $0<l \in \boldsymbol{Z}+1/2$.

We can prove them in the same way as in the Virasoro and $N=1$ superconformal cases.

From the above properties we obtain the degenerate conformal weights for the primary fields and the corresponding Kac formulae.

For the N-S sector we have

$$h_{(r,s)}=\frac{1-c_2}{8}(r^2-1)-\frac{rs}{2}+\frac{1}{2(1-c_2)}(s^2-\tau^2) \tag{4.22a}$$

$$h^{\pm}_{(l)}=\pm\tau l+\frac{1-c_2}{2}\left(l^2-\frac{1}{4}\right) \tag{4.22b}$$

where $0<r$, $s \in \boldsymbol{Z}$ and $0<l \in \boldsymbol{Z}+1/2$, and the Kac determinant formula for level N[27,36]

$$\begin{aligned}\det[F^{(N)}_{N-S}]=\text{const.}\times&\prod_{\substack{0<r,s\in \boldsymbol{Z}\\ 1\le rs\le N}}(h-h_{(r,s)})^{p^{N-S}(N-rs)}\\ \times&\prod_{\substack{0<l\in \boldsymbol{Z}+1/2\le N\\ A=\pm}}(h-h^A_{(l)})^{p^{N-S}_l(N-l)}\end{aligned} \tag{4.23}$$

where the number of partitions $p^{NS}(N)$ and $p^{N-S}_l(N)$ are defined by

$$\sum_N p^{N-S}(N)x^N=\frac{\prod_{0>k\in \boldsymbol{Z}+1/2}(1+x^k)^2}{\prod_{0<n\in \boldsymbol{Z}}(1-x^n)^2} \tag{4.24a}$$

$$\sum_N p^{N-S}_l(N)x^N=\frac{1}{1+x^l}\,\frac{\prod_{0<k\in \boldsymbol{Z}+1/2}(1+x^k)^2}{\prod_{0<n\in \boldsymbol{Z}}(1-x^n)^2} \tag{4.24b}$$

Similarly, we can obtain the degenerate highest weights and Kac derterminant for the R sector.

$$(4.25a)\qquad h_{(r,s)}=\frac{1-c_2}{8}(r^2-1)-\frac{rs}{2}+\frac{1}{2(1-c_2)}+\frac{1}{8}$$

$$(4.25b)\qquad h^{\pm}_{(l)}=\pm\tau l+\frac{1-c_2}{2}\left(l^2-\frac{1}{4}\right)+\frac{1}{8}$$

where $0<r$, $s\in Z$, $0\leq l\in Z$ and τ is the eigenvalue of I_0 which is given by $\tau=\pm 1/2-2i\lambda u$ depending on which subspace of the R-sector is considered (see the discussion below). And the determinant formula[27,36] becomes

$$(4.26)\qquad \begin{aligned}\det[F^{(N)}_R]=\text{const.}\times&\prod_{\substack{0<r,s\in Z\\ 1\leq rs\leq N}}(h-h_{(r,s)})^{p^R(N-rs)}\\ \times&\prod_{\substack{0\leq l\in Z\leq N\\ A=\pm}}(h-h^A_{(l)})^{p^R_l(N-l)}\end{aligned}$$

where $p^R(N)$ and $p^R_l(N)$ are defined by

$$(4.27a)\qquad \sum_N p^R(N)x^N=\frac{\prod_{0\leq k\in Z}(1+x^k)^2}{\prod_{0<n\in Z}(1-x^n)^2}$$

$$(5.27b)\qquad \sum_N p^R_l(N)x_N=\frac{1}{1+x^l}\,\frac{\prod_{0\leq k\in Z}(1+x^k)^2}{\prod_{0<n\in Z}(1-x^n)^2}$$

Notice that there is a degeneracy of the ground states in the R-sector. From the commutation relation $\{b^i_0, b^j_0\}=\delta^{ij}$, we see that the zero modes b^i_0 form a Clifford algebra. Thereby we take the spinor representation of the algebra for the ground states denoted by $|0;\,t+\lambda,\,u;\,\alpha\rangle$, where α is the two-component spinor index. Be careful that the state $|0;\,\lambda,\,0;\,\alpha\rangle$ is not the ket vacuum, but ground state with conformal weight 1/8. The ket vacuum belongs to the N-S sector. As shown in sec. 3–3 we can construct the spin field $S^\alpha(t, u, z)$ which generates the Ramond ground state from the ket vacuum:

$$(4.28)\qquad |0;\,t+\lambda,\,u;\,\alpha\rangle=S^\alpha(t,u,0)|0;\,\lambda,\,0\rangle$$

Finally, for the T sector we have only one series of the highest weights corresponding to the SVO Φ_r; there are no $\Theta^\pm$ type SVO since c_0 momentum is absent for the T sector. Hence we obtain[36]

$$(4.29)\qquad h_{(r,s)}=\frac{1-c_2}{8}(r^2-1)-\frac{rs}{2}+\frac{1}{2(1-c_2)}s^2+\frac{1}{8}$$

where $0<r\in Z$, $0<s\in Z+1/2$, and

$$\det[F_T^{(N)}] = \text{const.} \times \prod_{1/2 \leq rs \leq N} (h - h_{(r,s)})^{p^T(N-rs)} \left(h - \frac{c_2}{8}\right)^{p^T(N)} \tag{4.30}$$

where $p^T(N)$ is defined by

$$\sum p_N^T(N) x^N = \frac{\prod_{0<k\in Z}(1+x^k) \prod_{0<l\in Z+1/2}(1+x^l)}{\prod_{0<n\in Z}(1-x^n) \prod_{0<m\in Z+1/2}(1-x^m)} \tag{4.31}$$

The ground state in the T sector is also doubly degenerate and its structure is the same as that in the R sector of the $N=1$ case.

§ V. Conclusion

We have presented the null field construction in conformal and superconformal quantum field theories based on the oscillator representation of the corresponding conformal and superconformal algebras. Our method relies heavily on the vertex operator construction of primary fields in two dimensional conformal field theory. We have presented the oscillator representation with central charge extention for the cases of supercharge, $N=0$, 1 and 2. We have also given the construction of spin fields in our framework with variable central charge. The spin fields connect the Ramond sector with the Neveu-Schwarz sector, and are necessary and important entries for the unified treatment of the $N=1$ spin models.

The null field construction we have studied in the present paper for the case of $N=0$, 1 and 2 provides significant informations for the whole subject of two dimensional conformal field theories including statistical models at critical temperature as well as the representation theory of infinite dimensional Lie algebras: They are necessary ingredients for writing down the whole set of differential equations for correlation functions, and also they constitute another proof of the Kac determinant formulae for the superconformal representations of $N=0$, 1 and 2.

Superstring is a conformally invariant system. The whole conformal properties are embedded into the system. The machinary presented here is quite relevant to the development of the superstring game, particularly to the study of symmetry properties of the compactified space via Frenkel-Kac construction.

The extention of our approach to the superconformal algebra with supercharge $N=4$ is under current investigation.

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Appendix

Here we calculate the degeneracy of the zeros of the Kac determinant for the $N=1$ superconformal algebra. In the N-S sector, for example, we can see the restriction on indices r, s from the expression (3.25). Suppose $r=$ odd, then the level $N=rs/2$ must be half-integer since the odd numbers of $\gamma(z_i)$ factor in front of the exponential factor of bosonic operator in eq. (3.25b) contribute to the energy level. Therefore we have $s=$ odd for $r=$ odd. There is no restriction from the similar argument for $r=$ even. However we can see that the highest weights with $s=$ even for $r=$ even as well as those with $s=$ odd for $r=$ odd will saturate the zeros of the Kac determinant. Similarly to the discussion in section 2.3, the order of the Kac determinant is determined by the product of the diagonal elements of the Kac matrix. Thereby the order of the determinant becomes

$$d_N^{(N-S)}=\sum_{\substack{\{\delta_m,n_k\}\\ \Sigma_m m\delta_m+\Sigma_k kn_k=N}}\left(\sum_{m=1/2}^{\infty}\delta_m+\sum_{k=1}^{\infty}n_k\right) \tag{A-1}$$

where δ_m's take the values 0 and 1 while the n_k's run over non-negative integers. Consider the quantity

$$\begin{aligned}
\sum_{N>0}x^N d_N&=\sum_{N>0}\sum_{\{\delta_m,n_k\}}\delta_{N,\Sigma m\delta+\Sigma_m kn_k}\Big(\sum_m\delta_m+\sum_k n_k\Big)x^{\Sigma_m\delta_m+\Sigma_k kn_k}\\
&=\sum_{\delta_m}\sum_m\delta_m x^{m\delta_m}\cdot\prod_{m'\neq m}x^{m'\delta_{m'}}\cdot\sum_{n_k}\prod_k x^{kn_k}\\
&\quad+\sum_{\delta_m}\prod_m x^{m\delta_m}\cdot\sum_{n_k}\sum_k n_k x^{kn_k}\cdot\prod_{k'\neq k}x^{k'n_{k'}}\\
&=\sum_m\frac{x^m}{1+x^m}\prod_{m'}(1+x^{m'})\cdot\prod_k\frac{1}{1-x^k}\\
&\quad+\prod_m(1+x^m)\cdot\sum_k\frac{x^k}{1-x^k}\prod_{k'}\frac{1}{1-x^{k'}}\\
&=\sum_{k=1}^{\infty}\frac{x^{k-1/2}}{1+x^{k-1/2}}\sum_{l=0}^{\infty}x^l p_{N-S}(l)+\sum_{k=1}^{\infty}\frac{x^k}{1-x^k}\sum_{l=0}^{\infty}x^l p_{N-S}(l)\\
&=\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}[(-1)^{s-1}x^{s(k-1/2)+l}+x^{sk+l}]p_{N-S}(l)
\end{aligned} \tag{A-2}$$

$$= -\sum_{N\geq 1/2} x^N \sum_{s=1} \sum_{k=1} (-1)^s p_{N-S}\left(N - \frac{s(2k-1)}{2}\right)$$

$$+ \sum_{N\geq 1} x^N \sum_{s=1} \sum_{k=1} p_{N-S}\left(N - \frac{s\cdot 2k}{2}\right)$$

$$= \sum_{N>0} x^N \sum_{\substack{(s,r)=(\text{even, even})\\ \text{and (odd, odd)}}} p_{N-S}\left(N - \frac{sr}{2}\right)$$

Hence we have

$$d_N^{(N-S)} = \sum_{\substack{0<r,s\in Z\\ r-s=\text{even}}} p_{N-S}\left(N - \frac{rs}{2}\right) \tag{A-3}$$

This shows that the highest weights $h_{(r,s)}$ with positive integers r, s satisfying $r-s=$ even saturate the zeros of the Kac determinant and the proof of eq. (3.29) is completed.

In the same way we can show $d_N^{(R)} = \sum_{\substack{0<r,s\in Z\\ r-s=\text{odd}}} p_R(N-rs/2)$ for the R sector.

Here we may aslo add another argument which leads to the restriction $s=$ even for $r=$ even in the N-S sector. First note that the null field $\Phi_r^+(t_{(-r,s)}, z)$ is totally equivalent to the expression $\Phi_s^-(t_{(r,-s)}, z)$, that is, it is a matter of convenience or for simplicity of calculations which of the two expression is chosen for use to produce a null state with conformal weight $h_{(r,s)}^{N-S} + rs/2$ at level $rs/2$. In the first expression of $\Phi_r^+(t_{(-r,s)}, z)$ with $r=$ even s appears to presume even or odd integers. However, in the second expression of $\Phi_s^-(t_{(r,-s)}, z)$ $s=$ odd forbids $r=$ even. Consequently, with $r=$ even we must have even s. A similar argument for the restriction on r and s goes through in the R sector as well.

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