Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 215–236

Some Problems of Diophantine Approximation and a Kronecker's Limit Formula

Akio Fujii

§ 1. Introduction

Let α be a positive irrational number. It has been the subject of many mathematicians (e.g. Sierpinski [24], Lerch [20], Weyl [30], Hecke [12], Hardy-Littlewood [8]–[11], Behnke [4] [5], Ostrowski [21], Spencer [27], Sós [25] [26], Kesten [15], Erdös [6], Lang [19] and $\cdots$) to study as precise as possible the asymptotic behavior of the sum

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)$$

as X tends to ∞, where $\{y\}$ is the fractional part of y and n runs over the integers $\geqq 1$. It does not seem that even for a quadratic irrational α this sum is understood in a satisfactory way.

Towards this problem Hecke [12] has introduced and studied the zeta function defined by

$$Z_\alpha(s)=\sum_{n=1}^{\infty}\frac{\{\alpha n\}-\frac{1}{2}}{n^s}\qquad \text{for } \operatorname{Re}(s)>1.$$

If $\alpha=\sqrt{D}$ or $1/\sqrt{D}$, $D\equiv 2$ or $3 \pmod 4$ and D is a square free integer $\geqq 1$, then he has shown that $Z_\alpha(s)$ can be continued analytically to the whole complex plane with simple poles at most at the points

$$s=-2k\pm 2\pi i\frac{n}{\log \eta_D},\qquad k, n=0, 1, 2, \cdots$$

where η_D is the fundamental unit of the quadratic number field $Q(\sqrt{D})$ or the square of it. As a result, he has obtained an explicit formula for the Riesz mean of the second order. Precisely, he has shown that for the above α and for any positive δ,

Received April 24, 1986.

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)\left(\log\frac{X}{n}\right)^2$$
$$=\frac{1}{6}G_1(\alpha)\log^3 X+\frac{1}{2}G_2(\alpha)\log^2 X+G_3(\alpha)\log X$$
$$+\sum_{n=-\infty}^{\infty}C_n X^{(2\pi i n)/(\log\eta_D)}+O(X^{-1+\delta}),$$

where $G_1(\alpha)$, $G_2(\alpha)$ and $G_3(\alpha)$ are the coefficients in the Laurent's expansion

$$Z_\alpha(s)=\frac{G_1(\alpha)}{s}+G_2(\alpha)+G_3(\alpha)s+\cdots,$$

and C_n's are the constants which do not depend on X and satisfy

$$C_n=O(|n|^{-2+\delta})\qquad\text{for } n\neq 0.$$

Moreover, $G_1(\alpha)$ and $G_2(\alpha)$ have been evaluated, implicitly, as follows. We put after Hecke [12], when $N(\varepsilon_D)=1$,

$$\zeta(s;v_1)=\sum_{(\mu)}\frac{\operatorname{sgn}(\mu\mu')}{|N(\mu)|^s},$$

where (μ) runs over the non-zero principal integral ideals of $Q(\sqrt{D})$, $N(\mu)$ is the norm of μ, μ' is the conjugate of μ, $\operatorname{sgn}(\mu\mu')$ is the sign of $\mu\mu'$ and ε_D is the fundamental unit of $Q(\sqrt{D})$. Then

$$G_1(\sqrt{D})=\begin{cases}0 & \text{if } N(\varepsilon_D)=-1,\\ \dfrac{\zeta(1;v_1)\sqrt{D}}{\pi^2\log\varepsilon_D} & \text{if } N(\varepsilon_D)=1,\end{cases}$$

and

$$G_2(\sqrt{D})=\begin{cases}-\dfrac{1}{12}\sqrt{D} & \text{if } N(\varepsilon_D)=-1,\\ \dfrac{\zeta(1;v_1)\sqrt{D}}{\pi^2\log\varepsilon_D}(\gamma+\log 2\pi)-\dfrac{\sqrt{D}\zeta'(1;v)}{2\pi^2\log\varepsilon_D}-\dfrac{1}{12}\sqrt{D} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N(\varepsilon_D)=1,\end{cases}$$

where γ is the Euler constant and $\zeta'(s;v_1)$ is the derivative of $\zeta(s;v_1)$.

The purpose of the present paper is to improve and extend Hecke's result and to express $G_1(\alpha)$ and $G_2(\alpha)$ in a different form. Our extension is to get the explicit formulae for the Riesz mean of the first order and for the Cesàro mean of the first order. Namely, we shall show the following theorems.

Theorem 1. *Let* $\alpha=\sqrt{D}$ *or* $1/\sqrt{D}$, $D\equiv 2$ *or* 3 (mod 4) *and* D *is a square free integer* $\geqq 1$. *Then for any positive* δ,

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)\left(\log\frac{X}{n}\right)$$
$$=\frac{1}{2}G_1(\alpha)\log^2 X+G_2(\alpha)\log X+\sum_{n=-\infty}^{\infty}C_n X^{(2\pi i n)/(\log\eta_D)}+O(X^{-1/3+\delta}),$$

where $G_1(\alpha)$ *and* $G_2(\alpha)$ *are the same as above and* C_n*'s are the constants which do not depend on* X *and satisfy*

$$C_n=O(|n|^{-4/3+\delta})\qquad \text{for } n\neq 0.$$

Theorem 2. *Let* α *be the same as in Theorem* 1. *Then for any positive* δ,

$$\frac{1}{X}\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)(X-n)$$
$$=G_1(\alpha)\log X+\sum_{n=-\infty}^{\infty}C_n X^{(2\pi i n)/(\log\eta_D)}+O(X^{-1/3+\delta}),$$

where $G_1(\alpha)$ *is the same as above and* C_n*'s are the constants which do not depend on* X *and satisfy*

$$C_n=O(|n|^{-4/3+\delta})\qquad \text{for } n\neq 0.$$

Theorem 2 may be compared with Hardy-Littlewood's examples in pp. 247–248 of [10].

To prove Theorems 1 and 2, it is necessary to have a deeper study on $Z_\alpha(s)$ in $\mathrm{Re}(s)\leqq 1$. Hecke [12] has shown that for any positive δ and for σ in $-\sigma_1\leqq\sigma\leqq 1$,

$$H(s)Z_\alpha(s)\ll t^{1-\sigma+\delta},$$

where we put $s=\sigma+it$, $t>t_0$, $H(s)=\prod_{k=0}^{\infty}(1-\eta_D^{-s-2k})$ and σ_1 is any odd integer ($\geqq 1$). We need to improve this. For this purpose we shall estimate the order of

$$Z_\alpha\left(\frac{1}{2}+it\right)\qquad \text{as } t\to\infty.$$

We shall reduce our problem to the estimate of the sum

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)e^{it\log n}.$$

In fact, we shall treat this for any irrational α and we can show the following theorem. Let $\psi=\psi(t)$ be any positive non-decreasing function which is defined for $t\geqq 1$. An irrational number α is said to be of type $<\psi$ if

$$q\|q\alpha\|\geqq 1/\psi(q) \qquad \text{for all positive integers } q,$$

where we put

$$\|y\|=\mathrm{Min}\,(\{y\},\,1-\{y\}). \quad \text{(Cf. p. 121 of Kuipers-Niedereiter [18]).}$$

Theorem 3. *Suppose further that $\psi(y)y^{-1/2+\delta}$ is decreasing for an arbitrarily small positive δ.*

If α is irrational of type $<\psi$, then for $t>t_0$,

$$Z_\alpha\left(\frac{1}{2}+it\right)\ll t^{1/6}\log^2 t+\left(\psi(t)+\sum_{1\leqq k\leqq t}\frac{\psi(k)}{k}\right)\log t.$$

If one uses Kolesnik's method(*), then one might improve the constant 1/6 a little bit.

For a quadratic irrational α of the form stated above, we see, by the convexity argument, that for any positive δ

$$H(s)Z_\alpha(s)\ll\begin{cases} t^{2/3-\sigma+\delta} & \text{if } -\sigma_1\leqq\sigma\leqq\frac{1}{2}, \\ t^{(1-\sigma)/3+\delta} & \text{if } \frac{1}{2}\leqq\sigma\leqq 1. \end{cases}$$

This improves Hecke's estimate stated above and enables us to prove Theorems 1 and 2. As a by-product, we can also improve Hecke's explicit formula stated above in the following form.

Theorem 4. *Let α be the same as in Theorem* 1. *Then for any positive δ,*

$$\begin{aligned}&\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)\left(\log\frac{X}{n}\right)^2\\ &\quad=\frac{1}{6}G_1(\alpha)\log^3 X+\frac{1}{2}G_2(\alpha)\log^2 X+G_3(\alpha)\log X\\ &\qquad+\sum_{n=-\infty}^{\infty}C_n X^{(2\pi in)/\log\eta_D}+O(X^{-4/3+\delta})\end{aligned}$$

---

*) (added in proof) This method has been superseded by Bombieri-Iwaniec (cf. Ann. Scuola Normale Sup. Pisa vol XIII n. 3 (1986)).

where $G_1(\alpha)$, $G_2(\alpha)$ and $G_3(\alpha)$ are the same as above and C_n's are the constants which do not depend on X and satisfy

$$C_n = O(|n|^{-7/3+\delta}) \qquad \textit{for } n \neq 0.$$

We now turn our attentions to our evaluation of $G_1(\alpha)$ and $G_2(\alpha)$. We state our results for a general quadratic irrational α. For this purpose, we shall introduce some notations. We may suppose that $\alpha < 1$. If the continued fraction expansion of α is not purely periodic, we put

$$\alpha = \frac{1}{b_1+}\,\frac{1}{b_2+}\,\frac{1}{b_3+}\,\cdots+\frac{1}{b_M+\tilde{\alpha}} \qquad \text{and}$$

$$\tilde{\alpha} = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\cdots+\frac{1}{a_K+\tilde{\alpha}},$$

where $M, K, b_1, \cdots, b_M, a_1, \cdots$, and a_K are integers $\geqq 1$ and $((a_1, \cdots, a_K))$ is the cycle of the continued fraction expansion of $\tilde{\alpha}$. If the continued fraction expansion of α is purely periodic, we put $\tilde{\alpha} = \alpha$. When $K=1$, we suppose an obvious modification of the description below. We define $\tilde{\alpha}_j$ for $j=1, \cdots, K$ by

$$\tilde{\alpha} = \frac{1}{a_1+\tilde{\alpha}_1}, \quad \tilde{\alpha}_1 = \frac{1}{a_2+\tilde{\alpha}_2}, \quad \cdots, \quad \tilde{\alpha}_{K-1} = \frac{1}{a_K+\tilde{\alpha}}$$

and $\tilde{\alpha}_K = \tilde{\alpha}$. The cycle of $\tilde{\alpha}_j$ is $((a_{j+1}, \cdots, a_K, a_1, \cdots, a_j))$ for $j=1, \cdots, K-1$. We put

$$E_1 = \frac{1}{\tilde{\alpha}} \qquad \text{and}$$

$$E_j = \frac{1}{\tilde{\alpha}\tilde{\alpha}_1\cdots\tilde{\alpha}_{j-1}}(1 - \tilde{\alpha}_j\tilde{\alpha}_{j-1} + \tilde{\alpha}_j\tilde{\alpha}_{j-1}^2\tilde{\alpha}_{j-2} - \cdots + (-1)^{j+1}\tilde{\alpha}_j\tilde{\alpha}_{j-1}^2\tilde{\alpha}_{j-2}^2\cdots\tilde{\alpha}_2^2\tilde{\alpha}_1)$$

for $j=2, \cdots, K$. We put

$$E_2' = \frac{1}{\tilde{\alpha}_1} \qquad \text{and}$$

$$E_j' = \frac{1}{\tilde{\alpha}_1\cdots\tilde{\alpha}_{j-1}}(1 - \tilde{\alpha}_j\tilde{\alpha}_{j-1} + \tilde{\alpha}_j\tilde{\alpha}_{j-1}^2\tilde{\alpha}_{j-2} - \cdots + (-1)^{j-2}\tilde{\alpha}_j\tilde{\alpha}_{j-1}^2\cdots\tilde{\alpha}_3^2\tilde{\alpha}_2)$$

for $j=3, 4, \cdots, K+1$, where we put $\tilde{\alpha}_{K+1} = \tilde{\alpha}_1$. We put $A_K = \tilde{\alpha}\tilde{\alpha}_1\cdots\tilde{\alpha}_{K-1}$. For $j=1, 2, \cdots, K$, we put

$$D_j = E_j(A_K^{-1} + (-1)^{K+1}A_K) + (-1)^j\tilde{\alpha}_1\cdots\tilde{\alpha}_jE_K.$$

For $j=2, 3, \cdots, K+1$, we put

$$D'_j=E'_j(A_K^{-1}+(-1)^{K+1}A_K)+(-1)^{j-1}\tilde{\alpha}_2\cdots\tilde{\alpha}_j E'_{K+1},$$

and $D'_1=E'_{K+1}$. We put $D_0=E_k$ and $D_{K+1}=D_1$.

Let c and d be the denominators of the fractions

$$\frac{1}{b_1+}\;\frac{1}{b_2+}\;\cdots+\frac{1}{b_M} \quad\text{and}\quad \frac{1}{b_1+}\;\frac{1}{b_2+}\;\cdots+\frac{1}{b_{M-1}}$$

respectively, where we put $d=1$ when $M=1$. Let $B_2(x)=x^2-x+1/6$, $B_1(x)=x-1/2$, $B_2=B_2(0)$ and $B_1=B_1(0)$.

For a pair of positive numbers (w_1, w_2), let $\Gamma_2(z, (w_1, w_2))$ be the double gamma function introduced and studied by Barnes [2], [3] (Cf. also Shintani [22]), where z is a complex number. As in Barnes [2] [3] (Cf. also Shintani [22]) we define the positive constant $\rho_2((w_1, w_2))$ by

$$-\log\rho_2((w_1, w_2))=\lim_{a\to+0}\left(\frac{\partial}{\partial s}\sum_{m,n\geqq 1}(a+mw_1+nw_2)^{-s}\Big|_{s=0}+\log a\right).$$

We put

$$S(w_1, w_2)=\frac{1}{2w_1w_2}\left(B_2\left(\frac{w_1+w_2}{w_1}\right)w_1^2+2B_1\left(\frac{w_1+w_2}{w_1}\right)B_1w_1w_2+B_2w_2^2\right).$$

We put

$$c(j)=\begin{cases}1 & \text{if } j=1,\\ \tilde{\alpha}_1\cdots\tilde{\alpha}_{j-1} & \text{if } 2\leqq j\leqq K,\end{cases}\qquad c'(j)=\begin{cases}1 & \text{if } j=1,\\ \tilde{\alpha}_2\cdots\tilde{\alpha}_j & \text{if } 2\leqq j\leqq K,\end{cases}$$

$$\hat{c}(j)=(-1)^j c(j)(A_K^{-1}+(-1)^{K+1}A_K-E_K)\qquad\text{for } 1\leqq j\leqq K,$$

$$\begin{aligned}\tilde{c}(j)=(-1)^{j+1}(c\cdot c(j+1)&(A_K^{-1}+(-1)^{K+1}A_K-E_K)\\ &-d\cdot c'(j)(A_K^{-1}+(-1)^{K+1}A_K-E'_{K+1}))\qquad\text{for } 1\leqq j\leqq K,\end{aligned}$$

and $c(K+1)=1$.

We put further for real a and b,

$$F_1(a, b)=\sum_{l=1}^{\infty}\frac{b^l}{1+a^l}\quad\text{and}\quad F_2(a, b)=\sum_{l=1}^{\infty}\frac{b^l}{1-a^l}.$$

Under these notations our results may be described as follows.

Theorem 5. *For any purely periodic quadratic irrational α,*

$$G_1(\alpha)=\begin{cases}0 & \textit{if } K \textit{ is odd},\\ \dfrac{1}{\log(1/A_K)}\displaystyle\sum_{j=1}^{K}(-1)^j S(D_{j-1}, D_j) & \textit{if } K \textit{ is even}.\end{cases}$$

Theorem 6. *Let α be purely periodic quadratic irrational. If K is odd, then*

$$G_2(\alpha)=\frac{1}{2}\sum_{j=1}^{K}(-1)^j S(D_{j-1}, D_j)+\frac{1}{12}\sum_{j=1}^{K}(-1)^j\Big(F_1\Big(-A_K^2, -\frac{\tilde{\alpha}_j\hat{c}(j)}{D_j}\Big)$$
$$\times\Big(\frac{D_{j-1}}{D_j}+\frac{1}{\tilde{\alpha}_j}\Big)+F_1\Big(-A_K^2, \frac{\hat{c}(j)}{D_{j-1}}\Big)\Big(\frac{D_j}{D_{j-1}}+\tilde{\alpha}_j\Big)\Big)-\frac{1}{12}\alpha+\frac{1}{4}.$$

If K is even, then

$$G_2(\alpha)=\frac{1}{\log(1/A_K)}\sum_{j=1}^{K}(-1)^j\log\Big(\frac{\Gamma_2(D_{j-1}+D_j, (D_{j-1}, D_j))}{\rho_2((D_{j-1}, D_j))}\Big)$$
$$+\Big(\frac{1}{2}+\frac{1}{\log(1/A_K)}\log(A_K^{-1}+(-1)^{K+1}A_K)\Big)\sum_{j=1}^{K}(-1)^j S(D_{j-1}, D_j)$$
$$+\frac{1}{12}\sum_{j=1}^{K}(-1)^j\Big(F_2\Big(A_K^2, -\frac{\tilde{\alpha}_j\hat{c}(j)}{D_j}\Big)\Big(\frac{D_{j-1}}{D_j}+\frac{1}{\tilde{\alpha}_j}\Big)$$
$$+F_2\Big(A_K^2, \frac{\hat{c}(j)}{D_{j-1}}\Big)\Big(\frac{D_j}{D_{j-1}}+\tilde{\alpha}_j\Big)\Big)-\frac{1}{12}\alpha+\frac{1}{4}.$$

Theorem 5′. *If α is quadratic irrational and not purely periodic, then*

$$G_1(\alpha)=\begin{cases} 0 & \text{if } K \text{ is odd,}\\ \dfrac{(-1)^{M+1}}{\log(1/A_K)}\displaystyle\sum_{j=1}^{K}(-1)^j S(cD_j+dD'_j, cD_{j+1}+dD'_{j+1}) & \\ & \text{if } K \text{ is even.}\end{cases}$$

Theorem 6′. *Suppose that α is quadratic irrational and not purely periodic. If K is odd, then*

$$G_2(\alpha)=-\frac{1}{12}\alpha+\frac{1}{4}+\sum_{\nu=1}^{M+1}(-1)^\nu S(q_{\nu-1}, q_\nu)$$
$$+\frac{1}{2}(-1)^{M+1}\sum_{j=1}^{K}(-1)^j S(cD_j+dD'_j, cD_{j+1}+dD'_{j+1})$$
$$+\frac{1}{12}(-1)^{M+1}\sum_{j=1}^{K}(-1)^j\Big(F_1\Big(-A_K^2, -\frac{\tilde{\alpha}_{j+1}\tilde{c}(j)}{cD_{j+1}+dD'_{j+1}}\Big)$$
$$\times\Big(\frac{cD_j+dD'_j}{cD_{j+1}+dD'_{j+1}}+\frac{1}{\tilde{\alpha}_{j+1}}\Big)$$
$$+F_1\Big(-A_K^2, \frac{\tilde{c}(j)}{cD_j+dD'_j}\Big)\Big(\frac{cD_{j+1}+dD'_{j+1}}{cD_j+dD'_j}+\tilde{\alpha}_{j+1}\Big)\Big),$$

where q_ν is the denominator of the fraction

$$\frac{1}{b_1+}\ \frac{1}{b_2+}\ \cdots+\frac{1}{b_\nu}$$

for $\nu\leqq M$ *and* q_{M+1} *is the denominator of the fraction*

$$\frac{1}{b_1+}\ \frac{1}{b_2+}\ \cdots\ \frac{1}{b_M+}\ \frac{1}{a_1}.$$

If K is even, then

$$\begin{aligned}
G_2(\alpha)=&-\frac{1}{12}\alpha+\frac{1}{4}+\sum_{\nu=1}^{M+1}(-1)^\nu S(q_{\nu-1},q_\nu)\\
&+(-1)^{M+1}\left(\frac{1}{2}+\frac{1}{\log(1/A_K)}\log(A_K^{-1}+(-1)^{K+1}A_K)\right)\\
&\quad\times\sum_{j=1}^{K}(-1)^j S(cD_j+dD'_j,\ cD_{j+1}+dD'_{j+1})\\
&+\frac{(-1)^{M+1}}{\log(1/A_K)}\sum_{j=1}^{K}(-1)^j\\
&\times\log\left(\frac{\Gamma_2((cD_j+dD'_j)+(cD_{j+1}+dD'_{j+1}),(cD_j+dD'_j,\ cD_{j+1}+dD'_{j+1}))}{\rho_2((cD_{j+1}+dD'_{j+1},\ cD_{j+1}+dD'_{j+1}))}\right)\\
&+\frac{1}{12}(-1)^{M+1}\sum_{j=1}^{K}(-1)^j\left(F_2\left(A_K^2,\ -\frac{\tilde{\alpha}_{j+1}\tilde{c}(j)}{cD_{j+1}+dD'_{j+1}}\right)\right.\\
&\quad\times\left(\frac{cD_j+dD'_j}{cD_{j+1}+dD'_{j+1}}+\frac{1}{\tilde{\alpha}_{j+1}}\right)\\
&\left.+F_2\left(A_K^2,\ \frac{\tilde{c}(j)}{cD_j+dD'_j}\right)\left(\frac{cD_{j+1}+dD'_{j+1}}{cD_j+dD'_j}+\tilde{\alpha}_{j+1}\right)\right).
\end{aligned}$$

If we combine Hecke's evaluation stated before and ours of $G_1(\alpha)$ and $G_2(\alpha)$, then we get some new expressions of $\zeta(1;v_1)$ and $\zeta'(1;v_1)$. Let D be a square free integer $\geqq 1$, $D\equiv 2$ or 3 (mod 4) and let $\alpha=\sqrt{D}-[\sqrt{D}]$. Since $\alpha+2[\sqrt{D}]$ is reduced quadratic irrational, α has the following purely periodic continued fraction expansion

$$\frac{1}{a_1+}\ \frac{1}{a_2+}\ \cdots+\frac{1}{a_{K-1}+}\ \frac{1}{2[\sqrt{D}]+\alpha}$$

for $K\geqq 2$ and $\dfrac{1}{2[\sqrt{D}]+\alpha}$ for $K=1$.

Here, $A_K=1/\varepsilon_D$ and that $N(\varepsilon_D)=-1$ if and only if K is odd. Using the same notations as above, we get the following corollaries.

Corollary 1. *If* $N(\varepsilon_D)=1$, *then*

$$\zeta(1;v_1)=\frac{\pi^2}{\sqrt{D}}\sum_{j=1}^{K}(-1)^j S(D_{j-1}.\,D_j).$$

Corollary 2. *If* $N(\varepsilon_D)=1$, *then*

$$\zeta'(1;v_1)=-\frac{2\pi^2}{\sqrt{D}}\left(\sum_{j=1}^{K}(-1)^j\log\left(\frac{\Gamma_2(D_{j-1}+D_j,(D_{j-1},D_j))}{\rho_2((D_{j-1},D_j))}\right)\right.$$
$$+\left(-\gamma-\log 2\pi+\frac{1}{2}\log\varepsilon_D+\log(\varepsilon_D-\varepsilon_D^{-1})\right)\sum_{j=1}^{K}(-1)^j S(D_{j-1},D_j)$$
$$+\log\varepsilon_D\left(\frac{1}{12}[\sqrt{D}]+\frac{1}{4}\right)$$
$$+\frac{1}{12}\log\varepsilon_D\sum_{j=1}^{K}(-1)^j\left(F_2\left(\varepsilon_D^{-2},-\frac{\tilde{\alpha}_j\hat{c}(j)}{D_j}\right)\left(\frac{D_{j-1}}{D_j}+\frac{1}{\tilde{\alpha}_j}\right)\right.$$
$$\left.\left.+F_2\left(\varepsilon_D^{-2},\frac{\hat{c}(j)}{D_{j-1}}\right)\left(\frac{D_j}{D_{j-1}}+\tilde{\alpha}_j\right)\right)\right).$$

In view of Stark-Shintani's conjecture (Cf. Stark [28], Shintani [23] and also Arakawa [1]), one may formulate a conjecture on the values of the linear combination of $G_2(\alpha)$'s. In this context it is of interest to extend our results to any algebraic numbers of degree greater than 3.

We shall estimate the sum

$$\sum_{n\leqq X}\left(\{\alpha n\}-\frac{1}{2}\right)e^{it\log n}$$

in the section 2. We shall prove Theorem 3 in the section 3, Theorem 1, 2 and 4 in the section 4, Theorems 5 and 6 in the section 5 and Theorems 5′ and 6′ in the section 6.

§ 2. Hybrid estimates

Suppose that ψ is a positive non-decreasing function. Let α be any irrational number of type $<\psi$. Let $t>t_0$ and let $1\ll N\leqq N'\leqq 2N$. We shall estimate the sum

$$S\equiv\sum_{N\leqq n\leqq N'}\left(\{\alpha n\}-\frac{1}{2}\right)e^{it\log n}.$$

By the Fourier expansion of $\{\alpha n\}-1/2$, we get for any $M\geqq 1$,

$$S=-\sum_{1\leqq k\leqq M}\frac{1}{k\pi}\sum_{N\leqq n\leqq N'}\sin(2\pi k\alpha n)e^{it\log n}+O\Big(\frac{1}{M}\sum_{N\leqq n\leqq N'}\frac{1}{\|\alpha n\|}\Big)$$
$$=S_1+S_2,\ \text{say}.$$

$S_2\ll\dfrac{1}{M}N(\log N+\psi(2N))$ by 3.11 of p. 131 of Kuipers-Niedereiter [18].

We suppose first that $t/(2\pi N)<1$. Then

$$S_1=\Big(\sum_{1\leqq k\leqq M}{}'+\sum_{1\leqq k\leqq M}{}''\Big)\frac{1}{k\pi}\sum_{N\leqq k\leqq N'}{}''\sin(2\pi k\alpha n)e^{it\log n}$$
$$=S_1'+S_2'',\ \text{say},$$

where in S_1', k satisfies $\|k\alpha\|/2>(t/2\pi N)$ and in S_1'', k satisfies $\|k\alpha\|/2\leqq t/(2\pi N)$. We see first that for k in S_1', using van der Corput-Kusmin-Landau's lemma (Cf. p. 115 of Koksma [16]) for the first derivative of the function $2\pi k\alpha y\pm t\log y$,

$$\sum_{N\leqq n\leqq N'}\sin(2\pi k\alpha n)e^{it\log n}\ll 1/\|k\alpha\|.$$

Hence

$$S_1'\ll\sum_{1\leqq k\leqq M}\frac{1}{k\|k\alpha\|}\ll\log^2 M+\psi(M)+\sum_{k=1}^{M}\frac{\psi(k)}{k}$$

by 3.12 of p. 131 of of Kuipers-Niedereiter [18].

We see next that for k in S_1'', using van der Corpt's lemma (Cf. Theorem 5.9 of Titchmarsh [29]) for the second derivative of the function $2\pi k\alpha y\pm t\log y$,

$$\sum_{N\leqq n\leqq N'}\sin(2\pi k\alpha n)e^{it\log n}\ll\Big(N\frac{\sqrt{t}}{N}+\frac{N}{\sqrt{t}}\Big)$$
$$\ll\sqrt{t}+\sqrt{t}\frac{1}{\|k\alpha\|}\ll\sqrt{t}\frac{1}{\|k\alpha\|}.$$

Hence

$$S_1''\ll\sqrt{t}\sum_{1\leqq k\leqq M}\frac{1}{k\|k\alpha\|}\ll\sqrt{t}\Big(\log^2 M+\psi(M)+\sum_{k=1}^{M}\frac{\psi(k)}{k}\Big).$$

Taking $M=N$, we get

$$S\ll\sqrt{t}\Big(\log^2 N+\psi(2N)+\sum_{k=1}^{N}\frac{\psi(k)}{k}\Big).$$

We suppose next that $\sqrt{t} \ll N \leqq t/2\pi$. Using van der Corpt's lemma for the second derivative of the function $2\pi k\alpha y \pm t \log y$. we get

$$S_1 \ll \sum_{1\leqq k\leqq M} \frac{1}{k}\left(N\frac{\sqrt{t}}{N}+\frac{N}{\sqrt{t}}\right) \ll \sqrt{t} \sum_{1\leqq k\leqq M} \frac{1}{k} \ll \sqrt{t} \log M.$$

Taking $M=N$, we get

$$S \ll \sqrt{t} \log N + \psi(2N).$$

We suppose finally that $t^{1/3} \ll N \leqq t/2\pi$. If we use van der Corpt's lemma (Cf. Theorem 5.11 of Titchmarsh [29]) for the third derivative of the function $2\pi k\alpha y \pm t \log y$, we get

$$\sum_{N\leqq n\leqq N'} \sin(2\pi k\alpha n) e^{it\log n} \ll N\left(\frac{t}{N^3}\right)^{1/6} + N^{1/2}\left(\frac{N^3}{t}\right)^{1/6}.$$

Thus we get

$$S_1 \ll \sum_{1\leqq k\leqq M} \frac{1}{k}(N^{1/2}t^{1/6}+Nt^{-1/6})$$

and

$$S \ll (N^{1/2}t^{1/6}+Nt^{-1/6}) \log N + \psi(2N),$$

by taking $M=N$. Thus we have proved the following

Lemma 1. *Let $t>t_0$ and $1 \ll N \leqq N' \leqq 2N$. Suppose that ψ is a positive non-decreasing function and α is irrational of type $<\psi$. Then*

$$\sum_{N\leqq n\leqq N'} \left(\{\alpha n\}-\frac{1}{2}\right) e^{it\log n}$$
$$\ll \begin{cases} \sqrt{t}\left(\log^2 N + \psi(2N) + \sum_{k=1}^{N} \dfrac{\psi(k)}{k}\right) & \text{if } t/2\pi < N, \\ \sqrt{t} \log N + \psi(2N) \quad \text{if } \sqrt{t} \ll N \leqq t/2\pi, \\ (N^{1/2}t^{1/6}+Nt^{-1/6}) \log N + \psi(2N) & \text{if } t^{1/3} \ll N \leqq t/2\pi. \end{cases}$$

Using this we get if $\sqrt{t} \ll N \leqq t/2\pi$,

$$S' \equiv \sum_{N\leqq n\leqq N'} \frac{\{\alpha n\}-\frac{1}{2}}{\sqrt{n}} e^{it\log n} \ll (\sqrt{t} \log N + \psi(2N))/\sqrt{N}.$$

This is

$$\ll t^{1/6} \log t + \psi(t) t^{-1/3},$$

provided that $t^{2/3} \ll N \leqq t/2\pi$. If $t^{1/3} \ll N \leqq t/2\pi$, then

$$S' \ll (t^{1/6} + N^{1/2} t^{-1/6}) \log t + \psi(t) t^{-1/6}.$$

This is

$$\ll t^{1/6} \log t + \psi(t) t^{-1/6}, \text{ provided that } t^{1/3} \ll N \ll t^{2/3}.$$

Thus we get the following

Corollary 3. *Under the same assumptions as in Lemma* 1, *if* $1 \ll t^{1/3} \ll N \leqq t/2\pi$, *then*

$$\sum_{N \leqq n \leqq N'} \frac{\{\alpha n\} - \frac{1}{2}}{\sqrt{n}} e^{it \log n} \ll t^{1/6} \log t + \psi(t) t^{-1/6}.$$

§ 3. Proof of Theorem 3

We suppose further that $\psi(y) y^{-1/2+\delta}$ is decreasing for an arbitrarily small positive δ. Suppose that $U \gg t > t_0$. Then for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} Z_\alpha(s) &= \sum_{n \leqq U} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} + \sum_{n > U} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} \\ &= \sum_{n \leqq U} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} + O\left(\frac{|s|}{U^\sigma} \int_1^U \frac{\psi(y)}{y} dy \right), \end{aligned}$$

where we have used Lang's estimate [19]:

$$\sum_{n \leqq U} (\{\alpha n\} - \tfrac{1}{2}) \ll \int_1^U \frac{\psi(y)}{y} dy.$$

The last expression is valid for $\mathrm{Re}(s) > 0$. Hence,

$$\begin{aligned} Z_\alpha\left(\frac{1}{2} + it\right) &= \sum_{n << t^{1/3}} \frac{\{\alpha n\} - \frac{1}{2}}{n^{1/2+it}} + \sum_{t^{1/3} << n \leqq t/2\pi} \frac{\{\alpha n\} - \frac{1}{2}}{n^{1/2+it}} \\ &\quad + \sum_{t/2\pi \leqq n \leqq U} \frac{\{\alpha n\} - \frac{1}{2}}{n^{1/2+it}} + O\left(\frac{t}{U^{1/2}} \int_1^U \frac{\psi(y)}{y} dy \right) \\ &= S_3 + S_4 + S_5 + O(S_6), \text{ say.} \\ & S_3 \ll t^{1/6}, \\ & S_4 \ll (t^{1/6} \log t + \psi(t) t^{-1/6}) \log t, \end{aligned}$$

by Corollary 3 in the section 2.

Using the first part of Lemma 1, we get

$$S_5 \ll t^{1/2} U^{-1/2} \Big(\log^2 U + \psi(2U) + \sum_{1 \leqq k \leqq U} \frac{\psi(k)}{k} \Big)$$
$$+ \int_{t/2\pi}^{U} \frac{\sqrt{t} \log y}{y^{3/2}} \Big(\log^2 y + \psi(2y) + \sum_{1 \leqq k \leqq y} \frac{\psi(k)}{k} \Big) dy$$
$$\ll \log t \Big(\psi(t) + \log^2 t + \sum_{1 \leqq k \leqq t} \frac{\psi(k)}{k} \Big).$$

Letting U tend to ∞, we get our Theorem 3.

§ 4. Proof of Theorems 1, 2 and 4

In the present case $\psi(y) \ll 1$. Hence by Theorem 3 we get

$$Z_\alpha\Big(\frac{1}{2} + it\Big) \ll t^{1/6} \log^2 t.$$

Using this we get for each odd integer $\sigma_1 \geqq 1$ and for any $\delta > 0$,

$$H(s) Z_\alpha(s) \ll t^{((1+\sigma_1)\cdot(1/2-\sigma)/(1/2+\sigma_1)) + (1/6\cdot(\sigma+\sigma_1)/(1/2+\sigma_1)) + \delta}$$
$$\text{for } -\sigma_1 \leqq \sigma \leqq \tfrac{1}{2},$$

where $s = \sigma + it$ and we have used the convexity argument with Hecke's estimate stated in the introduction. Letting σ_1 tend to ∞, we get for each odd integer $\sigma_1' \geqq 1$ and for any positive δ

$$H(s) Z_\alpha(s) \ll t^{2/3 - \sigma + \delta} \qquad \text{for } -\sigma_1' \leqq \sigma \leqq \tfrac{1}{2}.$$

We shall prove only Theorem 1. Let $X > X_0$. Let k be an integer $> k_0$. We can take T_k such that

$$\frac{2\pi k}{\log \eta_D} < T_k < \frac{2\pi(k+1)}{\log \eta_D} \quad \text{and} \quad H(\sigma \pm iT_k)^{-1} \ll 1$$
$$\text{for } -1 \leqq \sigma \leqq 2.$$

We consider the integral

$$I \equiv \frac{1}{2\pi i} \int_{2-iT_k}^{2+iT_k} Z_\alpha(s) \frac{X^2}{s^2} ds.$$

Then

$$I = \sum_{n \leqq X} \Big(\{\alpha n\} - \frac{1}{2} \Big) \log \frac{X}{n} + O\Big(\frac{X^2}{T_k^2} \Big).$$

On the other hand, moving the line of the integration to

$$\left(-\frac{1}{3}+2\delta-iT_k,\ -\frac{1}{3}+2\delta+iT_k\right),$$

we get

$$I=\operatorname*{Res}_{s=0}\left(Z_\alpha(s)\frac{X^s}{s^2}\right)+\sum_{\substack{n=-k\\ n\neq 0}}^{k}\operatorname*{Res}_{s=2\pi in/\log\eta_D}(Z_\alpha(s))\frac{X^{2\pi in/\log\eta_D}}{\left(\dfrac{2\pi in}{\log\eta_D}\right)^2}$$
$$+O\left(\int_{-1/3+2\delta}^{2}X^\sigma\frac{|Z_\alpha(\sigma\pm iT_k)|}{|\sigma\pm iT_k|^2}d\sigma\right)$$
$$+O\left(\int_{-T_k}^{T_k}X^{-1/3+2\delta}\frac{|Z_\alpha(-\frac{1}{3}+2\delta+it)|}{|-\frac{1}{3}+2\delta+it|^2}dt\right).$$

Using the above estimate of $Z_\alpha(s)$ and letting k tend to ∞, we get our Theorem 1 with

$$C_n=\operatorname*{Res}_{s=2\pi in/\log\eta_D}(Z_\alpha(s))\frac{1}{\left(\dfrac{2\pi in}{\log\eta_D}\right)^2}\ll|n|^{-4/3+\delta}\qquad\text{for } n\neq 0.$$

The proofs of Theorems 2 and 4 are similar. For Theorems 2 and 4 we evaluate the integrals

$$\frac{1}{2\pi i}\int_{2-iT_k}^{2+iT_k}Z_\alpha(s)\frac{X^s}{s(s+1)}ds\quad\text{and}\quad\frac{1}{2\pi i}\int_{2-iT_k}^{2+iT_k}Z_\alpha(s)\frac{X^s}{s^3}ds$$

respectively. For Theorem 4 we move the line of the integration to

$$\left(-\frac{4}{3}+2\delta-iT_k,\ -\frac{4}{3}+2\delta+iT_k\right).$$

§ 5. Proof of Theorems 5 and 6

We use the same notations as in the introduction. We suppose in this section that α is purely periodic irrational <1, hence $\tilde{\alpha}=\alpha$ and we omit writing the tilder hereafter in this section. We denote the denominator of the ν-th principal convergent to α by q_ν for $\nu=1,2,\cdots$. We put $q_0=1$. Then,

$$q_1+\alpha_1 q_0=1/\alpha$$

and generally

$$q_\nu+\alpha_\nu q_{\nu-1}=1/(\alpha\alpha_1\cdots\alpha_{\nu-1}),$$

where we put $\alpha_\nu=\alpha_j$ if $\nu\equiv j \pmod K$ and $1\leqq j\leqq K$. Hence we get, by induction on j, that for $1 \leqq j \leqq K$,

$$q_j = E_j + (-1)^j \alpha_1 \alpha_2 \cdots \alpha_j,$$

where E_j is defined in the introduction. Generally, we get for $\nu = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, K$,

$$q_{K\nu+j} = A_K^{-\nu} E_j + (-1)^j \alpha_1 \alpha_2 \cdots \alpha_j q_{K\nu}.$$

In particular, we obtain for $\nu = 0, 1, 2, \cdots$

$$q_{K(\nu+1)} = A_K^{-\nu} E_K + (-1)^K A_K q_{K\nu}.$$

Hence, by induction on ν, we get for $\nu = 0, 1, 2, \cdots$,

$$q_{K\nu} = E_K \frac{A_K^{-\nu} + (-1)^{\nu K+1} A_K^{\nu}}{A_K^{-1} + (-1)^{K+1} A_K} + (-1)^{K\nu} A_K^{\nu}.$$

Now it is well known that for $\mathrm{Re}(s) > 2$,

$$Z_\alpha(s) = \alpha \zeta(s-1) - \frac{1}{2} \zeta(s) + \Phi(s),$$

where $\zeta(s)$ is the Riemann zeta function,

$$\Phi(s) = \sum_{\nu=1}^{\infty} (-1)^\nu \zeta_\nu(s) \quad \text{and} \quad \zeta_\nu(s) = \sum_{h,k=1}^{\infty} (hq_{\nu-1} + kq_\nu)^{-s}$$

(Cf. (4.1) and (4.3) of Hardy-Littlewood [8]).

We decomopse $\Phi(s)$ as follows.

$$\Phi(s) = \sum_{j=1}^{K} (-1)^j \sum_{\nu=0}^{\infty} (-1)^{K\nu} \zeta_{K\nu+j}(s).$$

We notice that for $1 \leqq j \leqq K$,

$$\begin{aligned} hq_{K\nu+j-1} + kq_{K\nu+j} &= \frac{A_K^{-\nu}}{A_K^{-1} + (-1)^{K+1} A_K} (hD_{j-1} + kD_j) + \frac{(-1)^{K\nu} A_K^{\nu}}{A_K^{-1} + (-1)^{K+1} A_K} \\ &\quad \times (-1)^{j-1} c(j) (A_K^{-1} + (-1)^{K+1} A_K - E_K)(h - k\alpha_j) \\ &= \frac{1}{A_K^{-1} + (-1)^{K+1} A_K} (A_K^{-\nu} P_j - (-1)^{K\nu} A_K^{\nu} Q_j), \quad \text{say,} \end{aligned}$$

where D_j for $0 \leqq j \leqq K$ and $c(j)$ for $1 \leqq j \leqq K$ are defined in the introduction. For even j,

$$D_j > 0,$$

since E_j and E_K are >0. When K is even and j is odd,

$$\begin{aligned} D_j=&\frac{1}{\alpha\alpha_1\cdots\alpha_{j-1}}((1-\alpha_j\alpha_{j-1})+\alpha_j\alpha_{j-1}^2\alpha_{j-2}(1-\alpha_{j-2}\alpha_{j-3}) \\ &+\cdots+\alpha_j\alpha_{j-1}^2\cdots\alpha_4^2\alpha_3(1-\alpha_3\alpha_2))(A_K^{-1}-A_K) \\ &+\frac{\alpha_1\cdots\alpha_j}{A_K}\left(\frac{1}{\alpha\alpha_1}-1\right)+\frac{\alpha_1\cdots\alpha_j}{A_K}(\alpha_K\alpha_{K-1}(1-\alpha_{K-1}\alpha_{K-2}) \\ &+\cdots+\alpha_K\alpha_{K-1}^2\cdots\alpha_4^2\alpha_3(1-\alpha_3\alpha_2)) \\ >&0. \end{aligned}$$

In a similar manner, we see that when K is odd and j is odd,

$$D_j>0,$$

We now get for $\mathrm{Re}(s)>2$,

$$\begin{aligned} \Phi(s)&=\sum_{j=1}^{K}(-1)^j\sum_{\nu=0}^{\infty}(-1)^{K\nu}(A_K^{-1}+(-1)^{K+1}A_K)^s \\ &\qquad\times\sum_{h,k=1}^{\infty}(A_K^{-\nu}P_j-(-1)^{K\nu}A_K^{\nu}Q_j)^{-s} \\ &=\sum_{j=1}^{K}(-1)^j\sum_{\nu=0}^{\infty}(A_K^{-1}+(-1)^{K+1}A_K)^s(-1)^{K\nu} \\ &\qquad\times\sum_{h,k=1}^{\infty}\sum_{l=0}^{\infty}(-1)^{K\nu l}\begin{bmatrix}s\\l\end{bmatrix}A_K^{(s+2l)\nu}\frac{Q_j^l}{P_j^{s+l}} \\ &=(A_K^{-1}+(-1)^{K+1}A_K)^s\sum_{j=1}^{K}(-1)^j\sum_{l=0}^{\infty}\frac{1}{1-(-1)^{(l+1)K}A_K^{s+2l}}\begin{bmatrix}s\\l\end{bmatrix}Z_{j,l}(s), \end{aligned}$$

where we put

$$\begin{bmatrix}s\\l\end{bmatrix}=\frac{s(s+1)\cdots(s+l+1)}{l!}\quad\text{and}\quad Z_{j,l}(s)=\sum_{h,k=1}^{\infty}\frac{Q_j^l}{P_j^{s+l}}$$

and the justification of the interchange of the summations is verified in a similar way as in p. 62 of Hardy-Littlewood [8]. For $l\neq 0$,

$$\begin{bmatrix}s\\l\end{bmatrix}Z_{j,l}(s)=\frac{\hat{c}^l(j)(-1)^l}{l!}\mathscr{D}_{j,l}Z_j(s),$$

where $\hat{c}(j)$ is defined in the introduction and we put

$$\mathscr{D}_{j,l}=\sum_{n=0}^{l}\binom{l}{n}(-\alpha_j)^{1-n}\frac{\partial^n}{\partial D_{j-1}^n}\frac{\partial^{l-n}}{\partial D_j^{l-n}}$$

and

$$Z_j(s)=Z_{j,0}(s).$$

Since $Z_j(s)$ is regular in the whole complex plane save for simple poles at $s=1$ and $s=2$, the same is true for $\begin{bmatrix} s \\ l \end{bmatrix} Z_{i,l}(s)$. Using the same argument as in pp. 63–68 of [8], we see that $Z_\alpha(s)$ is regular in the whole complex plane except at most at the points where

$$1-(-1)^{(l+1)K}A_K^{s+2l}=0.$$

In particular, we see that if K is odd, $s=0$ cannot be the pole of $Z_\alpha(s)$. When K is even, the residue of $\Phi(s)$ at $s=0$ is

$$\frac{1}{\log(1/A_K)}\sum_{j=1}^{K}(-1)^j Z_{j,0}(0).$$

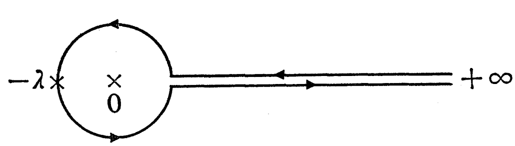
We now evaluate $Z_{j,0}(0)$.
For $\mathrm{Re}(s)>2$.

$$\begin{aligned} Z_{j,0}(s)\Gamma(s) &= \int_0^\infty \sum_{h,k=1}^{\infty} e^{-x(hD_{j-1}+kD_j)}x^{s-1}dx \\ &= \int_0^\infty \frac{x^{s-1}e^{-x(D_{j-1}+D_j)}}{(1-e^{-xD_{j-1}})(1-e^{-xD_j})}dx. \end{aligned}$$

Hence we get

$$Z_{j,0}(s)=\frac{\Gamma(1-s)e^{-\pi i s}}{2\pi i}\int_{I(\lambda)}\frac{z^{s-1}e^{-(D_{j-1}+D_j)z}}{(1-e^{-D_{j-1}z})(1-e^{-D_j z})}dz,$$

where $\log z$ is supposed to take its principal value, $I(\lambda)$ is the path indicated below and λ is a sufficiently small positive number.



This is valid for all complex s except positive integral values (Cf. p. 29 of Hardy-Littlewood [9] and Chapter 13 of Whittaker-Watson [31]). In particular, we get

$$Z_{j,0}(0)=\frac{1}{2\pi i}\int_{I(\lambda)}\frac{e^{-(D_{j-1}+D_j)z}}{(1-e^{-D_{j-1}z})(1-e^{-D_j z})}\frac{dz}{z}.$$

By Barnes [3] (and also 1.10 of Shintani [22]), we get

$$Z_{j,0}(0)=\frac{1}{2D_{j-1}D_j}\Big(B_2\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)D_{j-1}^2 +2B_1\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)B_1D_{j-1}D_j+B_2D_j^2\Big).$$

Thus we have completed the evaluation of the residue of $\Phi(s)$ at $s=0$, hence that of $Z_\alpha(s)$ at $s=0$.

We next evaluate the constant term of the Laurent expansion of $\Phi(s)$ at $s=0$. If K is odd, it is

$$\frac{1}{2}\sum_{j=1}^{K}(-1)^jZ_{j,0}(0)+\sum_{j=1}^{K}(-1)^j\sum_{l=1}^{\infty}\frac{\hat{c}^l(j)(-1)^l}{(1+(-1)^lA_K^{2l})l!}\mathscr{D}_{j,l}Z_j(0).$$

If K is even, it is

$$\begin{aligned}&\frac{1}{\log(1/A_K)}\sum_{j=1}^{K}(-1)^jZ'_{j,0}(0)\\&+\Big(\frac{1}{2}+\frac{1}{\log(1/A_K)}\log(A_K^{-1}+(-1)^{K+1}A_K)\sum_{j=1}^{K}(-1)^jZ_{j,0}(0)\\&+\sum_{j=1}^{K}(-1)^j\sum_{l=1}^{\infty}\frac{1}{1-A_K^{2l}}\frac{\hat{c}^l(j)(-1)^l}{l!}\mathscr{D}_{j,l}Z_j(0).\end{aligned}$$

For $l\geqq 1$ and for $1\leqq j\leqq K$,

$$\begin{aligned}\frac{(-1)^l}{l!}\mathscr{D}_{j,l}Z_j(0)&=\frac{(-1)^l}{12l!}\Big(\binom{l}{0}(-\alpha_j)^{l-0}(-1)^l\,l!\,D_{j-1}D_j^{-l-1}\\&\quad+\binom{l}{1}(-\alpha_j)^{l-1}(-1)^{l-1}(l-1)!\,D_j^{-l}\\&\quad+\binom{l}{l-1}(-\alpha_j)^{l-(l-1)}(-1)^{l-1}(l-1)!\,D_{j-1}^{-l}\\&\quad+\binom{l}{l}(-\alpha_j)^{l-l}(-1)^l\,l!\,D_jD_{j-1}^{-l-1}\\&=\frac{1}{12}\Big(-\frac{\alpha_j}{D_j}\Big)^l\Big(\frac{D_{j-1}}{D_j}+\frac{1}{\alpha_j}\Big)\\&\quad+\frac{1}{12}\Big(\frac{1}{D_{j-1}}\Big)^l\Big(\frac{D_j}{D_{j-1}}+\alpha_j\Big).\end{aligned}$$

Hence the last term of the above formula for the constant term for odd K is

$$=\frac{1}{12}\sum_{j=1}^{K}(-1)^j\Big(F_1\Big(-A_K^2,\ -\frac{\alpha_j\hat{c}(j)}{D_j}\Big)\Big(\frac{D_{j-1}}{D_j}+\frac{1}{\alpha_j}\Big)$$
$$+F_1\Big(-A_K^2,\ \frac{\hat{c}(j)}{D_{j-1}}\Big)\Big(\frac{D_j}{D_{j-1}}+\alpha_j\Big)\Big)$$

and that for even K is

$$=\frac{1}{12}\sum_{j=1}^{K}(-1)^j\Big(F_2\Big(A_K^2,\ -\frac{\alpha_j\hat{c}(j)}{D_j}\Big)\Big(\frac{D_{j-1}}{D_j}+\frac{1}{\alpha_j}\Big)$$
$$+F_2\Big(A_K^2,\ \frac{\hat{c}(j)}{D_{j-1}}\Big)\Big(\frac{D_j}{D_{j-1}}+\alpha_j\Big)\Big),$$

where F_1 and F_2 are introduced in the introduction and

$$0<A_K<1,\quad 0<\left|\frac{\alpha_j\hat{c}(j)}{D_j}\right|<1\quad\text{and}\quad 0<\left|\frac{\hat{c}(j)}{D_{j-1}}\right|<1.$$

Finally, we evaluate $Z'_{j,0}(0)$. By the above integral expression of $Z_{j,0}(s)$, we get

$$\frac{d}{ds}Z_{j,0}(s)\Big|_{s=0}=\frac{-\Gamma'(1)-\pi i}{2\pi i}\int_{I(\lambda)}\frac{e^{-(D_{j-1}+D_j)z}}{(1-e^{-D_{j-1}z})(1-e^{-D_jz})}\frac{dz}{z}$$
$$+\frac{\Gamma(1)}{2\pi i}\int_{I(\lambda)}\frac{e^{-(D_{j-1}+D_j)z}}{(1-e^{-D_{j-1}z})(1-e^{-D_jz})}\frac{\log z}{z}dz$$
$$=-(\Gamma'(1)+\pi i)\frac{1}{2D_{j-1}D_j}\Big(B_2\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)D_{j-1}^2$$
$$+2B_1\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)B_1D_{j-1}D_j+B_2D_j^2\Big)$$
$$+\log\Big(\frac{\Gamma_2(D_{j-1}+D_j,(D_{j-1},D_j))}{\rho_2((D_{j-1},D_j))}\Big)$$
$$-\frac{(\gamma-\pi i)}{2D_{j-1}D_j}\Big(B_2\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)D_{j-1}^2$$
$$+2B_1\Big(\frac{D_{j-1}+D_j}{D_{j-1}}\Big)B_1D_{j-1}D_j+B_2D_j^2\Big)$$
$$=\log\Big(\frac{\Gamma_2(D_{j-1}+D_j,(D_{j-1},D_j))}{\rho_2((D_{j-1},D_j))}\Big)$$

(Cf. Barnes [3] and also Proposition 2 of Shintani [22]).

This completes our proof of Theorems 5 and 6.

§ 6. Proof of Theorems 5′ and 6′

We shall describe only how to change the argument of the previous section. We suppose, as in the introduction, that

$$\alpha=\frac{1}{b_1+}\ \frac{1}{b_2+}\cdots+\frac{1}{b_M+\tilde{\alpha}}$$

and $\tilde{\alpha}$ is purely periodic. If we denote the n-th principal convergent to $\tilde{\alpha}$ by $\tilde{p}_n/\tilde{q}_n$, then the denominator q_{M+n} of the $(M+n)$-th principal convergent to α satisfies

$$q_{M+n}=c\tilde{q}_n+d\tilde{p}_n \qquad \text{for } n\geqq 1,$$

where c and d are introduced in the introduction.

We shall first get some formula for $\tilde{p}_\nu$. We have first that

$$\tilde{p}_\nu+\tilde{\alpha}_\nu\tilde{p}_{\nu-1}=\frac{1}{\tilde{\alpha}_1\cdots\tilde{\alpha}_{\nu-1}} \qquad \text{for } \nu\geqq 2,$$

where we put $\tilde{p}_1=1$. By induction on j, we get for $1\leqq j\leqq K$,

$$\tilde{p}_{j+1}=E'_{j+1}+(-1)^j\tilde{\alpha}_2\tilde{\alpha}_3\cdots\tilde{\alpha}_{j+1}\tilde{p}_1,$$

where E'_j is introduced in the introduction. Generally, we get for $\nu=0, 1, 2, \cdots$ and for $j=1, 2, \cdots, K$,

$$\tilde{p}_{K\nu+j+1}=A_K^{-\nu}E'_{j+1}+(-1)^j\tilde{\alpha}_2\tilde{\alpha}_3\cdots\tilde{\alpha}_{j+1}\tilde{p}_{K\nu+1}.$$

In particular, we get

$$\tilde{p}_{K(\nu+1)+1}=A_K^{-\nu}E'_{K+1}+(-1)^K A_K\tilde{p}_{K\nu+1}.$$

Hence, by induction on ν, we get for $\nu=0, 1, 2, \cdots$

$$\tilde{p}_{K\nu+1}=E'_{K+1}\frac{A_K^{-\nu}+(-1)^{\nu K+1}A_K^{\nu}}{A_K^{-1}+(-1)^{K+1}A_K}+(-1)^{K\nu}A_K^{\nu}.$$

Now, we decompose $\Phi(s)$ as follows.

$$\begin{aligned}\Phi(s)&=\sum_{\nu=1}^{M+1}(-1)^\nu\zeta_\nu(s)+\sum_{j=1}^{K}\sum_{\nu=0}^{\infty}(-1)^{M+K\nu+j+1}\zeta_{M+K\nu+j+1}(s)\\&=\Phi_1(s)+\Phi_2(s), \quad \text{say.}\end{aligned}$$

$\Phi_1(s)$ is regular except simple poles at most at $s=1$ and 2. To treat $\Phi_2(s)$, we note first that for $2\leqq j\leqq K$,

$$h\tilde{p}_{K\nu+j}+k\tilde{p}_{K\nu+j+1}$$
$$=A_K^{-\nu}\left(hE'_j+kE'_{j+1}+(-1)^{j-1}\frac{\tilde{\alpha}_2\cdots\tilde{\alpha}_jE'_{K+1}}{A_K^{-1}+(-1)^{K+1}A_K}(h-k\tilde{\alpha}_{j+1})\right)$$
$$+A_K^{\nu}(-1)^{K\nu+j-1}\frac{\tilde{\alpha}_2\cdots\tilde{\alpha}_j}{A_K^{-1}+(-1)^{K+1}A^K}(A_K^{-1}+(-1)^{K+1}A_K-E'_{K+1})$$
$$\times(h-k\tilde{\alpha}_{j+1}).$$

For $j=1$, we put $E'_1=0$ and $\tilde{\alpha}_2\cdots\tilde{\alpha}_j=1$, then the above formula holds also in this case. Hence, we get for $1\leqq j\leqq K$,

$$h\tilde{p}_{K\nu+j}+k\tilde{p}_{K\nu+j+1}=\frac{A_K^{-\nu}(hD'_j+kD'_{j+1})}{A_K^{-1}+(-1)^{K+1}A_K}+\frac{A_K^{\nu}(-1)^{K\nu+j-1}c'(j)}{A_K^{-1}+(-1)^{K+1}A_K}$$
$$\times(A_K^{-1}+(-1)^{K+1}A_K-E'_{K+1})(h-k\tilde{\alpha}_{j+1}),$$

where D'_j and $c'(j)$ are introduced in the introduction. Using this we get for $1\leqq j\leqq K$ and for $\nu=0, 1, 2, \cdots$

$$hq_{M+K\nu+j}+kq_{M+K\nu+j+1}$$
$$=c(h\tilde{q}_{K\nu+j}+k\tilde{q}_{K\nu+j+1})+d(h\tilde{p}_{K\nu+j}+k\tilde{p}_{K\nu+j+1})$$
$$=(A_K^{-1}+(-1)^{K+1}A_K)^{-1}(cA_K^{-\nu}(hD_j+kD_{j+1})$$
$$+c(-1)^{K\nu}A_K^{\nu}(-1)^jc(j+1)(A_K^{-1}+(-1)^{K+1}A_K-E_K)$$
$$\times(h-k\tilde{\alpha}_{j+1})+dA_K^{-\nu}(hD'_j+kD'_{j+1})+d(-1)^{K\nu}$$
$$\times A_K^{\nu}(-1)^{j-1}c'(j)(A_K^{-1}+(-1)^{K+1}A_K-E'_{K+1})(h-k\tilde{\alpha}_{j+1}).$$

The rest is clear and we may omit writing it.

References

[1] Arakawa, T., Generalized eta function and certain ray class invariants of real quadratic fields, Math. Ann., **260** (1982), 475–494.
[2] Barnes, E. W., The genesis of the double gamma function, Proc. Lond. Math. Soc., **31** (1899), 358–381.
[3] ——, The theory of the double gamma function, Philos. Trans. Roy. Soc., (A) **196** (1901), 265–388.
[4] Behnke, H., Über die Verteilung von Irrationalitaten mod 1, Abh. Math. Sem. Hamburg, **1** (1922), 252–267.
[5] ——, Zur Theorie der diophntischen Approximationen. I, Abh. Math. Sem. Hamburg, **3** (1924), 261–318.
[6] Erdös, P., Problems and results on diophantine approximations, Compositio Math., **16** (1964), 52–65.
[7] Fujii, A., Some explicit formulae in the theory of numbers, Proc. Japan Acad. **57** Ser. A (1981), 326–330.
[8] Hardy, G. H. and Littlewood, J. E., Some problems of Diophnatine approximation; The analytic character of the sum of a Dirichlet series considered by Hecke, Abh. Math. Sem. Hamburg, **3** (1923), 57–68.

[9] ——, Some problems of Diophantine approximation. The lattice points of a right angled triangle. Proc. Lond. Math. Soc., **20**(2), (1921), 15–36.
[10] ——, Some problems of Diophantine approximation, The lattice points of a right angled triangle (second memoir), Abh. Math. Sem. Hamburg, **1** (1922), 212–249.
[11] ——, Some problems of Diophantine approximation, Proc. of the fifth international congress of math., (1912), 223–229.
[12] Hecke, E., Über analytische Funktionen und die Verteilung von Zahlen mod eins, (Werke 313–335).
[13] ——, Über die Kroneckersche Grenzformel fur reelle quadratische Körper und die Klassenzahl relative Abelscher Körper (Werke 249–289).
[14] ——, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, Zweite Mitteilung (Werke 249–289).
[15] Kesten, H., Uniform distribution mod 1, Ann of Math., (2) **71** (1960), 445–471.
[16] Koksma, J. F., Diophantische Approximationen, Springer, Berlin, 1936.
[17] Kolesnik, G., On the estimation of multiple exponential sums, Recent progress in analytic number theory, **1** (1981), 231–246, Academic Press.
[18] Kuipers, L. and Niedereiter, H., Uniform distribution of sequences, John Wiley, 1974.
[19] Lang, S., Asymptotic diophantine approximations, Proc. Nat. Acad. Sci. USA, **55** (1966), 31–33.
[20] Lerch, M., l'Intermediaire des Mathematiciens, **11** (1904), 145–146.
[21] Ostrowski, A., Bemerkungen zur Theorie der Diophantischen Approximationen. Abh. Math. Sem. Hamburg, **1** (1921), 77–98.
[22] Shintani, T., On a Kronecker limit formula for real quadratic number fields, J. Fac. Sci. Univ. Tokyo, 1A **24** (1977), 167–199.
[23] ——, On certain ray class invariants of real quadratic fields, J. Math. Soc. Japan, **30** 1, (1978), 139–167.
[24] Sierpinski, W., Jahrbuch über die Fortschritte der Math. 1909, p. 221.
[25] Sos, V. T., On the theory of Diophntine approximations I, Acta Math. Acad. Sci. Hungar., **8** (1957), 461–472.
[26] ——, On the distribution mod 1 of the sequence αn, Ann. Univ. Sci. Budapest Eotvos, **1** (1958), 127–134.
[27] Spencer, D. C., On a Hardy-Littlewood problem of Diophantine approximation. Proc. Cambridge Philos. Soc., **35** (1939), 527–547.
[28] Stark, H. M., Hilbert's twelfth problem and L-series, Bull. of Amer. Math. Soc., **83** (1977), 1072–1074.
[29] Titchmarsh, E. C., The theory of the Riemann zeta function, Clarendon, Oxford 1951.
[30] Weyl, H., Uber die Gleichverteilung von Zahlen mod Eins, Math. Ann., **77** (1916), 313–352.
[31] Whittaker, E. T. and Watson, G. N., A course of modern analysis, Cambridge, 1927.
[32] Zagier, D., A Kronecker limit formula for real quadratic fields, Math. Ann., **213** (1975), 153–184.

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro
Tokyo 171, Japan