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Gauss Sums and Generalized Theta Series

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Let K be a totally imaginary number field containing the n-th roots of unity. Following Kubota theory [3], we define generalized theta series for K as residues of metaplectic Eisenstein series. The Fourier coefficients of this Eisenstein series are Dirichlet series whose coefficients are n-th Gauss sums. Our problem is to evaluate the coefficients of generalized theta series, i.e., the residues of the above Dirichlet series. This problem is deeply connected with the distribution of Gauss sums. In case of $n=3$, for $K=\boldsymbol{Q}(\sqrt{-3})$, it is completely solved ([5]). In [6], we obtained some informations in case of $n=4$ for $K=\boldsymbol{Q}(i)$. In this paper, we shall apply the method of [6] to the case of odd prime l such that the cyclotomic number field $\boldsymbol{Q}(\zeta)$, $\zeta=e^{2\pi i/l}$, has class number 1 (i.e. $l=3, 5, 7, 11, 13, 17, 19$).

We consider a family of Dirichlet series of the following type:

$$\psi(s, \mu)=\sum_{(c)}\left(\frac{\lambda^{(l+1)/2}}{c}\right)g((c), \mu)N(c)^{-s} \quad (s \in \boldsymbol{C})$$

where $\mu \in \vartheta^*-\{0\}$ (ϑ^*: the inverse ideal of the different of K), $\lambda=1-\zeta$, (c) runs over all the integral ideals of K such that $c\equiv 1\ (\lambda^{(l+1)/2})$, $(-)$ is the l-th power residue symbol and $g((c), \mu)$ is a Gauss sum (See Section 1). Then, $\psi(s, \mu)$ is holomorphically continued to the region $\mathrm{Re}(s)>1$ except possibly a simple pole at $s=(l+1)/l$. We put $\psi(\mu)=\mathrm{Res}_{s=(l+1)/l}\psi(s, \mu)$. We see that $\psi(\mu)$ ($\mu \in \vartheta^*-\{0\}$) are coefficients of the theta series for K. Our results are stated as follows:

(1) $\psi(m^l\mu)=\psi(\mu)$ for any integer m,

(2) if $(m, \mu\lambda)=1$, then $\psi(m^{l-1}\mu)=0$,

(3) if m is a prime such that $m\equiv 1\ (\lambda^{(l+1)/2})$, $(m, \mu)=1$, then

$$\psi(m^{l-t-2}\mu)=\left(\frac{\lambda^{(l+1)/2}}{m}\right)^{-t-1}\bar{g}_{t+1}((m), \mu)N(m)^{-1+(t+1)/l}\psi(m^t\mu)$$

for $t=0, 1, 2, \cdots, (l-3)/2$ ($g_{t+1}((m), \mu)$ is also a Gauss sum, see Section 6).

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S. J. Patterson informed the author that he obtained similar results and announced them without proof in his paper: On the distribution of general Gauss sums, Recent Progress in analytic number theory, vol. 2, Academic Press, 1981. His method depends on the representation theory but essentially on the same ideas of ours (for details see Kazhdan and Patterson's Metaplectic forms I, to appear).

§ 1. Cyclotomic fields and the reciprocity law

Let l denote a rational prime number such that $5 \leqq l \leqq 19$. Put $\zeta = e^{2\pi i/l} \in C$, and consider the cyclotomic field $K = Q(\zeta) \subset C$; K is an algebraic number fields with a unique factorization, of degree $l-1$. Let ϑ be the ring of integers in K, i.e., $\vartheta = Z[\zeta]$. Put $\lambda = 1-\zeta$; λ is the prime divisor of l in K. Let U be the group of all units in K; U is generated by $-\zeta$ and the so-called circular units. Let D denote the set of all integral divisors in K; we identify D with a set of representatives of ϑ'/U ($\vartheta' = \vartheta - \{0\}$), which will be specified later. We take a primitive root g of l. The Galois group of the extension K/Q is generated by s which is defined by $\zeta^s = \zeta^g$. For the sake of brevity, we put $l^* = (l-1)/2$ and $\lambda^* = \lambda^{(l+1)/2}$.

Lemma 1. *We put*

$$\varepsilon_k = \varepsilon_0^{(s-1)(s-g^2)(s-g^4)\cdots(s-g^{k-2})(s-g^{k+2})\cdots(s-g^{l-3})} \quad (k=2, 4, 6, \cdots, l-3)$$

where

$$\varepsilon_0 = \left(\frac{(1-\zeta^g)(1-\zeta^{-g})}{(1-\zeta)(1-\zeta^{-1})} \right)^{l^*}.$$

Then, ε_k $(k=2, 4, \cdots, l-3)$ *satisfy*

$$\varepsilon_k \equiv 1 + b_k \lambda^k \ (\lambda^{k+1}), \qquad \varepsilon_k^{s-g^k} \equiv 1 \ (\lambda^{l+1})$$

where b_k $(k=2, 4, \cdots, l-3)$ *are rational integers prime to* l. *Furthermore, if* ε *is a unit satisfying* $\varepsilon \equiv 1$ (λ), *then there exist rational integers* t_k $(k=0, 2, 4, \cdots, l-3)$ *such that*

$$\varepsilon \equiv \zeta^{t_0} \varepsilon_2^{t_2} \varepsilon_4^{t_4} \cdots \varepsilon_{l-3}^{t_{l-3}} \ (\lambda^{l+1}).$$

Proof. For the first part, see Hilfssatz 29 of [2]. We prove the second assertion. Choose t_0 such that $\varepsilon\zeta^{-t_0}$ is real. There exist rational integers r prime to l and $t_2', t_4', \cdots, t_{l-3}'$ such that $(\varepsilon\zeta^{-t_0})^r = \varepsilon_2^{t_2'} \varepsilon_4^{t_4'} \cdots \varepsilon_{l-3}^{t_{l-3}'}$ (see the proof of Satz 154 of [2]). Since $(\varepsilon\zeta^{-t_0}) \equiv 1$ (λ^{l+1}), the second assertion follows.

We take

$$\eta_1=\zeta, \qquad \eta_{l-1}=1+l, \qquad \eta_l=1-\lambda^l,$$
$$\eta_k=(1-\lambda^k)^{-g^k(s-1)(s-g)(s-g^2)\cdots(s-g^{k-1})(s-g^{k+1})\cdots(s-g^{l-2})}$$
$$(k=3, 5, \cdots, l-2),$$
$$\eta_k=\varepsilon_k^{b'_k} \quad (k=2, 4, \cdots, l-3)$$

where b'_k is a rational integer such that $b_k b'_k \equiv -1$ (l) $(k=2, 4, \cdots, l-3)$. By Lemma 1, η_k $(k=1, 2, 3, \cdots, l)$ satisfy

$$\eta_k \equiv 1-\lambda^k \ (\lambda^{k+1}), \qquad \eta_k^{s-g^k} \equiv 1 \ (\lambda^{l+1}).$$

Let $(-)$ be the l-th power residue symbol for K. If $\alpha \equiv \beta \equiv 1$ (λ),

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^{-1}=\left(\frac{\beta, \alpha}{\lambda}\right) \tag{1.1}$$

where $\left(\frac{\ ,\ }{\lambda}\right)$ is the norm residue symbol for K. If

$$\alpha \equiv \eta_1^{t_1(\alpha)}\eta_2^{t_2(\alpha)}\cdots\eta_l^{t_l(\alpha)} \ (\lambda^{l+1}), \qquad \beta \equiv \eta_1^{t_1(\beta)}\eta_2^{t_2(\beta)}\cdots\eta_l^{t_l(\beta)} \ (\lambda^{l+1}),$$

then

$$\left(\frac{\beta, \alpha}{\lambda}\right)=\zeta^{-\Sigma_{k=1}^{l-1} k t_k(\alpha) t_{l-k}(\beta)}, \tag{1.2}$$

$$\left(\frac{\lambda}{\alpha}\right)=\zeta^{t_l(\alpha)+\Sigma_{k=1}^{l-2}((-1)^k/k!)B_k t_{l-k}(\alpha)}, \tag{1.3}$$

where B_k is the so-called Bernoulli number (see [1]). Especially, if $\alpha \equiv \beta \equiv 1(\lambda^*)$, then

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^{-1}=1. \tag{1.4}$$

For $N \in D$, we define

U_N = the group of units congruent to 1 modulo N in K,
D_N = a set of representatives of ϑ'/U_N,
$U(N)=[U: U_N]$,
$A(N)$ = a set of invertible residues of ϑ (mod N) which are inequivalent to each other under multiplication of any unit of ϑ,
$\tilde{\Phi}(N)$ = the cardinal number of $A(N)=\Phi(N)/U(N)$
($\Phi(N)$: Euler function on K).

Lemma 2. *Let η_0 be a unit satisfying $\eta_0 \equiv g$ (λ^*), then a complete set of representatives of U/U_{λ^*} is given by*

(1.5) $\{\eta_0^{t_0}\eta_1^{t_1}\eta_2^{t_2}\eta_4^{t_4}\cdots\eta_{l'}^{t_{l'}};\ 0\leqq t_0\leqq l-2,\ 0\leqq t_k\leqq l-1\ (k=1, 2, 4, 6, \cdots, l')\}$

where

$$l'=\begin{cases}\dfrac{l-1}{2} & \textit{if } l\equiv 1\ (4),\\ \dfrac{l-3}{2} & \textit{if } l\equiv 3\ (4).\end{cases}$$

Proof. This follows from Lemma 1.

Now, by Lemma 2, we can choose $A(\lambda^*)$ once for all as follows;

(1.6) $A(\lambda^*)=\{\eta_3^{t_3}\eta_5^{t_5}\cdots\eta_{l''}^{t_{l''}};\ 0\leqq t_k\leqq l-1\ (k=3, 5, \cdots, l'')\}$

where

$$l''=\begin{cases}\dfrac{l-3}{2} & \text{if } l\equiv 1\ (4),\\ \dfrac{l-1}{2} & \text{if } l\equiv 3\ (4).\end{cases}$$

For the above fixed $A(\lambda^*)$ we can choose the set D of representatives for ϑ'/U (once for all) satisfying that if $\gamma\in D$, $(\gamma, \lambda)=1$, then $\gamma\equiv r_0\ (\lambda^*)$ for $r_0\in A(\lambda^*)$.

Lemma 3. *Let $\gamma\in D$, then $\gamma\equiv 1\ (\lambda^*)$ if and only if*

$$\left(\frac{\gamma, \varepsilon}{\lambda}\right)=1 \quad \textit{for all } \varepsilon\in U_{\lambda^*}.$$

Proof. This lemma follows from Lemma 1, (1.2) and (1.6).

For a given prime divisor m $(\neq\lambda)$, we see that $U(\lambda^*)|U(\lambda^*m)$. We can assume that every element of $A(\lambda^*m)$ is a residue of some element of D.

We consider the product $\boldsymbol{C}^{l^*}$ of l^* copies of $\boldsymbol{C}$ an $\boldsymbol{R}$-algebra. For $z=(z_1, z_2, \cdots, z_{l^*})\in\boldsymbol{C}^{l^*}$, we put $e(z)=\exp(2\pi i\sum_{k=1}^{l^*}(z_k+\bar{z}_k))$. Then $e(z)$ is a character of the additive group of $\boldsymbol{C}^{l^*}$. The cyclotomic field $K/\boldsymbol{Q}$ is considered *a* $\boldsymbol{Q}$-subalgebra of $\boldsymbol{C}^{l^*}$ by the map

$$c\longmapsto(c, c^s, c^{s^2}, \cdots, c^{s^{l^*-1}})\in\boldsymbol{C}^{l^*}\quad(c\in K).$$

Then, $e(c)=\exp(2\pi i\,Tr_{K/\boldsymbol{Q}}(c))$ for $c\in K$. Considering ϑ a lattice in $\boldsymbol{C}^{l^*}$, let ϑ^* be the dual lattice of ϑ in $\boldsymbol{C}^{l^*}$ with respect to $e(\ )$.

Now we define Gauss sums as follows. For $c\in D$, $c\equiv 1\ (\lambda^*)$, and $\mu\in\vartheta^*$ $(\mu\neq 0)$, we put

(1.7)
$$g(c, \mu)=\sum_{\delta \bmod c}\left(\frac{\delta}{c}\right)e\left(\frac{\delta}{c}\mu\right)$$

where δ runs through a set of invertible residues (mod c). By Lemma 3, $g(c, \mu)$ is well-defined and satisfies $g(c, \varepsilon\mu)=g(c, \mu)$ for $\varepsilon \in U_{\lambda*}$. We define the Dirichlet series $\psi(s, \mu)$ for $\mu \in \vartheta^*$-$\{0\}$, which is our object to investigate:

(1.8)
$$\psi(s, \mu)=\sum_{\substack{c\equiv 1(\lambda^*)\\ c\in D}}\left(\frac{\lambda^*}{c}\right)g(c, \mu)N(c)^{-s}$$

where $N(c)=N_{K/Q}(c)$ and $s \in \boldsymbol{C}$. It is easily seen that:

(1.9) $\psi(s, \varepsilon\mu)=\psi(s, \mu)$ for every $\varepsilon \in U_{\lambda*}$,

(1.10) $\psi(s, \lambda^l\mu)=\psi(s, \mu)$.

§ 2. $\Gamma(N)$

For $z=(z_1, z_2, \cdots, z_{l*}) \in \boldsymbol{C}^{l*}$, we put $\bar{z}=(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{l*})$. We consider matrices of the form $u=\begin{pmatrix} z & -v \\ v & \bar{z}\end{pmatrix}$ where $z \in \boldsymbol{C}^{l*}$, $v=(v_1, v_2, \cdots, v_{l*}) \in (\boldsymbol{R}_+^\times)^{l*}$. Then, the space of all matrices of this form is identified with the product H^{l*} of l^* copies of the upper half space H. For $w \in \boldsymbol{C}^{l*}$, we put $\tilde{w}=\begin{pmatrix} w & 0 \\ 0 & \bar{w}\end{pmatrix}$. Then, $SL_2(\boldsymbol{C}^{l*})\cong SL_2(\boldsymbol{C})\times\cdots\times SL_2(\boldsymbol{C})$ acts on H^{l*} by

(2.1)
$$\sigma(u)=(\tilde{a}u+\tilde{b})(\tilde{c}u+\tilde{d})^{-1}$$

where $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\boldsymbol{C}^{l*})$, $a, b, c, d \in \boldsymbol{C}^{l*}$. The action is transitive and the stabilizer of a point is isomorphic to $SU_2(\boldsymbol{C})\times\cdots\times SU_2(\boldsymbol{C})$.

For $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_{l*}) \in GL_2(\boldsymbol{C}^{l*})\cong GL_2(\boldsymbol{C})\times\cdots\times GL_2(\boldsymbol{C})$ ($\sigma_k \in GL_2(\boldsymbol{C})$, $k=1, 2, \cdots, l^*$), we put

$$\det\sigma=(\det\sigma_1, \det\sigma_2, \cdots, \det\sigma_{l*}),$$
$$(\det\sigma)^{-1/2}=(\pm(\det\sigma_1)^{-1/2}, \cdots, \pm(\det\sigma_{l*})^{-1/2}).$$

Then, $GL_2(\boldsymbol{C}^{l*})\cong GL_2(\boldsymbol{C})\times\cdots\times GL_2(\boldsymbol{C})$ acts on H^{l*} by

(2.2)
$$\sigma(u)=((\det\sigma)^{-1/2}\sigma)(u), \qquad u \in H^{l*}$$

where $\sigma \in GL_2(\boldsymbol{C}^{l*})$ and $(\det\sigma)^{-1/2}\sigma \in SL_2(\boldsymbol{C}^{l*})$.

For $z=(z_1, z_2, \cdots, z_{l*}) \in \boldsymbol{C}^{l*}$, put $\|z\|=\prod_{k=1}^{l*}|z_k|$. We write $v(u)=\|v\|$ for $u=\begin{pmatrix} z & -v \\ v & \bar{z}\end{pmatrix} \in H^{l*}$. Then, for $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\boldsymbol{C}^{l*})$ such that $\|\det\sigma\|=1$, (2.3) $$v(\sigma u)=j(\sigma, u)^{-1}v(u)$$

where $j(\sigma, u)=\|\det(\tilde{c}u+\tilde{d})\|$.

We can consider $GL_2(\vartheta)$ a discontinuous subgroup of $GL_2(C^{l^*})$. Then $GL_2(\vartheta)$ acts on H^{l^*} discontinuously, and has a fundamental domain whose volume with respect to the invariant measure of H^{l^*} is finite. Let N denote either λ^* or λ^*m once for all where m is a prime divisor ($\neq\lambda$) of ϑ. We put

(2.4) $$\Gamma(N)=\left\{\sigma\in GL_2(\vartheta);\ \sigma\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \bmod N\right\}.$$

For the sake of brevity, we put $(b/a)=1$ for $a\equiv 1, 2, \cdots, l-1$ (λ^*) and $b\equiv 0$ (a). We put

(2.5) $$\chi(\sigma)=\left(\frac{c}{a}\right) \qquad \text{for } \sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(N).$$

Lemma 4. *Let* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2(\vartheta)$. *If* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(\lambda^*)$ *and* $\alpha\delta-\beta\gamma\equiv 1$ (λ^*), *then*

$$\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{c}{\alpha a+\beta c}\right).$$

Furthermore,

$$\begin{aligned}\chi\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)&=\left(\frac{\beta c}{\alpha}\right)^{-1}\left(\frac{c}{\alpha a+\beta c}\right) && \text{if } \alpha\equiv 1, 2, \cdots, l-1\ (\lambda^*),\\ &=\left(\frac{\alpha}{\beta c}\right)^{-1}\left(\frac{c}{\alpha a+\beta c}\right) && \text{if } \alpha\equiv a\equiv 0\ (\lambda^*),\\ &=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{\alpha a}{c}\right) && \text{if } c\equiv 1, 2, \cdots, l-1\ (\lambda^*),\\ &=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{c}{\alpha a}\right) && \text{if } c\equiv\beta\equiv 0\ (\lambda^*).\end{aligned}$$

Proof. By the definition of χ and the reciprocity law ((1.1), (1.2), (1.3)),

$$\begin{aligned}\left(\frac{\gamma a+\delta c}{\alpha a+\beta c}\right)&=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{\alpha\gamma a+\alpha\delta c}{\alpha a+\beta c}\right)\\ &=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{(\alpha\delta-\beta\gamma)c}{\alpha a+\beta c}\right)\end{aligned}$$

$$=\left(\frac{\alpha}{\alpha a+\beta c}\right)^{-1}\left(\frac{c}{\alpha a+\beta c}\right).$$

If $\alpha\equiv 1, 2, \cdots, l-1$ (λ^*), then $\alpha^{l-1}\equiv 1$ (λ^*), $(\alpha/(\alpha a+\beta c))^{l-1}=(\beta c/\alpha)^{l-1}$ and so $(\alpha/(\alpha a+\beta c))=(\beta c/\alpha)$. If $\alpha\equiv a\equiv 0$ (λ^*), then $\alpha a+\beta c\equiv\beta c$ (λ^{l+1}) and so $(\alpha/(\alpha a+\beta c))=(\alpha/\beta c)$. Now the lemma follows.

From Lemma 4, we have

Proposition 1. *χ is a character on $\Gamma(N)$.*

The set of all cusps of $GL_2(\vartheta)$ is identified with $K\cup\{\infty\}$ on which $GL_2(\vartheta)$ acts transitively. Two cusps $\kappa=\alpha/\gamma$, $\kappa'=\alpha'/\gamma'$ are $\Gamma(N)$-equivalent if and only if there is a unit $\varepsilon\in U$ such that

$$\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\equiv\varepsilon\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix} \bmod N.$$

For a cusp κ, let $\Gamma_\kappa=\{\sigma\in\Gamma(N);\ \sigma\kappa=\kappa\}$ and put

$$\sigma_\kappa=\begin{cases}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \text{if } \kappa=\infty,\\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & \text{if } \kappa=0,\\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\vartheta) & \text{if } \kappa=\alpha/\gamma\neq 0.\end{cases}$$

Then, we have

(2.6) $$\kappa=\sigma_\kappa(\infty),$$

(2.7) $$\Gamma_\infty=\left\{\begin{pmatrix}\varepsilon_1 & b\\ 0 & \varepsilon_2\end{pmatrix}:\ \varepsilon_1, \varepsilon_2\in U_N,\ b\in N\vartheta\right\},$$

(2.8) $$\Gamma_\kappa=\sigma_\kappa\Gamma_\infty\sigma_\kappa^{-1}.$$

If the character χ is trivial on Γ_κ, then κ is said to be an essential cusp of $\Gamma(N)$. This notion depends only on the $\Gamma(\lambda^*)$-equivalence class of κ. So the set of all essential cusps of $\Gamma(\lambda^* m)$ is equal to that of $\Gamma(\lambda^*)$. It is easily seen that $0, 1, 2, \cdots, l-1$ and ∞ are essential cusps of $\Gamma(\lambda^*)$. Put $P(\lambda^*)=\{0, 1, 2, \cdots, l-1, \infty\}$.

Proposition 2. *$P(\lambda^*)$ is a complete set of $\Gamma(\lambda^*)$-inequivalent essential cusps.*

Proof. Using Lemma 4, we see that $\kappa=\alpha/\gamma$ is an essential cusp if and only if

$$(2.9) \qquad \left(\frac{\alpha}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)^{-1}\left(\frac{\gamma}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)=1$$

for all ε_1, $\varepsilon_2 \in U_{\lambda^*}$ and $b \in \lambda^*\vartheta$. First, let $\kappa=\alpha/\gamma$ be an essential cusp such that $\lambda|\alpha$. Then $\alpha\equiv 0$ (λ^*). For, if $\lambda^k|\alpha$ and $\lambda^{k+1}\nmid\alpha$ $(k\leqq l^*)$, and if we take ε_1, ε_2, b such that $\varepsilon_1=1$, $\varepsilon_2=1$, $1-\gamma\alpha b\equiv 1+\lambda^l$ (λ^{l+1}), then

$$\left(\frac{\alpha}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)^{-1}\left(\frac{\gamma}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)=\left(\frac{\alpha}{1-\gamma\alpha b}\right)^{-1}\left(\frac{\gamma}{1-\gamma\alpha b}\right)$$
$$=\left(\frac{\lambda^k}{1+\lambda^l}\right)\neq 1.$$

Now we may assume that $\gamma \in D$ and $\sigma_\kappa=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ where $\alpha\delta\equiv 0$ (λ^{l+1}), $\beta\equiv -\gamma^{-1}$ (λ^{l+1}). Then, $\varepsilon_2^{-1}(\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b)\equiv 1$ (λ^{l+1}) and

$$\left(\frac{\alpha}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)^{-1}\left(\frac{\gamma}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)=\left(\frac{\gamma}{\varepsilon_2^{-1}(\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b)}\right)$$
$$=\left(\frac{\varepsilon_2^{-1}\varepsilon_1}{\gamma}\right)=\left(\frac{\varepsilon_2\varepsilon_1^{-1},\gamma}{\lambda}\right).$$

So, by Lemma 3, we have $\gamma\equiv 1$ (λ^*). This means α/γ is $\Gamma(\lambda^*)$-equivalent to 0.

Similarly we have: if $\kappa=\alpha/\gamma$ is an essential cusp such that $\lambda|\gamma$, then κ is $\Gamma(\lambda^*)$-equivalent to ∞.

Next, let $\kappa=\alpha/\gamma$ be an essential cusp such that $(\alpha, \lambda)=(\gamma, \lambda)=1$. We may assume that $\gamma \in D$, $\sigma_\kappa=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ where $\delta\equiv 0$ (λ^{l+1}), $\beta\equiv -\gamma^{-1}$ (λ^{l+1}). Then, $\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b\equiv\varepsilon_2-\gamma\alpha b$ (λ^{l+1}) and

$$\left(\frac{\alpha}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)^{-1}\left(\frac{\gamma}{\alpha\delta\varepsilon_1-\gamma\beta\varepsilon_2-\gamma\alpha b}\right)$$
$$=\left(\frac{\varepsilon_2-\gamma\alpha b,\alpha}{\lambda}\right)^{-1}\left(\frac{\varepsilon_2}{\alpha}\right)^{-1}\left(\frac{\varepsilon_2-\gamma\alpha b,\gamma}{\lambda}\right)\left(\frac{\varepsilon_1}{\gamma}\right)$$
$$=\left(\frac{1-\gamma\alpha\varepsilon_2^{-1}b,\alpha}{\lambda}\right)^{-1}\left(\frac{\varepsilon_2\varepsilon_1^{-1}-\gamma\alpha\varepsilon_1^{-1}b,\gamma}{\lambda}\right).$$

Taking $b=0$, we get $\gamma\equiv 1$ (λ^*) from Lemma 3. Taking $\varepsilon_1\equiv\varepsilon_2\equiv 1$ (λ^*), we have $\alpha^{l-1}\equiv 1$ (λ^*), i.e., $\alpha\equiv 1, 2, \cdots, l-1$ (λ^*). So $\kappa=\alpha/\gamma$ is $\Gamma(\lambda^*)$-equivalent to one of $1, 2, \cdots, l-1$.

We classify all the essential cusps of $\Gamma(\lambda^*)$ into three types and choose σ_κ once for all in the following way:

type	$\kappa=\alpha/\gamma$	$\sigma_\kappa=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$
A-type	$\alpha\equiv 0\ (\lambda^*)$, $\gamma\equiv 1\ (\lambda^*)$	$\alpha\delta\equiv 0\ (\lambda^{l+1})$
B-type	$\alpha\equiv 1, 2, \cdots, l-1\ (\lambda^*)$, $\gamma\equiv 1\ (\lambda^*)$	$\delta\equiv 0\ (\lambda^{l+1})$
C-type	$\alpha\equiv 1\ (\lambda^*)$, $\gamma\equiv 0\ (\lambda^*)$	$\beta\gamma\equiv 0\ (\lambda^{l+1})$

In case of $N=\lambda^*m$, we classify them into nine types; an essential cusp κ is said to be of $(A,\ )$-type (resp. of $(B,\ )$-type, of $(C,\ )$-type) if κ is of A-type (resp. of B-type, of C-type) in the above sense; $(\ , A)$-type, $(\ , B)$-type and $(\ , C)$-type are defined in the following table:

type	$\kappa=\alpha/\gamma$
$(\ , A)$-type	$\alpha\equiv 0\ (m)$
$(\ , B)$-type	$(\alpha, m)=(\gamma, m)=1$
$(\ , C)$-type	$\gamma\equiv 0\ (m)$

In particular, if $\kappa=\alpha/\gamma$ is of (A, A)-type, then $\alpha\equiv 0\ (\lambda^*m)$ and $\gamma\equiv 1\ (\lambda^*)$.

§ 3. Eisenstein series and theta series

The greater part of this section is taken from [3] and [4]. See for details [3] and [4].

Let $\{\kappa_0, \kappa_1, \kappa_2, \cdots\}$ be the set of essential cusps for $\Gamma(\lambda^*)$. For the sake of brevity, we put $\sigma_{\kappa_i}=\sigma_i$, $\Gamma_{\kappa_i}=\Gamma_i$ $(i=0, 1, 2, \cdots)$ and $\kappa_0=\infty$, $\sigma_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. For each cusp κ_i, the Eisenstein series

$$(3.1)\qquad E_i(u, s, \Gamma(N))=\sum_{\sigma\in\Gamma_i\backslash\Gamma(N)}\bar{\chi}(\sigma)v(\sigma_i^{-1}\sigma(u))^s \quad (u\in H^{l^*},\ s\in \boldsymbol{C})$$

is defined; it is absolutely convergent for $\mathrm{Re}(s)>2$. If we take $\sigma\kappa_i$ $(\sigma\in\Gamma(N))$ instead of κ_i, then $E_i(u, s, \Gamma(N))$ changes to $\bar{\chi}(\sigma)E_i(u, s, \Gamma(N))$. As a function of u, $E_i(u, s, \Gamma(N))$ is an eigenfunction of all Laplacians of H^{l^*} and satisfies

$$(3.2)\qquad E_i(\sigma u, s, \Gamma(N))=\chi(\sigma)E_i(u, s, \Gamma(N)) \quad \text{for every } \sigma\in\Gamma(N).$$

We consider the Fourier expansion of $E_i\ (u, s, \Gamma(N))$ at every cusp. Let

$$K(w, s)=\int_{C^{l*}} j\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}z & -1\\ 1 & \bar{z}\end{pmatrix}\right)^{-s} e(-wz)dV(z) \tag{3.3}$$
$$(w \in C^{l*}, s \in C)$$

where $dV(z)$ is the Euclidian measure of C^{l*}; then we have

$$K(w, s)=(2\pi)^{l*s}|w|^{s-1}\Gamma(s)^{-l*}\tilde{K}_{s-1}(4\pi w) \tag{3.4}$$

where $\tilde{K}_{s-1}(w)$ is a product of the modified Bessel function K_{s-1}; $\tilde{K}_{s-1}(w) = \prod_{k=1}^{l*} K_{s-1}(|w_k|)$ $(w=(w_1, w_2, \cdots, w_{l*}) \in C^{l*})$. Denote by $V(N)$ the volume of $C^{l*}/N\vartheta$ with respect to $dV(z)$. For any two cusps κ_i, κ_j, let $M_{ij}(N)$ be a set of pairs $(c, d) \in D_N \times \vartheta$ such that $\sigma_i\begin{pmatrix}* & *\\ c & d\end{pmatrix}\sigma_j^{-1} \in \Gamma(N)$ and, for a fixed c, d runs through a set of residues mod Nc prime to c. For $(c, d) \in M_{ij}(N)$, we define

$$\chi_{ij}(c, d)=\chi\left(\sigma_i\begin{pmatrix}* & *\\ c & d\end{pmatrix}\sigma_j^{-1}\right). \tag{3.5}$$

Then we have

Proposition 3. *$E_i(\sigma_j, u, s, \Gamma(N))$ has a Fourier expansion of the form*

$$\begin{aligned} E_i(\sigma_j u, s, \Gamma(N)) &= \delta_{ij}v(u)^s + \tilde{\psi}_{ij}(s, \Gamma(N))v(u)^{2-s} \\ &+ \sum_{\substack{\mu\in\vartheta^* \\ \mu\neq 0}} \psi_{ij}(s, u, \Gamma(N))v(u)^{2-s}K(\mu v/N, s)e(\mu z/N) \end{aligned} \tag{3.6}$$

with

$$\delta_{ij}=\begin{cases}1 & \text{if } \kappa_j \text{ is } \Gamma(N)\text{-equivalent to } \kappa_i,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{\psi}_{ij}(s, \Gamma(N))=\left(\frac{\pi}{s-1}\right)^{l*} V(N)^{-1} \sum_{(c,d)\in M_{ij}(N)} \chi_{ij}(c, d)N(c)^{-s},$$

$$\psi_{ij}(s, \mu, \Gamma(N))=V(N)^{-1} \sum_{(c,d)\in M_{ij}(N)} \chi_{ij}(c, d)e\left(\frac{d\mu}{cN}\right)N(c)^{-s}.$$

Here $\tilde{\psi}_{ij}(s, \Gamma(N))$ does not depend on the choice of σ_j such that $\sigma_j(\infty)=\kappa_j$.

Proof. See [3].

Definition 1. For $\mu \in \vartheta^*-\{0\}$, we put

$$\psi_i(s, \mu, \Gamma(N))=\psi_{i0}(s, \mu, \Gamma(N));$$

we call it the μ-th coefficient of $E_i(u, s, \Gamma(N))$.

Definition 2. For an essential cusp κ_j, $\tilde{\psi}_{ij}(s, \Gamma(N))$ is said to be the singular value of $E_i(u, s, \Gamma(N))$ at κ_j.

As we see later, $\tilde{\psi}_{ij}(s, \Gamma(N))$'s are expressed by the Dedekind zeta function of K. So we get the fact that $\tilde{\psi}_{ij}(s, \Gamma(N))$'s are holomorphic in the region $\mathrm{Re}(s)>1$ except possibly a simple pole at $s=(l+1)/l$. Moreover we have

Proposition 4. *$E_i(u, s, \Gamma(N))$ and $\psi_i(s, \mu, \Gamma(N))$ are holomorphically continued to the region* $\mathrm{Re}(s)>1$ *except possibly a simple pole at* $s=(l+1)/l$.

Proof. See [3] and [4].

Remark. It is well known that $E_i(u, s, \Gamma(N))$'s are meromorphically continued to the whole plane $\boldsymbol{C}$ and satisfy functional equations. But we do not use this fact.

Let $E(u, s, \Gamma(N))$ be a linear combination $\sum_i c_i E_i(u, s, \Gamma(N))$ ($c_i \in \boldsymbol{C}$) of Eisenstein series.

Definition 1′. We define the μ-th coefficient of $E(u, s, \Gamma(N))$ by

$$\psi(s, \mu, \Gamma(N))=\sum_i c_i \psi_i(s, \mu, \Gamma(N)).$$

Definition 2′. We define the singular value of $E(u, s, \Gamma(N))$ at an essential cusp κ_j by

$$\tilde{\psi}(s, \kappa_j, \Gamma(N))=\sum_i c_i \tilde{\psi}_{ij}(s, \Gamma(N)).$$

By Proposition 3, we can take the residue $\theta(u, \Gamma(N))$ of $E(u, s, \Gamma(N))$ at $s=(l+1)/l$; we call it a theta series for $\Gamma(N)$. The theta series $\theta(u, \Gamma(N))$ is a square-integrable automorphic (with respect to the character χ) function of $\Gamma(N)$ and is an eigenfunction of all Laplacians of H^{l*}.

Definition 1″. We define the μ-th coefficient of $\theta(u, \Gamma(N))$ by

$$\psi(\mu, \Gamma(N))=\mathrm{Res}_{s=(l+1)/l}\, \psi(s, \mu, \Gamma(N)).$$

Definition 2″. We define the singular value of $\theta(u, \Gamma(N))$ by

$$\tilde{\psi}(\kappa_j, \Gamma(N))=\mathrm{Res}_{s=(l+1)/l}\, \tilde{\psi}(s, \kappa_j, \Gamma(N)).$$

Then $\theta(u, \Gamma(N))$ has the following Fourier expansion:

$$\begin{aligned}\theta(u, \Gamma(N))=&\tilde{\psi}(\infty, \Gamma(N))v(u)^{(l-1)/l}+(2\pi)^{(l^2-1)/l}\Gamma\left(\frac{l+1}{l}\right)^{-l*}\|N\|^{-1/l}\\ &\times\sum_{\substack{\mu\in\vartheta^*\\ \mu\neq 0}}\psi(\mu, \Gamma(N))\|\mu\|^{1/l}v(u)\tilde{K}_{1/l}(4\pi\mu v/N)e(\mu z/N).\end{aligned} \tag{3.7}$$

Let $\theta(u, \Gamma(\lambda^*))$ be a theta series for $\Gamma(\lambda^*)$ which has μ-th coefficients $\psi(\mu, \Gamma(\lambda^*))$. Regarding $\theta(u, \Gamma(\lambda^*))$ as a theta series for $\Gamma(\lambda^* m)$, let $\psi(\mu, \Gamma(\lambda^* m))$ be its μ-th coefficient. Then, by (3.7),

$$\psi(\mu, \Gamma(\lambda^*)) = \psi(\mu m, \Gamma(\lambda^* m)), \tag{3.8}$$

$$\psi(\mu, \Gamma(\lambda^* m)) = 0 \qquad \text{if } (\mu, m) = 1. \tag{3.9}$$

Our argument in this paper is based on the following lemma.

Lemma 5. *$\theta(u, \Gamma(N)) = 0$ if and only if $\tilde{\psi}(\kappa_j, \Gamma(N)) = 0$ for all κ_j.*

Proof. See [4], Theorem 4.1.2.

§ 4. Eisenstein series for $\Gamma(\lambda^*)$ and their residues at $s = (l+1)/l$

For each $\kappa_i \in P(\lambda^*)$, i.e., $\kappa_i = 0, 1, 2, \cdots, l-1, \infty$, put

$$E(u, s, \kappa_i) = V(\lambda^*) E_i(u, s, \Gamma(\lambda^*)). \tag{4.1}$$

Let $\psi(s, \mu, \kappa_i)$ be the μ-th coefficient of $E(u, s, \kappa_i)$. Now we calculate them.
If $\kappa_i = \alpha$ ($\alpha = 0, 1, 2, \cdots, l-1$), then

$$\begin{aligned} \psi(s, \mu, \alpha) &= \sum_{\substack{c \equiv -1(\lambda^*) \\ c \in D_{\lambda^*}}} \sum_{\substack{d \bmod \lambda^* c \\ d \equiv \alpha(\lambda^*)}} \left(\frac{d}{c}\right) e\left(\frac{d\mu}{c\lambda^*}\right) N(c)^{-s} \\ &= e(-\alpha\mu/\lambda^*) \psi(s, \mu), \end{aligned} \tag{4.2}$$

where $\psi(s, \mu)$ is defined by (1.8).
Meanwhile,

$$\psi(s, \mu, \infty) = \sum_{\substack{c \equiv 0(\lambda^*) \\ c \in D_{\lambda^*}}} \sum_{\substack{d \bmod \lambda^* c \\ d \equiv 1(\lambda^*)}} \left(\frac{c}{d}\right) e\left(\frac{d\mu}{c\lambda^*}\right) N(c)^{-s}. \tag{4.3}$$

Here we can write $c = \varepsilon\lambda^b c'$ where $\varepsilon \in U/U_{\lambda^*}$, $b \geqq (l+1)/2$, $c' \in D$, $(c', \lambda) = 1$. There exist $A, B \in \mathfrak{O}$ such that $\lambda^*\lambda^b A + c'B = 1$. Then d can represented by

$$d = \lambda^*\lambda^b A d_1 + c'B\delta$$

where d_1 runs through a complete set of invertible residues (mod c') and δ runs through a set of residues (mod $\lambda^*\lambda^b$) subject to $\delta \equiv 1$ (λ^*). Now we get

$$\left(\frac{c}{d}\right) = \left(\frac{\varepsilon\lambda^b}{d}\right)\left(\frac{c'}{d}\right) = \left(\frac{\varepsilon\lambda^b}{\delta}\right)\left(\frac{d_1}{c'}\right)\left(\frac{\delta, c'}{\lambda}\right) \tag{4.5}$$

and so

$$
(4.6)\qquad
\begin{aligned}
\sum_{\substack{d \bmod \lambda^* c\\ d\equiv 1(\lambda^*)}} \left(\frac{c}{d}\right) e\left(\frac{d\mu}{c\lambda^*}\right) &= \sum_{\substack{\delta \bmod \lambda^*\lambda^b\\ \delta\equiv 1(\lambda^*)}} \left(\frac{\varepsilon\lambda^b}{\delta}\right)\left(\frac{\delta, c'}{\lambda}\right) e\left(\frac{B\delta\mu}{\varepsilon\lambda^*\lambda^b}\right)\\
&\quad\times \sum_{d_1 \bmod c'} \left(\frac{d_1}{c'}\right) e\left(\frac{Ad_1\mu}{c'}\right)\\
&= \sum_{\substack{\delta \bmod \lambda^*\lambda^b\\ \delta\equiv c'^{-1}(\lambda^*)}} \left(\frac{\varepsilon\lambda^b}{\delta}\right)\left(\frac{\delta, c'}{\lambda}\right) e\left(\frac{\delta\mu}{\varepsilon\lambda^*\lambda^b}\right)\\
&\quad\times \left(\frac{\varepsilon^2\lambda^{2b}\lambda^*}{c'}\right) \sum_{d_1 \bmod c'} \left(\frac{d_1}{c'}\right) e\left(\frac{d_1\mu}{c'}\right).
\end{aligned}
$$

For $r \in A(\lambda^l)$, we put

$$
(4.7)\qquad \psi_r(s, \mu) = \sum_{\substack{c\in D\\ c\equiv r(\lambda^l)}} \left(\frac{\lambda^*}{c}\right) g(c, \mu) N(c)^{-s},
$$

$$
(4.8)\qquad \Gamma_r(\varepsilon\lambda^b, \mu) = \sum_{\substack{\delta \bmod \lambda^*\lambda^b\\ \delta\equiv r^{-1}(\lambda^*)}} \left(\frac{\varepsilon\lambda^b}{\delta}\right)\left(\frac{\delta, r}{\lambda}\right) e\left(\frac{\delta\mu}{\varepsilon\lambda^*\lambda^b}\right).
$$

Then we have

$$
(4.9)\qquad \psi(s, \mu, \infty) = \sum_{\substack{r\in A(\lambda^l)\\ \varepsilon\in U/U_{\lambda^*}\\ b\geqq (l+1)/2}} \Gamma_r(\varepsilon\lambda^b, \mu) l^{-bs} \psi_r(s, \varepsilon^{-2}\lambda^{(l-2)b}\mu).
$$

Let $\tilde{\psi}(s, \kappa_i, \kappa_j)$ be the singular value of $E(u, s, \kappa_i)$ at an essential cusp κ_j of $\Gamma(\lambda^*)$; it is given by

$$
(4.10)\qquad \tilde{\psi}(s, \kappa_i, \kappa_j) = \left(\frac{\pi}{s-1}\right)^{l^*} \sum_{(c,d)\in M_{ij}(\lambda^*)} \bar{\chi}_{ij}(c, d)\, N(c)^{-s}.
$$

We put $\zeta(s) = \sum_{\substack{c\in D\\ (c,\lambda)=1}} N(c)^{-s}$.

Lemma 6. *We have*

$$
\sum_{\substack{c\equiv 0(\lambda^*)\\ c\in D_{\lambda^*}}} \sum_{\substack{d \bmod \lambda^* c\\ d\equiv 1(\lambda^*)}} \left(\frac{c}{d}\right) N(c)^{-s} = \Phi(\lambda)(l^{ls-l}-1)^{-1}\zeta(ls-l)\zeta(ls-l+1)^{-1}.
$$

Proof. If we write $c = \varepsilon\lambda^b c'$ as the above argument for $\tilde{\psi}(s, \mu, \infty)$, we get

$$
\sum_{\substack{d \bmod \lambda^* c\\ d\equiv 1(\lambda^*)}} \left(\frac{c}{d}\right) = \sum_{\substack{\delta \bmod \lambda^*\lambda^b\\ \delta\equiv 1(\lambda^*)}} \left(\frac{\varepsilon\lambda^b}{\delta}\right)\left(\frac{\delta, c'}{\lambda}\right) \sum_{d_1 \bmod c'} \left(\frac{d_1}{c'}\right)
$$

$$=\begin{cases} l^b\Phi(c') & \text{if } \varepsilon\equiv 1, 2, \cdots, l-1\ (\lambda^*),\ l\mid b \text{ and} \\ & (c') \text{ is a } l\text{-th power,} \\ 0 & \text{otherwise.} \end{cases}$$

This proves Lemma 6.

Lemma 7. *If* $\gamma\equiv 1, 2, \cdots, l-1\ (\lambda^*)$, *then*

$$\sum_{\substack{c\equiv\gamma(\lambda^*)\\ c\in D_{\lambda^*}}}\ \sum_{\substack{d \bmod \lambda^* c\\ d\equiv\alpha(\lambda^*)}}\left(\frac{d}{c}\right)N(c)^{-s}=\zeta(ls-l)\zeta(ls-l+1)^{-1}.$$

for any α.

Proof. It is enough to prove the case $\gamma\equiv 1\ (\lambda^*)$. We may assume $c\in D$, $c\equiv 1\ (\lambda^*)$. Since

$$\sum_{\substack{d \bmod \lambda^* c\\ d\equiv\alpha(\lambda^*)}}\left(\frac{d}{c}\right)=\begin{cases}\Phi(c) & \text{if } (c) \text{ is a } l\text{-th power,}\\ 0 & \text{otherwise,}\end{cases}$$

our assertion follows.

Now let us calculate $\tilde{\psi}(s, \kappa_i, \kappa_j)$. We devide it into seven cases [1]–[7]. [1] the case of $\kappa_i=0$ and $\kappa_j=\alpha'/\gamma'$ (A-type). In this case

$$M_{ij}(\lambda^*)=\{(c, d)\in D_{\lambda^*}\times\vartheta;\ c\equiv 0\ (\lambda^*),\ d\equiv 1\ (\lambda^*),\ d \bmod c\lambda^*\}$$

and, if $(c, d)\in M_{ij}(\lambda^*)$,

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha'}{\gamma'}\right)\left(\frac{c}{d}\right).$$

So, by Lemma 6, we have

$$(4.11)\quad \tilde{\psi}(s, 0, \alpha'/\gamma')=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)\Phi(\lambda)(l^{ls-l}-1)^{-1}\zeta(ls-l)\zeta(ls-l+1)^{-1}.$$

[2] the case of $\kappa_i=0$ and $\kappa_j=\alpha'/\gamma'$ (B-type or C-type). If $(c, d)\in M_{ij}(\lambda^*)$, then $c\equiv -\alpha'\ (\lambda^*)$ $(\alpha'=1, 2, \cdots, l-1)$ and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{d}{c}\right).$$

So, by Lemma 7, we have

$$(4.12)\qquad \tilde{\psi}(s, 0, \alpha'/\gamma')=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\zeta(ls-l)\zeta(ls-l+1)^{-1}.$$

[3] the case of $\kappa_i=\alpha$ ($\alpha=1, 2, \cdots, l-1$) and $\kappa_j=\alpha'/\gamma'$ (A-type). If $(c, d) \in M_{ij}(\lambda^*)$, then $c\equiv\alpha$ (λ^*) and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha'}{\gamma'}\right)\left(\frac{d}{c}\right).$$

So, by Lemma 7, we have

$$\tilde{\psi}(s, \alpha, \alpha'/\gamma')=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)\zeta(ls-l)\zeta(ls-l+1)^{-1}. \tag{4.13}$$

[4] the case of $\kappa_i=\alpha$ ($\alpha=1, 2, \cdots, l-1$) and $\kappa_j=\alpha'/\gamma'$ (B-type). If $\alpha\equiv\alpha'$ (λ^*), then

$$M_{ij}(\lambda^*)=\{(c, d) \in D_{\lambda^*}\times\vartheta;\ c\equiv 0\ (\lambda^*),\ d\equiv 1\ (\lambda^*),\ d \bmod c\lambda^*\},$$

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{c}{d}\right) \quad ((c, d) \in M_{ij}(\lambda^*)).$$

If $\alpha\not\equiv\alpha'$ (λ^*), then for $(c, d) \in M_{ij}(\lambda^*)$, $c\equiv 1, 2, \cdots, l-1$ (λ^*) and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{d}{c}\right).$$

So, by Lemma 6 and Lemma 7, we get

$$\tilde{\psi}(s, \alpha, \alpha'/\gamma')=\begin{cases}\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\Phi(\lambda)(l^{ls-l}-1)^{-1}\zeta(ls-l)\zeta(ls-l+1)^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha\equiv\alpha'\ (\lambda^*), \\ \left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\zeta(ls-l)\zeta(ls-l+1)^{-1} \quad \text{if } \alpha\not\equiv\alpha'\ (\lambda^*).\end{cases} \tag{4.14}$$

[5] the case of $\kappa_i=\alpha$ ($\alpha=1, 2, \cdots, l-1$) and $\kappa_j=\alpha'/\gamma'$ (C-type). If $(c, d) \in M_{ij}(\lambda^*)$, then $c\equiv -1$ (λ^*) and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{d}{c}\right).$$

So, by Lemma 7, we have

$$\tilde{\psi}(s, \alpha, \alpha'/\gamma')=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\zeta(ls-l)\zeta(ls-l+1)^{-1}. \tag{4.15}$$

[6] the case $\kappa_i=\infty$ and $\kappa_j=\alpha'/\gamma'$ (A-type or B-type). If $(c, d) \in M_{ij}(\lambda^*)$, then $c\equiv 1$ (λ^*) and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha'}{\gamma'}\right)\left(\frac{d}{c}\right).$$

So, by Lemma 7, we have

$$\tilde{\psi}(s, \infty, \alpha'/\gamma') = \left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)\zeta(ls-l)\zeta(ls-l+1)^{-1}. \tag{4.16}$$

[7] the case of $\kappa_i=\infty$ and $\kappa_j=\alpha'/\gamma'$ (C-type). In this case

$$M_{ij}(\lambda^*)=\{(c, d) \in D_{\lambda^*}\times\vartheta\colon c\equiv 0\ (\lambda^*),\ d\equiv 1\ (\lambda^*),\ d \bmod c\lambda^*\},$$

$$\chi_{ij}(c, d)=\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{c}{d}\right).$$

So, by Lemma 6, we have

$$\tilde{\psi}(s, \infty, \alpha'/\gamma') = \left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\Phi(\lambda)(l^{ls-l}-1)\zeta(ls-l)^{-1}\zeta(ls-l+1)^{-1}. \tag{4.17}$$

Next, we consider the residue $\theta(u, \kappa_i)$ of $E(u, s, \kappa_i)$ at $s=(l+1)/l$. The singular value of $\theta(u, \kappa_i)$ at κ_j is given by $\mathrm{Res}_{s=(l+1)/l}\,\tilde{\psi}(s, \kappa_i, \kappa_j)$ which is easily evaluated by (4.11), $\cdots$, (4.17). Noting that $(\gamma'/\alpha')=(\alpha'/\gamma')$ when α'/γ' is of B-type and that the value of $\Phi(\lambda)(l^{ls-l}-1)^{-1}$ at $s=(l+1)/l$ is 1, we see that the singular values of $\theta(u, \kappa_i)$ at each essential cusp are equal to each other. Hence, by Lemma 5, $\theta(u, \kappa_i)$ are equal to each other. Now we write it simply $\theta(u)$; we call it the l-th power theta series for K. We restate what we have proved as

Proposition 5. *Let $\tilde{\psi}(\kappa_j)$ be the singular value of $\theta(u)$ at an essential cusp κ_j. Then*

$$\tilde{\psi}(\kappa_j)=\begin{cases}(l\pi)^{l^*}\left(\dfrac{\alpha'}{\gamma'}\right)\zeta(2)^{-1}\,\mathrm{Res}_{s=1}\,\zeta(s) & \text{if } \kappa_j=\alpha'/\gamma',\ \gamma'\equiv 1\ (\lambda^*),\\[2ex] (l\pi)^{l^*}\left(\dfrac{\gamma'}{\alpha'}\right)\zeta(2)^{-1}\,\mathrm{Res}_{s=1}\,\zeta(s) & \text{if } \kappa_j=\alpha'/\gamma',\ \alpha'\equiv 1\ (\lambda^*).\end{cases}$$

Now consider the μ-th coefficient $\psi(\mu)$ of $\theta(u)$. Since $\theta(u)=\theta(u, \kappa_i)$, it follows that $\psi(\mu)=\mathrm{Res}_{s=(l+1)/l}\,(s, \mu, \kappa_i)$ for every κ_i. So, by (4.2), we have

Proposition 6. *$\psi(\mu)=0$ unless $e(\mu/\lambda^*)=1$; and $\psi(\mu)=\mathrm{Res}_{s=(l+1)/l}\,\psi(s, \mu, \infty)$ where $\psi(s, \mu, \infty)$ is given by* (4.9).

Remark. If we calculate (4.9) explicitly, we can get some relations among $\psi(\varepsilon\lambda^k\mu)$ ($\varepsilon \in U$, $(\mu, \lambda)=1$, $\mu \in D$). But it is complicated even in case of $l=5$.

§ 5. Eisenstein series for $\Gamma(\lambda^* m)$

As in the second section m is a prime divisor $\neq \lambda$ of K ($m \in D$). Let $P(\lambda^* m)$ be a complete set of $\Gamma(\lambda^* m)$-inequivalent essential cusps. We consider Eisenstein series of $\Gamma(\lambda^* m)$ for $\kappa_i \in P(\lambda^* m)$ such that $\kappa_i = \alpha/\gamma$ is of (A, A)-type, i.e., $\gamma \in D$, $\gamma \equiv 1$ (λ^*), $\alpha \equiv 0$ $(\lambda^* m)$: We put

$$(5.1) \qquad E(u, s, \gamma) = V(\lambda^* m)\left(\frac{\alpha}{\gamma}\right) E_i(u, s, \Gamma(\lambda^* m)).$$

This does not depend upon the choice of α but on γ (mod m).

Let $\psi(s, \mu, \gamma)$ be the μ-th coefficient of $E(u, s, \gamma)$. Then we have

$$(5.2) \qquad \psi(s, \mu, \gamma) = \sum_{\substack{c \equiv -\gamma(\lambda^* m) \\ c \in D_{\lambda^* m}}} \sum_{\substack{d \bmod \lambda^* mc \\ d \equiv \alpha(\lambda^* m)}} \left(\frac{d}{c}\right) e\left(\frac{d\mu}{c\lambda^* m}\right) N(c)^{-s}$$
$$= \sum_{\substack{c \in D \\ c \equiv \gamma(\lambda^* m)}} \left(\frac{\lambda^* m}{c}\right) g(c, \mu) N(c)^{-s}.$$

Let $\mathfrak{a}$ be an ideal character of ϑ defined modulo (m). We put

$$(5.3) \qquad E(u, s, \mathfrak{a}) = \sum_{\substack{\gamma \in A(\lambda^* m) \\ \gamma \equiv 1(\lambda^*)}} \mathfrak{a}(\gamma) E(u, s, \gamma).$$

Let $\psi(s, u, \mathfrak{a})$ be the μ-th coefficient of $E(u, s, \mathfrak{a})$. Then we have

Proposition 7.

$$\psi(s, \mu, \mathfrak{a}) = \sum_{\substack{c \in D \\ c \equiv 1(\lambda^*) \\ (c, m) = 1}} \mathfrak{a}(c)\left(\frac{\lambda^* m}{c}\right) g(c, \mu) N(c)^{-s};$$

especially

$$\psi(s, \mu, 1_m) \sum_{\substack{c \in D \\ c \equiv 1(\lambda^*) \\ (c, m) = 1}} \left(\frac{\lambda^* m}{c}\right) g(c, \mu) N(c)^{-s}$$

where 1_m *is the trivial character modulo* (m).

In order to calculate the singular values of $E(u, s, \gamma)$, we need the following two lemmas.

Lemma 8. *Suppose* $\gamma \equiv 0$ (λ^*), $\alpha \equiv 1$ (λ^*), $(\gamma, \alpha) = 1$. *Then*

$$\sum_{\substack{c \in D_{\lambda^* m} \\ c \equiv \gamma(\lambda^* m)}} \sum_{\substack{d \bmod \lambda^* mc \\ d \equiv \alpha(\lambda^* m)}} \left(\frac{c}{d}\right) N(c)^{-s}$$

$$=\begin{cases} U(\lambda^*m)U(\lambda^*)^{-1}\Phi(\lambda)(l^{ls-l}-1)^{-1}\sum_{k=1}^{\infty}\left(\frac{\alpha}{m}\right)^k N(m)^{k(1-s)} \\ \qquad\qquad \times \sum_{\substack{c\in D\\(c,m\lambda)=1}} \Phi(c)N(c)^{l-1-ls} & \textit{if } \gamma\equiv 0\ (m), \\ \sum_{\substack{\varepsilon\lambda^{lb}c^l\equiv\gamma(m)\\ b\geqq 1, c\in D\\ (c,m\lambda)=1\\ \varepsilon\in U/U_{\lambda^*m}, \varepsilon\equiv 1,2,\cdots,l-1(\lambda^*)}} l^{(l-ls)b}\Phi(c)N(c)^{l-1-ls} & \textit{if } \gamma\not\equiv 0\ (m). \end{cases}$$

Proof. We write $c=\varepsilon\lambda^b m^k c'$ where $\varepsilon\in U/U_{\lambda^*m}$, $b\geqq(l+1)/2$, $k\geqq 0$, $c'\in D$, $(c', m\lambda)=1$. Then there exist A_1, A_2, $A_3\in\vartheta$ such that

$$m^{k+1}c'A_1+\lambda^*\lambda^b c'A_2+\lambda^*\lambda^b m^{k+1}A_3=1.$$

The representations of d can be chosen as follows:

$$d=m^{k+1}c'A_1d_1+\lambda^*\lambda^b c'A_2d_2+\lambda^*\lambda^b m^{k+1}A_3d_3$$

where d_1 runs through a set of residues (mod $\lambda^*\lambda^b$) subject to $d_1\equiv 1$ (λ^*), d_2 runs through a set of residues (mod m^{k+1}) subject to $d_2\equiv\alpha$ (m) and d_3 runs through a complete set of invertible residues (mod c'). Then

$$\left(\frac{c}{d}\right)=\left(\frac{\varepsilon\lambda^b}{b}\right)\left(\frac{m^k}{d}\right)\left(\frac{c'}{d}\right)=\left(\frac{\varepsilon\lambda^b}{d_1}\right)\left(\frac{d_2}{m}\right)^k\left(\frac{d_3}{c'}\right)\left(\frac{d_1, m^k c'}{\lambda}\right)$$

and so

$$\sum_{\substack{d \bmod \lambda^*mc\\ d\equiv\alpha(\lambda^*m)}}\left(\frac{c}{d}\right)=\sum_{\substack{d_1 \bmod \lambda^*\lambda^b\\ d_1\equiv 1(\lambda^*)}}\left(\frac{\varepsilon\lambda^b}{d_1}\right)\left(\frac{d_1, m^k c'}{\lambda}\right)\sum_{\substack{d_2 \bmod m^{k+1}\\ d_2\equiv\alpha(m)}}\left(\frac{d_2}{m}\right)^k\sum_{d_3 \bmod c'}\left(\frac{d_3}{c'}\right)$$

$$=\begin{cases} l^b\left(\frac{\alpha}{m}\right)^k N(m)^k\Phi(c') & \text{if } \varepsilon\equiv 1,2,\cdots,l-1\ (\lambda^*),\ l\,|\,b, \\ & m^k\equiv 1\ (\lambda^*) \text{ and } (c') \text{ is a } l\text{-th power}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we may write $c=\varepsilon\lambda^{lb}m^k c_0^l$ where $\varepsilon\in U/U_{\lambda^*m}$, $\varepsilon\equiv 1, 2, \cdots, l-1$ (λ^*), $b\gtreqless 1$, $k\geqq 0$, $m^k\equiv 1$ (λ^*), $c_0\in D$, $(c_0, m\lambda)=1$. If $\gamma\equiv 0$ (m), then $c=\varepsilon\lambda^{lb}m^k c_0^l$, $k\geqq 1$. If $\gamma\not\equiv 0$ (m), then $c=\varepsilon\lambda^{lb}c_0^l$, $\varepsilon\lambda^{lb}c_0^l\equiv\gamma$ (m). So our lemma follows.

Lemma 9. *Suppose* $\gamma\equiv 1, 2, \cdots, l-1$ (λ^*), $(\alpha, \gamma)=1$. *Then*

$$\sum_{\substack{c\equiv\gamma(\lambda^*m)\\ c\in D\lambda^*m}}\sum_{\substack{d \bmod \lambda^*mc\\ d\equiv\alpha(\lambda^*m)}}\left(\frac{d}{c}\right)N(c)^{-s}$$

$$=\begin{cases} U(\lambda^*m)U(\lambda^*)^{-1}\sum_{\substack{k=1\\ m^k\equiv 1(\lambda^*)}}^{\infty}\left(\frac{\alpha}{m}\right)^k N(m)^{k(1-s)}\sum_{\substack{c\in D\\ (c,m\lambda)=1}}\Phi(c)N(c)^{l-1-ls} \\ \qquad\qquad\qquad\qquad \text{if } \gamma\equiv 0(m). \\ \sum_{\substack{\varepsilon c^l\equiv\gamma(m)\\ c\in D,(c,m\lambda)=1\\ \varepsilon\in U/U_{\lambda^*m}, \varepsilon\equiv 1(\lambda^*)}}\Phi(c)N(c)^{l-1-ls} \quad \text{if } \gamma\not\equiv 0(m). \end{cases}$$

Proof. It is enough to prove the case $\gamma\equiv 1\ (\lambda^*)$. We can write $c=\varepsilon m^k c'$ where $\varepsilon \in U/U_{\lambda^* m}$, $k\geqq 0$, $c' \in D$, $(c', m\lambda)=1$. Since $c\equiv 1\ (\lambda^*)$, we have $\varepsilon\equiv 1\ (\lambda^*)$, $m^k c'\equiv 1\ (\lambda^*)$. Then

$$\sum_{\substack{d \bmod \lambda^* mc \\ d\equiv\alpha(\lambda^* m)}} \left(\frac{d}{c}\right)=\left(\frac{d}{m}\right)^k N(m)^k \Phi(c') \qquad \text{if } (c') \text{ is a } l\text{-th power,}$$
$$=0 \qquad \text{otherwise.}$$

Hence we may write $c=\varepsilon m^k c_0^l$ where $\varepsilon \in U/U_{\lambda^* m}$, $\varepsilon\equiv 1\ (\lambda^*)$, $k\geqq 0$, $m^k\equiv 1(\lambda^*)$, $c_0 \in D$, $(c_0, m\lambda)=1$. If $\gamma\equiv 0\ (m)$, then $c=\varepsilon m^\kappa c_0^l$, $k\geqq 1$, $m^k\equiv 1\ (\lambda^*)$. If $\gamma\not\equiv 0\ (m)$, then $c=\varepsilon c_0^l$, $\varepsilon c_0^l\equiv\gamma\ (m)$. So our assertion follows.

Let $\tilde{\psi}(s, \gamma, \kappa_j)$ be the singular value of $E(u, s, \gamma)$ at an essential cusp κ_j; it is given by

$$\tilde{\psi}(s, \gamma, \kappa_j)=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha}{\gamma}\right) \sum_{(c,d)\in M_{ij}(\lambda^* m)} \bar{\chi}_{ij}(c, d)N(c)^{-s} \tag{5.4}$$

where $\kappa_i=\alpha/\gamma \in P(\lambda^* m)$. We take $\sigma_i=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\sigma_j=\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$, then

$$\begin{aligned} M_{ij}(\lambda^* m)=\{(c, d) \in D_{\lambda^* m}\times\vartheta;\ & c\equiv -\gamma\alpha'(\lambda^* m),\\ & d\equiv -\gamma\beta'(\lambda^* m),\ d \bmod c\lambda^* m\}\end{aligned} \tag{5.5}$$

Let us calculate $\tilde{\psi}(s, \gamma, \kappa_j)$ for each type of κ_j.

[1] $\kappa_j=\alpha'/\gamma'$: (A, A)-type. If $(c, d) \in M_{ij}(\lambda^* m)$, then $c\equiv 0\ (\lambda^* m)$, $d\equiv \gamma\gamma'^{-1}\ (\lambda^* m)$ and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha}{\gamma}\right)^{-1}\left(\frac{\alpha'}{\gamma'}\right)\left(\frac{c}{d}\right).$$

Hence, by Lemma 8, we have

$$\begin{aligned}\tilde{\psi}(s, \gamma, \kappa_j)=&\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)U(\lambda^* m)U(\lambda^*)^{-1}\Phi(\lambda)(l^{ls-l}-1)^{-1}\\ &\times \sum_{\substack{k=1\\ m^k\equiv 1(\lambda^*)}} \left(\frac{\gamma}{m}\right)^k\left(\frac{\gamma'}{m}\right)^{-k} N(m)^{k(1-s)} \sum_{\substack{c\in D\\ (c, m\lambda)=1}}^{\infty} \Phi(c)N(c)^{l-1-ls}.\end{aligned} \tag{5.6}$$

[2] $\kappa_j=\alpha'/\gamma'$: (A, B)-type or (A, C)-type. If $(c, d) \in M_{ij}(\lambda^* m)$, then $c\equiv 0\ (\lambda^*)$, $c\equiv -\gamma\alpha'\ (m)$, $d\equiv 1\ (\lambda^*)$ and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha}{\gamma}\right)^{-1}\left(\frac{\alpha'}{\gamma'}\right)\left(\frac{c}{d}\right).$$

Hence, by Lemma 8, we have

$$\tilde{\psi}(s, \gamma, \kappa_j)=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right) \tag{5.7}$$
$$\times \sum_{\substack{\varepsilon\lambda^{lb}c^l\equiv\gamma\alpha'(m)\\ b\geqq 1, c\in D, (c, m\lambda)=1\\ \varepsilon\in U/U_{\lambda^*m}, \varepsilon\equiv 1,2,\cdots,l-1(\lambda^*)}} l^{lb(1-s)}\Phi(c)N(c)^{l-1-ls}.$$

[3] $\kappa_j=\alpha'/\gamma'$: (B, A)-type or (C, A)-type. If $(c, d)\in M_{ij}(\lambda^* m)$, then $c\equiv 1, 2, \cdots, l-1\ (\lambda^*)$, $c\equiv 0\ (m)$, $d\equiv\gamma\gamma'^{-1}\ (m)$ and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha}{\gamma}\right)^{-1}\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{d}{c}\right).$$

Hence, by Lemma 9, we have

$$\tilde{\psi}(s, \gamma, \kappa_j)=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)U(\lambda^* m)U(\lambda^*)^{-1} \tag{5.8}$$
$$\times \sum_{\substack{k=1\\ m^k\equiv 1(\lambda^*)}}^{\infty}\left(\frac{\gamma}{m}\right)^k\left(\frac{\gamma'}{m}\right)^{-k}N(m)^{k(1-s)}\sum_{\substack{c\in D\\ (c, m\lambda)=1}}\Phi(c)N(c)^{l-1-ls}.$$

[4] $\kappa_j=\alpha'/\gamma'$: (B, B)-type, (B, C)-type (C, B)-type or (C, C)-type. If $(c, d)\in M_{ij}(\lambda^* m)$, then $c\equiv 1, 2, \cdots, l-1\ (\lambda^*)$, $c\equiv -\gamma\alpha'\ (m)$ and

$$\bar{\chi}_{ij}(c, d)=\left(\frac{\alpha}{\gamma}\right)^{-1}\left(\frac{\gamma'}{\alpha'}\right)\left(\frac{d}{c}\right).$$

Hence, by Lemma 9, we have

$$\tilde{\psi}(s, \gamma, \kappa_j)=\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\sum_{\substack{\varepsilon c^l\equiv -\gamma\alpha'(m)\\ \varepsilon\in U/U_{\lambda^* m}, \varepsilon\equiv 1(\lambda^*)\\ c\in D, (c, m\lambda)=1}}\Phi(c)N(c)^{l-1-ls}. \tag{5.9}$$

Let $\tilde{\psi}(s, \mathfrak{a}, \kappa_j)$ be the singular value of $E(u, s, \mathfrak{a})$ at an essential cusp κ_j; it is given by

$$\tilde{\psi}(s, \mathfrak{a}, \kappa_j)=\sum_{\substack{\gamma\in A(\lambda^* m)\\ \gamma\equiv 1(\lambda^*)}}\mathfrak{a}(\gamma)\tilde{\psi}(s, \gamma, \kappa_j). \tag{5.10}$$

We define

$$\zeta(s, \mathfrak{a})=\sum_{\substack{c\in D\\ (c, m\lambda)=1}}\mathfrak{a}(c)N(c)^{-s}.$$

Then, by (5.6), $\cdots$, (5.9) and (5.10), we have

Proposition 8. *Suppose* $\mathfrak{a}^l\neq 1_m$. *Then*

$\tilde{\psi}(s, \mathfrak{a}, \kappa_j)$

$$=\begin{cases} \left(\dfrac{\pi}{s-1}\right)^{l^*}\left(\dfrac{\alpha'}{\gamma'}\right)\bar{\mathfrak{a}}(\alpha')\Phi(\lambda)((\bar{\mathfrak{a}}(\lambda)l)^{l-ls}-1)^{-1}\zeta(ls-l, \mathfrak{a}^l)\zeta(ls-l+1, \mathfrak{a}^l)^{-1} \\ \qquad \text{if } \kappa_j=\alpha'/\gamma' \textit{ is of } (A, B)\textit{-type or of } (A, C)\textit{-type}, \\ \left(\dfrac{\pi}{s-1}\right)^{l^*}\left(\dfrac{\gamma'}{\alpha'}\right)\bar{\mathfrak{a}}(\alpha')\zeta(ls-l, \mathfrak{a}^l)\zeta(ls-l+1, \mathfrak{a}^l)^{-1} \\ \qquad \textit{if } \kappa_j=\alpha'/\gamma' \textit{ is of } (B, B)\textit{-type, of } (B, C)\textit{-type,} \\ \qquad \textit{of } (C, B)\textit{-type or of } (C, C)\textit{-type,} \\ 0 \qquad \textit{otherwise.} \end{cases}$$

Proposition 9. *$\tilde{\psi}(s, 1_m, \kappa_j)$ is given by: if $\kappa_j=\alpha'/\gamma'$ is of (A, A)*-type,

$$\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)\Phi(\lambda)U(\lambda^*m)U(\lambda^*)^{-1}(l^{ls-l}-1)^{-1}\tilde{\Phi}(\lambda^*m)$$
$$\times\tilde{\Phi}(\lambda^*)^{-1}(N(m)^{ls-l}-1)^{-1}\zeta(ls-l, 1_m)\zeta(ls-l+1, 1_m)^{-1};$$

if $\kappa_j=\alpha'/\gamma'$ is of (A, B)-type or (A, C)-type,

$$\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\alpha'}{\gamma'}\right)\Phi(\lambda)(l^{ls-l}-1)^{-1}\zeta(ls-l, 1_m)\zeta(ls-l+1, 1_m)^{-1};$$

if $\kappa_j=\alpha'/\gamma'$ is of (B, A)-type or of (C, A)-type,

$$\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)U(\lambda^*m)U(\lambda^*)^{-1}\tilde{\Phi}(\lambda^*m)\tilde{\Phi}(\lambda^*)^{-1}(N(m)^{ls-l}-1)^{-1}$$
$$\times\zeta(ls-l, 1_m)\zeta(ls-l+1, 1_m)^{-1};$$

if $\kappa_j=\alpha'/\gamma'$ is of (B, B)-type, (B, C)-type, (C, B)-type or (C, C)-type,

$$\left(\frac{\pi}{s-1}\right)^{l^*}\left(\frac{\gamma'}{\alpha'}\right)\zeta(ls-l, 1_m)\zeta(ls-l+1, 1_m)^{-1}.$$

§ 6. Main results

We put

$$\theta(u, \mathfrak{a})=\operatorname{Res}_{s=(l+1)/l}E(u, s, \mathfrak{a}). \tag{6.1}$$

Then the singular value $\tilde{\psi}(\mathfrak{a}, \kappa_j)$ of $\theta(u, \mathfrak{a})$ at an essential cusp κ_j is given by $\operatorname{Res}_{s=(l+1)/l}\tilde{\psi}(s, \mathfrak{a}, \kappa_j)$.

Proposition 10. *Suppose $\mathfrak{a}^l\neq 1_m$, then*

$$\theta(u, \mathfrak{a})=0.$$

Proof. Since $\mathrm{Res}_{s=(l+1)/l}\zeta(ls-l, \mathfrak{a}^l)\zeta(ls-l+1, \mathfrak{a}^l)^{-1}=0$, we get, by Proposition 8,

$$\tilde{\psi}(\mathfrak{a}, \kappa_j)=0 \qquad \text{for all } \kappa_j.$$

This implies $\theta(u, \mathfrak{a})=0$ from Lemma 5.

Theorem 1. *Suppose* $\mathfrak{a}^l \neq 1_m$, *then the Dirichlet series*

$$\sum_{\substack{c\in D \\ (c,m)=1 \\ c\equiv 1(\lambda^*)}} \mathfrak{a}(c)g(c, \mu)N(c)^{-s}$$

is holomorphically continued to the region $\mathrm{Re}(s)>1$.

Proof. This theorem follows from Proposition 7 and Proposition 10.

We consider $\theta(u)$ a theta series for $\Gamma(\lambda^* m)$. Then we have

Proposition 11.

$$\theta(u)=(1+N(m)^{-1})\theta(u, 1_m).$$

Proof. The singular values of $\theta(u)$ are given in Proposition 5, and those of $\theta(u, 1_m)$ are derived from Proposition 9. Meanwhile, the values of $\Phi(\lambda)(l^{ls-l}-1)^{-1}$ and $U(\lambda^* m)U(\lambda^*)^{-1}\tilde{\Phi}(\lambda^* m)\tilde{\Phi}(\lambda^*)^{-1}(N(m)^{ls-l}-1)^{-1}$ at $s=(l+1)/l$ are equal to 1, and that of $\zeta(ls-l)\zeta(ls-l+1)^{-1}\zeta(ls-l+1, 1_m)$ $\zeta(ls-l, 1_m)^{-1}$ at $s=(l+1)/l$ is $1+N(m)^{-1}$. So the singular values of $\theta(u)$ and $(1+N(m)^{-1})\theta(u, 1_m)$ at each κ_j are equal to each other. This proves Proposition 11.

Proposition 12. *If* $(\mu, m)=1$, *then*

$$\mathrm{Res}_{s=(l+1)/l}\,\psi(s, \mu, 1_m)=0.$$

In other words, if $(\mu, m)=1$, *then the Dirichlet series*

$$\sum_{\substack{c\equiv 1(\lambda^*) \\ (c,m)=1 \\ c\in D}} \left(\frac{m}{c}\right) g(c, \mu)N(c)^{-s}$$

is holomorphically continued to the region $\mathrm{Re}(s)>1$.

Proposition 13.

$$\psi(\mu)=(1+N(m)^{-1})\,\mathrm{Res}_{s=(l+1)/l}\,\psi(s, m\mu, 1_m).$$

Proof of Proposition 12 *and Proposition* 13. These follow from Proposition 11 and (3.8), (3.9).

Theorem 2. *We have*

$$\psi(m^l\mu)=\psi(\mu).$$

Proof. Since $\psi(s, m^{l+1}\mu, 1_m)=\psi(s, m\mu, 1_m)$, our assertion follows from Proposition 13.

Theorem 3. *If* $(\mu, m)=1$, *then*

$$\psi(m^{l-1}\mu)=0.$$

Proof. Since $\psi(s, \mu, 1_m)=\psi(s, m^{l-1}m\mu, 1_m)$, our assertion follows from Proposition 12 and Proposition 13.

In (1.8) of Section 1, we replace μ by $m^t\mu$ $(t\geqq 0, (\mu, m)=1)$ and rearrange the right hand side in the following way:

$$\psi(s, m^t\mu)=\sum_{k=0}^{\infty} N(m)^{-ks} \sum_{\substack{c\in D\\ m^k c\equiv 1(\lambda^*)\\ (c,m)=1}} \left(\frac{\lambda^*}{m^k c}\right) g(m^k c, m^t\mu)N(c)^{-s}. \tag{6.2}$$

We put

$$g_k(m, \mu)=\sum_{\delta \bmod m}\left(\frac{\delta}{m}\right)^k e\left(\frac{\delta\mu}{m}\right) \tag{6.3}$$

for $k=1, 2, \cdots, l-1$. It is easy to see that, for $t=0, 1, 2, \cdots, l-2$,

$$g(m^k c, m^t\mu)=\begin{cases}\left(\dfrac{m, c}{\lambda}\right)^k g_k(m, \mu)N(m)^{k-1}g(c, m^{l-k-1}\mu) & \text{if } k=t+1,\\ g(c\ m^t\mu) & \text{if } k=0,\\ 0 & \text{otherwise.}\end{cases} \tag{6.4}$$

Then (6.2) is written in the following form:

$$\begin{aligned}\psi(s, m^t\mu)=&\sum_{\substack{c\in D\\ c\equiv 1(\lambda^*)\\ (c,m)=1}} \left(\frac{\lambda^*}{c}\right) g(c, m^t\mu)N(c)^{-s}\\ &+\left(\frac{\lambda^*}{m}\right)^{t+1} g_{t+1}(m, \mu)N(m)^{t-(t+1)s}\\ &\times \sum_{\substack{c\in D\\ m^{t+1}c\equiv 1(\lambda^*)\\ (c,m)=1}} \left(\frac{\lambda^*}{m}\right)\left(\frac{m, c}{\lambda}\right)^{t+1} g(c, m^{l-t-2}\mu)N(c)^{-s}\end{aligned} \tag{6.5}$$

for $t=0, 1, 2, \cdots, l-2$. If $m\equiv 1$ (λ^*), then

$$\begin{aligned}\psi(s, m^t\mu)=&\psi(s, m^{t+1}\mu, 1_m)\\ &+\left(\frac{\lambda^*}{m}\right)^{t+1} g_{t+1}(m, \mu)N(m)^{t+(t+1)s}\psi(s, m^{l-t-1}\mu, 1_m)\end{aligned} \tag{6.6}$$

for $t=0, 1, 2, \cdots, l-2$. Taking residues of both sides of (6.6) and using Proposition 13, we have

Theorem 4. *If* $m\equiv 1\ (\lambda^*)$, $(m, \mu)=1$, *then*

$$\psi(m^{l-t-2}\mu)=\left(\frac{\lambda^*}{m}\right)^{-t-1}\bar{g}_{t+1}(m, \mu)N(m)^{-1+(t+1)/l}\psi(m^t\mu)$$

for $t=0, 1, \cdots, (l-3)/2$.

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