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Some Results on Algebraic Groups with Involutions

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§ 1. Introduction

Let G be a reductive linear algebraic group over the algebraically closed field F of characteristic not 2. Assume given an involution θ of G, i.e. an automorphism (in the sense of algebraic groups) of order 2. Denote by K the fixed point group of θ and by B a Borel subgroup. Then K has finitely many orbits in the flag manifold G/B (first proved for $F=\mathbf{C}$ in [6]). The geometry of these orbits is of importance in the study of Harish-Chandra modules, as is shown by the results of [11].

In the present paper we shall establish a number of basic elementary facts about these orbits or, equivalently, the double cosets BxK.

After assembling a number of known results in n° 2, we discuss in n° 3 twisted involutions in Weyl groups. These are needed for the description of double cosets given in n° 4. That section also contains a fairly explicit description of the double cosets as algebraic varieties. N° 5 deals with the open double coset and with those of codimension one. As an application we deduce a result (5.6) about K-fixed vectors in G-modules, which is well-known in characteristic 0 ([4], [12]). Finally, n° 6 contains some information about orbit closures. Similar results have recently been found by Matsuki for $F=\mathbf{C}$ [7].

The results about double cosets BxK established here bear some resemblance to the familiar results about the Bruhat decomposition into double cosets BxB, but are somewhat more complicated.

§ 2. Notations and recollections

2.1. In the sequel, F denotes an algebraically closed field of characteristic not 2. Let G be a linear algebraic group over F, provided with an automorphism θ of order 2. Denote by $\mathfrak{g}$ the Lie algebra of G. The automorphism of $\mathfrak{g}$ induced by θ will also be denoted by θ. We shall mainly be interested in the case that G is connected and reductive, but we do not yet assume this.

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In this situation we denote by $K=\{x\in G|\theta x=x\}$ the group of fixed points of θ. It is a closed subgroup of G. Further $\tau x=x(\theta x)^{-1}$ defines a morphism of G. We write S (or S_G) for the image τG. G acts transitively on S by $g*x=gx(\theta g)^{-1}$.

2.2. Proposition. *S is a closed subvariety of G and the map $g\mapsto g*e$ induces an isomorphism of affine G-varieties $G/K\xrightarrow{\sim}S$.*

This is proved in [8, 2. 4] for the case that G is reductive. The same proof can be given in the general case. For a proof that S is closed see for example [9, 4.4.4].

2.3. According to a result of Steinberg [10, §7] there exists a θ-stable Borel subgroup B of G and a θ-stable maximal torus T contained in B. We shall call such a pair (T, B) a *standard* pair, and T a *standard* maximal torus. The unipotent radical U of B is also θ-stable, and B is the semi-direct product $B=T.U$.

Next assume G to be reductive. Denote by $N=N_G(T)$ the normalizer of T in G and by $W=N/T$ the Weyl group. The root system of G with respect to T will be denoted by Φ. It is contained in the vector space $V=X^*(T)\otimes_{\mathbf{Z}}\mathbf{R}$ (where $X^*(T)$ denotes the character group of T). The Borel group B defines a system of positive roots $\Phi^+\subset\Phi$ and a corresponding basis Δ of Φ. Clearly, θ operates on Φ and W. We denote the induced action also by θ. Notice that $\theta\Phi^+=\Phi^+$ and that θ permutes the roots of Δ.

2.4. *θ-stable tori and their root systems.*

More generally (still assuming G reductive), let T be any θ-stable maximal torus, with Weyl group W and root system Φ. Again, θ operates on these and it is clear that T is standard if and only if θ fixes a set of positive roots (or a basis) of Φ.

If $\alpha\in\Phi$, let x_α be the one parameter additive subgroup of G defined by α. This is an isomorphism of the additive group onto a closed subgroup U_α of G, normalized by T, such that

$$tx_\alpha(\xi)t^{-1}=x_\alpha(\alpha(t)\xi)\qquad(t\in T,\ \xi\in F).$$

They may and shall be chosen such that

$$n_\alpha=x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)\tag{1}$$

lies in $N(=N_G(T))$ for all $\alpha\in\Phi$. In that case we have, if $\xi\in F^*$,

$$x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi)=\alpha^\vee(\xi)n_\alpha;\tag{2}$$

here $\alpha^\vee$ is the coroot of α, which is a one parameter multiplicative subgroup of T. If x_α is given, then $x_{-\alpha}$ is uniquely determined by requiring (1) or (2) to hold. Moreover, $n_\alpha T$ is the reflection $s_\alpha \in W$ defined by α. We have $n_\alpha^2=\alpha^\vee(-1)=t_\alpha$ and $n_{-\alpha}=t_\alpha n_\alpha=n_\alpha t_\alpha$.

We denote by $X_\alpha \in \mathfrak{g}$ the root vector dx_α (1). Then

$$\mathrm{Ad}\,(t)X_\alpha=\alpha(t)X_\alpha \qquad (t \in T).$$

For these facts see e.g. [9, Ch. 11].

Since θ stabilizes T, it follows that there exists $c_\alpha \in F^*$ such that for $\xi \in k$,

$$\theta(x_\alpha(\xi))=x_{\theta\alpha}(c_\alpha\xi).$$

We then have $\theta X_\alpha=c_\alpha X_{\theta\alpha}$, whence $c_\alpha c_{\theta\alpha}=1$. We shall normalize the x_α. Several cases are possible.

(a) $\theta\alpha\neq\pm\alpha$. In this case we say that α is *complex* (relative to θ). We may, by changing $x_{\theta\alpha}$, assume that $c_\alpha=1$. Using (1) and (2) one sees that then also $c_{-\alpha}=1$, from which we see that $\theta(n_\alpha)=n_{\theta\alpha}$, $\theta(n_{-\alpha})=n_{-\theta\alpha}$. In this case, let $\Phi_1\subset\Phi$ be the smallest subsystem containing α and $\theta\alpha$.

2.5. Lemma. *Φ_1 is either of type $A_1\times A_1$ or of type A_2.*

Introduce a symmetric positive definite bilinear form $\langle\ ,\ \rangle$ on the vector space V underlying Φ, which is W- and θ-invariant, such that $\langle\alpha,\alpha\rangle=2$. Clearly, α and $\theta\alpha$ have the same length. Hence (see [2, p. 148]) if $\alpha+i\theta\alpha$ is a root, we have $|i|\leqslant 1$. So if Φ_1 is not of type $A_1\times A_1$, it must consist of the roots $\pm\alpha$, $\pm\theta\alpha$, $\pm(\alpha+\theta\alpha)$ resp. of the roots $\pm\alpha$, $\pm\theta\alpha$, $\pm(\alpha-\theta\alpha)$. This proves the lemma. In the first case we have $\langle\alpha, \theta\alpha\rangle=-1$, in the second case $\langle\alpha, \theta\alpha\rangle=1$.

(b) Next assume $\theta\alpha=-\alpha$. In this case we say that α is *real.* Choose d_α, $d_{-\alpha}$ such that $d_{\pm\alpha}^2=c_{\pm\alpha}^{-1}$, $d_\alpha d_{-\alpha}=1$. Then $\theta(x_\alpha(d_\alpha\xi))=x_{-\alpha}(d_{-\alpha}\xi)$, from which we conclude that we may again assume $c_\alpha=c_{-\alpha}=1$. We now have $\theta(n_\alpha)=n_{-\alpha}=n_\alpha t_\alpha$.

Notice that if α is complex and $\alpha-\theta\alpha$ is a root, it is a real one.

(c) Finally consider the case that $\theta\alpha=\alpha$. Then we say that α is *imaginary.* We now have $c_\alpha^2=1$. If $c_\alpha=1$ then α is *compact imaginary* and if $c_\alpha=-1$ it is *noncompact imaginary.* In the first case we have $\theta(n_\alpha)=n_\alpha$, in the second case $\theta(n_\alpha)=n_{-\alpha}$.

2.6. Lemma. *If α is complex and $\alpha+\theta\alpha \in \Phi$, then $\alpha+\theta\alpha$ is noncompact imaginary.*

Let X_α and $X_{\theta\alpha}=\theta X_\alpha$ be root vectors. We may assume (using 2.5) that $[X_\alpha, X_{\theta\alpha}]=X_{\alpha+\theta\alpha}$. Then $\theta(X_{\alpha+\theta\alpha})=[X_{\theta\alpha}, X_\alpha]=-X_{\alpha+\theta\alpha}$, whence the lemma.

Notice that if T is standard, there are only complex and imaginary roots.

2.7. *Split tori.*

Still assume G to be reductive. A θ-stable subtorus A of G is called *split* (relative to θ or θ-split) if $\theta x=x^{-1}$ for all $x \in A$. (In [8] and [12] such tori are called anisotropic. We do not use terminology, as it is too much in conflict with the terminology about roots adapted above, which is the one of [11]). Nontrivial split tori exist [12, §1], so there are maximal ones.

2.8. Proposition. *Two maximal split tori are conjugate by an element of* K°.

This is proved in [loc. cit.]. As usual, K° denotes the identity component of K.

Let A be a maximal split torus. Fix a maximal torus T containing A, it is automatically θ-stable. Let Φ be the root system of T.

A parabolic subgroup P of G is *split* relative to θ or θ-split (called anisotropic in [8] and [12]) if P and θP are opposite, i.e. if $P \cap \theta P$ is a Levi subgroup of both P and θP.

Assume for simplicity G to be connected (and reductive).

2.9. Proposition. (i) *Let A be a maximal split torus. There exists a minimal split parabolic subgroup with Levi subgroup $Z_G(A)$. The derived group $Z_G(A)'$ is contained in K;*

(ii) *Two minimal split parabolic subgroups are conjugate by an element of* K°.

This is also contained in [12, § 1]. We shall obtain a description of minimal split parabolic subgroups in n° 5.

We shall say that θ is *split* (replative to θ) or that G is θ-split if there is a maximal split torus which is a maximal torus of G, i.e. if there is a maximal torus T of G such that $\theta x=x^{-1}$ for all $x \in T$. We recall the following result.

2.10. Proposition. *Let G be a connected reductive group. There exists an automorphism θ of G of order 2 such that (G, θ) is split. If θ' is another such automorphism, there is $x \in G$ such that $x\theta(g)x^{-1}=\theta'(xgx^{-1})$, for all $g \in G$.*

This follows readily from the familiar results about existence of isomorphisms of reductive groups (see e.g. [9, Ch. 11]).

§ 3. Twisted involution in Weyl groups

3.1. In this section we establish some results about Weyl groups, to be used later. We use some familiar facts, which can be found in [2].

Let Φ be a reduced root system in a euclidean vector space E, with scalar product $\langle\ ,\ \rangle$. Assume given a system of positive roots Φ^+, with basis Δ. We denote by W the Weyl group of Φ, and we assume our scalar product to be W-invariant. We also assume given a linear transformation θ of E with $\theta^2=1$, which stabilizes Φ and Φ^+. Then θ induces an automorphism of W, also denoted by θ. So $\theta(w)=\theta\circ w\circ\theta^{-1}$, if $w\in W$. If $s_\alpha\in W$ is the reflection defined by $\alpha\in\Phi$ then $\theta(s_\alpha)=s_{\theta\alpha}$. The scalar product is assumed to be θ-invariant.

A *twisted involution* in W (relative to θ) is an element $w\in W$ with $\theta(w)=w^{-1}$. We denote by $\mathscr{T}=\mathscr{T}_\theta$ the set of these elements. If $\theta=\mathrm{id}$, then these are involutions of W in the ordinary sense.

W is generated by $\Sigma=(s_\alpha)_{\alpha\in\Delta}$. We denote by l the length function of W with respect to Σ. Notice that Δ and Σ are stable under the actions of θ.

If Π is a subset of Δ, it is a basis of a subsystem Φ_Π, with Weyl group W_Π. We denote its longest element by w_Π°. In particular, w_Δ° is the longest element of W.

Finally, $\leqslant$ denotes the Bruhat order on W (relative to the set of generators Σ).

3.2. Lemma. *Let $w\in\mathscr{T}_\theta$, $s\in\Sigma$.*

(i) *If $sw<w$ then either $l(sw\theta(s))=l(w)-2$ or $sw\theta(s)=w$ and $s=s_\alpha$, with $\alpha\in\Delta$ and $w\theta\alpha=-\alpha$;*

(ii) *If $sw>w$ then either $l(sw\theta(s))=l(w)+2$ or $sw\theta(s)=w$ and $s=s_\alpha$, with $\alpha\in\Delta$ and $w\theta\alpha=\alpha$.*

Assume $sw<w$. Then either $l(sw\theta(s))=l(w)-2$ or $l(sw\theta(s))=l(w)$. Assume the second alternative and write $w=s_1\cdots s_h$, with $s_i\in\Sigma$, $s_1=s$ and $l(w)=h$. Then also $w=\theta(s_h)\cdots\theta(s_1)$. Since $sw<w$, we have by the exchange condition that $sw=\theta(s_h)\cdots\widehat{\theta(s_i)}\cdots\theta(s_1)$, for some i with $1\leqslant i\leqslant l$. If $i>1$ then $l(sw\theta(s))<l(w)$. Hence $sw\theta(s)=w$ and (i) readily follows.

The proof of (ii) is similar: Assume again $w=s_1\cdots s_h$, with $s_i\in\Sigma$, $l(w)=h$. Then $sw=ss_1\cdots s_h$. Either $l(sw\theta(s))=l(w)+2$, or $l(sw\theta(s))<l(sw)$, in which case the exchange condition implies that $sw\theta(s)=s_1\cdots s_h=w$, from which one infers (ii).

3.3. Proposition. *Let $w \in \mathcal{T}_\theta$. There exists a θ-stable subset Π of Δ and $s_1, \cdots, s_h \in \Sigma$ such that:*

(a) $w = s_1 \cdots s_h w_\Pi^\circ \theta(s_h) \cdots \theta(s_1)$ *and* $l(w) = l(w_\Pi^\circ) + 2h$;

(b) $w_\Pi^\circ \alpha = -\theta\alpha$ *for all* $\alpha \in \Phi_\Pi$;

(c) *if* $s \in \Sigma$ *and* $sw < w$, $sw = w\theta(s)$, *then* $ss_1 \cdots s_h = s_1 \cdots s_h s_\alpha$ *with* $\alpha \in \Pi$.

We prove the existence of Π and (a), (b) by induction on $l(w)$, starting with the trivial case $w=1$. Assume $l(w)>0$. If there is $s \in \Sigma$ with $l(sw\theta(s)) = l(w) - 2$, these statements come readily from the induction. By 3.2 (i), it remains to deal with the case that $sw\theta(s) = w$ for all $s \in \Sigma$ with $sw < w$. In that case let $\Pi = \{\alpha \in \Delta \mid w\theta\alpha = -\alpha\}$. Then $\Pi \neq \emptyset$, by 3.2(i). Clearly, $w\theta\beta = -\beta$ for all $\beta \in \Phi_\Pi$.

If $\alpha \in \Pi$, we have $w_{\theta\Pi}^\circ w^{-1}\alpha = -w_{\theta\Pi}^\circ \theta\alpha > 0$. If $\alpha \in \Delta - \Pi$, then $w^{-1}\alpha > 0$ by the definition of Π, and our assumption on w. Moreover, we then have $w^{-1}\alpha \notin \theta\Phi_\Pi^+$ (because $w\theta\beta = -\beta$ for all $\beta \in \Phi_\Pi$). Hence $w_{\theta\Pi}^\circ w^{-1}\alpha > 0$ for all $\alpha \in \Delta$, whence $w = w_{\theta\Pi}^\circ$. This implies that for $\alpha \in \Pi$ we have $-\alpha = w\theta\alpha = w_{\theta\Pi}^\circ \theta\alpha = \theta(w_\Pi^\circ \alpha)$, i.e. $w_\Pi^\circ \alpha = -\theta\alpha$. So $\theta\alpha \in \Pi$, and Π is θ-stable, $w = w_\Pi^\circ$. This proves (a) and (b).

We finally prove (c). Let $s \in \Sigma$ be such that $sw < w$, $sw\theta(s) = w$. By 3.2(i) we have $s = s_\alpha$ with $\alpha \in \Delta$ and $w\theta\alpha = -\alpha$. This implies that $w_\Pi^\circ(s_h \cdots s_1\alpha) = -\theta(s_h \cdots s_1\alpha)$, which shows that $s_h \cdots s_1\alpha \in \Phi_\Pi$. If $ss_1 \cdots s_h < s_1 \cdots s_h$ then $sw\theta(s) < w$, which is not the case. Hence $ss_1 \cdots s_h > s_1 \cdots s_h$, whence $\beta = s_h \cdots s_1\alpha \in \Phi_\Pi^+$. Moreover, we have $ss_1 \cdots s_h = s_1 \cdots s_h s_\beta$. Now $l(ss_1 \cdots s_h) = h+1$. It follows that there exist $w' \leqslant s_1 \cdots s_h$ and $w'' \leqslant s_\beta$ such that $ss_1 \cdots s_h = w'w''$, and $l(w') + l(w'') = h+1$. Notice that since $s_\beta \in W_\Pi$, we have $w'' \in W_\Pi$. Then

$$w = ss_1 \cdots s_h w_\Pi^\circ \theta(s_h) \cdots \theta(s_1)\theta(s) = w'w''w_\Pi^\circ \theta(w'')^{-1}\theta(w')^{-1}.$$

But we have $w''w_\Pi^\circ = w_\Pi^\circ \theta(w'')$ for all $w'' \in W_\Pi$, in fact this is so for the generators s_α of W_Π ($\alpha \in \Pi$) because $w_\Pi^\circ \alpha = -\theta\alpha$. Hence $w = w'w_\Pi^\circ \theta(w')^{-1}$ and $l(w) = 2h + l(w_\Pi^\circ) \leqslant 2l(w') + l(w_\Pi^\circ)$. But $l(w') \leqslant h$. So we must have $w' = s_1 \cdots s_h$ and $l(w'') = 1$, whence (c).

Remark. 3.3 can be generalized to Coxeter groups, leading to an extension and refinement of the results in [3].

In the course of the proof the following was established.

3.4. Corollary. *Let $w \in \mathcal{T}_\theta$ be such that $sw\theta(s) = w$ for all $s \in \Sigma$ with $sw < w$. Then $h = 0$ and there is Π such as in 3.3 with $w = w_\Pi^\circ$.*

We shall need another result of the same kind.

3.5. Proposition. *Let $w \in \mathcal{T}_\theta$ be such that $sw\theta(s)=w$ for all $s \in \Sigma$ with $sw>w$. Then $w=w_\Pi^\circ w_\Delta^\circ$, where $\Pi=\{\alpha \in \Delta \mid w\theta\alpha=\alpha\}$.*

The proof of this is quite similar to that of 3.4, using 3.2 (ii) instead of 3.2 (i), and will be omitted.

3.6. If $w \in W$, denote by Φ_w^+ the set of $\alpha \in \Phi^+$ with $w^{-1}\alpha<0$. One knows that $\alpha \in \Phi^+$ lies in Φ_w^+ if and only if $s_\alpha w<w$. Also, Card $(\Phi_w^+)=l(w)$.

Next let $w \in \mathcal{T}_\theta$. We shall say that $\alpha \in \Phi$ is *complex* relative to w if $w\theta\alpha \neq \pm\alpha$, and that it is *real* (resp. *imaginary*) relative to w if $w\theta\alpha=-\alpha$ (resp. $w\theta\alpha=\alpha$).

We write

$$C_w'=\{\alpha \in \Phi_w^+ \mid w\theta\alpha \neq -\alpha\}, \qquad C_w''=\{\alpha \in \Phi^+-\Phi_w^+ \mid w\theta\alpha \neq \alpha\},$$
$$R_w=\{\alpha \in \Phi_w^+ \mid w\theta\alpha=-\alpha\}, \qquad I_w=\{\alpha \in \Phi^+-\Phi_w^+ \mid w\theta\alpha=\alpha\}.$$

We write $w=w'w_\Pi^\circ\theta(w')^{-1}$, where $l(w)=2l(w')+l(w_\Pi^\circ)$, as in 3.3.

3.7. Lemma. (i) *C_w' is the disjoint union of $\Phi_{w'}^+$ and $-w\theta(\Phi_{w'}^+)$;*
(ii) *$R_w=w'(\Phi_\Pi^+)$;*
(iii) *$l(w'w'')=l(w')+l(w'')$ for all $w'' \in W_\Pi$.*
In particular, Card $(C_w')=2l(w')$, Card $(R_w)=l(w_\Pi^\circ)=$ Card (Φ_Π^+).

If $\alpha \in \Phi_{w'}^+$, then $s_\alpha w'<w'$ whence $s_\alpha w\theta(s_\alpha)<w$, so $w\theta\alpha \neq -\alpha$. Since $s_\alpha w<w$, we have $\alpha \in C_w'$ and so $\Phi_{w'}^+ \subset C_{w'}'$. We also have $-w\theta(\Phi_{w'}^+) \subset C_w'$. Now if $\alpha \in \Phi_{w'}^+$ then $-(w')^{-1}w\theta(\alpha)=-w_\Pi^\circ\theta((w')^{-1}\alpha)>0$, since $-\theta((w')^{-1}\alpha) \in \Phi^+-\Phi_\Pi^+$. Consequently, $\Phi_{w'}^+ \cap -w\theta(\Phi_{w'}^+)=\emptyset$. Next, if $\alpha \in \Phi_\Pi^+$, then $w\theta(w'(\alpha)) =-w'(\alpha)$, so either $w'(\alpha)$ or $-w'(\alpha)$ lies in R_w. But if $\alpha \in \Phi_\Pi^+$ and $w'(\alpha)<0$, then $w's_\alpha<w'$. Since $s_\alpha w_\Pi^\circ\theta(s_\alpha)=w_\Pi^\circ$, we would get the contradiction $w= w''w_\Pi^\circ\theta(w'')^{-1}<w$ with $w''=w's_\alpha$. Hence $\Phi_w^+ \supset w'(\Phi_\Pi^+)$. We have shown, in particular, that Card $(\Phi_w^+) \geqslant 2$ Card $(\Phi_{w'}^+)+$Card $(\Phi_\Pi^+)=2l(w')+l(w_\Pi^\circ)= l(w)$. Since Card $(\Phi_w^+)=l(w)$, the assertions (i) and (ii) follow. The argument used to prove that $w'(\Phi_\Pi^+) \subset R_w$ also gives (iii) (see [10, 1.16]).

3.8. Lemma. *Let $\alpha \in \Phi^+$. Then $\alpha \in I_w$ if and only if $(w')^{-1}\alpha$ is a root orthogonal to Π which is fixed by θ and w_Π°.*

Orthogonality is meant in the sense of our euclidean metric. Clearly, $w\theta\alpha=\alpha$ if and only if $\beta=(w')^{-1}\alpha$ is an eigenvector for θw_Π°, for the eigenvalue 1. Since Π consists of eigenvectors for the eigenvalue -1, it is clear that if β is such an eigenvector it must be orthogonal to Π. But then $w_\Pi^\circ\beta=\beta$, hence also $\theta\beta=\beta$. This proves one half of the lemma, and the other half is trivial.

3.9. Lemma. (i) *If* $s \in \Sigma$ *we have* $\mathrm{Card}\,(C'_{sw\theta(s)})+\mathrm{Card}\,(C''_{sw\theta(s)})=\mathrm{Card}\,(C'_w)+\mathrm{Card}\,(C''_w)$;

(ii) *If* $s \in \Sigma$, $sw\theta(s)<w$ *then* $\mathrm{Card}\,(C'_{sw\theta(s)})=\mathrm{Card}\,(C'_w)-2$;

(iii) *If* $s \in \Sigma$, $sw\theta(s)>w$ *then* $\mathrm{Card}\,(C''_{sw\theta(s)})=\mathrm{Card}\,(C''_w)-2$.

(i) follows from the observation that

$$2\,(\mathrm{Card}\,(C'_w)+\mathrm{Card}\,(C''_w))=\mathrm{Card}\,(\Phi)-\mathrm{Card}\,\{\alpha \in \Phi \mid w\theta\alpha=\pm\alpha\}.$$

To prove (ii), observe that if $sw\theta(s)<w$ we may take $s_1=s$ (cf. the proof of 3.3) and use 3.7. Then (iii) follows from (i) and (ii).

§ 4. Double coset decomposition

We return to the situation of 2.1. We assume G to be reductive. We fix a standard pair (T, B) and use the notations of 2.3. The set S of 2.1 is contained in $S'=\{x \in G \mid \theta x=x^{-1}\}$, and the action $(g, x)\mapsto g*x$ of G on S extends to an action of G on S', defined in the same way. In particular, B operates on S'.

4.1. Lemma (i) *Any B-orbit in S' meets N;*

(ii) *The number of such orbits is finite.*

Let $x \in S'$ and write it, according to Bruhat's lemma, in the form $x=unu'$, with $u, u' \in U$, $n \in N$. Then $\theta(x)=\theta(u)\theta(n)\theta(u')=(u')^{-1}n^{-1}u^{-1}$. The uniqueness of the Bruhat decomposition shows that $\theta n=n^{-1}$. Put $w=nT$, so $\theta(w)=w^{-1}$. As in 2.4, let $U_\alpha\subset U$ be the one parameter subgroup defined by $\alpha \in \Phi^+$ and denote by U_w (resp. U'_w) the subgroup of U generated by the U_α with $\alpha \in \Phi^+$, $w\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^+$, $w\alpha \notin \Phi^+$). In the decomposition $x=unu'$ we may take $u' \in U'_w$, and then this decomposition is unique. Since the product map $U_w\times U'_w\to U$ is bijective, we can write $(\theta u)^{-1}=u_1u'_1$, with $u_1 \in U_w$, $u'_1 \in U'_w$. Then

$$(u'_1)^{-1}u_1^{-1}n^{-1}\theta(u')=(u')^{-1}n^{-1}\theta(u_1)nn^{-1}\theta(u'_1).$$

Now $\theta(U_w)=U_{w^{-1}}$, whence $n^{-1}\theta(U_w)n=U_w$. The uniqueness of the Bruhat decomposition now shows that $u'=u'_1$, and it follows that the B-orbit of x in S' contains $\theta(u_1)^{-1}n$. It also follows that $n^{-1}\theta(u_1)n=u_1^{-1}$. Now $\psi(u)=n^{-1}\theta(u)n$ defines an automorphism of U_w of period 2, such that $\psi(u_1)=u_1^{-1}$. By a familiar result [1, p. 230] there exists $u_2 \in U_w$ such that $u_1=\psi(u_2)u_2^{-1}$. But then $nu_1=\theta(u_2)nu_2^{-1}$, and we see that the B-orbit of x contains $n \in N$. This proves (i). To establish (ii), it suffices to prove that for any $n \in N\cap S'$, the set $nT\cap S'$ intersects only finitely many B-orbits in S'. Now if $t \in T$, $n \in N\cap S'$ then $tn \in N\cap S'$ if and only if $n\theta(t)n^{-1}=t^{-1}$, i.e. if and only if

$w\theta(t)=t^{-1}$, where $w=nT$. Also, it follows from the uniqueness part of Bruhat's lemma that n and tn lie in the same B-orbit on S' if and only if there is $t_1 \in T$ such that $t=t_1(w\theta(t_1))^{-1}$. Writing endomorphisms of T additively, we see that the number of distinct B-orbits in S' intersecting $nT(n \in N \cap S')$ equals the order of $\mathrm{Ker}(w\theta+1)/\mathrm{Im}(w\theta-1)$, which is well-known to be a finite abelian group, of exponent 2. This implies (ii).

The group $T\times K$ acts on $\tau^{-1}N=\{x \in G|x(\theta x)^{-1} \in N\}$, by $(t, k)x=txk^{-1}$ and the number of orbits of $T\times K$ in $\tau^{-1}N$ is finite, as a consequence of the proof of 4.1 (ii). Let V be the set of these orbits. If $v \in V$, we denote by $\dot{v} \in G$ a representative. If $x \in \tau^{-1}N$, $n \in N$, then clearly $nx \in \tau^{-1}N$, whence an action of W on V (which we write as $(w, v)\mapsto w\cdot v$).

4.2. Theorem. (i) *G is the disjoint union of the double cosets BvK, with $v \in V$;*

(ii) *Each set BvK is a locally closed subset of G, and its closure is a union of similar sets.*

$\tau G=S$ is a B-stable subset of the set S' of 3.1. Hence B has finitely many orbits in S, represented by elements of $N\cap S$. This implies that G is the union of finitely many cosets KxB, with $x \in \tau^{-1}N$. If $x, x' \in \tau^{-1}N$ and $x' \in BxK$, then $(x')(\theta x')^{-1}=bx(\theta x)^{-1}(\theta b)^{-1}$ for some $b \in B$ and the uniqueness part of Bruhat's lemma implies that there is $t \in T$ with $(x')(\theta x')^{-1}=tx(\theta x)^{-1}(\theta t)^{-1}$. It follows that the double cosets BxK are parametrized by the $T\times K$-orbits in $\tau^{-1}N$, whence (i). Then (ii) is an immediate consequence of the fact that $B\times K$, acting in the obvious way in G, has finitely many orbits.

4.3. Corollary. (i) *K has finitely many orbits in G/B;*

(ii) *B has finitely many orbits in G/K.*

Let $v \in V$ and put $T_1=\dot{v}^{-1}T\dot{v}$, $B_1=\dot{v}^{-1}B\dot{v}$. Then $\theta(T_1)=T_1$ and B_1 is a Borel subgroup containing T_1 (but not necessarily θ-stable). Conversely, if T_1 is a θ-stable maximal torus we can write $T_1=x^{-1}Tx$, and then clearly $x(\theta x)^{-1} \in N$, i.e. $x \in \tau^{-1}N$. From these remarks one deduces readily the following result, due to Matsuki for $F=\mathbf{C}$ (see [6]).

4.4. Corollary. *There is a bijection of the set of double cosets KxB onto the set of K-orbits of pairs (T_1, B_1), where T_1 is a θ-stable maximal torus and B_1 a Borel subgroup containing T_1.*

4.5. As in 3.1., let $\mathscr{T}_\theta=\{w \in W|\theta(w)=w^{-1}\}$. We define a map $\varphi : V\to\mathscr{T}_\theta$ by $\varphi(v)=\dot{v}(\theta\dot{v})^{-1}T$. This map is not necessarily surjective. For example, if $\theta(x)=axa^{-1}$, with $a \in T$, $a^2=1$, then, denoting the conjugacy class of $a \in G$ by $C(a)$, we see that $N \cap S = (N \cap C(a))a^{-1}$, whence $\varphi(V) = N \cap C(a) \bmod T$. If follows that $\varphi(V)$ consists of involutions in W which can be lifted to involutions in N. It is known that such a lifting is not always possible.

With the notations of the proof of 4.1 (ii) we have for $w \in \mathscr{T}_\theta$ that $\mathrm{Card}\, \varphi^{-1}(w) \leqslant \mathrm{Card}\, (\mathrm{Ker}\, (w\theta + 1)/\mathrm{Im}\, (w\theta - 1))$.

4.6. Examples. (a) Let G_1 be a reductive group and put $G = G_1 \times G_1$, $\theta(x, y) = (y, x)$, for $(x, y) \in G$. Then $K \simeq G_1$. Denote by T_1, B_1 a maximal torus and a Borel subgroup of G_1 with $T_1 \subset B_1$. Let W_1 be the Weyl group of T_1. We may then take $T = T_1 \times T_1$, $B = B_1 \times B_1$, $W = W_1 \times W_1$. It is easy to check that $V \simeq W_1$ and that the W-action on V is given by $(w_1, w_2)w = w_1 w (w_2)^{-1}$.

(b) $G = \mathrm{SL}_2$. We put

$$T = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \middle|\, \xi \in F^* ,\ a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\} \quad (i^2 = -1)$$

and $\theta x = axa^{-1}$ $(x \in G)$. Then T and both Borel groups containing T are θ-stable and $K = T$. We have $S = \left\{ \begin{pmatrix} \xi & \eta \\ \zeta & \xi \end{pmatrix} \middle|\, \xi^2 - \eta\zeta = 1 \right\}$ and $S' = S$. Now V has three elements v, v' and v_1, represented by the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of S respectively. The generator s of W acts by $s.v = v'$, $s.v' = v$, $s.v_1 = v_1$.

(c) $G = \mathrm{PSL}_2$, with the automorphism induced by the one of the previous example. We deduce from it that now V has two elements, on which W acts trivially.

4.7. We shall now discuss the orbits BvK in more detail. Fix $v \in V$ and put $\dot{v}(\theta\dot{v})^{-1} = n$, $w = nT$. Then $\theta(w) = w^{-1}$. We write $w = w' w_\Pi^\circ (w')^{-1}$ as in 3.3. Clearly, the variety BvK is isomorphic to $B \times K/\dot{v}^{-1}B\dot{v} \cap K$ (the latter group being imbedded in an obvious way).

Assume $b = tu \in B$ and $\theta(\dot{v}^{-1}b\dot{v}) = \dot{v}^{-1}b\dot{v}$. Then $n\theta(t)n^{-1} = t$, $n\theta(u)n^{-1} = u$. Hence $w\theta(t) = t$, and $t \in \mathrm{Ker}\, (w\theta - 1) \simeq \mathrm{Ker}\, (w_\Pi^\circ \theta - 1)$,

It also follows that $u \in U \cap nUn^{-1} = U_{w^{-1}}$ (notations as in the proof of 4.1). Now $\psi(x) = n\theta(x)n^{-1}$ defines an automorphism of period 2 of $U_{w^{-1}}$ and it is known [1, p. 230] that its group of fixed points is a connected subgroup of $U_{w^{-1}}$. To determine its dimension, we look at the action of ψ on the Lie algebra of $U_{w^{-1}}$, which is spanned by the root vectors X_α with $\alpha \in \Phi^+$, $w^{-1}\alpha > 0$. If $w\theta\alpha \neq \alpha$, i.e. if $\alpha \in C_w''$ (with the notations of 3.6) we can assume that $\psi(X_\alpha) = X_{\theta\alpha}$. If however $w\theta\alpha = \alpha$, i.e. $\alpha \in I_w$, then

$\psi(X_\alpha)=c_\alpha X_\alpha$, $\psi(X_\alpha)=c_\alpha X_\alpha$, with $c_\alpha=\pm 1$. Denote by I_v^c (resp. I_v^n) the set of $\alpha \in I_w$ with $c_\alpha=1$ (resp. $c_\alpha=-1$). These are the *compact imaginary roots* (resp. *noncompact imaginary roots*) relative to v. We shall also write C_v', C_v'', R_v, I_v for the sets C_w', C_w'', R_w, I_w of 3.6 and call roots complex$\cdots$relative to v if they are so relative to w. The roots of the various kinds relative to v correspond to roots of the same kinds for the θ-stable torus $\dot{v}^{-1}T\dot{v}$, according to 2.4. It then follows that the fixed point set of ψ in Lie $(U_{w^{-1}})$ has dimension $\frac{1}{2}\operatorname{Card}(C_v'')+\operatorname{Card}(I_v^c)$.

The preceding analysis proves

4.8. Proposition. *$K\cap \dot{v}^{-1}B\dot{v}$ is isomorphic to the product of* $\operatorname{Ker}(w_\Pi^\circ\theta-1)$ *and a connected unipotent group of dimension* $\frac{1}{2}\operatorname{Card}(C_v'')+\operatorname{Card}(I_v^c)$.

4.9. Corollary.

$$\dim BvK=\dim B+\dim K-\dim \operatorname{Ker}(w_\Pi^\circ-1)$$
$$-\tfrac{1}{2}\operatorname{Card}(C_v'')-\operatorname{Card}(I_v^c).$$

The proposition can be used to obtain topological information about BvK and similar varieties. As an example, take $F=\mathbf{C}$ and consider the K-orbits in G/B. They are parametrized by V, let $O(v)$ be the orbit containing $\dot{v}^{-1}B$. Assume for simplicity that G is connected, semi-simple and simply connected. Then K is connected by a theorem of Steinberg [10, 8.1]. We put $\Gamma(v)=\operatorname{Ker}(w_\Pi^\circ\theta-1)/\operatorname{Im}(w_\Pi^\circ\theta+1)$, this is a finite abelian group of exponent 2. In this situation we have:

4.10. Corollary. (i) *The fundamental group $\pi_1(O(v))$ is an extension of $\Gamma(v)$ by a quotient of $\pi_1(K)$;*

(ii) *There is a bijection of the set of isomorphism classes of locally constant K-equivariant sheaves of one dimensional vector spaces on $O(v)$, whose pullback to K is trivial, onto the character group of $\Gamma(v)$.*
(The fundamental groups are relative to appropriate base points).

We have $O(v)\simeq K/K_v$, where $K_v=\dot{v}^{-1}B\dot{v}\cap K$. Now, K_v° denoting the identity component of K_v, we have that K/K_v° is a Galois covering of $O(v)$ with group $\operatorname{Ker}(w_\Pi^\circ\theta-1)/\operatorname{Ker}(w_\Pi^\circ\theta-1)^\circ\simeq\Gamma(v)$. Moreover, there is an exact sequence

$$\pi_1(K_v^\circ)\longrightarrow\pi_1(K)\longrightarrow\pi_1(K/K_v^\circ)\longrightarrow\pi_0(K_v^\circ)=1.$$

Hence $\pi_1(K/K_v^\circ)$ is a quotient of $\pi_1(K)$, whence (i). Then (ii) is a consequence of standard results.

§ 5. The big cell

Assume G to be connected and reductive. We keep the notations of the preceding section. It follows from 4.2 that there is a unique $v^\circ \in V$ such that $Bv^\circ K$ is open and dense in G. We write $n^\circ = \dot{v}^\circ(\theta\dot{v}^\circ)^{-1}$, $w^\circ = \varphi(v^\circ)$. In the present section we shall deduce some properties of v° and w°. We start with an auxiliary result. The notations are as in 4.7, with $v \in V$ arbitrary. We put $O(v) = BvK$.

5.1. Lemma. *If $\alpha \in I_v^n \cap \Delta$ (resp. $\alpha \in C_v'' \cap \Delta$) there is $v_1 \in V$ such that $O(v) \subset \overline{O(v_1)}$, $\dim O(v_1) = \dim O(v) + 1$ and $\varphi(v_1) = s_\alpha w$ (resp. $\varphi(v_1) = s_\alpha w\theta(s_\alpha)$).*

Let $\alpha \in I_v^n \cap \Delta$. Denote by P_α the parabolic subgroup generated by B and $U_{-\alpha}$ and by G_α the subgroup generated by U_α and $U_{-\alpha}$ (which is isomorphic to SL_2 or PSL_2). If τ is as in 2.1, then $Z = \tau(\overline{O(v)})$ is an irreducible closed subvariety of $S = \tau G$. which is B-stable for the action $*$. Now $P_\alpha * Z$ is also closed and irreducible, because P_α/B is a complete variety (by a familiar argument, see e.g. [9, p. 114, ex. (9)] for a similar result). We have $xn\theta(x)^{-1} \in P_\alpha * Z$ for all $x \in G_\alpha$. Now $\psi(x) = n\theta(x)n^{-1}$ defines an automorphism of G_α which is nontrivial (since α is noncompact imaginary we have $\psi(x_\alpha(\xi) = x_\alpha(-\xi))$. From the formulas of 2.4 one deduces that $\psi(n_\alpha) = n_\alpha^{-1}$. But then there exists $x \in G_\alpha$ with $x\psi(x)^{-1} = n_\alpha$ (see 4.6 (b) for the case $G_\alpha = \mathrm{SL}_2$, the case $G_\alpha = \mathrm{PSL}_2$ is similar), from which we see that $n_\alpha n \in P_\alpha * Z$. But $n_\alpha n \notin Z$: all elements of Z lie in double cosets $Bw'B$ with $w' \leqslant w$, whereas $n_\alpha n \in Bs_\alpha wB$ and $w < s_\alpha w$. It follows that there is a unique dense B-orbit $\tau(O(v_1))$ in $P_\alpha * Z$. The assertions now readily follow (observe that $\dim P_\alpha * Z = \dim Z + 1$).

In the case that $C_v'' \cap \Delta \neq \emptyset$ the argument is quite similar and even a bit easier. If $\alpha \in C_v'' \cap \Delta$ we have $n_\alpha n\theta(n_\alpha)^{-1} \in P_\alpha * Z$ and $s_\alpha w\theta(s_\alpha) > w$. We omit the details.

We can now deal with v°. If $\Pi \subset \Delta$ denote by $P_\Pi \supset B$ the corresponding standard parabolic subgroup (so $P_\phi = B$, $P_\Delta = G$).

5.2. Theorem. (i) *$v = v^\circ$ if and only if $C_v'' \cap \Delta = \emptyset$, $I_v^n \cap \Delta = \emptyset$;*

(ii) *There is a subset $\Pi \subset \Delta$ such that $w^\circ = w_\Pi^\circ w_\Delta^\circ$. We have $w_\Pi^\circ \alpha = w_\Delta^\circ \theta\alpha$ for all $\alpha \in \Pi$, and $\Pi \subset I_{v^\circ}^c$;*

(iii) *$(\dot{v}^\circ)^{-1} P_\Pi \dot{v}^\circ$ is a minimal θ-split parabolic subgroup of G.*

It follows from 5.1 that $C_{v^\circ}'' \cap \Delta = \emptyset$, $I_{v^\circ}^n \cap \Delta = \emptyset$. Next let $v \in V$ be such that $C_v'' \cap \Delta = \emptyset$. If $w = \varphi(v)$, then w has the property of 3.5, so $w = w_\Pi^\circ w_\Delta^\circ$, as in 3.5,

Still assuming only $C_v'' \cap \Delta = \emptyset$, put $P = \dot{v}^{-1} P_\Pi \dot{v}$ (Π being as in 3.5). Then $\theta P \cap P$ is generated by $\dot{v}^{-1} T \dot{v}$ and the groups $\dot{v}^{-1} U_\alpha \dot{v}$ with $\alpha \in \Phi_\Pi \cup (\Phi^+ \cap w\theta\Phi^+)$, i.e. by those with $\alpha \in \Phi$, $w\theta\alpha = \alpha$ (as a consequence of

3.5). It follows that $\theta P \cap P$ must be a Levi subgroup of P and θP, i.e. θP is θ-split. Now P is minimal θ-split if and only if the derived group of $P \cap \theta P$ lies in K (as follows from [12, § 1]) i.e. if and only if $\dot{v}^{-1} U_\alpha \dot{v} \subset K$ for all $\alpha \in \Pi$ or if and only if $\Pi \subset I_v^c$. Assuming this to be the case, we have that PK is dense in G [loc. cit.]. But then $P_\Pi v K$ is also dense and $P_\Pi v K = BvK$. Hence we must have $v = v^\circ$, and we have proved the if-part of (i). The other part of (i) was already established, and (ii) and (iii) were proved in the course of the argument.

5.3. Corollary. *G is θ-split if and only if $w^\circ = w_\Delta^\circ$, and $\theta_\alpha = -w_\Delta^\circ \alpha$ for all $\alpha \in \Delta$.*

This follows from 5.2, using that if G is split a minimal θ-split parabolic subgroup is a Borel group.

An example is $G = \mathrm{SL}_n$ and $\theta x = ({}^t x)^{-1}$.

5.4. Proposition. (i) *If* $\dim O(v) = \dim G - 1$ *then either* $\varphi(v) = s_\alpha w^\circ \theta(s_\alpha)$ *with* $\alpha \in C'_{v^\circ} \cap \Delta$ *or* $\varphi(v) = s_\alpha w^\circ$ *with* $\alpha \in R_{v^\circ} \cap \Delta$;

(ii) *Conversely, if* $\alpha \in C'_{v^\circ} \cap \Delta$ *(resp.* $\alpha \in R_{v^\circ} \cap \Delta$*) there exists* $v \in V$ *with* $\dim O(v) = \dim G - 1$ *and* $\varphi(v) = s_\alpha w^\circ \theta(s_\alpha)$ *(resp.* $\varphi(v) = s_\alpha w^\circ$*).*

Assume $\dim O(v) = \dim G - 1$. We cannot have $C''_v \cap \Delta = \emptyset$, $I_v^n \cap \Delta = \emptyset$, otherwise we had $v = v^\circ$, by 5.2 (i). Assume $\alpha \in C''_v \cap \Delta$ and apply 5.1. We see that the element v_1 of 5.1 must be v°, and we conclude that $\varphi(v) = s_\alpha w^\circ \theta(s_\alpha)$. It is then also clear that $\alpha \in C'_{v^\circ}$. If $C''_v \cap \Delta = \emptyset$, take $\alpha \in I_{v^\circ}^n \cap \Delta$. Then 5.1 gives that $\varphi(v) = s_\alpha w^\circ$, and it is readily checked that $\alpha \in R_{v^\circ} \cap \Delta$. This proves (i). To prove (ii), take $v = n_\alpha v^\circ$ if $\alpha \in C'_{v^\circ} \cap \Delta$. Then $\alpha \in C''_v \cap \Delta$, and the element v_1 of 5.1 equals v°, whence $\dim O(v) = \dim G - 1$. Similarly, in the case that $\alpha \in R_{v^\circ} \cap \Delta$, one takes $v = x \dot{v}^\circ (\theta x)^{-1}$, with a suitable $x \in G_\alpha$, as in the proof of 5.1 and applies again 5.1.

5.5. As a consequence of 5.4 we can deduce a result which is well-known in characteristic 0 (see [4, p. 79] and [12, § 3]). We shall sketch a proof. Assume now that G is connected and semi-simple.

Denote by $F[G]$ the algebra of regular functions on G. Let χ be a rational character of T and extend it to a character of B by $\chi(tu) = \chi(t)$ (for $t \in T$, $u \in U$). Put

$$V_\chi = \{ f \in F[G] \mid f(xb) = \chi(b) f(x),\ x \in G,\ b \in B \}.$$

It is known that $V_\chi \neq \{0\}$ if and only if χ lies in the set P^+ of dominant weights (of T relative to B). So, denoting by $\langle\, , \rangle$ the canonical pairing between characters of T and one parameter subgroups of T, we have $\langle \chi, \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta$. The group G acts by right translations on V_χ

and it is well-known that if char $F=0$ one obtains in this manner the irreducible rational representation of G with highest weight χ (assuming $\chi \in P^+$). The result gives a criterion for K to contain a nonzero K-fixed vector. We formulate it as follows. Denote by $\tilde{T}$ the subgroup of T of elements fixed by $w^\circ\theta$ (where w° is as before).

5.6. Theorem. *Assume $\chi \in P^+$. There is a nonzero $f \in V_\chi$ fixed by K if and only if $\chi(\tilde{T})=1$. Such an f is unique up to a scalar.*

If $f \in V_\chi$ is fixed by K, we have

$$f(k(\dot{v}^\circ)^{-1}b)=\chi(b)f((\dot{v}^\circ)^{-1}) \qquad (k \in K, b \in B). \tag{1}$$

Since $Bv^\circ K$ is dense in G this implies the uniqueness statement. Also, if $t \in \tilde{T}$ we have $(\dot{v}^\circ)^{-1}t(v^\circ) \in K$, whence

$$f((\dot{v}^\circ)^{-1})=f((\dot{v}^\circ)^{-1}t)=\chi(t)f((\dot{v}^\circ)^{-1}),$$

from which we see that the condition of the theorem is necessary.

Assume the condition to hold. We then can define a nonzero rational function on G by the formula (1). Clearly, the set of points of G where f is defined is a union of orbits $O(v)$ and since G is normal it suffices to prove that f is defined along all orbits $O(v)$ of codimension 1.

Via τ, we may transfer the problem to a similar one for $S=\tau G$. Put $\Omega(v)=\tau(O(v))$, this is a B-orbit in S, for the action of 2.1. The rational function f on G defines a nonzero rational function h on S such that

$$h(bn^\circ(\theta b)^{-1})=\chi(b)^{-1}h(n^\circ).$$

We have to prove that h is defined along the orbits $\Omega(v)$, with $\dim \Omega(v)= \dim S-1$. Let $\Omega'=\Omega(v)$ be such an orbit, and put $\Omega=\Omega(v^\circ)$. We have the two cases of 5.4: either $w^\circ=s_\alpha\varphi(v)$, and $\alpha \in R_{v^\circ} \cap \Delta$, or $w^\circ=s_\alpha\varphi(v)\theta(s_\alpha)$ with $\alpha \in C'_{v^\circ} \cap \Delta$. In the first case we have, as in the proof of 5.1, that

$$\Omega \cup \Omega'=\{x\Omega'(\theta x)^{-1} \mid x \in G_\alpha\},$$

where G_α is as in that proof. Let U' be the unipotent radical of P_α and T' the connected center of the Levi group of P_α containing T. Putting $n=\dot{v}(\theta\dot{v})^{-1}$, we write the elements of $\Omega \cup \Omega'$ as

$$y=utgn\,\theta(utg)^{-1} \qquad (u \in U',\ t \in T',\ g \in G_\alpha). \tag{2}$$

Denote by ψ the automorphism $g \mapsto n\theta(g)n^{-1}$ of G_α, which has order 2. We then have

$$h(y)=\chi(t)^{-1}h(g\psi(g)^{-1}n)$$

and we see that we are reduced to proving the theorem in the case of (G_α, ψ), i.e. in the case that $G=\mathrm{SL}_2$ or PSL_2, with a (unique up to conjugacy) nontrivial involutorial automorphism. This is easy, and we leave the proof to the reader.

In the second case ($w^\circ=s_\alpha w\theta(s_\alpha)$, where $w=\varphi(v)$) we have $w\theta\alpha\neq\alpha$ and $s_\alpha w>w$. A similar argument now gives a reduction to the case of the group H generated by G_α and $nG_\alpha n^{-1}$, which is either of type A_2 or of type $A_1\times A_1$. In the first case we have $H\simeq\mathrm{SL}_3$, with $\theta x=({}^t x)^{-1}$, or a similar situation in PSL_3. It follows from 5.3 that we are in the situation which has been already dealt with. The last possibility is type $A_1\times A_1$, with an automorphism exchanging the factors. We leave the discussion of this case to the reader.

§ 6. Orbit closures

6.1. We keep the same notations. We shall now discuss the closures of the orbits $O(v)=BvK$. Fix $v\in V$ and put $\dot v(\theta\dot v)^{-1}=n$, $\varphi(v)=w$. So $\theta(w)=w^{-1}$. We first assume, with the notations of 3.3, that $w=w_\Pi^\circ$. So Π is a θ-stable subset of Δ such that $w_\Pi^\circ\theta\alpha=-\alpha$ for $\alpha\in\Pi$. Denote by $P_\Pi\supset B$ the standard parabolic subgroup defined by Π. Write $P=\dot v^{-1}P_\Pi\dot v$, this is a parabolic subgroup of G of type Π. If Q is any parabolic subgroup, we denote by $H(Q)$ the derived group of Q/R_uQ, this is a connected semi-simple group.

6.2. Lemma. (i) *P is a θ-stable parabolic subgroup and $H(P)$ is split, relative to the involution induced by θ;*

(ii) *Conversely, if P is a parabolic subgroup of G with the properties of* (i), *then there is a θ-stable subset Π of Δ such that $w_\Pi^\circ\theta\alpha=-\alpha$ for $\alpha\in\Pi$, and an element $v\in V$ with $\varphi(v)=w_\Pi^\circ$ such that $P=\dot v^{-1}P_\Pi\dot v$. The orbit $W_\Pi\cdot v$ in V is uniquely determined by P and Π.*

As before, W_Π denotes the subgroup of W generated by the s_α with $\alpha\in\Pi$. It is the Weyl group of $H(P_\Pi)$.

(i) is immediate. If P is as in (i), choose a θ-stable maximal torus T_1 of P such that the identity component of the intersection of its image in P/R_uP with $H(P)$ is split, for the involution induced by θ in $H(P)$. There is a unique subset $\Pi\subset\Delta$, and $x\in G$ with $P=x^{-1}P_\Pi x$, $T_1=x^{-1}Tx$. Then $x(\theta x)^{-1}\in P_\Pi\cap N$, let w be its image in W. The assumption about P implies that $w\theta\alpha=-\alpha$ for all $\alpha\in\Pi$. Also, since $x(\theta x)^{-1}$ normalizes R_uP_Π, we see that $w^{-1}\alpha>0$ for all $\alpha\in\Delta-\Pi$. It follows that we must have $w=w_\Pi^\circ$ and that Π is θ-stable (as in the proof of 3.3). This proves (ii), except for the last point. If $v_1 \in V$ is another element with similar properties, then $\dot{v}_1^{-1}T\dot{v}_1$ is a maximal torus of P which on account of 2.8 is conjugate to T_1 by an element of $P \cap K$. We may then assume that $\dot{v}_1^{-1}T\dot{v}_1 = T_1$. But then $n = \dot{v}_1\dot{v}^{-1}$ normalizes P_Π and T, hence lies in $N_{P_\Pi}(T) = N_{L_\Pi}(T)$ (L_Π denoting the Levi subgroup of P_Π containing T). This shows that $v_1 = w \cdot v$ for some $w \in W_\Pi$. Since $ww_\Pi^\circ = w_\Pi^\circ\theta(w)$ for all $w \in W_\Pi$ (this is so for the generators s_α of W_Π, with $\alpha \in \Pi$) we have $\varphi(w \cdot v) = w_\Pi^\circ$ for all $w \in W_\Pi$.

We now fix $v \in V$ with $\varphi(v) = w_\Pi^\circ$, as in 6.2. Let P be a parabolic subgroup with the properties of 6.2 (i). We denote its unipotent radical by R and we fix a θ-stable Levi subgroup L of P. The morphism τ and the variety S_G are as in 2.1. As in the proof of 5.7 write $\Omega(v) = \tau(O(v))$, this is a B-orbit in S_G.

6.3. Lemma. (i) *τP is a closed, irreducible, smooth subvariety of S_G. We have $\tau P = \{xS_L(\theta x)^{-1} | x \in R\}$;*

(ii) *$O(v) = \dot{v}(\tau^{-1}(\tau P))$ and $\overline{\Omega(v)} = \{xn(\theta x)^{-1} | x \in P_\Pi\}$.*

τP is closed in G, hence in S_G by 2.2 (here we use 2.2 for a nonreductive group). Since τP is isomorphic to $P/P \cap K$ (see 2.2) it is an irreducible and smooth variety. The last assertion of (i) is immediate from the definitions.

The two statements of (ii) are equivalent, so it suffices to prove the second one. The set A with which we want to identify $\overline{\Omega(v)}$ is closed and irreducible by (i). Also, $\Omega(v) \subset A$. It then corresponds to an orbit in S_L (for a suitable Borel group) which contains the element $\dot{v}^{-1}n(\theta\dot{v})$. But by 5.3 this must be the open orbit in S_L. It follows that $\Omega(v)$ is the open orbit in A, whence $\overline{\Omega(v)} = A$. (Another way to prove the last result would be to show that $\dim \Omega(v) = \dim A$.)

6.4. Now consider an arbitrary element $v \in V$. We write $\varphi(v) = w$ in the form of 3.3:

$$w = s_1 \cdots s_h w_\Pi^\circ \theta(s_h) \cdots \theta(s_1),$$

with $l(w) = l(w_\Pi^\circ) + 2h$. We put $w' = s_1 \cdots s_h$ and write $v' = (w')^{-1}.v$. So $\varphi(v') = w_\Pi^\circ$. We shall describe now the orbit closure $\overline{\Omega(v)}$ in S_G. Since $\overline{O(v)} = \tau^{-1}(\overline{\Omega(v)})$ we then also have a description of orbit closures in G and of closures of K-orbits in G/B (these have been studied recently over $\mathbf{C}$ by Matsuki [7]). If $\alpha \in \Delta$, we denote as before by P_α the parabolic subgroup generated by B and $U_{-\alpha}$. We write $P_i = P_{\alpha_i}$, where $s_i = s_{\alpha_i}$ ($1 \leqslant i \leqslant h$). With these notations we have:

6.5. Theorem. $\overline{\Omega(v)} = P_1 * P_2 * \cdots * P_h * \overline{\Omega(v')}$.

$\overline{\Omega(v')}$ has been described in 6.3. We prove the theorem by induction on h, starting with $h=0$. Put $v_1=s_1\cdot v$. We can then assume that

$$\overline{\Omega(v_1)}=P_2*\cdots*P_h*\overline{\Omega(v')}.$$

So the right hand side contains $\Omega(v_1)$ as an open subset. Now $P_1*\overline{\Omega(v_1)}$ is closed and irreducible (by an argument used before, in the proof of 5.1), and $\dim P_1*\overline{\Omega(v_1)}\leqslant\dim\overline{\Omega(v_1)}+1$. But $P_1*\Omega(v_1)$ contains $\Omega(v)$ and $\dim\Omega(v)=\dim\Omega(v_1)+1$, as one readily deduces from 4.9 (notice that $\operatorname{Card} I_v^c=\operatorname{Card} I_{v_1}^c$). It follows that $\dim\overline{\Omega(v)}=\dim P_1*\overline{\Omega(v_1)}$, from which one concludes that the two sets $\overline{\Omega(v)}$ and $P_1*\overline{\Omega(v_1)}$ must coincide. This proves the theorem.

As a first consequence of 6.5 we shall characterize the closed orbits.

6.6. Corollary. (i) *$\Omega(v)$ is closed if and only if $\varphi(v)=1$;*
(ii) *The closed orbits $\Omega(v)$ correspond to the K-conjugacy classes of standard pairs (T, B) (see* 2.3).

If $\varphi(v)=1$, then putting $T_1=\dot{v}^{-1}T\dot{v}$, $B_1=\dot{v}^{-1}B\dot{v}$, it is clear that (T_1, B_1) is a standard pair. Hence [8, 5.1] $B_1\cap K^\circ$ is a Borel subgroup of K°. It follows that $K^\circ\dot{v}^{-1}B$ is closed in G/B, whence one concludes that $B\dot{v}K^\circ$ and $B\dot{v}K$ are closed in G, consequently $\Omega(v)$ is closed in $G/K\simeq S$.

Conversely, let $\Omega(v)$ be closed. With the notations of the proof of 6.5, we must have $h=0$. Otherwise we had $\Omega(v_1)\subset\overline{\Omega(v)}$ and $\Omega(v_1)\neq\Omega(v)$, because $\dim\Omega(v_1)<\dim\Omega(v)$. So we are now in the situation of 6.3, and there is only one orbit $\Omega_L(v)$ in S_L. But if L is not a torus, there are at least two such orbits, viz. the open one and $\Omega_L(1)$. Hence in our case L is a torus, i.e. $\Pi=\emptyset$ and $\varphi(v)=1$. We have proved (i), and (ii) now readily follows. $G=\mathrm{SL}_2$, with a nontrivial involution, provides already an example where there are several closed orbits.

Theorem 6.5 can be viewed as giving an inductive description of the closures of the $\Omega(v)$, depending on the knowledge of such closures in smaller groups L (except possibly for the case of the maximal orbit $\Omega(v^\circ)$, of course, but then $\overline{\Omega(v^\circ)}=S$). One can formalize this description a bit.

Define a "Bruhat order" on V by: $v'\leqslant v$ if and only if $\Omega(v')\leqslant\overline{\Omega(v)}$. Let $s=s_\alpha\in\Sigma$ be a generator of W. We also define (provisionally) a relation $\leqslant_s$ on V by: $v'\leqslant_s v$ if and only if $\Omega(v')\subset P_\alpha*\Omega(v)$. Then the proof of 6.5 implies the following property of the Bruhat order of V.

6.6. Corollary. *Let $v\in V$, $\alpha\in C_v'$. Then $v'\leqslant v$ if and only if $v'\leqslant_s v''$ for some $v''\leqslant_s v$.*

This is useless if $C_v'=\emptyset$, but then we are in the situation of 6.3. To make 6.6 more concrete, we shall now describe the relation $\leqslant_s$ in more detail.

6.7. *The relation* $\leqslant_s$.

Let $v \in V$ and put $\dot{v}(\theta\dot{v})^{-1}=n$, $\varphi(v)=w$. Fix $s=s_\alpha \in \Sigma$ $(\alpha \in \Delta)$. We shall describe $P_\alpha * \Omega(v)$. Notice that by Bruhat's lemma for the group G_α, we have $P_\alpha * \Omega(v)=\Omega(v) \cup \Omega'$, where

$$\Omega'=\{un_\alpha x\theta(un_\alpha)^{-1} | s \in \Omega(v),\ u \in U_\alpha\},$$

with the usual notations. We distinguish several cases.

(a)$''$ $\alpha \in C''_v$, so $l(sw\theta(s))=l(w)+2$. Then $\Omega'=\Omega(s\cdot v)$, as a consequence of 5.1. So now $v' \leqslant_s v$ means $v'=v$ or $v'=s\cdot v$.

(a)$'$ $\alpha \in C'_v$. By the previous case we have

$$\Omega(v)=\{un_\alpha x\theta(un_\alpha)^{-1} | x \in \Omega(s\cdot v),\ u \in U_\alpha\},$$

and familiar properties of rank one groups show that $P_\alpha * \Omega(v)= \Omega(v) \cup \Omega(s\cdot v)$. Again, $v' \leqslant_s v$ means $v'=v$ or $v'=s\cdot v$.

We are left with the cases where $sw\theta(s)=w$. Then we proceed as follows. Write $x \in \Omega(v)$ in the form $x=bx_\alpha(\xi)n\theta(bx_\alpha(\xi))^{-1}$, with b in the radical of the parabolic subgroup P_α, and $\xi \in F$. Then

$$n_\alpha x\theta(n_\alpha)^{-1}=b'n_\alpha x_\alpha(\xi)n\theta(n_\alpha x_\alpha(\xi))^{-1}n^{-1}\cdot n\theta(b')^{-1},$$

with b' similar to b. Denote by ψ the involutorial automorphism of G_α with $\psi(g)=n\theta(g)n^{-1}$. We then have to study the elements of G_α of the form $n_\alpha x_\alpha(\xi)\psi(n_\alpha x_\alpha(\xi))^{-1}$, and we are reduced to a problem in rank one groups.

(b) $\alpha \in I^c_v$. Then $\psi=\mathrm{id}$, and $v' \leqslant_s v$ means $v'=v$.

(c) $\alpha \in I^n_v$. Now $\psi(x_{\pm\alpha}(\xi))=x_{\pm\alpha}(-\xi)$, $\psi(n_{\pm\alpha})=n^{-1}_{\pm\alpha}=\alpha^\vee(-1)n_{\pm\alpha}$. An example of this case is $G_\alpha=\mathrm{SL}_2$ with $U_\alpha(U_{-\alpha})$ the group of upper (resp. lower) unipotent matrices and ψ the inner automorphism defined by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. A calculation, which can be carried out in SL_2, shows that the element $n_\alpha x_\alpha(\xi)\psi(n_\alpha x_\alpha(\xi))^{-1}$ is either $\alpha^\vee(-1)$ (if $\xi=0$) or has the form $yn_\alpha\psi(y)^{-1}$, with $y \in B \cap G_\alpha$. There exists $z \in G_\alpha$ such that $z\psi(z)^{-1}=n_\alpha$. Then $z\dot{v}$ defines an element of V, which we denote by $\sigma_\alpha(v)$, and which is well-defined. We have $\varphi(\sigma_\alpha(v))=s\cdot w$. Notice also that now $sn\theta(s)= \alpha^\vee(-1)n$. From the preceding facts one readily deduces that now $\Omega'= \Omega(s\cdot v) \cup \Omega(\sigma_\alpha(v))$. Hence $v' \leqslant_s v$ means now $v'=v$ or $v'=s\cdot v$ or $v'=\sigma_\alpha(v)$.

(d) $\alpha \in R_v$. We now can take $\psi(x_\alpha(\xi))=x_{-\alpha}(\xi)$, $\psi(n_\alpha)=n_{-\alpha}=n_\alpha^{-1}$. An example is $G_\alpha=\mathrm{SL}_2$, with U_α and $U_{-\alpha}$ as before and ψ the inner automorphism defined by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Defining $\sigma_\alpha(v)$ as before, we find in a similar manner that now $v' \leqslant_s v$ means either $v'=v$ or $v'=\sigma_\alpha(v)$ or $s\sigma_\alpha(v)$. Notice that in this case we have $s \cdot v = v$. This need not be so in the previous case.

We have thus completely described $\leqslant_s$. This description provides an inductive definition of the Bruhat order on V.

The cases just discussed are similar to the ones occurring in [5, p. 371-372] and [11, p. 397]. It is to be expected that our description of orbits can be useful for the matters discussed in these papers. We shall not enter into these matters here.

It should be pointed out that it is not true that $v' \leqslant v$ is equivalent to $\varphi(v') \leqslant \varphi(v)$ (for the Bruhat order on W). It is easy to give examples of four elements v, v_1, v', v'_1 of V such that $\varphi(v)=\varphi(v_1)$, $\varphi(v')=\varphi(v'_1)$ and $\varphi(v') \leqslant \varphi(v)$, $v' \leqslant v$, $v'_1 \leqslant v_1$ but $v \not\leqslant v_1$, $v'_1 \not\leqslant v$.

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