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On the Generalized Springer Correspondence for Exceptional Groups

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Let G be a connected reductive algebraic group defined over an algebraically closed field k. Let $\mathfrak{B}^G$ be the variety of Borel subgroups of G, and for $x \in G$ let $\mathfrak{B}^G_x = \{B \in \mathfrak{B}^G | B \ni x\}$. Springer [28] has shown that the Weyl group W of G acts naturally on the ℓ-adic cohomology groups $H^i(\mathfrak{B}^G_x; \overline{\mathbf{Q}}_\ell)$ (ℓ a prime, $\ell \neq \mathrm{char}(k)$). We shall consider here the action of W defined by Lusztig [13] rather than that defined originally by Springer (they agree up to tensoring by the sign representation of W [10]; moreover no restriction on the characteristic is needed in [13]).

If H is a finite group, let $H^\wedge$ be the set of all isomorphism classes of irreducible $\overline{\mathbf{Q}}_\ell$-representations of H. For $x \in G$ the finite group $A_G(x) = C_G(x)/C^0_G(x)$ acts on $H^*(\mathfrak{B}^G_x)$ (we shall omit the $\overline{\mathbf{Q}}_\ell$), and this action commutes with that of W. Let $d_x = \dim \mathfrak{B}^G_x$. The action of $W \times A_G(x)$ in $H^{2d_x}(\mathfrak{B}^G_x)$ turns out to be particularly interesting. For any $\rho \in W^\wedge$ there exist a unipotent element $u \in G$ and $\phi \in A_G(u)^\wedge$ such that $\rho \otimes \phi$ occurs in $H^{2d_u}(\mathfrak{B}^G_u)$. Moreover the pair (u, ϕ) is unique up to G-conjugation, and $\rho \otimes \phi$ occurs with multiplicity one in $H^{2d_u}(\mathfrak{B}^G_u)$. We write then $\rho = \rho^G_{u,\phi}$. This injective map from $W^\wedge$ to the set $\mathcal{N}_G$ of conjugacy classes of pairs (u, ϕ) ($u \in G$ unipotent, $\phi \in A_G(u)^\wedge$) is explicitly known in most cases. If the characteristic is good, it is described by Shoji [22] [23] (G classical, F_4), Springer [28] (G_2), Alvis, Lusztig and the author [3] (E_n, $n = 6, 7, 8$). Classical groups in characteristic 2 are treated in [17]. We shall consider here exceptional groups in bad characteristic.

We shall also be concerned with Lusztig's generalization of Springer's correspondence [14]. Consider a pair (u, ϕ) ($u \in G$ unipotent, $\phi \in A_G(u)^\wedge$). Lusztig attaches to (u, ϕ) a 4-tuple (L, v, ψ, ρ) (or rather a G-conjugacy class of such objects), where L is a Levi factor of some parabolic subgroup of G, $v \in L$ is unipotent, $\psi \in A_L(v)^\wedge$, $\rho \in (N_G(L)/L)^\wedge$. If $L = G$, then (v, ψ) is conjugate to (u, ϕ) and ρ is automatically trivial, and we say that (u, ϕ) is cuspidal for G. Lusztig's construction has in particular the following properties. It defines a bijection between $\mathcal{N}_G$ and the set $\mathcal{X}_G$ of G-conjugacy classes of 4-tuples (L, v, ψ, ρ) as above for which (v, ψ) is cuspidal

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for L. When $L=T$ is a maximal torus, we have $v=1$, $\psi=1$ and $\rho \in (N_G(T)/T)^\wedge$. The subset of $\mathscr{X}_G$ corresponding to $L=T$ can therefore be identified with $W^\wedge$, and the restriction of Lusztig's bijection to this subset is precisely Springer's correspondence. In general, we shall write $\rho=\rho^G_{u,\phi}$ if (u, ϕ) corresponds to (L, v, ψ, ρ).

We shall compute the generalized Springer correspondence for groups of type G_2, F_4, E_6, E_7, E_8. The results will however not be complete. In the case where $N_G(L)/L$ is dihedral of order 12 there are two irreducible representations of degree 2 and the methods used in this paper do not allow to distinguish them.

Most of the theoretical results which we shall need concerning Springer's correspondence and its generalization are proved by Lusztig in [14] (without any assumption on the characteristic).

I wish to thank G. Lusztig for helpful conversations.

§ 1.

We shall proceed essentially as in [3]. As the characteristic is now arbitrary, we must however be more careful. We shall use the following properties.

1.1. The Weyl group W can be considered in a natural way as a reflection group in a $\overline{\mathbf{Q}}_\ell$-vector space V, and it acts on the space of all polynomial functions on V. For $\rho \in W^\wedge$, let $a_\rho=\min\{d \in \mathbf{N} \mid \rho$ occurs in the space of homogeneous polynomial functions of degree d on $V\}$. Suppose $\rho=\rho^G_{u,\phi}$. Then

(I) $a_\rho \geq d_u$; moreover $a_\rho=d_u$ if $\phi=1$.

This follows from the results of Borho and MacPherson [6]. Here we have to use the identification of the action of W on $H^*(\mathfrak{B}^G_1)$ with the classical action of W on $H^*(\mathfrak{B}^G)$. This identification is stated and used in [6], but the proof is left to the reader. For completeness a proof is given in paragraph 2.

1.2. Let P be a parabolic subgroup of G with unipotent radical U, and let M be a Levi factor of P. Let $u \in G$, $u' \in M$ be unipotent and let $Y_{u,u'}=\{x \in G \mid x^{-1}ux \in u'U\}$. It is known that $\dim Y_{u,u'} \leq \frac{1}{2}(\dim C_G(u) + \dim C_M(u')) + \dim U$ [29]. The group $C_G(u) \times C_M(u')U$ acts on $Y_{u,u'}$ by $(g, m)\cdot x=gxm^{-1}$ ($g \in C_G(u)$, $m \in C_M(u')U$, $x \in Y_{u,u'}$). In particular $A_G(u) \times A_M(u')$ acts on the set $X_{u,u'}$ of all irreducible components of $Y_{u,u'}$ of dimension $\frac{1}{2}(\dim C_G(u) + \dim C_M(u')) + \dim U$, affording a permutation representation $\varepsilon_{u,u'}$ (over $\overline{\mathbf{Q}}_\ell$). Let $L \subset M$ be a Levi factor of some parabolic subgroup of M (or equivalently of G). Let $v \in L$ be unipotent and $\psi \in A_L(v)^\wedge$ be such that (v, ψ) is cuspidal for L. Let $W_0=N_G(L)/L$, $W'_0=$

$N_M(L)/L$, $\rho \in W_0^\wedge$, $\rho' \in W_0'^\wedge$, $\phi \in A_G(u)^\wedge$, $\phi' \in A_M(u')^\wedge$. Suppose that (u, ϕ), (u', ϕ') correspond respectively to (L, v, ψ, ρ) (for G) and (L, v, ψ, ρ') (for M). Then

(II) $\langle \rho', \mathrm{Res}^{W_0}_{W_0'}(\rho)\rangle_{W_0'} = \langle \phi \otimes \phi'^*, \varepsilon_{u,u'}\rangle_{A_G(u)\times A_M(u')}$, where ϕ'^* is the dual of ϕ'. Moreover the irreducible representations of $A_G(u)\times A_M(u')$ which occur in $\varepsilon_{u,u'}$ are all of the form described above.

This is a reformulation of a result of Lusztig [14, Thm 8.3]. It generalizes a formula of Springer [29].

Property (II) is especially useful in two cases where we have a good grip on the set $X_{u,u'}$.

1.3. Suppose first that in (1.2) we are in the situation where the class C of u in G is obtained by the process of induction from the class of u' in M [16], that is, C contains a dense open subset of $u'U$. Without loss of generality we may then assume that $u \in u'U$. Then $C_P^0(u) = C_G^0(u) \subset C_M(u')U$ and $C_P(u)$ meets all irreducible components of $C_M(u')U$ [16]. Let $N = C_P(u)/C_G^0(u) \subset A_G(u)$, $H = C_{C_M^0(u')U}(u)/C_G^0(u) \subset N$. Then H is normal in N and N/H is naturally isomorphic to $A_M(u')$. Now $A_G(u)\times(N/H)$ acts on $A_G(u)/H$ by $(a, nH)\cdot(xH) = axn^{-1}H$ $(a, x \in A_G(u), n \in N)$ and this defines also an action of $A_G(u)\times A_M(u')$ on $A_G(u)/H$. We have then:

(III) In this case $X_{u,u'}$ and $A_G(u)/H$ are isomorphic as sets with $A_G(u)\times A_M(u')$-actions.

This follows from results in [16] (this was already used in [3]).

1.4. Suppose now that in (1.2) we have $u = u'$. Let S be the connected centre of M. Then $M = C_G(S)$ and therefore $C_M(u) = C_{C_G(u)}(S)$. As the centralizer of a torus in a connected group is connected, we find that $C_M^0(u) = C_{C_G^0(u)}(S)$. This implies that $A_M(u)$ can be considered in a natural way as a subgroup of $A_G(u)$, and $A_G(u)\times A_M(u)$ acts on $A_G(u)$ by $(a, b)\cdot x = axb^{-1}$ $(a, x \in A_G(u), b \in A_M(u))$. We have then:

(IV) In this case $A_G(u)$ and $X_{u,u'}$ are isomorphic as sets with $A_G(u) \times A_M(u)$-actions.

This follows immediately from:

Lemma. *In the situation above, we have*:

(i) $Y_{u,u} = C_G(u)U$.

(ii) $C_G(u)\,U$ *is of pure dimension* $\frac{1}{2}(\dim C_G(u) + \dim C_M(u)) + \dim U$.

(iii) $C_U(u)$ *is connected.*

The connectedness of $C_U(u)$ is proved in [25]. It is known that $\dim C_G(u) = \dim C_M(u) + 2\dim C_U(u)$ [14], [25]. Therefore $\dim C_G(u)U = \dim C_G(u) + \dim U - \dim C_U(u) = \frac{1}{2}\dim C_G(u) + (\frac{1}{2}\dim C_G(u) - \dim C_u(u)) + \dim U = \frac{1}{2}\dim C_G(u) + \frac{1}{2}\dim C_M(u) + \dim U$. Thus $C_G(u)U$ has the required dimension, as well as each of its irreducible components since it is a homogeneous space for $C_G(u)\times U$. This proves (ii). It remains to prove (i). Clearly $C_G(u)U\subset Y_{u,u}$. As $C_G(u)U$ is a single $C_G(u)\times U$-orbit in $Y_{u,u}$ and $\dim C_G(u)U=\dim Y_{u,u}$, we see that $C_G(u)U$ is open in $Y_{u,u}$. Let C be the conjugacy class of u in G. The morphism π: $G\to C$, $g\mapsto g^{-1}ug$ is open, and so is therefore its restriction to $Y_{u,u}\to u'U\cap C$. In particular $\pi(C_G(u)U)$ is open in $u'U\cap C$. Notice also that $C_G(u)U$ is a union of fibres of π. Let now $x\in Y_{u,u}$. We must show that $x\in C_G(u)U$. We have $\pi(x)=uy$ for some $y\in U$. Let S be the connected centre of M. Then $xS\subset Y_{u,u}$. Moreover $\pi(xS)=\{us^{-1}ys|s\in S\}$, and therefore $u\in\overline{\pi(xS)}$. Thus $\pi(xS)$ meets $\pi(C_G(u)U)$. Therefore $xs\in C_G(u)U$ for some $s\in S$. So $Y_{u,u}=C_G(u)US=C_G(u)SU=C_G(u)U$.

1.5. Let Q be a parabolic subgroup of G with unipotent radical U_Q, and let L be a Levi factor of Q. Suppose that there exists a unipotent element $v\in L$ and $\psi\in A_L(v)$ such that (v,ψ) is cuspidal for L. Then the conjugacy class of Q is determined by L [14]; it follows that $N_G(L)/L$ can be considered in a natural way as a Coxeter group [15] and we can therefore talk about its sign representation ε. Let now $\rho\in(N_G(L)/L)^\wedge$ and let (L, v, ψ, ρ) correspond to (u, ϕ). Let C be the conjugacy class of u in G. We have then:

(V) If $\rho=\varepsilon$, then $C\ni v$. If $\rho=1$, then C is induced from the class of v in L, i.e. C contains a dense open subset of vU_Q.

This is proved in [14].

Remark. Property (V) is very important in the case where $|N_G(L)/L|=2$. It is the starting point to apply (II) in a non-trivial way.

§ 2. Action of W on $H^*(\mathfrak{B}^G)$

This paragraph is devoted to a proof of the result of Borho and MacPherson referred to in (1.1). We have two actions of W on $H^*(\mathfrak{B}^G)$. One of them is the action of W on $H^*(\mathfrak{B}_x^G)$ in the case where $x=1$. The other one, which we shall call the classical action of W on $H^*(\mathfrak{B}^G)$, has been studied quite extensively and has the properties required in (1.1) (see e.g. [28, 7.1]).

Theorem. (Borho-MacPherson). *These two actions of W on $H^*(\mathfrak{B}^G)$ coincide.*

2.1. It is convenient here to define the Weyl group W of G as in [7]. The essential point is that for every pair $B\supset T$ consisting of a Borel subgroup and a maximal torus, there is a canonical isomorphism between W and $N_G(T)/T$.

2.2. The classical action of W on $H^*(\mathfrak{B}^G)$ is obtained as follows. Let $B_0 \supset T_0$ be a Borel subgroup and a maximal torus of G. Then $\mathfrak{B}^G \cong G/B_0$ and the natural morphism $G/T_0 \to G/B_0$ induces an isomorphism $H^*(G/B_0) \cong H^*(G/T_0)$. Right multiplication gives a right action of $N_G(T_0)/T_0$ on G/T_0, hence a left action of W on $H^*(\mathfrak{B}^G)$. It is easily checked that this action is independent of the choice of B_0 and T_0.

2.3. We describe now Lusztig's construction [13]. For simplicity we consider first the analogous situation in the Lie algebra case (as is done in [6]).

Let $\mathfrak{g} = \mathrm{Lie}(G)$. We assume that the open subset $\mathfrak{g}_{rs} = \{x \in \mathfrak{g} \mid x$ is regular and semisimple$\}$ is dense in $\mathfrak{g}$ (this is certainly the case if $\mathrm{char}(k) \neq 2$ or if G is adjoint). If X is a subvariety of $\mathfrak{g}$, let $\tilde{X} = \{(x, B) \in X \times \mathfrak{B}^G \mid x \in \mathrm{Lie}(B)\}$ and let $\pi_X \colon \tilde{X} \to X$, $p_X \colon \tilde{X} \to \mathfrak{B}^G$ be the projections. Let also $\pi = \pi_{\mathfrak{g}}$, $p = p_{\mathfrak{g}}$, $\pi_{rs} = \pi_{\mathfrak{g}_{rs}}$. For $x \in \mathfrak{g}$, $\pi^{-1}(x)$ can be identified with $\mathfrak{B}_x^G = \{B \in \mathfrak{B}^G \mid x \in \mathrm{Lie}(B)\}$.

If $x \in \mathfrak{g}_{rs}$, W acts simply transitively on $\mathfrak{B}_x^G$. Let $B \in \mathfrak{B}_x^G$, $w \in W$. As $x \in \mathfrak{g}_{rs}$, $T = C_G^0(x)$ is a maximal torus of B. Let $\dot{w} \in N_G(T)/T$ represent w. The image of B under the action of w is by definition ${}^{\dot{w}}B$. This defines a right action. If $g \in N_G(T)/T$, the image of ${}^{g}B$ under w is ${}^{g\dot{w}}B$.

This action turns $\tilde{\mathfrak{g}}_{rs}$ into a principal W-bundle over $\mathfrak{g}_{rs}$. It follows that W acts on the local system $E = \pi_{rs*}\overline{\mathbf{Q}}_\ell$ over $\mathfrak{g}_{rs}$, and hence also on the intersection cohomology complex $\mathscr{S}^\bullet$ on $\mathfrak{g}$ which extends E. But one can show in this case that $\mathscr{S}^\bullet$ is quasi-isomorphic to $R\pi_*\overline{\mathbf{Q}}_\ell$. In particular for every $x \in \mathfrak{g}$ the stalk at x of the cohomology $\mathscr{H}^i\mathscr{S}^\bullet$ is isomorphic to $H^i(\mathfrak{B}_G^x)$, and this defines an action of W on $H^*(\mathfrak{B}_x^G)$.

2.4. Following a suggestion of Lusztig, we shall use the following obvious result.

Lemma. *Let $X \subset Y$ be subvarieties of $\mathfrak{g}$. Then the natural map $\mathbf{H}^*(Y; \mathscr{S}^\bullet) \to \mathbf{H}^*(X; \mathscr{S}^\bullet)$ is W-equivariant.*

Notice that for any subvariety X of $\mathfrak{g}$ we have $\mathbf{H}^*(X; \mathscr{S}^\bullet) \cong H^*(\tilde{X})$. In particular this turns $H^*(\tilde{X})$ into a W-module, and in the situation of the lemma the natural map $H^*(\tilde{Y}) \to H^*(\tilde{X})$ is W-equivariant.

2.5. If X is a subvariety of $\mathfrak{g}$, let $i_X \colon \tilde{X} \to \tilde{\mathfrak{g}}$ be the inclusion map. Let $i_0 = i_{\{0\}}$. From (2.4) we get immediately:

Lemma. *Let X be a subvariety of $\mathfrak{g}$. Then $p_X^* \colon H^*(\mathfrak{B}^G) \to H^*(\tilde{X})$ is W-equivariant.*

Proof. Since $p_X = p \circ i_X$, we have $p_X^* = i_X^* \circ p^*$. But p turns $\tilde{\mathfrak{g}}$ into a vector bundle over $\mathfrak{B}^G$. Thus p^* is an isomorphism and $p^*=(i_0^*)^{-1}$. Therefore $p_X^*=i_X^* \circ (i_0^*)^{-1}$ is W-equivariant since by (2.4) both i_X^* and i_0^* are so.

2.6. If $x \in \mathfrak{g}$, $g \in G$, let $g \cdot x = \mathrm{Ad}(g)(x)$. Choose $x_0 \in \mathfrak{g}_{rs}$ and $B_0 \in \mathfrak{B}^G_{x_0}$, and let $C=\{g \cdot x_0 | g \in G\}$ be the conjugacy class of x_0. Let $T_0=C_G^0(x_0) \subset B_0$. Identify W with $N_G(T_0)/T_0$. If $\dot{w} \in N_G(T_0)$ represents $w \in W$, write ${}^wB_0$ for ${}^{\dot{w}}B_0$.

Let $\overline{W}=C_W(x_0)\backslash W$ and for $v \in W$ let $\bar{v}=C_W(x_0)v$. Using the fact that $\pi: \tilde{\mathfrak{g}} \to \mathfrak{g}$ is G-equivariant, one finds easily that for every $v \in W$ there is a unique irreducible component $\tilde{C}_{\bar{v}}$ of $\tilde{C}$ which contains $(x_0, {}^vB_0)$, and that $\tilde{C}$ is the disjoint union of $(\tilde{C}_{\bar{v}})_{\bar{v} \in \overline{W}}$. Let $p_{\bar{v}}: \tilde{C}_{\bar{v}} \to \mathfrak{B}^G$ be the projection.

Lemma. *$p_{\bar{v}}^*: H^*(\mathfrak{B}^G) \to H^*(\tilde{C}_{\bar{v}})$ is an isomorphism.*

Proof. The fiber of $p_{\bar{v}}$ over ${}^vB_0$ is $\{b \cdot x_0 | b \in {}^vB\}$. As $x_0 \in \mathfrak{g}_{rs}$ this is isomorphic to an affine space. Combined with the G-equivariance of $p_{\bar{v}}$, this gives the result.

2.7. Let ρ denote the representation of W in $H^*(\mathfrak{B}^G)$ given by Lusztig's construction. For each $w \in W$ let $t(w): \tilde{C} \to \tilde{C}$ be the restriction of the action of w on $\tilde{\mathfrak{g}}_{rs}$. Let $v \in W$. Then $\tilde{C}_{\bar{v}}=\{(g \cdot x_0, {}^{gv}B_0) | g \in G\}$, and $t(w)(g \cdot x_0, {}^{gv}B_0)=(g \cdot x_0, {}^{gvw}B_0)$. Let $t_{\bar{v}}(w)$ be the restriction of $t(w)$ to $\tilde{C}_{\bar{v}} \to \tilde{C}_{\overline{vw}}$.

Lemma. $p_{\bar{v}}^* \circ \rho(w) = t_{\bar{v}}(w)^* \circ p_{\overline{vw}}^*$.

Proof. This is a restriction of the formula $p_C^* \circ \rho(w)=t(w)^* \circ p_C^*$ which expresses the fact that p_C^* is W-equivariant.

2.8. We can now prove the analogue of the theorem in the Lie algebra case.

Let σ be the right action of W on $\tilde{C}_{\bar{1}}$ defined by $\sigma(w)\,(g \cdot x_0, {}^gB_0)=(gw \cdot x_0, {}^{gw}B_0)$. It is clear that $\tilde{C}_{\bar{1}}$ is isomorphic to G/T_0 and that σ corresponds to the action of W on G/T_0 which gives rise to the classical action of W on $H^*(\mathfrak{B}^G)$.

It is therefore enough to prove that the isomorphism $p_{\bar{1}}^*: H^*(\mathfrak{B}^G) \to H^*(\tilde{C}_{\bar{1}})$ is W-equivariant with respect to the action ρ on $H^*(\mathfrak{B}^G)$ and the action induced by σ on $H^*(\tilde{C}_{\bar{1}})$. That is, we must check that $p_{\bar{1}}^* \circ \rho(w) = \sigma(w)^* \circ p_{\bar{1}}^*$, and this follows from (2.7) and the fact that $p_{\bar{w}} \circ t_{\bar{1}}(w)=p_{\bar{1}} \circ \sigma(w)$.

2.9. We turn now to the proof of the theorem itself. The idea is essentially the same. In particular one can define in a similar way $G_{rs} \subset G$, and for a subvariety X of G one can define $\tilde{X}$, π_X, p_X, etc. We mention first two minor differences.

a) G_{rs} is always open dense in G.

b) G always contains strongly regular semisimple elements, that is, elements s such that $C_G(s)$ is a torus. If the class C is chosen carefully enough in the analogue of (2.6), it is not necessary to introduce $\overline{W}$. The existence of strongly regular semisimple elements in $\mathfrak{g}$ is discussed in [28].

There is however one more serious difference. I don't know if the analogue of (2.5) holds for arbitrary subvarieties of G. However (2.5) is used only in the case where $X=C$, and in order to prove the theorem the following result is sufficient.

Lemma. *Let X be a subvariety of G which is contained in a single conjugacy class of G. Then p_X^*: $H^*(\mathfrak{B}^G)\to H^*(\tilde{X})$ is W-equivariant.*

Proof. If Y is a subvariety of G, let p_Y^{even} be the restriction of p_Y^* to $\bigoplus_{i\geq 0} H^{2i}(\mathfrak{B}^G)\to\bigoplus_{i\geq 0} H^{2i}(\tilde{Y})$. As $\mathfrak{B}^G$ has no odd cohomology, the method used in (2.5) shows that p_X^* is W-equivariant if X is contained in a subvariety Y of G for which p_Y^{even} is bijective.

Let $B\supset T$ be a Borel subgroup and a maximal torus of G, let U be the unipotent radical of B and let ϕ: $B\to T$ be the natural projection. Let $t\in T$ be conjugate to the semisimple part of some element of X. Let Z_1 be an irreducible curve in T containing both t and 1, let $Z=\bigcup_{w\in W} {}^wZ_1$ and let $Y=\bigcup_{g\in G} g(ZU)g^{-1}$. Then $Y\supset X$. We need now only to show that p_Y^{even} is an isomorphism.

Using the inclusion i: $\mathfrak{B}^G\to\tilde{Y}$ given by $\{1\}\subset Y$ and the fact that $p_Y\circ i$: $\mathfrak{B}^G\to\mathfrak{B}^G$ is the identity, we see that we need only to check that $H^{2i}(\mathfrak{B}^G)$ and $H^{2i}(\tilde{Y})$ have the same dimension for all $i\geq 0$. The elements of $\tilde{Y}$ are all of the form $(x, {}^gB)$ with $g^{-1}xg\in ZU$. Consider the map $\delta:\tilde{Y}\to Z\times\mathfrak{B}^G$ given by $\delta(x, {}^gB)=(\phi(g^{-1}xg), {}^gB)$. It is locally trivial, with fibres isomorphic to U. It follows that δ^*: $H^*(Z\times\mathfrak{B}^G)\to H^*(\tilde{Y})$ is an isomorphism. Now $H^0(Z)\cong\mathbf{Q}_\ell$ and $H^i(Z)=0$ for $i\geq 2$. Since $\mathfrak{B}^G$ has no odd cohomology, Künneth formula shows that $H^{2i}(\mathfrak{B}^G)$ and $H^{2i}(\tilde{Y})$ are isomorphic vector spaces for all $i\geq 0$. This proves the lemma, and also the theorem.

§ 3.

We need information about unipotent classes and characters of Weyl groups.

3.1. The irreducible characters of the Weyl groups of type E_n ($n=6, 7, 8$) were computed by Frame [8], [9]. We shall denote an irreducible representation ρ of one of these groups as d_a, where d is the degree of ρ and $a=a_\rho$ (1.1).

For F_4 the characters of the Weyl group were obtained by Kondo [12]. In Kondo's tables there are 3 "isolated" characters of degrees 4, 12, 16. We shall denote them χ_4, χ_{12}, χ_{16} respectively. The remaining characters occur in families and we label $\chi_{i,j}$ the j^{th} character in the family of characters of degree i in Kondo's table.

The integers a_ρ are given in [5].

3.2. Let W be a Weyl group of exceptional type and let W' be a parabolic subgroup of W. Let $\rho \in W$, $\rho' \in W'^\wedge$. In order to use property (II) we need to know $\langle \rho', \mathrm{Res}^W_{W'}(\rho)\rangle_{W'}$. This has been computed by Alvis [2].

3.3. The results we shall need about the unipotent classes of groups of exceptional types and about the groups $A_G(u)$ are all contained in the papers by Mizuno [18] (E_6), [19] (E_7, E_8), Shoji [24] (F_4, char $(k)\neq 2$), Shinoda [21] (F_4, char $(k)=2$). In characteristic 0 the groups $A_G(u)$ are also described by Alekseevski [1].

We shall use Mizuno's notation for the unipotent classes of groups of type E_n ($n=6, 7, 8$). For F_4 we shall use the Bala-Carter notation [4], [20] when $p\neq 2$. We adapt it to the case of characteristic 2 as in [26, p. 29].

3.4. We need also an explicit knowledge of induction of unipotent classes in the case of exceptional group. It was computed by Elashvili in characteristic 0. The results are listed in [26], together with the extra cases occuring in bad characteristic. Another method to work out induction explicitly for exceptional groups is described in [25].

§ 4.

The computation of the generalized Springer correspondence involves the same kind of computations as those done in [3]. The first step is the determination of the representations of the form $\rho^G_{u,1} \in W^\wedge$. This can be used to get information on the sets $X_{u,u'}$ of (1.2) which is useful for the remaining computations.

4.1. Suppose that the class of u in G is induced. We can assume that we are in the same situation as in (1.3), with $M\neq G$. Then $\rho=\rho^G_{u,1}$ can be determined from $\rho^M_{u',1}$ by the generalized Macdonald construction described in [16]. Indeed, by (II) and (III) we know that ρ occurs in the representation σ of W obtained by inducing ρ'. But by (I) used in M and G we have $a_\rho=a_{\rho'}$, and it is known that ρ is the only component of σ with this property.

Remark. This relation between $\rho^G_{u,1}$ and $\rho^M_{u',1}$ was established in [16] by a different method.

4.2. For the remaining cases (I) limits already strongly the possibilities for $\rho^G_{u,1}$. When (II) and (IV) are also taken into account, we are left with only a very small number of cases. For E_8 we have to chose between 1344_{38} and 350_{38} for the unipotent class A_3+A_1, and between 1344_{19} and 5600_{19} for the class $D_5(a_1)+A_2$.

We use the following consequence of (II). In the situation of (1.2), let C be the conjugacy class of u in G and let W' be the Weyl group of M. If $Y_{u,u'}\neq\emptyset$, it is clear that $u'\in\bar{C}$. Thus $u'\in\bar{C}$ if for some $\phi'\in A_M(u')$ we have $\rho^M_{u',\phi'}\in W'^\wedge$ and $\langle\rho^M_{u',\phi'}, \mathrm{Res}^W_{W'}(\rho^G_{u,1})\rangle_{W'}\neq 0$. Notice that for $x\in\bar{C}$ we must have $d_x\geq d_u$.

We use this with M of type E_7. The restriction of 350_{38} to W' involves $189_{17}\in W'^\wedge$. But 189_{17} corresponds to the class $D_4(a_1)+A_1$ of M. In G, for x in the class $D_4(a_1)+A_1$, we have $d_x=32<38$. Thus $\rho^G_{u,1}\neq 350_{38}$ if u is in the class A_3+A_1 of G, and therefore $\rho^G_{u,1}=1344_{38}$. In a similar way the restriction of 5600_{19} involves $280_9\in W'^\wedge$. But in M, 280_9 corresponds to the class $D_6(a_2)+A_1$. Let x be an element of this class. In G the class of x is denoted A_5+A_2 by Mizuno, and we have $d_x=17<19$. Thus for u in the class $D_5(a_1)+A_2$ of G we have $\rho^G_{u,1}\neq 5600_{19}$, hence $\rho^G_{u,1}=1344_{19}$.

Remarks. a) The case of the class A_3+A_1 in a group of type E_8 is treated in a similar way in [3], but the argument is based on a formula, the proof of which is only sketched. Unfortunately Lusztig's proof of this result uses Kazhdan's theorem [11] and requires some restrictions on the characteristic. We shall therefore not use it here. The information we could gain from it here can actually also be deduced from (II) and (IV).

b) In [3] a stronger form of (I) is used, namely it is stated also that a representation of the form $\rho^G_{u,1}$ occurs with multiplicity one in the space of polynomial functions of degree d_u on V. In the case of E_8 this rules 5600_{19} out as a possible $\rho^G_{u,1}$.

This stronger property holds if and only if the closure of the orbit of u is not branched at the origin. The corresponding statement for nilpotent orbits certainly holds since the closure of a nilpotent orbit is a cone, and therefore it holds for unipotent orbits too when the characteristic is good. It turns out eventually to be always true, but in bad characteristic there doens't seem to be such an a priori reason for it.

4.3. The determination of the representations $\rho^G_{u,\phi}$ with $\phi\neq 1$ is also carried out as in [3], where explicit examples of computations are given. In our case there are some additional difficulties due to the fact that we can have many more conjugacy classes of pairs (u, ϕ) than irreducible representations of the Weyl group. But we dispose now of property (IV) which was not used in [3], and it turns out eventually that we have enough information to get the desired result.

Notice also that in (1.2) we can now decide whether $X_{u,u'}$ is empty. This can be tested with $\phi=1$, $\phi'=1$ in property (II). When we look at $\rho^G_{u,\phi}$ with ϕ arbitrary, we can then use property (II) and the sets $X_{u,u'}$ which are empty to rule out many possibilities.

Remark. In [3] an argument related to the characteristic 0 theory is used for the class $D_8(a_3)$ when G is of type E_8. This can be avoided as follows. Let C be this conjugacy class and let $u \in C$. The difficulty occurs only when char $(k)\neq 3$. In this case $A_G(u)\cong S_3$. Let ε (resp. θ) be the sign representation (resp. the 2-dimensional irreducible representation) of $A_G(u)$. Then $\{\rho^G_{u,\varepsilon}, \rho^G_{u,\theta}\}=\{840_{13}, 175_{12}\}$, and we must find the right bijection.

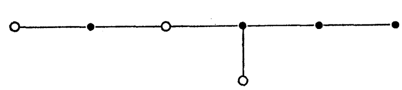
Take a parabolic subgroup of G with a Levi factor M of type D_5+A_2. Then C is induced from the class $(A_2; \emptyset)$ of M. We can therefore use (III). This gives subgroups $H \lhd N$ of $A_G(u)$ with $N/H\cong A_M(u')\cong Z_2$. Up to conjugation there are only two possibilities. If $N=A_G(u)$, then $\rho^G_{u,\varepsilon}=840_{13}$. The characteristic 0 theory was invoked to eliminate the case where $N\neq A_G(u)$. But this second case gives multiplicities which do not work (in particular because the restriction of 175_{12} to the Weyl group of M has no component of the form $\rho^M_{u',\phi'}$) and this can also be used to reject it.

§ 5.

We review now the various groups under consideration and state the results. We start with some notation.

5.1. Let Q be a parabolic subgroup of G, L a Levi factor of Q and v a unipotent element of L. Suppose there exists $\psi \in A_L(v)^\wedge$ such that (v, ψ) is cuspidal for L. Let $Q'\supset Q$ be a parabolic subgroup of G which is minimal among those containing Q strictly. The group $N_{Q'}(L)/L$ has order 2. Let $s_{Q'}$ be its non-trivial element. Then $N_G(L)/L$, together with the set of its elements of the form $s_{Q'}$, is a Coxeter group [15].

We shall use the following kind of notation. If G, L are respectively of types F_4, B_2, we write $s(B_3)$, $s(C_3)$ for the two generators of $N_G(L)/L$. This notation is somewhat ambiguous for example when G is of type E_7 and L of type $(3A_1)''$, that is when L corresponds to the following subdiagram:



In this case $N_G(L)/L$ is isomorphic to a Weyl group of type F_4. Of the four generators, two are of the form $s(4A_1)$ and two of the form $s((A_3+A_1)'')$. But it still makes sense to decide that $s(4A_1)$ corresponds to a short root and $s((A_3+A_1)'')$ to a long root. We can then consider Kondo's tables [12] as the character table of $N_G(L)/L$.

Let $p=\mathrm{char}(k)$. The cases where L is neither G nor a maximal torus are listed in the table below. The third column gives the conditions under which there exists a unipotent element $v \in L$ and $\psi \in A_L(v)^\wedge$ with (v, ψ) cuspidal for L.

G	L	conditions on G and p	$N_G(L)/L$	long roots	short roots
F_4	B_2	$p=2$	B_2	$s(B_3)$	$s(C_3)$
E_6	$2A_2$	simply connected $p\neq 3$	G_2	$s(A_5)$	$s(2A_2+A_1)$
E_6	D_4	$p=2$	A_2	—	—
E_7	$(3A_1)''$	simply connected $p\neq 2$	F_4	$s((A_3+A_1)'')$	$s(4A_1)$
E_7	D_4	$p=2$	B_3	$s(D_5)$	$s(D_4+A_1)$
E_7	E_6	$p=3$	A_1	—	—
E_8	D_4	$p=2$	F_4	$s(D_5)$	$s(D_4+A_1)$
E_8	E_7	$p=2$	A_1	—	—
E_8	E_6	$p=3$	G_2	$s(E_7)$	$s(E_6+A_1)$

5.2. In addition to that introduced in (3.1), we use the following notation for $(N_G(L)/L)^\wedge$.

If $N_G(L)/L$ is of type B_3, we use pairs of partitions to parametrize $(N_G(L)/L)^\wedge$, with $(3, 0)$ for the trivial representation and $(0, 1^3)$ for the sign representation.

Suppose now that $N_G(L)/L$ is of type A_1, A_2, B_2 or G_2. The sign representation is denoted ε. If there are long roots and short roots, let s_ℓ, $s_c \in N_G(L)/L$ be the generators corresponding respectively to a long root and to a short root. Let then ε_ℓ, ε_c be the representations of degree 1 defined by $\varepsilon_\ell(s_\ell)=\varepsilon_c(s_c)=-1$, $\varepsilon_\ell(s_c)=\varepsilon_c(s_\ell)=1$. If $N_G(L)/L$ is of type A_2 or B_2, let θ be the irreducible representation of degree 2. If $N_G(L)/L$ is of type G_2, there are two irreducible representations of degree 2. We shall denote them θ', θ'' without trying to distinguish them (and the θ' arising from $2A_2\subset E_6$ doesn't need to be the same as the θ' arising from $E_6\subset E_8$).

5.3. Unless otherwise stated, we shall assume now that G is simple and simply connected. This assumption holds in particular for the tables.

The corresponding results for adjoint groups of type E_6 or E_7 can be easily recovered.

5.4. We fix now the notation for $A_G(u)^\wedge$. The possibilities for $A_G(u)$ are the following:

$$1,\ Z_2,\ Z_3,\ Z_4,\ Z_5,\ Z_6 \cong Z_2 \times Z_3,\quad Z_2 \times Z_2,\quad S_3,\ S_4,\ S_5,\ Z_2 \times S_3,\ D_8,$$

where Z_n is cyclic of order n, S_n is the permutation group of n letters and D_8 is dihedral of order 8.

If $A_G(u)$ is cyclic of even order, let -1 be the unique irreducible representation which has $\{1, -1\}$ as image (exceptionally this representation will also be denoted ε, when no confusion may arise). For S_3 and D_8 let ε be the sign representation and θ the unique irreducible representation of degree 2. For D_8 there are two additional representations of degree 1. We shall denote them ε', ε'' without trying to distinguish them further. For S_4 (resp. S_5) we use partitions of 4 (resp. 5) to parametrize $A_G(u)^\wedge$, with 1^4 (resp. 1^5) for the sign representation.

When $A_G(u)$ is isomorphic to $Z_2 \times Z_2$, the action of $A_G(u)$ on the set of irreducible components of $\mathfrak{B}_u^G$ factors through Z_2. This defines an irreducible representation ε. The remaining irreducible representations will be denoted ε', ε''.

If $A_G(u)$ is cyclic of order 4, let η be a faithful irreducible representation of $A_G(u)$. Then $A_G(u)^\wedge = \{1, -1, \eta, -\eta\}$.

If $A_G(u)$ is isomorphic to Z_3 or Z_5, let ζ be a non-trivial irreducible representation. Then $A_G(u)^\wedge = \{1, \zeta, \zeta^2\}$ or $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. If $A_G(u) \cong Z_6 \cong Z_3 \times Z_2$, let ζ be a non-trivial irreducible representation which factors through Z_3. Then $A_G(u)^\wedge = \{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$. Suppose now that G is of type E_6 with $\operatorname{char}(k) \neq 3$. Then the centre Z of G is cyclic of order 3. Let $\bar{G} = G/Z$, $\bar{u} = uZ$. If $A_G(u) \neq A_{\bar{G}}(\bar{u})$, then $A_G(u)$ is canonically isomorphic to $A_{\bar{G}}(\bar{u}) \times Z$ (this is an empirical observation). This shows that the choice of $\zeta \in A_G(u)^\wedge$ can be done uniformly.

It remains the case where $A_G(u) \cong Z_2 \times S_3$. Combining the notation above, we have $A_G(u)^\wedge = \{1, \theta, \varepsilon, -1, -\theta, -\varepsilon\}$. This description of $A_G(u)^\wedge$ depends on the isomorphism with $Z_2 \times S_3$. The Z_2 factor is well defined, but there are two possibilities for S_3, and to interchange them has the effect to permute -1 and $-\varepsilon$. We have $A_G(u) \cong Z_2 \times S_3$ in two cases (G of type E_7, u in the class $D_6(a_2)+A_1$, $p \neq 2$; G of type E_8, u in the class $E_7(a_2)+A_1$, $p=2$). In both cases there is a unique $\phi \in A_G(u)^\wedge$ for which (u, ϕ) is cuspidal, and the isomorphism $A_G(u) \cong Z_2 \times S_3$ can be arranged to give $\phi = -\varepsilon$.

5.5. When $N_G(L)/L$ is of type A_1, the result is given directly by property (V).

Suppose now that G is of type E_8, L of type E_6 and $\mathrm{char}(k)=3$. If v is a regular unipotent element of L, then $A_L(v)$ is cyclic of order 3 and (v, ζ), (v, ζ^2) are cuspidal for L. The group $N_G(L)/L$ is of type G_2. Assuming that in (5.4) the choice of $\zeta \in A_G(u)^\wedge$ is made carefully enough when $A_G(u)$ is cyclic of order 3 or 6, the part of the generalized Springer correspondence pertaining to (v, ζ) is given by the following table.

class of u	$A_G(u)$	ϕ	$\rho^G_{u,\phi}$
E_8	Z_3	ζ	1
$E_8(a_1)$	Z_3	ζ	ε_c
E_7+A_1	Z_6	ζ	θ'
E_7	Z_3	ζ	θ''
E_6+A_1	Z_3	ζ	ε_ℓ
E_6	Z_3	ζ	ε

The remaining parts of the generalized Springer correspondence are given by the tables below.

Remarks. a) In comparing with Shoji's results [23] for F_4, the reader should remember to tensor the irreducible representations of W by the sign representation.

b) For F_4, $\mathrm{char}(k)=2$, the properties listed in paragraph 1 are actually not sufficient to compute Springer's correspondence. Let u be an element of the class $F_4(a_3)$. Then $A_G(u)\cong S_3$, $\rho^F_{u,1}=\chi_{12}$, $\chi_{6,2}$ is of the form $\rho^G_{u,\phi}$ for some $\phi \in A_G(u)^\wedge$ and the remaining element of $A_G(u)^\wedge$ gives a cuspidal pair. We have to decide whether $\phi=\theta$ or $\phi=\varepsilon$. A similar problem was encountered in [27]. The same method works here. One can start with the representative x_{17} given by Shinoda [21].

c) For E_6, E_7 and E_8 the computations depend to a large extent on Mizuno's results [18], [19]. For E_8 in characteristic 2, Mizuno's tables give 149 conjugacy classes of pairs (u, ϕ), and following Lusztig there should be only 146. For the class $E_8(a_2)$ (resp. D_4+A_2) Mizuno gives Z_4 (resp. Z_2) for $A_G(u)$, but one can show that actually $A_G(u)=Z_2$ (resp. $A_G(u)=1$). The tables give the corrected values.

G_2

class of u	d_u	$A_G(u)$	$\emptyset$	
			φ	ρ
G_2	0	$Z_{(6,p)}$	1	1
$G_2(a_1)$	1	S_3 $(p\neq 3)$	1	θ'
		Z_2 $(p=3)$	θ	ε_ℓ $(p\neq 3)$
$\tilde{A}_1$	2	1	1	θ''

A_1	3	1	1	ε_c
$(\tilde{A}_1)_3$	3	1	1	ε_ℓ $(p=3)$
$\emptyset$	6	1	1	ε

Here θ' is the standard representation of W.

F_4

class of u	d_u	$A_G(u)$	$\emptyset$ φ	$\emptyset$ ρ	B_2 $(p=2)$ φ	B_2 $(p=2)$ ρ
F_4	0	$Z_{(12,\,p^2)}$	1	$\chi_{1,1}$	-1	1
$F_4(a_1)$	1	Z_2	1	$\chi_{4,1}$	——	
			-1	$\chi_{2,3}$ $(p\neq 2)$	——	
$F_4(a_2)$	2	Z_2 $(p\neq 2)$	1	$\chi_{9,1}$	θ	θ
		D_8 $(p=2)$	-1	$\chi_{2,1}$ $(p\neq 2)$	——	
			ε'	$\chi_{2,1}$ $(p=2)$	——	
			ε''	$\chi_{2,3}$ $(p=2)$	——	
B_3	3	$Z_{(2,\,p)}$	1	$\chi_{8,1}$	-1	ε_c
C_3	3	$Z_{(2,\,p)}$	1	$\chi_{8,3}$	-1	ε_ℓ
$F_4(a_3)$	4	S_4 $(p\neq 2)$	(4)	χ_{12} $(p\neq 2)$	——	
		S_3 $(p=2)$	(31)	$\chi_{9,3}$ $(p\neq 2)$	——	
			(22)	$\chi_{6,2}$ $(p\neq 2)$	——	
			(211)	$\chi_{1,3}$ $(p\neq 2)$	——	
			1	χ_{12} $(p=2)$	——	
			θ	$\chi_{6,2}$ $(p=2)$	——	
$C_3(a_1)$	5	$Z_{(2,\,p-1)}$	1	χ_{16}	——	
			-1	$\chi_{4,3}$ $(p\neq 2)$	——	
B_2	6	$Z_{(2,\,p-1)}$	1	$\chi_{9,2}$	——	
			-1	χ_4 $(p\neq 2)$	——	
$\tilde{A}_2+A_1$	6	1	1	$\chi_{6,1}$	——	
$C_3(a_1)_2$	6	1	1	$\chi_{9,3}$ $(p=2)$	——	
$A_2+\tilde{A}_1$	7	1	1	$\chi_{4,2}$	——	
$(\tilde{A}_2+A_1)_2$	7	1	1	$\chi_{4,3}$ $(p=2)$	——	
$(B_2)_2$	8	Z_2	1	χ_4 $(p=2)$	-1	ε
$\tilde{A}_2$	9	$Z_{(2,\,p)}$	1	$\chi_{8,2}$	——	
			-1	$\chi_{1,3}$ $(p=2)$	——	
A_2	9	Z_2	1	$\chi_{8,4}$	——	
			-1	$\chi_{1,2}$	——	
$A_1+\tilde{A}_1$	10	1	1	$\chi_{9,4}$	——	
$\tilde{A}_1$	13	$Z_{(2,\,p-1)}$	1	$\chi_{4,4}$	——	
			-1	$\chi_{2,2}$ $(p\neq 2)$	——	
$(\tilde{A}_1)_2$	16	1	1	$\chi_{2,2}$ $(p=2)$	——	
A_1	16	1	1	$\chi_{2,4}$	——	
$\emptyset$	24	1	1	$\chi_{1,4}$	——	

E_6

class of u	d_u	$A_G(u)$	$\emptyset$ φ	ρ	$2A_2$ $(p\neq 3)$ φ	ρ	D_4 $(p=2)$ φ	ρ
E_6	0	$Z_{3(2,p)}$	1	1_0	ζ	1	-1	1
$E_6(a_1)$	1	$Z_{(3,p^2-1)}$	1	6_1	ζ	ε_c	—	
D_5	2	$Z_{(2,p)}$	1	20_2	—		-1	φ
A_5+A_1	3	$Z_{2(3,p^2-1)}$	1	30_3	ζ	θ'	—	
			-1	15_5	—		—	
A_5	4	$Z_{(3,p^2-1)}$	1	15_4	ζ	θ''	—	
$D_5(a_1)$	4	1	1	64_4	—		—	
A_4+A_1	5	1	1	60_5	—		—	
D_4	6	$Z_{(2,p)}$	1	24_6	—		-1	ε
A_4	6	1	1	81_6	—		—	
$D_4(a_1)$	7	S_3	1	80_7	—		—	
			θ	90_8	—		—	
			ε	20_{10}	—		—	
A_3+A_1	8	1	1	60_8	—		—	
$2A_2+A_1$	9	$Z_{(3,p^2-1)}$	1	10_9	ζ	ε_l	—	
A_3	10	1	1	81_{10}	—		—	
A_2+2A_1	11	1	1	60_{11}	—		—	
$2A_2$	12	$Z_{(3,p^2-1)}$	1	24_{12}	ζ	ε	—	
A_2+A_1	13	1	1	64_{13}	—		—	
A_2	15	Z_2	1	30_{15}	—		—	
			-1	15_{17}	—		—	
$3A_1$	16	1	1	15_{16}	—		—	
$2A_1$	20	1	1	20_{20}	—		—	
A_1	25	1	1	6_{25}	—		—	
$\emptyset$	36	1	1	1_{36}	—		—	

E_7

class of u	d_u	$A_G(u)$	$\emptyset$ φ	ρ	$3A_1''$ $(p\neq 2)$ φ	ρ	D_4 $(p=2)$ φ	ρ
E_7	0	$Z_{2(6,p)}$	1	1_0	-1	$\chi_{1,1}$	-1	$(3,0)$
$E_7(a_1)$	1	Z_2	1	7_1	-1	$\chi_{2,1}$	-1	$(0,3)$
$E_7(a_2)$	2	Z_2	1	27_2	-1	$\chi_{4,1}$	-1	$(2,1)$
D_6+A_1	3	$Z_2\times Z_{(2,p-1)}$	1	56_3	ε'	$\chi_{8,1}$	—	
			ε	21_6	ε''	$\chi_{1,2}$	—	
E_6	3	$Z_{(6,p)}$	1	21_3	—		-1	$(1,2)$
$E_6(a_1)$	4	Z_2	1	120_4	—		—	

			-1	105_5	——		——	
D_6	4	Z_2	1	35_4	-1	$\chi_{9,1}$	-1	$(21, 0)$
$D_6(a_1)+A_1$	5	$Z^2_{(2,p-1)}$	1	189_5	ε'	$\chi_{4,2}$	——	
			ε	15_7 $(p\neq 2)$	ε''	$\chi_{8,3}$	——	
A_6	6	$Z_{(2,p)}$	1	105_6	——		——	
			-1	15_7 $(p=2)$	——		——	
$D_6(a_1)$	6	Z_2	1	210_6	-1	$\chi_{9,2}$	-1	$(0, 21)$
D_5+A_1	6	Z_2	1	168_6	-1	$\chi_{2,3}$	-1	$(1^2, 1)$
$D_6(a_2)+A_1$	7	$S_3\times Z_{(2,p-1)}$	1	315_7	-1	χ_{12}	——	
			θ	280_9	$-\theta$	$\chi_{6,2}$	——	
			ε	35_{13}	——		——	
D_5	7	$Z_{(2,p)}$	1	189_7	——		-1	$(1, 1^2)$
$(A_5+A_1)'$	8	Z_2	1	405_8	——		——	
			-1	189_{10}	——		——	
$D_6(a_2)$	8	$Z_{(2,p-1)}$	1	280_8	-1	χ_{16}	——	
$(A_5+A_1)''$	9	$Z_{(2,p-1)}$	1	70_9	-1	$\chi_{6,1}$	——	
A_5'	9	1	1	216_9	——		——	
$D_5(a_1)+A_1$	9	$Z_{(2,p-1)}$	1	378_9	-1	χ_4	——	
$D_5(a_1)$	10	Z_2	1	420_{10}	——		——	
			-1	336_{11}	——		——	
A_4+A_2	10	1	1	210_{10}	——		——	
A_4+A_1	11	Z_2	1	512_{11}	——		——	
			-1	512_{12}	——		——	
A_5''	12	$Z_{(2,p-1)}$	1	105_{12}	-1	$\chi_{8,4}$	——	
D_4+A_1	12	Z_2	1	84_{12}	-1	$\chi_{9,3}$	-1	$(1^3, 0)$
A_4	13	Z_2	1	420_{13}	——		——	
			-1	336_{14}	——		——	
$A_3+A_2+A_1$	13	$Z_{(2,p-1)}$	1	210_{13}	-1	$\chi_{4,3}$	——	
A_3+A_2	14	$Z_{(2,p-1)}$	1	378_{14}	——		——	
			-1	84_{15} $(p\neq 2)$	——		——	
$(A_3+A_2)_2$	15	1	1	84_{15} $(p=2)$	——		——	
D_4	15	$Z_{(2,p)}$	1	105_{15}	——		-1	$(0, 1^3)$
$D_4(a_1)+A_1$	15	$Z_2\times Z_{(2,p-1)}$	1	405_{15}	ε'	$\chi_{9,4}$	——	
			ε	189_{17}	ε''	$\chi_{2,4}$	——	
A_3+2A_1	16	$Z_{(2,p-1)}$	1	216_{16}	-1	$\chi_{8,2}$	——	
$D_4(a_1)$	16	S_3	1	315_{16}	——		——	
			θ	280_{18}	——		——	
			ε	35_{22}	——		——	

$(A_3+A_1)'$	17	1	1	208_{17}	——		——
$2A_2+A_1$	18	1	1	70_{18}	——		——
$(A_3+A_1)''$	20	$Z_{(2,p-1)}$	1	189_{20}	-1	$\chi_{4,4}$	——
A_3	21	1	1	210_{21}	——		——
$2A_2$	21	1	1	168_{21}	——		——
A_2+3A_1	21	$Z_{(2,p-1)}$	1	105_{21}	-1	$\chi_{1,3}$	——
A_2+2A_1	22	1	1	189_{22}	——		——
A_2+A_1	25	Z_2	1	120_{25}	——		——
			-1	105_{26}	——		——
$4A_1$	28	$Z_{(2,p-1)}$	1	15_{28}	-1	$\chi_{2,2}$	——
A_2	30	Z_2	1	56_{30}	——		——
			-1	21_{33}	——		——
$3A_1'$	31	1	1	35_{31}	——		——
$3A_1''$	36	$Z_{(2,p-1)}$	1	21_{36}	-1	$\chi_{1,4}$	——
$2A_1$	37	1	1	27_{37}	——		——
A_1	46	1	1	7_{46}	——		——
$\emptyset$	63	1	1	1_{63}	——		——

E_8

class of u	d_u	$A_G(u)$	$\emptyset$		D_4 $(p=2)$	
			φ	ρ	φ	ρ
E_8	0	$Z_{(60,p^2)}$	1	1_0	-1	$\chi_{1,1}$
$E_8(a_1)$	1	$Z_{(12,p^2)}$	1	8_1	-1	$\chi_{2,1}$
$E_8(a_2)$	2	$Z_{(2,p)}$	1	35_2	-1	$\chi_{4,1}$
E_7+A_1	3	$Z_2\times Z_{(6,p)}$	1	112_3	ε'	$\chi_{8,1}$
			ε	28_8	ε''	$\chi_{1,2}$
E_7	4	$Z_{(12,p^2)}$	1	84_4	-1	$\chi_{9,1}$
D_8	4	Z_2	1	210_4	——	
			-1	160_7	——	
$E_7(a_1)+A_1$	5	Z_2	1	560_5	-1	$\chi_{4,2}$
			-1	50_8 $(p\neq 2)$	——	
$E_7(a_1)$	6	$Z_{(2,p)}$	1	567_6	-1	$\chi_{9,2}$
$D_8(a_1)$	6	Z_2 $(p\neq 2)$	1	700_6	θ	$\chi_{8,3}$
		D_8 $(p=2)$	-1	300_8 $(p\neq 2)$	——	
			ε'	300_8 $(p=2)$	——	
			ε''	50_8 $(p=2)$	——	
D_7	7	$Z_{(2,p)}$	1	400_7	-1	$\chi_{2,3}$
$E_7(a_2)+A_1$	7	$S_3\times Z_{(2,p)}$	1	1400_7	-1	χ_{12}

			θ	1008_9	$-\theta$	$\chi_{6,2}$
			ε	56_{19}	——	
A_8	8	S_3	1	1400_8	——	
			θ	1575_{10}	——	
			ε	350_{14}	——	
$E_7(a_2)$	8	$Z_{(2,p)}$	1	1344_8	-1	χ_{16}
E_6+A_1	9	$Z_{(6,p)}$	1	448_9	-1	$\chi_{6,1}$
$D_7(a_1)$	9	$Z_{(2,p-1)}$	1	3240_9	——	
			-1	1050_{10} $(p\neq 2)$	——	
$(D_7(a_1))_2$	10	Z_2	1	1050_{10} $(p=2)$	-1	χ_4
$D_8(a_3)$	10	S_3 $(p\neq 3)$	1	2240_{10}	——	
		Z_2 $(p=3)$	θ	175_{12} $(p\neq 3)$	——	
			ε	840_{13}	——	
D_6+A_1	10	Z_2	1	2268_{10}	——	
			-1	1296_{13}	——	
$E_6(a_1)+A_1$	11	Z_2	1	4096_{11}	——	
			-1	4096_{12}	——	
A_7	11	1	1	1400_{11}	——	
$(A_7)_3$	12	1	1	175_{12} $(p=3)$	——	
E_6	12	$Z_{(6,p)}$	1	525_{12}	-1	$\chi_{8,4}$
$D_7(a_2)$	12	Z_2	1	4200_{12}	——	
			-1	3360_{13}	——	
D_6	12	$Z_{(2,p)}$	1	972_{12}	-1	$\chi_{9,3}$
$E_6(a_1)$	13	Z_2	1	2800_{13}	——	
			-1	2100_{16}	——	
D_5+A_2	13	$Z_{(2,p-1)}$	1	4536_{13}	——	
			-1	840_{14} $(p\neq 2)$	——	
$(D_5+A_2)_2$	14	Z_2	1	840_{14} $(p=2)$	-1	$\chi_{4,3}$
$D_6(a_1)+A_1$	14	$Z_{(2,p-1)}$	1	6075_{14}	——	
			-1	700_{16} $(p\neq 2)$	——	
A_6+A_1	14	1	1	2835_{14}	——	
A_6	15	$Z_{(2,p)}$	1	4200_{15}	——	
			-1	700_{16} $(p=2)$	——	
$D_6(a_1)$	15	$Z_2\times Z_{(2,p)}$	1	5600_{15}	ε'	$\chi_{9,4}$
			ε	2400_{17}	ε''	$\chi_{2,4}$
$2A_4$	16	S_5	(5)	4480_{16}	——	
			(41)	5670_{18}	——	
			(32)	4536_{18}	——	

			(31^2)	1680_{22}	——	
			(2^21)	1400_{20}	——	
			(21^3)	70_{32}	——	
D_5+A_1	16	$Z_{(2,p)}$	1	3200_{16}	-1	$\chi_{8,2}$
A_5+A_2	17	S_3	1	7168_{17}	——	
			θ	5600_{19}	——	
			ε	448_{25}	——	
$D_6(a_2)$	18	Z_2	1	4200_{18}	——	
			-1	2688_{20}	——	
A_5+2A_1	18	Z_2	1	3150_{18}	——	
			-1	1134_{20}	——	
$(A_5+A_1)'$	19	1	1	2016_{19}	——	
$D_5(a_1)+A_2$	19	1	1	1344_{19}	——	
D_5	20	$Z_{(2,p)}$	1	2100_{20}	-1	$\chi_{4,4}$
A_4+A_3	20	1	1	420_{20}	——	
$(A_5+A_1)''$	21	Z_2	1	5600_{21}	——	
			-1	2400_{23}	——	
D_4+A_2	21	$Z_{(2,p-1)}$	1	4200_{21}	——	
			-1	168_{24} $(p\neq 2)$	——	
A_5	22	1	1	3200_{22}	——	
$D_5(a_1)+A_1$	22	1	1	6075_{22}	——	
$A_4+A_2+A_1$	22	1	1	2835_{22}	——	
A_4+A_2	23	1	1	4536_{23}	——	
$(D_4+A_2)_2$	24	Z_2	1	168_{24} $(p=2)$	-1	$\chi_{1,3}$
A_4+2A_1	24	Z_2	1	4200_{24}	——	
			-1	3360_{25}	——	
$D_5(a_1)$	25	Z_2	1	2800_{25}	——	
			-1	2100_{28}	——	
A_4+A_1	26	Z_2	1	4096_{26}	——	
			-1	4096_{27}	——	
$2A_3$	26	1	1	840_{26}	——	
D_4+A_1	28	$Z_{(2,p)}$	1	700_{28}	-1	$\chi_{2,2}$
$D_4(a_1)+A_2$	28	Z_2	1	2240_{28}	——	
			-1	840_{31}	——	
$A_3+A_2+A_1$	29	1	1	1400_{29}	——	
A_4	30	Z_2	1	2268_{30}	——	
			-1	1296_{33}	——	
A_3+A_2	31	$Z_{(2,p-1)}$	1	3240_{31}	——	

			-1	972_{32} $(p\neq 2)$		——
$(A_3+A_2)_2$	32	1	1	972_{32} $(p=2)$		——
$D_4(a_1)+A_1$	32	S_3	1	1400_{32}		——
			θ	1575_{34}		——
			ε	350_{38}		——
A_3+2A_1	34	1	1	1050_{34}		——
D_4	36	$Z_{(2,p)}$	1	525_{36}	-1	$\chi_{1,4}$
$2A_2+2A_1$	36	1	1	175_{36}		——
$D_4(a_1)$	37	S_3	1	1400_{37}		——
			θ	1008_{39}		——
			ε	56_{49}		——
A_3+A_1	38	1	1	1344_{38}		——
$2A_2+A_1$	39	1	1	448_{39}		——
$2A_2$	42	Z_2	1	700_{42}		——
			-1	300_{44}		——
A_2+3A_1	43	1	1	400_{43}		——
A_3	46	1	1	567_{46}		——
A_2+2A_1	47	1	1	560_{47}		——
A_2+A_1	52	Z_2	1	210_{52}		——
			-1	160_{55}		——
$4A_1$	56	1	1	50_{56}		——
A_2	63	Z_2	1	112_{63}		——
			-1	28_{68}		——
$3A_1$	64	1	1	84_{64}		——
$2A_1$	74	1	1	35_{74}		——
A_1	91	1	1	8_{91}		——
$\emptyset$	120	1	1	1_{120}		——

Unipotent cuspidal pairs (u, φ)

G	class of u	$A_G(u)$	φ	condition on p	condition on G
G_2	$G_2(a_1)$	S_3	ε	$p\neq 3$	——
G_2	G_2	Z_2	-1	$p=2$	——
G_2	G_2	Z_3	ζ, ζ^2	$p=3$	——
G_2	$G_2(a_1)$	Z_2	-1	$p=3$	——
F_4	$F_4(a_3)$	S_4	(1^4)	$p\neq 2$	——
F_4	F_4	Z_3	ζ, ζ^2	$p=3$	——
F_4	F_4	Z_4	$\eta, -\eta$	$p=2$	——
F_4	$F_4(a_1)$	Z_2	-1	$p=2$	——

F_4	$F_4(a_2)$	D_8	ε	$p=2$	——
F_4	$F_4(a_3)$	S_3	ε	$p=2$	——
E_6	A_5+A_1	Z_6	$-\zeta, -\zeta^2$	$p\neq 3$	simply connected
E_6	E_6	Z_6	$-\zeta, -\zeta^2$	$p=2$	simply connected
E_6	E_6	Z_3	ζ, ζ^2	$p=3$	——
E_7	$D_6(a_2)+A_1$	$S_3\times Z_2$	$-\varepsilon$	$p\neq 2$	simply connected
E_7	E_7	Z_6	$-\zeta, -\zeta^2$	$p=3$	simply connected
E_7	E_7	Z_4	$\eta, -\eta$	$p=2$	——
E_8	$2A_4$	S_5	(1^5)	——	——
E_8	E_8	Z_5	$\zeta, \zeta^2, \zeta^3, \zeta^4$	$p=5$	——
E_8	E_7+A_1	Z_6	$-\zeta, -\zeta^2$	$p=3$	——
E_8	$E_8(a_1)$	Z_4	$\eta, -\eta$	$p=2$	——
E_8	$D_8(a_1)$	D_8	ε	$p=2$	——
E_8	$E_7(a_2)+A_1$	$S_3\times Z_2$	$-\varepsilon$	$p=2$	——

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