Advanced Studies in Pure Mathematics 6, 1985
Algebraic Groups and Related Topics
pp. 289–316

On the Generalized Springer Correspondence for Classical Groups

G. Lusztig and N. Spaltenstein

In [4], the first named author defined a generalization of Springer's correspondence [10] between unipotent classes of connected reductive algebraic groups and Weyl groups representations. Several explicit computations are also carried out in [4]. These computations extend in particular Shoji's earlier determination [7] of the usual Springer correspondence for $SO_n(k)$ and $Sp_{2n}(k)$, char $(k)\neq 2$. We consider here the following cases.

1) $Sp_{2n}(k)$, char $(k)=2$.
2) $SO_{2n}(k)$, char $(k)=2$.
3) $\mathrm{Spin}_n(k)$, char $(k)\neq 2$, the part of the correspondence which doesn't survive in $SO_n(k)$.
4) $SL_n(k)$ (only for reference purpose, since it is not treated explicitly elsewhere).

In view of the results in [4], this completes the explicit determination of the generalized Springer correspondence for groups of classical type, and almost completes it in general since for exceptional groups it is described in [9] up to minor indeterminacies.

§ 0. Notation and recollections

0.1. In this paper k is an algebraically closed field and G is a connected reductive algebraic group defined over k.

0.2. Let ℓ be a prime, $\ell\neq \mathrm{char}(k)$, and let $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of $\mathbf{Q}_\ell$. If H is a finite group, $H^\wedge$ denotes the set of all isomorphism classes of irreducible representations of H over $\overline{\mathbf{Q}}_\ell$.

0.3. For $x\in G$, let $C_G(x)$ be the centralizer of x in G and let $A_G(x)=C_G(x)/C_G^0(x)$ be its group of components. Let $\mathfrak{N}_G$ be the set of all G-conjugacy classes of pairs (u, φ) with $u\in G$ unipotent and $\varphi\in A_G(u)^\wedge$.

0.4. Let P be a parabolic subgroup of G with unipotent radical U

Received March 28, 1984.

and let M be a Levi factor of P. Let $u \in G$, $u' \in M$ be unipotent. The group $C_G(u) \times C_M(u')U$ acts on

$$Y_{u,u'} = \{x \in G \mid x^{-1}ux \in u'U\}$$

by

$$(g, m) \cdot x = gxm^{-1} \ (g \in C_G(u),\ m \in C_M(u')U,\ x \in Y_{u,u'}).$$

Let $d_{u,u'} = \frac{1}{2}(\dim C_G(u) + \dim C_M(u')) + \dim U$. It is known that $\dim Y_{u,u'} \leqslant d_{u,u'}$. The group $A_G(u) \times A_M(u')$ acts on the set $X_{u,u'}$ of all irreducible components of $Y_{u,u'}$ of dimension $d_{u,u'}$; the corresponding permutation representation is denoted $\varepsilon_{u,u'}$.

We review now some of the results of [4].

The pair (u, φ) is said to be cuspidal if

(1) $$\langle \varphi, \varepsilon_{u,u'} \rangle_{A_G(u)} \neq 0 \Longrightarrow P = G.$$

Given a pair $(u, \varphi) \in \mathfrak{N}_G$, there exist P, M, u' as above and $\varphi' \in A_M(u')^\wedge$ such that

i) (u', φ') is cuspidal for M.

ii) $\langle \varphi \otimes \varphi'^*, \varepsilon_{u,u'} \rangle \neq 0$, where φ'^* is the dual of φ'.

Moreover (P, M, u', φ') is unique up to conjugacy, and one associates to (u, φ) an irreducible representation $\rho^G_{u,\varphi} \in (N_G(M)/M)^\wedge$. This gives a map

$$\rho^G : \mathfrak{N}_G \longrightarrow \coprod (N_G(M)/M)^\wedge$$

where the disjoint union is over all the conjugacy classes of 4–tuples (P, M, u', φ') with P, M, u' as above and $\varphi' \in A_M(u')^\wedge$ such that (u', φ') is cuspidal for M, or equivalently, as it turns out, over all the conjugacy classes of triples (M, u', φ') with (u', φ') cuspidal for M and M a Levi factor of some parabolic subgroup of G. The generalized Springer correspondence is this map ρ^G. It is a bijection.

We shall use in particular the following properties of ρ^G [4, 9.5].

(2) If $(u, \varphi) \in \mathfrak{N}_G$ maps to the trivial representation $1 \in (N_G(M)/M)^\wedge$ in the copy corresponding to (M, u', φ'), then the class of u in G is induced from the class of u' in M in the sense of [6].

(3) If $(u, \varphi) \in \mathfrak{N}_G$ maps to the sign representation $\varepsilon \in (N_G(M)/M)^\wedge$ in the copy corresponding to (M, u', φ'), then u is conjugate to u'.

Our main tool for the identification of the generalized Springer correspondence is the restriction formula [4, 8.3]. Let L be a Levi factor of some parabolic subgroup of M (and hence of G), let $v \in L$ be unipotent and let $\psi \in A_L(v)^\wedge$. Assume that (v, ψ) is cuspidal for L and that both $(u, \varphi) \in \mathfrak{N}_G$ and $(u', \varphi') \in \mathfrak{N}_M$ correspond to (L, v, ψ). Then

$$(4) \qquad \langle \varphi\otimes\varphi'^{*},\ \varepsilon_{u,u'}\rangle_{A_G(u)\times A_M(u')}=\langle \mathrm{Res}\ \rho^{G}_{u,\varphi},\ \rho^{M}_{u',\varphi'}\rangle_{N_M(L)/L},$$

where Res $\rho^G_{u,\varphi}$ is the restriction of $\rho^G_{u,\varphi}$ from $N_G(L)/L$ to $N_M(L)/L$ and φ'^* is the dual of φ'. Moreover every irreducible representation of $A_G(u)\times A_M(u')$ which occurs in $\varepsilon_{u,u'}$ can be obtained in this way.

0.5. Let $n\in\mathbf{N}$. We write partitions of n as sequences

$$\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_m) \quad\text{with } \lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_m \quad (\lambda_i\geqslant 0).$$

0.6. For $n\geqslant 1$ let W_n be a Weyl group of type B_n. Let also $W_0=1$. For $n\geqslant 2$ let $W'_n\subset W_n$ be a Weyl group of type D_n. Let also $W'_0=W'_1=1$. For $n<0$ let $W_n^\wedge=\emptyset$.

The set $W_n^\wedge$ is parametrized in a natural way by ordered pairs of partitions (α, β) with $\Sigma\alpha_i+\Sigma\beta_i=n$. We use the convention that the trivial representation corresponds to (α, β) with $\alpha=(n)$, and that the sign representation corresponds to (α, β) with $\beta=(1, 1, \cdots, 1)$ (n terms).

Let $W_n^{\wedge\prime}$ be the quotient of $W_n'^{\wedge}$ by the natural action of W_n/W'_n. The parametrization of $W_n^\wedge$ by pairs of partitions induces a parametrization of $W_n^{\wedge\prime}$ by unordered pairs of partitions $\{\alpha, \beta\}$. Moreover $\{\alpha, \beta\}$ corresponds to exactly one (resp. two) element(s) of $W_n'^{\wedge}$ if and only if $\alpha\neq\beta$ (resp. $\alpha=\beta$), and we say in this case that $\{\alpha, \beta\}$ and the corresponding elements of $W_n^{\wedge\prime}$ and $W_n'^{\wedge}$ are non-degenerate (resp. degenerate).

0.7. Let $G=\mathrm{Sp}_{2n}$, char$(k)=2$ and let M be a Levi factor of some parabolic subgroup of G. Then M has a cuspidal pair (v, ψ) ($v\in M$ unipotent, $\psi\in A_M(u)^\wedge$) if and only if it is of type C_m for some m of the form $d(d-1)$ $(d\geqslant 1)$, and up to conjugacy it has at most one. As $\{d(d-1)|d\geqslant 1\}$ $\{d(d-1)|d\in\mathbf{Z},\ d \text{ odd}\}$, the generalized Springer correspondence for G is a bijection [4, 10.5]

$$\mathfrak{N}_G\longrightarrow \bigcup_{\substack{d\in\mathbf{Z}\\ d\text{ odd}}} W^\wedge_{n-d(d-1)}.$$

0.8. Let $G=\mathrm{SO}_{2n}$, char$(k)=2$ and let M be a Levi factor of some parabolic subgroup of G. Then M has a cuspidal pair (v, ψ) ($v\in M$ unipotent, $\psi\in A_M(v)^\wedge$) if and only if M is of type D_m for some m of the form d^2 with $d\geqslant 0$ even, and up to conjugacy it has at most one. (In [4, 10.7] the condition that d is even was erroneously omitted.) The generalized Springer correspondence for G is thus a bijection [4, 10.7]

$$(1) \qquad \mathfrak{N}_G\longrightarrow W_n'^{\wedge}\cup\Big(\bigcup_{\substack{d>0\\ d\text{ even}}} W^\wedge_{n-d^2}\Big).$$

Let $\tilde{G}=\mathrm{O}_{2n}$. Then $\tilde{G}/G$ acts on $\mathfrak{N}_G$ and on the set of all unipotent classes of G. We say that an element of $\mathfrak{N}_G$ or a unipotent class of G is non-degenerate (resp. degenerate) if it is fixed (resp. not fixed) by this action. It is known that $(u,\varphi) \in \mathfrak{N}_G$ is degenerate if and only if u is degenerate, and in this case $A_G(u)=1$ and $\varphi=1$. Let $\mathfrak{N}'_G$ be the quotient of $\mathfrak{N}_G$ by $\tilde{G}/G$. Then (1) induces a bijection

$$\mathfrak{N}'_G \longrightarrow W_n^{\wedge\prime} \cup \Big(\bigcup_{\substack{d>0\\ d \text{ even}}} W_{n-d^2}^{\wedge}\Big). \tag{2}$$

We shall actually describe (2) instead of (1). The passage from (2) to (1) is easily done in terms of Richardson classes and Macdonald representations, using [6, 3.5] and [8, II.7.6].

0.9. For Sp_{2n} (resp. SO_{2n}), char $(k)=2$, the generalized Springer correspondence gives natural partitions

$$\mathfrak{N}_G = \bigcup_{\substack{d\in\mathbf{Z}\\ d \text{ odd}}} \mathfrak{N}_d, \qquad A_G(u)^\wedge = \bigcup_{\substack{d\in\mathbf{Z}\\ d \text{ odd}}} A_G(u)_d^\wedge$$

$$\text{(resp. } \mathfrak{N}_G = \bigcup_{\substack{d\in\mathbf{N}\\ d \text{ even}}} \mathfrak{N}_d, \qquad A_G(u)^\wedge = \bigcup_{\substack{d\in\mathbf{N}\\ d \text{ even}}} A_G(u)_d^\wedge).$$

0.10. Let $G=\mathrm{Spin}_n$, char $(k)\neq 2$. There is a natural isogeny $\pi: G \to \mathrm{SO}_n$. The unipotent classes in G are mapped bijectively to the unipotent classes in SO_n. If $u \in G$ is unipotent, the sizes of the Jordan blocks of $\pi(u)$ give a partition λ of n. The partitions λ which occur in this way are exactly those which for each even integer $m \geqslant 2$ have an even number of parts equal to m. Moreover such a partition λ characterizes a single unipotent class in G, except if all its parts are even. In this last case λ corresponds to two unipotent classes of G, and the union of the images under π of these classes is a single conjugacy class in O_n.

Let Z be the centre of G and let Z_0 be the kernel of π. There are natural partitions [4, 14.2]

$$\mathfrak{N}_G = \bigcup_{\chi\in Z^\wedge} \mathfrak{N}_\chi, \qquad A_G(u)^\wedge = \bigcup_{\chi\in Z^\wedge} A_G(u)_\chi^\wedge.$$

The restriction of the generalized Springer correspondence to $\bigcup_{\chi(Z_0)=1} \mathfrak{N}_\chi$ is essentially the same as the generalized Springer correspondence for SO_n. We fix now a representation $\chi \in Z^\wedge$ with $\chi|_{Z_0}\neq 1$. The subset $\mathfrak{N}_\chi$ of $\mathfrak{N}_G$ is described in [4, 14.4]. In particular the map which associated to $(u, \varphi) \in \mathfrak{N}_G$ the partition $\lambda(u)$ gives a bijection between $\mathfrak{N}_\chi$ and the set X_n of all partitions λ of n such that:

i) for each even integer $m \geqslant 2$, λ has an even number of parts equal to m.

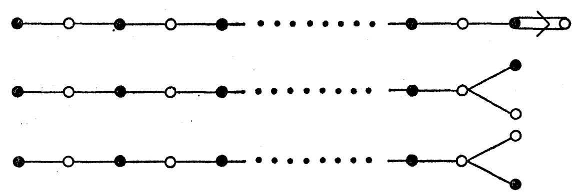
ii) for each odd integer m, λ has at most one part equal to m.

Let M be a Levi factor of a parabolic subgroup of G. Then up to conjugacy there exists at most one pair $(u, \psi) \in \mathfrak{N}_M$ which is cuspidal and such that the subset of $\mathfrak{N}_G$ corresponding to (M, v, ψ) is contained in $\mathfrak{N}_\chi$. Moreover up to conjugacy there exists exactly one such triple (M, v, ψ) for each integer $d \in \mathbf{Z}$ such that $d \equiv n \pmod 4$ and $d(2d-1) \leqslant n$. If $d \in \mathbf{Z}$ has these properties, the corresponding subgroup M of G is of type $B_{(d-1)(2d+1)/2}+A_1+A_1+\cdots$ if n is odd, $D_{d(2d-1)/2}+A_1+A_1+\cdots$ if n is even, where the number of factors A_1 is $\frac{1}{4}(n-d(2d-1))$. For $d \in \{0, 1\}$ these expressions must be understood as follows. The Dynkin diagram of M consists of the dots marked in black in the Dynkin diagram of G:

$d=1$

$d=0$

or



(if $d=0$ only one of these diagrams actually occurs; which one depends on χ).

The restriction to $\mathfrak{N}_\chi$ of the generalized Springer correspondence for G is therefore a bijection [4, 14.6]

$$\mathfrak{N}_\chi \longrightarrow \bigcup_{\substack{d \in \mathbf{Z} \\ 4 \mid n-d}} W^{\wedge}_{(n-d(2d-1))/4}.$$

We shall describe combinatorially the corresponding bijection

$$X_n \longrightarrow \bigcup_{\substack{d \in \mathbf{Z} \\ 4 \mid n-d}} W^{\wedge}_{(n-d(2d-1))/4}. \tag{1}$$

0.11. If we want to check whether a pair $(v, \varphi) \in \mathfrak{N}_G$ is cuspidal, it is usually not necessary to test 0.4 (1) with all parabolic subgroups of G. For example, if $G = \mathrm{Sp}_{2n}$ (resp. SO_{2n}), choose M of type C_{n-1} (resp. D_{n-1}). Suppose that $\langle \varepsilon_{u,u'}, \varphi \rangle_{A_G(u)} = 0$ for every unipotent element $u' \in M$. Then (u, φ) is cuspidal. Similarly, if $G = \mathrm{Spin}_{2n}$ (resp. Spin_{2n+1}) and $(u, \varphi) \in \mathfrak{N}_\chi$, with χ as in 0.10 satisfying $\chi|_{Z_0} \neq 1$, let M be of type $D_{n-2}+A_1$ (resp. $B_{n-2}+A_1$). Suppose that $\langle \varepsilon_{u,u'}, \varphi \rangle_{A_G(u)} = 0$ for every unipotent element $u' \in M$ which contributes to the part pertaining to χ of the generalized Springer correspondence for M. Then (u, φ) is cuspidal.

These are special cases of the following general fact. Let $(M_i)_{i \in I}$ be a family of Levi factors of proper parabolic subgroups of G. Assume that this family is such that whenever L is a Levi factor of some proper parabolic subgroup of G for which $\mathfrak{N}_L$ contains a cuspidal pair (pertaining to some fixed $\chi \in Z^\wedge$ [4, 14.2]), then M is contained in a conjugate of one of the M_i's. Then for a pair $(u, \varphi) \in \mathfrak{N}_G$ (pertaining to χ) to be cuspidal, it is sufficient that it satisfies the condition $\langle \varphi, \varepsilon_{u,u'} \rangle_{A^G(u)} = 0$ for every unipotent element $u' \in M$ when M runs over the family $(M_i)_{i \in I}$.

In each of the cases discussed above, we have chosen one special M, and it follows immediately from the results in [4, §10, 14] that we can take for $(M_i)_{i \in I}$ the family consisting of this single subgroup M.

§ 1. Some combinatorics

1.1. Let $r, s, n \in \mathbf{N}$, $d \in \mathbf{Z}$, $e = \left[\frac{d}{2}\right] \in \mathbf{Z}$, and let $\tilde{X}^{r,s}_{n,d}$ be the set of all ordered pairs (A, B) of finite sequences of natural integers A: $a_1, \cdots, a_{m+d}$ and B: $b_1, \cdots, b_m$ (for some m) satisfying the following conditions:

(1) $$a_i - a_{i-1} \geqslant r+s \qquad (1 < i \leqslant m+d)$$

(2) $$b_i - b_{i-1} \geqslant r+s \qquad (1 < i \leqslant m)$$

(3) $$b_1 \geqslant s$$

(4) $$\Sigma a_i + \Sigma b_i = n + r(m+e)(m+d-e-1) + s(m+e)(m+d-e)$$

Notice that (3) is void if $s=0$. If d is odd, (4) becomes

(4′) $$\Sigma a_i + \Sigma b_i = n + r(m+e)^2 + s(m+e)(m+e+1).$$

If d is even and $s=0$, (4) becomes

(4″) $$\Sigma a_i + \Sigma b_i = n + r(m+e)(m+e-1).$$

The set $\tilde{X}^{r,s}_{n,d}$ is equipped with a shift $\sigma_{r,s}$. If (A, B) is as above, then $\sigma_{r,s}(A, B) = (A', B')$ where A': $a'_1, \cdots, a'_{m+d+1}$ and B': $b'_1, \cdots, b'_{m+1}$ are defined as follows:

$$a'_i = \begin{cases} 0 & \text{if } i=1 \\ a_{i-1}+r+s & \text{if } 2 \leqslant i \leqslant m+d+1 \end{cases}$$

$$b'_i = \begin{cases} s & \text{if } i=1 \\ b_{i-1}+r+s & \text{if } 2 \leqslant i \leqslant m+1 \end{cases}$$

Let $X^{r,s}_{n,d}$ be the quotient of $\tilde{X}^{r,s}_{n,d}$ by the equivalence relation generated by the shift and let

$$X_n^{r,s} = \bigcup_{d\,\text{odd}} X_{n,d}^{r,s}.$$

The equivalence class of (A, B) is still denoted (A, B).

Consider now the special case where $s=0$. It is clear that $(A, B) \mapsto (B, A)$ defiines a bijection from $X_{n,d}^{r,0}$ to $X_{n,-d}^{r,0}$, and an involution of each of the following sets:

$$X_{n,\text{even}}^r = \bigcup_{d\,\text{even}} X_{n,d}^{r,0}, \quad X_{n,\text{odd}}^r = \bigcup_{d\,\text{odd}} X_{n,d}^{r,0} \quad (=X_n^{r,0}).$$

Let $Y_{n,\text{even}}^r$ (resp. $Y_{n,\text{odd}}^r$) be the quotient of $X_{n,\text{even}}^r$ (resp. $X_{n,\text{odd}}^r$) by this involution. For $d \geqslant 0$, the image of $X_{n,\pm d}^{r,0}$ in $Y_{n,\text{even}}^r$ or $Y_{n,\text{odd}}^r$ is denoted $Y_{n,d}^r$, and the image of (A, B) is denoted $\{A, B\}$. When we consider one of the sets $Y_{n,\text{even}}^r$ or $Y_{n,\text{odd}}^r$, we set conventionally $s=0$.

When we consider simultaneously two elements $(A, B) \in X_{n,d}^{r,s}$ and $(A', B') \in X_{n',d'}^{r',s'}$ (with $d-d'$ even), with

$$A\colon a_1, \cdots, a_{m+d}, \qquad B\colon b_1, \cdots, b_m$$
$$A'\colon a'_1, \cdots, a'_{m'+d'}, \qquad B'\colon b'_1, \cdots, b'_{m'},$$

we assume always that we have chosen representatives such that $2m+d=2m'+d'$. In particular $m=m'$ if $d=d'$. The same convention holds for pairs $\{A, B\} \in Y_{n,d}^r$ and $\{A', B'\} \in Y_{n',d'}^{r'}$, with $d, d' \geqslant 0$ and $d-d'$ even.

1.2. Examples. a) $X_{n,d}^{0,0}$ is obviously in bijection with the set of all pairs of partitions (α, β) with $\Sigma\alpha_i + \Sigma\beta_i = n$, and therefore with $W_n^\wedge$. The same holds for $Y_{n,d}^0$ if $d \geqslant 1$.

b) $Y_{n,0}^0$ is obviously in bijection with the set of all unordered pairs of partitions $\{\alpha, \beta\}$ with $\Sigma\alpha_i + \Sigma\beta_i = n$, and therefore with $W_n^{\wedge\prime}$.

c) $Y_{n,d}^1$ is the set of all symbols of rank n and defect d, as defined in [5] ($d \geqslant 0$).

d) $X_n^{1,1}$, $Y_{n,\text{even}}^2$ and $Y_{n,\text{odd}}^2$ are used in [4] to describe the generalized Springer correspondence for Sp_{2n}, SO_{2n} and SO_{2n+1} respectively, in the case where char$(k) \neq 2$.

e) Let $n_0 = n_0(d, r, s) = re(d-e) + se(d-e-1)$. Then $X_{n_0,d}^{r,s}$ consists of a single element $\Lambda_{n_0,d}^{r,s}$ which is represented by the following pair (A, B). If $d>0$, A is the sequence $0, r+s, \cdots, (d-1)(r+s)$ and B is the empty sequence. If $d<0$, A is the empty sequence and B is the sequence $s, s+(r+s), \cdots, s+(-d-1)(r+s)$. If $d=0$, both A and B are the empty sequence.

If moreover $d \geqslant 0$ and $s=0$, then $Y_{n_0,d}^r$ also consists of a single element $\Lambda_{n_0,d}^r$ which is the image of $\Lambda_{n_0,d}^{r,0}$.

1.3. There is an obvious addition

$$X_{n,d}^{r,s}\times X_{n',d}^{r',s'}\longrightarrow X_{n+n',d}^{r+r',s+s'}$$

with $(A'', B'')=(A, B)+(A', B')$ defined by $a_i''=a_i+a_i', b_i''=b_i+b_i'$.

If $d>0$, the same formula defines an addition

$$Y_{n,d}^{r}\times Y_{n',d}^{r'}\longrightarrow Y_{n+n',d}^{r+r'}.$$

In particular we have a map

(1)
$$\begin{aligned} X_{n-n_0,d}^{0,0} &\longrightarrow X_{n,d}^{r,s} \\ \Lambda &\longmapsto \Lambda+\Lambda_{n_0,d}^{r,s} \end{aligned}$$

where $n_0=n_0(d, r, s)$ (1.2 (e)). Similarly, for $d\geqslant 0$ we get a map

(2)
$$\begin{aligned} Y_{n-n_0,d}^{0} &\longrightarrow Y_{n,d}^{r} \\ \Lambda &\longmapsto \Lambda+\Lambda_{n_0,d}^{r} \end{aligned}$$

where $n_0=n_0(d, r, 0)$. It is easily checked that (1) and (2) are bijections. Combined with 1.2(a), this gives bijections

(3) $$W_{n-n_0}^{\wedge}\longrightarrow X_{n,d}^{r,s}$$

(4) $$W_{n-n_0}^{\wedge}\longrightarrow Y_{n,d}^{r}\ (d>0),\qquad W_{n-n_0}^{\wedge\prime}\longrightarrow Y_{n,d}^{r}\ (d=0),$$

with $n_0=n_0(d, r, s)$ for (3) and $d\geqslant 0$, $n_0=n_0(d, r, 0)$ for (4).

1.4. We say that $(A, B)\in X_{n,d}^{r,s}$ is *distinguished* if $d=0$ and $a_1\leqslant b_1\leqslant a_2\leqslant b_2\leqslant\cdots\leqslant a_m\leqslant b_m$, or $d=1$ and $a_1\leqslant b_1\leqslant a_2\leqslant b_2\leqslant\cdots\leqslant b_m\leqslant a_{m+1}$. If $d\geqslant 0$, we say that $\{A, B\}\in Y_{n,d}^{r}$ is distinguished if (A, B) or (B, A) is distinguished. Let $D_n^{r,s}$, $D_{n,\mathrm{even}}^{r}$, $D_{n,\mathrm{odd}}^{r}$ be the sets of all distinguished elements in $X_n^{r,s}$, $Y_{n,\mathrm{even}}^{r}$, $Y_{n,\mathrm{odd}}^{r}$ respectively.

Assume now $r\geqslant 1$. If $(A, B)\in \tilde{X}_{n,d}^{r,s}$, we can think of A and B as subsets of $\mathbf{N}$. We say then that the elements (A, B), (C, D) of $X_n^{r,s}$ are similar if $A\cup B=C\cup D$ and $A\cap B=C\cap D$. Similarity in $Y_{n,\mathrm{even}}^{r}$ or $Y_{n,\mathrm{odd}}^{r}$ is defined in the same way.

These two notions are related as follows.

Lemma. *Assume $r\geqslant 1$, and let $\mathscr{C}$ be a similarity class in $X_n^{r,s}$, $Y_{n,\mathrm{even}}^{r}$ or $Y_{n,\mathrm{odd}}^{r}$. Then*:

(a) *$\mathscr{C}$ contains a unique distinguished element Λ.*

(b) *$\mathscr{C}$ is in a natural way a vector space over $\mathbf{F}_2$, with origin Λ.*

Let (A, B) or $\{A, B\}$ be an element of $\mathscr{C}$. For (a), write the elements of A and B in increasing order

$$(1) \qquad c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{2m+d}$$

and let $A'=\{c_i \mid i \text{ odd}\}$, $B'=\{c_i \mid i \text{ even}\}$. Then $\Lambda=(A', B')$ (if $\mathscr{C} \subset X_n^{r,s}$) or $\Lambda=\{A', B'\}$ (if $\mathscr{C}$ is a similarity class in $Y_{n,\mathrm{even}}^r$ or $Y_{n,\mathrm{odd}}^r$) is clearly the only distinguished element in $\mathscr{C}$.

For (b) we introduce the notion of interval. Let $S=(A \cup B)\backslash(A \cap B)$. A non-empty subset I of S is an *interval* of (A, B) if it satisfies the following conditions.

i) If $i<j$ are consecutive elements of I, then $j-i<r+s$.

ii) If $i \in I$, $j \in S$ and $|i-j|<r+s$, then $j \in I$.

We say that I is an initial interval if there exists $i \in I$ with $i<s$, and that I is a proper interval otherwise. If $s=0$ all intervals are proper. If $s>0$, after a shift there exists always an initial interval, which is obviously unique.

If I is an interval and $(C, D) \in \mathscr{C}$, then either

$$(2) \qquad A \cap I = C \cap I \quad \text{and} \quad B \cap I = D \cap I$$

or

$$(3) \qquad A \cap I = D \cap I \quad \text{and} \quad B \cap I = C \cap I.$$

Moreover both possibilities occur if I is proper, and only (2) occurs if I is an initial interval. Let E be the set of all proper intervals of (A, B) or $\{A, B\}$. The set $\mathfrak{P}(E)$ of all subsets of E is a vector space over $\mathbf{F}_2$. If $\mathscr{C} \subset X_n^{r,s}$, it operates simply transitively on $\mathscr{C}$ as follows. The image of (A, B) under $F \subset E$ is the pair $(C, D) \in C$ defined by the requirement that for every $I \in E$, (3) holds if and only if $I \in F$. This turns $\mathscr{C}$ into a vector space over $\mathbf{F}_2$ with origin Λ. If $\mathscr{C} \subset Y_{n,\mathrm{even}}^r$ (resp. $Y_{n,\mathrm{odd}}^r$), the same formula defines a simply transitive action of $\mathfrak{P}(E)$ on the preimage of $\mathscr{C}$ in $X_{n,\mathrm{even}}^r$ (resp. $X_{n,\mathrm{odd}}^r$). As E itself transforms (A, B) into (B, A), we get a simply transitive action of $\mathfrak{P}(E)/\{\emptyset, E\}$ on $\mathscr{C}$, which turns $\mathscr{C}$ into a vector space over $\mathbf{F}_2$ with origin Λ. This proves (b).

If Λ is the element of $\mathscr{C}$ given by (a) and $\mathscr{C} \subset X_n^{r,s}$ (resp. $Y_{n,\mathrm{even}}^r \cup Y_{n,\mathrm{odd}}^r$), let $\mathscr{V}_\Lambda^{r,s}$ (resp. $\mathscr{V}_\Lambda^r$) denote the similarity class $\mathscr{C}$ equipped with its structure of vector space. Moreover, if E is as in the proof, let $V_\Lambda^{r,n}$ (resp. V_Λ^r) denote the vector space $\mathfrak{P}(E)$ (resp. $\mathfrak{P}(E)/\{\emptyset, E\}$), and for $F \in V_\Lambda^{r,s}$ (resp. V_Λ^r) let Λ_F be the image of Λ under the action of F.

§ 2. Symplectic groups in characteristic 2

In this paragraph $\mathrm{char}(k)=2$ and $G=\mathrm{Sp}_{2n}$. We describe combinatorially the map (0.7)

$$\mathfrak{N}_G \longrightarrow \bigcup_{\substack{d \in \mathbf{Z} \\ d \text{ odd}}} W^{\wedge}_{n-d(d-1)}$$

2.1. Let $u \in G$ be unipotent. The conjugacy class C of u is characterized by the following data ([8]; see also [3], [11]).

a) The sizes of the Jordan blocks of u give a partition λ of $2n$. We write it as $(\lambda_1, \lambda_2, \cdots, \lambda_{2m+1})$, with $\lambda_1 = 0$.

b) For each even integer $i \geqslant 2$, if there exists j such that $\lambda_j = i$, we must specify a number $\varepsilon(i) \in \{0, 1\}$; however, we must take $\varepsilon(i) = 1$ if the number of j's such that $\lambda_j = i$ is odd.

Let ω be some fixed object, $\omega \notin \{0, 1\}$. It is convenient to set $\varepsilon(0) = 1$ and $\varepsilon(i) = \omega$ if $i \geqslant 1$ and $\varepsilon(i)$ is not given by (b).

We can partition $\{1, 2, \cdots, 2m+1\}$ in a unique way into blocks of length 1 or 2 in such a way that the following hold.

i) If $\varepsilon(\lambda_i) = 1$, then $\{i\}$ is a block.

ii) All the other blocks consist of two consecutive integers.

Notice that if $\{i, i+1\}$ is a block, then $\lambda_i = \lambda_{i+1}$.

We attach to C the sequence $c_1, c_2, \cdots, c_{2m+1}$ defined as follows.

(1) If $\{i\}$ is a block, then

$$c_i = \lambda_i/2 + 2(i-1).$$

(2) If $\{i, i+1\}$ is a block and $\varepsilon(\lambda_i) = \omega$, then

$$c_i = (\lambda_i + 1)/2 + 2(i-1)$$
$$c_{i+1} = (\lambda_{i+1} - 1)/2 + 2i = c_i + 1.$$

(3) If $\{i, i+1\}$ is a block and $\varepsilon(\lambda_i) = 0$, then

$$c_i = (\lambda_i + 2)/2 + 2(i-1)$$
$$c_{i+1} = (\lambda_{i+1} - 2)/2 + 2i = c_i.$$

Taking $a_i = c_{2i-1}$ $(1 \leqslant i \leqslant m+1)$, $b_i = c_{2i}$ $(1 \leqslant i \leqslant m)$, we get a pair of sequences (A, B) which gives a well-defined element of $X^{2,2}_{n,1}$. We denote it $\rho(u)$, $\rho(C)$ or $\rho_G(u)$, depending on the needs of the context.

2.2. Lemma. (a) *$C \mapsto \rho(C)$ is a bijection from the set of all unipotent classes in* Sp_{2n} *to* $D^{2,2}_n$.

(b) *$A_G(u)^{\wedge}$ is canonically isomorphic to* $V^{2,2}_{\rho(u)}$.

(a) That $\rho(C) \in D^{2,2}_n$ is easily checked from the definitions, as well as the injectivity of $C \mapsto \rho(C)$. Suppose now that $(A, B) \in D^{2,2}_n$. The corresponding unipotent class of G is obtained as follows. Let $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{2m+1}$ denote the sequence $a_1 \leqslant b_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2m+1}$. If $c_i = c_{i+1}$, then

$\{i, i+1\}$ is a block with $\varepsilon(\lambda_i)=0$, and λ_i, λ_{i+1} are recovered from 2.1 (3). If $c_i=c_{i+1}-1$, then $\{i, i+1\}$ is a block with $\varepsilon(\lambda_i)=\omega$, and λ_i, λ_{i+1} are recovered from 2.1 (2). All the blocks of length 2 are obtained in this way, and for the others we use 2.1 (1).

For (b) we describe first $A_G(u)$. It is canonically isomorphic to the abelian group generated by the set $\{s_i | \varepsilon(\lambda_i)\neq 0\}$ with the following relations:

(1) $s_i^2=1$

(2) $s_i=s_j$ if $\lambda_i=\lambda_j$, or if λ_i is even and $|\lambda_i-\lambda_j|\leqslant 2$.

(3) $s_i=1$ if $\lambda_i=0$.

Let A, B and $c_1, \cdots, c_{2m+1}$ be as in 2.1. As in 1.4 let $S=(A\cup B)\setminus(A\cap B)$. Then $\varepsilon(\lambda_i)\neq 0$ if and only if $c_i\in S$. Relation (2) says that if c_i, c_j belong to the same interval of (A, B), then s_i and s_j have the same image in $A_G(u)$. Thus we get an element σ_I of $A_G(u)$ for each interval I of (A, B), and $\sigma_I^2=1$ by (1). Moreover (3) means that $\sigma_I=1$ if I is the initial interval.

The required isomorphism $V^{2,2}_{\rho(u)}\to A_G(u)^\wedge$ is then defined as follows. If F is a set of proper intervals, we associate to F the linear character of $A_G(u)$ which takes the value -1 on σ_I if and only if $I\in F$.

2.3. Remark. If K is a finite subfield of k and u, C are as in 2.2, it is known that $\mathrm{Sp}_{2n}(K)\cap C$ contains exactly $|A_G(u)|$ conjugacy classes of $\mathrm{Sp}_{2n}(K)$. Combined with 1.3, the lemma shows thus that the number of unipotent classes in $\mathrm{Sp}_{2n}(K)$ is $\sum_{d \text{ odd}}|W^\wedge_{n-d(d-1)}|$, a result which is due to Andrews [1]. The proof given here seems simpler.

2.4. By 1.3 we have a bijection

$$\bigcup_{d \text{ odd}} W^\wedge_{n-d(d-1)} \longrightarrow \bigcup_{d \text{ odd}} X^{2,2}_{n,d}=X^{2,2}_n.$$

It follows that for $G=\mathrm{Sp}_{2n}$, char$(k)=2$, the generalized Springer correspondence can be described by a bijection

$$\mathfrak{N}_G \longrightarrow X^{2,2}_n.$$

Consider a pair (u, φ) with $u\in G$ unipotent and $\varphi\in A_G(u)^\wedge$. In 2.1 we have defined $\rho(u)\in D^{2,2}_n$. Let ρ denote also the bijection $A_G(u)^\wedge\to V^{2,2}_{\rho(u)}$ defined in 2.2 and the bijection $\mathfrak{N}_G\to X^{2,2}_n$ $(u, \varphi)\mapsto \rho(u)_{\rho(\varphi)}$.

Theorem. *The map* ρ: $\mathfrak{N}_G\to X^{2,2}_n$, $(u, \varphi)\mapsto\rho(u)_{\rho(\varphi)}$ *represents the generalized Springer correspondence for* Sp_{2n} *in characteristic* 2.

The remaining part of this paragraph is devoted to a proof of this statement.

2.5. We consider simultaneously all the maps ρ: $\mathfrak{N}_{\mathrm{Sp}_{2n}}\to X^{2,2}_n$ $(n\in \mathbf{N})$

defined in 2.4. The main step in the proof of 2.4 is to check that they are compatible with the following case of the restriction formula 0.4 (4).

Let M, u, u', $Y_{u,u'}$, $X_{u,u'}$, $\varepsilon_{u,u'}$, U be as in 0.4, with M of type C_{n-1}. Recall that $A_G(u)^{\wedge}=\bigcup_{d\text{ odd}} A_G(u)^{\wedge}_d$, and similarly $A_M(u')^{\wedge}=\bigcup_{d\text{ odd}} A_M(u')^{\wedge}_d$. Let $\varphi\in A_G(u)^{\wedge}_d$, $\varphi'\in A_M(u')^{\wedge}_{d'}$. We have then

(1) If $\langle\varphi\otimes\varphi', \varepsilon_{u,u'}\rangle\neq 0$, then $d=d'$.

(2) If $d=d'$, then $\langle\varphi\otimes\varphi', \varepsilon_{u,u'}\rangle=\langle\operatorname{Res}^{W_m}_{W_{m-1}}\rho^G_{u,\varphi}, \rho^M_{u',\varphi'}\rangle_{W_{m-1}}$ where $m=n-d(d-1)$.

2.6. In the special case considered here, the set $X_{u,u'}$ and the action of $A_G(u)\times A_M(u')$ are essentially described in [8]. We reformulate here the required results.

If $Y_{u,u'}\neq\emptyset$, then $Y_{u,u'}$ is a single orbit under the action of $C_G(u)\times C_M(u')U$. It follows that $X_{u,u'}$ is either empty or a single $A_G(u)\times A_M(u')$-orbit, hence of the form $A_G(u)\times A_M(u')/H_{u,u'}$ for some subgroup $H_{u,u'}$ of $A_G(u)\times A_M(u')$.

Let $\Lambda=\rho_G(u)$, $\Lambda'=\rho_M(u')$. Then $X_{u,u'}\neq\emptyset$ if and only if Λ' can be deduced from Λ by decreasing one of the entries of Λ by 1.

If $X_{u,u'}\neq\emptyset$, we have a subgroup $H_{u,u'}$ of $A_G(u)\times A_M(u')$. We can describe it as follows. If A and B are groups, a subgroup C of $A\times B$ is characterized by the triple (A_0, B_0, f), where $A_0=\mathrm{pr}_1(C)$, $B_0=B\cap C$ and f: $A_0\to N_B(B_0)/B_0$ is defined by $a\mapsto bB_0$ if $(a, b)\in C$. Let then $A=A_G(u)$, $B=A_M(u')$ and $C=H_{u,u'}$. Let $c_1\leqslant\cdots\leqslant c_{2m+1}$ (resp. $c'_1\leqslant\cdots\leqslant c'_{2m+1}$) correspond to Λ (resp. Λ') as in 1.4(1). Recall that $A_G(u)$ is generated by $\{s_i\mid c_i\neq c_j \text{ for all } j\neq i\}$. Similarly $A_M(u')$ is generated by $\{s'_i\mid c'_i\neq c'_j \text{ for all } j\neq i\}$. Then A_0 is the subgroup of $A_G(u)$ generated by $\{s_i\mid c_i\neq c_j \text{ and } c'_i\neq c'_j \text{ for all } j\neq i\}$. Moreover f is defined by

(1) $s_i\mapsto s'_i$ (i such that $c_i\neq c_j$ and $c'_i\neq c'_j$ for $j\neq i$) and B_0 is the smallest subgroup of $A_M(u')$ for which (1) actually defines a morphism from A_0 to $A_M(u')/B_0$.

2.7. This description of $X_{u,u'}$ allows to compute $\varepsilon_{u,u'}$ and

$$(1)\qquad \{(\varphi, \varphi')\in A_G(u)^{\wedge}\times A_M(u')^{\wedge}\mid\langle\varphi\otimes\varphi', \varepsilon_{u,u'}\rangle\neq 0\}.$$

There are various cases to consider. An example is given in 2.10. We find in particular

Corollary. *If $n=d(d-1)$, with d odd, then (u, φ) is cuspidal if and only if $\rho(u, \varphi)=\Lambda^{2,2}_{n,d}$ (1.2(e)), and in this case u corresponds to the partition $2n=4+8+12+\cdots$.*

2.8. Let $(A, B)\in X^{2,2}_{n,d}$ correspond to $\chi\in W^{\wedge}_{n-d(d-1)}$. In view of 1.3

and 2.5 (1) we look at the pairs $(A', B') \in X^{2,2}_{n-1,d}$ which correspond to the components of the restriction of χ to $W_{n-1-d(d-1)}$. They are exactly those which can be deduced from (A, B) by decreasing one of the entries a_i or b_i by 1. Notice that since we want $(A', B') \in X^{2,2}_{n-1}$, we are allowed to decrease a_i (resp. b_i) only if $i \geqslant 2$ and $a_i - a_{i-1} > 4$ (resp. $b_i - b_{i-1} > 4$) or $i=1$ and $a_1 > 0$ (resp. $b_1 > 2$). We write $(A, B) \to (A', B')$ if (A, B) and (A', B') are related in this way.

Suppose that $(A, B) \to (A', B')$. It is easily checked that if (A, B) and (A', B') are similar to $\Lambda \in D^{2,2}_n$ and $\Lambda' \in D^{2,2}_{n-1}$ respectively, then $\Lambda \to \Lambda'$. We are thus led to look at

$$\{(F, F') \in V^{2,2}_{\Lambda} \times V^{2,2}_{\Lambda'} \mid \Lambda_F \longrightarrow \Lambda'_{F'}\}. \tag{1}$$

2.9. We can now check that the system of maps ρ ($n \in \mathbf{N}$) is compatible with 2.5 (1) and 2.5 (2). Notice first that the condition to have $X_{u,u'} \neq \emptyset$ in 2.6 is precisely $\Lambda \to \Lambda'$, where $\Lambda = \rho_G(u)$, $\Lambda' = \rho_M(u')$. It follows that we need only to check that 2.8 (1) is the image under ρ of 2.7 (1).

Let $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{2m+1}$ be as in 2.6, and suppose that Λ' is obtained by decreasing c_i by 1. Both for 2.7 (1) and 2.8 (1) we have to see what effect this has on the intervals of Λ and Λ'. There are various cases to consider, and in each of them the required equality is easy to check. We describe now one of them and leave the others to the reader.

2.10. Suppose $i \geqslant 3$, $c_{i-2} < c_{i-1} = c_i - 4$ and $c_{i+1} = c_i + 3$. Let I (resp. I') be the interval of Λ (resp. Λ') containing $c_{i-1} = c'_{i-1}$, and let J (resp. J') be the interval of Λ (resp. Λ') containing $c_{i+1} = c'_{i+1}$. There are two possibilities.

a) I is a proper interval of Λ. Then $\Lambda_F \to \Lambda'_{F'}$ if and only if (F, F') satisfies the following conditions:

i) $F \backslash \{I, J\} = F' \backslash \{I', J'\}$.

ii) $F \cap \{I, J\} = F' \cap \{I', J'\} = \emptyset$ or $\{I, J\} \subset F$ and $\{I', J'\} \subset F'$.

On the other hand, $A_G(u)$ (resp. $A_M(u')$) is an $\mathbf{F}_2$-vector space with one basis element σ_K (resp. σ'_K) for each proper interval K of Λ (resp. Λ'), and $X_{u,u'}$ is the quotient of $A_G(u) \times A_M(u')$ by the subspace $H_{u,u'}$ generated by $\sigma_I \sigma'_{I'}$, $\sigma_I \sigma'_{J'}$, $\sigma_J \sigma'_{J'}$ and the elements of the form $\sigma_K \sigma'_K$ with K a proper interval of both Λ and Λ'. The compatibility between 2.7 (1) and 2.8 (1) is then clear.

b) I is the initial interval of Λ. Then $\Lambda_F \to \Lambda'_{F'}$ if and only if $F = F'$ (and in particular $J \notin F$, $J' \notin F'$). To get $A_G(u)$, $A_M(u')$, $X_{u,u'}$, we set $\sigma_I = 1$, $\sigma'_{I'} = 1$ in (a). Again the compatibility between 2.7(1) and 2.8(1) is clear.

2.11. It is easily checked that ρ is also compatible with 0.4 (2) and 0.4 (3). Using this, we find in particular that the restriction of ρ to $\mathfrak{N}_d \to$

$W^{\wedge}_{n-d(d-1)}$ is the correct map if $n-d(d-1)\leqslant 2$. But for $m\geqslant 3$ it is known that an irreducible representation of W_m is entirely characterized by its restriction to W_{m-1}. Thus the compatibility of ρ with the restriction formula proves the theorem.

§ 3. Even orthogonal groups in characteristic 2

In this paragraph char$(k)=2$, $G=\mathrm{SO}_{2n}$ and $\tilde{G}=\mathrm{O}_{2n}$. We describe combinatorially the map

$$\mathfrak{N}'_G \longrightarrow W^{\wedge\prime}_n \cup \Big(\bigcup_{\substack{d>0\\ d\ \text{even}}} W^{\wedge}_{n-d^2}\Big)$$

defined in 0.8(2). As the results and the demonstrations are similar to those in paragraph 2, we shall be more sketchy.

3.1. As char$(k)=2$, we have $\tilde{G}\subset \mathrm{Sp}_{2n}$. If $u\in\tilde{G}$ is unipotent, we can therefore attach to u a pair (λ, ε) as in 2.1. We modify however the convention for $\varepsilon(0)$. We take now $\varepsilon(0)=0$, and we write λ in the form $(\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_{2m})$. It is known that $u\in G$ if and only if λ has an even number of non-zero parts. The class of u in $\tilde{G}$ is entirely determined by the pair (λ, ε).

We divide $\{1, 2, \cdots, 2m\}$ into blocks as in 2.1, and we attach to the $\tilde{G}$-class C of $u\in G$ the sequence $c_1, c_2, \cdots, c_{2m}$ defined as follows.

(1) If $\{i\}$ is a block, then

$$c_i=(\lambda_i-6)/2+2i.$$

(2) If $\{i, i+1\}$ is a block with $\varepsilon(\lambda_i)=\omega$, then

$$c_i=(\lambda_i-5)/2+2i$$
$$c_{i+1}=(\lambda_{i+1}-7)/2+2(i+1)=c_i+1.$$

(3) If $\{i, i+1\}$ is a block with $\varepsilon(\lambda_i)=0$, then

$$c_i=(\lambda_i-4)/2+2i$$
$$c_{i+1}=(\lambda_{i+1}-8)/2+2(i+1)=c_i.$$

Taking $a_i=c_{2i-1}$, $b_i=c_{2i}$ $(1\leqslant i\leqslant m)$, we get an element $\{A, B\}\in Y^4_{n,0}$ which doesn't depend on the choice of m. We call it $\rho(u)$ or $\rho(C)$ (or $\rho_G(u)$ if this is required by the context).

3.2. Lemma. (a) *$C\mapsto\rho(C)$ is a bijection from the set of all unipotent O_{2n}-classes in SO_{2n} to $D^4_{n,\mathrm{even}}$.*

(b) *$A_G(u)^\wedge$ is canonically isomorphic to $V^4_{\rho(u)}$.*

In this case $A_G(u)$ can be described as follows. Consider the abelian group generated by the set $\{s_i | \varepsilon(\lambda_i) \neq 0\}$, subject to the relations 2.2 (1) and 2.2 (2) (notice that 2.2 (3) is now irrelevant since $\varepsilon(0)=0$). This group is naturally isomorphic to $A_{\tilde{G}}(u) = C_{\tilde{G}}(u)/C^0_{\tilde{G}}(u)$. If $\varepsilon(\lambda_i) \neq 0$ the image of s_i in $A_{\tilde{G}}(u)$ depends only on the interval I of $\rho(u)$ which contains c_i. Call it σ_I. Then $A_{\tilde{G}}(u)$ is a vector space over $\mathbf{F}_2$ with basis $(\sigma_I)_{I \in E}$, where E is the set of all intervals of $\rho(u)$, and we can identify $\mathfrak{P}(E)$ with $A_{\tilde{G}}(u)^\wedge$ in the obvious way. The subgroup $A_G(u)$ of $A_{\tilde{G}}(u)$ consists of the elements which can be written as a product of an even number of generators, and from the identification $A_{\tilde{G}}(u)^\wedge = \mathfrak{P}(E)$ we get the required isomorphism $A_G(u)^\wedge \cong \mathfrak{P}(E)/\{\emptyset, E\} = V^4_{\rho(u)}$.

3.3. In addition to the map $C \mapsto \rho(C)$, let ρ (or ρ_G) denote also the following ones:

(i) the bijection $A_G(u)^\wedge \to V^4_{\rho(u)}$ defined in 3.2.

(ii) the map $\mathfrak{N}'_G \to Y^4_{n,\,\mathrm{even}}$, $(u, \varphi) \mapsto \rho(u)_{\rho(\varphi)}$. By 1.4 and 3.2 the map (ii) is also a bijection. We want to describe explicitly the bijection 0.8 (2)

$$\mathfrak{N}'_G \longrightarrow W_n^{\wedge\prime} \cup \Big(\bigcup_{\substack{d>0\\ d\ \mathrm{even}}} W^\wedge_{n-d^2}\Big).$$

By 1.3 we have a natural bijection

$$Y^4_{n,\,\mathrm{even}} \longrightarrow W_n^{\wedge\prime} \cup \Big(\bigcup_{\substack{d>0\\ d\ \mathrm{even}}} W^\wedge_{n-d^2}\Big).$$

Thus the generalized Springer correspondence for G can be represented by a bijection

$$\mathfrak{N}'_G \longrightarrow Y^4_{n,\,\mathrm{even}}.$$

The main result of this paragraph is:

Theorem. *The bijection ρ: $\mathfrak{N}'_G \to Y^4_{n,\,\mathrm{even}}$, $(u, \varphi) \mapsto \rho(u)_{\rho(\varphi)}$ represents the generalized Springer correspondence for $G = \mathrm{SO}_{2n}$ in characteristic 2.*

We get also

Corollary. *Let $n = d^2$, with $d > 0$ even. Then $(u, \varphi) \in \mathfrak{N}_G$ is cuspidal if and only if $\rho(u, \varphi) = \Lambda^4_{n,d}$ (1.2 (e)). In particular the partition λ of $2n$ corresponding to the class of u is $(2, 6, 10, \cdots, 4d-2)$.*

The proofs run along the same lines as those of 2.4, 2.7. We comment briefly on some points.

3.4. We use the restriction formula 0.4(4) with M of type D_{n-1}.

If $\{A, B\} \in Y^4_{n,\text{even}}$, $\{A', B'\} \in Y^4_{n-1,\text{even}}$, let $\{A, B\} \to \{A', B'\}$ denote the following relation: $\{A', B'\}$ can be deduced from $\{A, B\}$ by decreasing one term of A or B by 1. Let $\Lambda \in D^4_{n,\text{even}}$, $\Lambda' \in D^4_{n-1,\text{even}}$ be similar to $\{A, B\}$ and $\{A', B'\}$ respectively. If $\{A, B\} \to \{A', B'\}$, then $\Lambda \to \Lambda'$. When using the restriction formula, we have to look at the set

$$\{(F, F') | F \in V^4_\Lambda,\ F' \in V^4_{\Lambda'} \text{ and } \Lambda_F \to \Lambda'_{F'}\}$$

which can be computed easily in each case.

The representation $\varepsilon_{u,u'}$ is also essentially given by [8]. Let $\tilde{Y}_{u,u'} = \{x \in \tilde{G} | x^{-1}ux \in u'U\}$. Then $\dim \tilde{Y}_{u,u'} = \dim Y_{u,u'}$. Let $\tilde{P} = N_{\tilde{G}}(P)$, $\tilde{M} = N_{\tilde{P}}(M)$. If $\tilde{Y}_{u,u'} \neq \emptyset$, then $C_{\tilde{G}}(u) \times C_{\tilde{M}}(u')U$ acts transitively on $\tilde{Y}_{u,u'}$. Moreover $\dim \tilde{Y}_{u,u'} = d_{u,u'}$ if and only if $\rho_G(u) \to \rho_M(u')$. In this case $A_{\tilde{G}}(u) \times A_{\tilde{M}}(u')$ acts transitively on the set $\tilde{X}_{u,u'}$ of all irreducible components of maximal dimension of $\tilde{Y}_{u,u'}$. In particular $\tilde{X}_{u,u'}$ is isomorphic to $A_{\tilde{G}}(u) \times A_{\tilde{M}}(u')/\tilde{H}_{u,u'}$ for some subgroup $\tilde{H}_{u,u'}$ of $A_{\tilde{G}}(u) \times A_{\tilde{M}}(u')$. Let $A = A_{\tilde{G}}(u)$, $B = A_{\tilde{M}}(u')$, $C = \tilde{H}_{u,u'}$. We describe $C \subset A \times B$ as in 2.6 by a triple (A_0, B_0, f). Let $c_1, \cdots, c_{2m}$ be as in 3.1 and associate in a similar way a sequence $c'_1, \cdots, c'_{2m}$ to $u' \in M$. Then A (resp. B) is generated by elements s_i (resp. s'_i) for i such that $c_j \neq c_i$ (resp. $c'_j \neq c'_i$) for every $j \neq i$. Now A_0 is the subgroup of A generated by

$$\{s_i | c_i \neq c_j \text{ and } c'_i \neq c'_j \text{ for every } j \neq i\},$$

the homomorphism $f\colon A_0 \to N_B(B_0)/B_0 = B/B_0$ is defined by $s_i \mapsto s'_i$ (i such that $c_i \neq c_j$ and $c'_i \neq c'_j$ for $j \neq i$) and B_0 is the smallest subgroup of B for which f is actually defined. The subset $X_{u,u'}$ of $\tilde{X}_{u,u'}$ is the image in $\tilde{X}_{u,u'}$ of the subgroup of $A_{\tilde{G}}(u) \times A_{\tilde{M}}(u')$ consisting of the elements which can be written as a product of an even number of generators. It is also the image of $A_G(u) \times A_M(u')$.

This allows to compute $\varepsilon_{u,u'}$ and to work out the set

$$\{(\varphi, \varphi') \in A_G(u)^\wedge \times A_M(u')^\wedge | \langle \varphi \otimes \varphi', \varepsilon_{u,u'} \rangle \neq 0\}.$$

In each case it is easy to check the compatibility of ρ with the special case of the restriction formula considered here. When Λ or Λ' is degenerate one should pay attention to the difference between $W_n^{\wedge\prime}$ and $W_n'^{\wedge}$.

§ 4. Spin groups

In this paragraph $G = \mathrm{Spin}_n$, char $(k) \neq 2$ and χ is a one dimensional representation of the center of G which is non-trivial on the kernel of the natural isogeny $\pi\colon G \to \mathrm{SO}_n$. Let X_n be as in 0.10. We describe in combinatorial terms the bijection 0.10 (1)

$$\rho_n \colon X_n \longrightarrow \bigcup_{\substack{d \in \mathbf{Z} \\ d \mid n-d}} W^{\wedge}_{(n-d(2d-1))/4}$$

which represents the part of the generalized Springer correspondence pertaining to χ.

4.1. If $\lambda \in X_n$ for some n and $\rho_n(\lambda) \in W^{\wedge}_{(n-d(2d-1))/4}$, set $d(\lambda)=d$. If $m \in \mathbf{Z}$, set

$$d(m)=\begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \equiv 1 \pmod 4 \\ -1 & \text{if } m \equiv -1 \pmod 4 \end{cases}$$

If $\lambda=(\lambda_1) \in X_n$ has only one part (necessarily odd), it follows from 0.4 (2) that $d(\lambda)=d(\lambda_1)$.

We shall show in general that $d(\lambda)=\sum_i d(\lambda_i)$.

4.2. Lemma. *Let $\lambda \in X_n$, $\mu \in X_m$. Assume that one of the following holds.*

(a) *λ and μ have the same odd parts.*

(b) *for some integer s,*

$$\mu_i=\begin{cases} \lambda_i & \text{if } i<s \\ \lambda_i-4 & \text{if } i \geqslant s. \end{cases}$$

Then $d(\lambda)=d(\mu)$.

We use the restriction formula 0.4(4). For (a) we can assume that all the non-zero parts of μ are odd. Choose $M \ni u$ in such a way that u is distinguished in M in the sense of Bala-Carter [2]. Then M is isogenous to $SO_m \times GL_{\ell_1} \times GL_{\ell_2} \times \cdots$, where $\ell_1, \ell_1, \ell_2, \ell_2, \cdots$ are the even parts of λ. Take $u'=u$. In this case $X_{u,u'}$ can be described explicitly [9], and the restriction formula forces $d(\lambda)=d(\mu)$.

For (b) let $n-m=4r$. Choose M isogenous to $SO_m \times GL_{2r}$ and let $u' \in M$ correspond to the partitions μ in SO_m and $(2, 2, \cdots, 2)$ (r terms) in GL_{2r}. Then the class of u in G is induced from that of u' in M, in the sense of [6], and $X_{u,u'}$ can be described explicitly [9] (notice that here $|A_G(u)|=|A_M(u')|$). Again the restriction formula gives $d(\lambda)=d(\mu)$.

4.3. Corollary. *If λ has at most two parts, then $d(\lambda)=\sum d(\lambda_i)$.*

This is already known if $\lambda=(\lambda_1, \lambda_2)$ with $\lambda_1=0$. If $\lambda_1>0$, using 4.2 (b) we can find $\mu \in \{(1, 3), (2, 2), (1, 5), (3, 5), (4, 4), (3, 7)\}$ such that $d(\lambda)=d(\mu)$ and $d(\lambda_1)+d(\lambda_2)=d(\mu_1)+d(\mu_2)$. Let $m=\mu_1+\mu_2$. If $m=4$ or $m=8$, we have automatically $d(\mu)=0$. If $m=6$, we must have $d=2$. For $m=10$,

X_m has 3 elements, two of which correspond to $d=2$ and one to $d=-2$. By 4.2, $d(1,9)=d(1,2,2,5)=2$. Thus $d(3,7)=-2$. This proves the lemma.

4.4. Our main tool is again the restriction formula. We use the notation introduced in 0.4, and we choose M such that $\pi(M)\cong \mathrm{SO}_{n-4}\times \mathrm{GL}_2$. In the restriction formula, we can get non-zero contributions only if $u'\in M$ has the following properties.

1) The projection of $\pi(u')$ on GL_2 is regular.

2) The projection of $\pi(u')$ on SO_{n-4} corresponds to a unipotent element of Spin_{n-4} which contributes to the part of the generalized Springer correspondence pertaining to χ (for Spin_{n-4}).

The relevant unipotent classes of M are therefore parametrized by X_{n-4}. Notice also that the parts pertaining to χ of the generalized Springer correspondences for M and Spin_{n-4} are essentially the same.

Let $\lambda\in X_n$ correspond to the class of u in G and let $m=\frac{1}{4}(n-d(\lambda)\times(2d(\lambda)-1))$. The restriction of $\rho_n(\lambda)$ to W_{m-1} is multiplicity free and it is therefore characterized by the set $\mathrm{Res}\,(\rho_n(\lambda))\subset W_{m-1}^{\wedge}$, where for $\theta\in W_m^{\wedge}$ we define

$$\mathrm{Res}\,(\theta)=\{\theta'\in W_{m-1}^{\wedge}\,|\,\langle \mathrm{Res}_{W_{m-1}}^{W_m}(\theta),\theta'\rangle\neq 0\},$$

or equivalently by the set

$$\mathrm{Res}\,(\lambda)=\{\lambda'\in X_{n-4}\,|\,d(\lambda')=d(\lambda)\quad\text{and}\quad \rho_{n-4}(\lambda')\in \mathrm{Res}\,(\rho_n(\lambda))\}.$$

Notice that by definition

$$\mathrm{Res}\,(\rho_n(\lambda))=\rho_{n-4}(\mathrm{Res}\,(\lambda)).$$

If u' gives a non-zero contribution to the restriction of $\rho_n(\lambda)$, we must have $\dim Y_{u,u'}=d_{u,u'}$. But $\dim Y_{u,u'}=\dim \pi(Y_{u,u'})$. Hence $\dim\pi(Y_{u,u'})=d_{u,u'}$. Using results in [8, II.6] we find then easily:

4.5. Lemma. *Let $\lambda\in X_n$, $\lambda'\in \mathrm{Res}\,(\lambda)$. Then exactly one of statements* (a), (b) *below holds.*

(a) *There exists an integer i such that one of the following conditions is satisfied.*

a$_1$) *λ_i is odd, $\lambda_i>\lambda_{i-1}+4$, and*

$$\lambda'_j=\begin{cases}\lambda_j-4 & \text{if } j=i\\ \lambda_j & \text{otherwise}\end{cases}$$

a$_2$) *$\lambda_i=\lambda_{i+1}\geqslant\lambda_{i-1}+2$ and*

$$\lambda'_j=\begin{cases}\lambda_j-2 & \text{if } j\in\{i, i+1\}\\ \lambda_j & \text{otherwise}\end{cases}$$

a$_3$) $\lambda_i=\lambda_{i+1}\geqslant\lambda_{i-1}+4$ *and*

$$\lambda'_j=\begin{cases}\lambda_j-3 & \text{if } j=i\\ \lambda_j-1 & \text{if } j=i+1\\ \lambda_j & \text{otherwise}\end{cases}$$

a$_4$) $\lambda_{i+1}-2=\lambda_i\geqslant\lambda_{i-1}+1$ *and*

$$\lambda'_j=\begin{cases}\lambda_j-1 & \text{if } j=i\\ \lambda_j-3 & \text{if } j=i+1\\ \lambda_j & \text{otherwise}\end{cases}$$

a$_5$) $\lambda_{i+2}=\lambda_{i+1}=\lambda_i+1$ *and*

$$\lambda'_j=\begin{cases}\lambda_i-1 & \text{if } j\in\{i, i+2\}\\ \lambda_j-2 & \text{if } j=i+1\\ \lambda_j & \text{otherwise}\end{cases}$$

(b) *There exist distinct integers i, i' such that λ_i and $\lambda_{i'}$ are odd and*

$$\lambda'_j=\begin{cases}\lambda_j-2 & \text{if } j\in\{i, i'\}\\ \lambda_j & \text{otherwise.}\end{cases}$$

4.6. Corollary. *Let n be even and let $\lambda=(\lambda_1, \lambda_2)$ have exactly two parts. Let $\rho_n(\lambda)=(\alpha, \beta)$. Then*:

(a) *If $d(\lambda)=0$, then $\alpha=(\alpha_1)$ and $\beta=(\beta_1)$ where*

$$\alpha_1=\begin{cases}\frac{1}{4}(\lambda_2+1) & \text{if } d(\lambda_1)=1\\ \frac{1}{4}(\lambda_1-3) & \text{if } d(\lambda_1)=-1\\ [\frac{1}{4}\lambda_1] & \text{if } d(\lambda_1)=0\end{cases}\qquad \beta_1=\begin{cases}\frac{1}{4}(\lambda_1-1) & \text{if } d(\lambda_1)=1\\ \frac{1}{4}(\lambda_2+3) & \text{if } d(\lambda_1)=-1\\ [\frac{1}{4}(\lambda_1+2)] & \text{if } d(\lambda_1)=0\end{cases}$$

(b) *If $d(\lambda)=2$, then $\alpha=(\alpha_1, \alpha_2)$ and $\beta=(0)$, where*

$$\alpha_1=\tfrac{1}{4}(\lambda_1-1),\qquad \alpha_2=\tfrac{1}{4}(\lambda_2-5)$$

(c) *If $d(\lambda)=-2$, then $\alpha=(\alpha_1, \alpha_2)$, $\beta=(0)$, where*

$$\alpha_1=\tfrac{1}{4}(\lambda_1-3),\qquad \alpha_2=\tfrac{1}{4}(\lambda_2-7).$$

Let $m=n-d(\lambda)(2d(\lambda)-1)$. The result is obvious if $m=0$, and in the following cases where we can use 0.4 (2):

$$d(\lambda)=0, \qquad \lambda_1=1$$
$$d(\lambda)=2, \qquad \lambda_1=1$$
$$d(\lambda)=-2, \qquad \lambda_1=3.$$

For the remaining cases we use induction on n, the restrictions on $\mathrm{Res}(\lambda)$ given by 4.3 and 4.5, and the representation theory of W_m. For example suppose that $d(\lambda)=2$. We have then by 4.3 and 4.5

$$\emptyset \neq \mathrm{Res}(\lambda) \subset \begin{cases} \{(\lambda_1-4, \lambda_2)\} & \text{if } \lambda_1=\lambda_2-4 \\ \{(\lambda_1-4, \lambda_2), (\lambda_1, \lambda_2-4)\} & \text{if } 1 \neq \lambda_1 \neq \lambda_2-4. \end{cases}$$

Notice that $\lambda'=(\lambda_1-2, \lambda_2-2)$, which is allowed by 4.5 (b), does not belong to $\mathrm{Res}(\lambda)$ since by 4.3 $d(\lambda')=-2\neq d(\lambda)$.

If $\lambda_1=\lambda_2-4$, we have therefore $\mathrm{Res}(\lambda)=\{(\lambda_1-4, \lambda_2)\}$, and the result follows. If $1\neq\lambda_1\neq\lambda_2-4$, the result is also clear unless $\mathrm{Res}(\lambda)$ has only one element. As $\rho_{n-4}(\mathrm{Res}(\lambda))$ is of the form $\mathrm{Res}(\theta)$ for some $\theta \in W_m^{\wedge}$, this can happen only in the following cases:

1) $\lambda_1=5$ and $\mathrm{Res}(\lambda)=\{(1, \lambda_2)\}$.
2) $\lambda_1=\lambda_2-12$ and $\mathrm{Res}(\lambda)=\{(\lambda_1, \lambda_2-4)\}$.
3) $\lambda=(5, 13)$ and $\mathrm{Res}(\lambda)=\{(5, 9)\}$.

But (1) gives $\rho_n(\lambda)=\rho_n(1, \lambda_2+4)$, (2) gives $\rho_n(\lambda)=\rho_n(\lambda_1+4, \lambda_1+8)$, and, using 4.2 and 4.5, (3) gives $\rho_n(\lambda)=\rho_n(2, 2, 5, 9)$. Thus each of (1), (2), (3) contradicts the injectivity of ρ_n.

Similar arguments apply if $d(\lambda)=0$ or $d(\lambda)=-2$. If $d(\lambda)=0$ we cannot however rule out a priori a contribution of 4.5 (b) to $\mathrm{Res}(\lambda)$, but this comes out of the computation.

4.7. We go back to the situation considered in 4.4, and we assume that $\dim Y_{u,u'}=d_{u,u'}$. By restriction, π gives a double covering $Y_{u,u'}\to \pi(Y_{u,u'})$. The group $A_G(u)\times A_M(u')$ acts on the set of all irreducible components of $\pi(Y_{u,u'})$ of dimension $d_{u,u'}$. This gives a permutation representation $\bar{\varepsilon}_{u,u'}$ and a surjective morphism of representations $f\colon \varepsilon_{u,u'}\to\bar{\varepsilon}_{u,u'}$. In the restriction formula, $\bar{\varepsilon}_{u,u'}$ corresponds to the part of the generalized Springer correspondence coming from SO_n, and $\mathrm{Ker}(f)$ to the part corresponding to χ (resp. to χ and its conjugate) if n is odd (resp. even).

Let $Y^0_{u,u'}=\{y \in Y_{u,u'} | \pi^{-1}(\pi(y))$ is contained in some irreducible component of $Y_{u,u'}\}$. We have shown that $\lambda' \in \mathrm{Res}(\lambda)$ if and only if the complement of $Y^0_{u,u'}$ in $Y_{u,u'}$ has dimension $d_{u,u'}$.

4.8. Lemma. *Let* $\lambda \in X_n$, $\lambda' \in \mathrm{Res}(\lambda)$. *Then condition* (a) *of* 4.5 *holds.*

If λ has exactly two parts, this follows from 4.6 (and even from 4.3 if $d(\lambda)\neq 0$).

In the general case we suppose that condition (b) of 4.5 holds and we show that in the situation of 4.7 the complement of $Y^0_{u,u'}$ in $Y_{u,u'}$ has positive codimension, contradicting $\lambda' \in \mathrm{Res}(\lambda)$.

In the definition of $Y_{u,u'}$ we used a parabolic subgroup P. Let $\mathfrak{P}$ be the variety of all conjugates of P and let $\mathfrak{P}_{u,u'}=\{{}^xP \mid x \in Y_{u,u'}\}$. We have a natural map $p\colon Y_{u,u'}\to\mathfrak{P}_{u,u'}$, $x\mapsto {}^xP$, and the fibers of p are the $C_M(u')U$-orbits in $Y_{u,u'}$. In particular all the fibers have the same dimension, and if we set $\mathfrak{P}^0_{u,u'}=p(Y^0_{u,u'})$, then $Y^0_{u,u'}=p^{-1}(\mathfrak{P}^0_{u,u'})$. Thus it is sufficient to prove that $\mathfrak{P}^0_{u,u'}$ is dense in $\mathfrak{P}_{u,u'}$.

We think of $\mathfrak{P}$ as the variety of all 2-dimensional subspaces of k^n which are totally isotropic with respect to the quadratic form used to define G. Let $v=\pi(u)\in \mathrm{SO}_n$. Then $\mathfrak{P}_{u,u'}$ becomes the set of all subspaces $E\in\mathfrak{P}$ such that

1) E is v-stable.

2) $v|_E\neq 1$.

3) The partition of the unipotent endomorphism of $E^{\perp}/E$ induced by v is λ'.

Let $i>i'$ be the integers given by 4.5 (b) and let $r=\lambda_i$, $s=\lambda_{i'}$. Let $L=\mathrm{Ker}(v-1|_E)$ and let $e_1\in L$, $e_1\neq 0$. For some h, $e_1\in\mathrm{Im}(v-1)^{h-1}$, $e_1\notin\mathrm{Im}(v-1)^h$. Choose $e_2,\cdots,e_h$ such that $(v-1)(e_j)=e_{j-1}$ $(1<j\leqslant h)$. Let $V_1=\sum_{j=1}^h ke_j$, $V_1'=V_1^{\perp}$ and let v_1' be the restriction of v to V_1'. Then $k^n=V_1\oplus V_1'$ and $h\in\{r,s\}$. There are two cases to consider.

(i) $E\subset\mathrm{Im}(v-1)^{h-2}$. Then $h=r$ and $s=r-2$.

(ii) $E\not\subset\mathrm{Im}(v-1)^{h-2}$. Then $e_2\notin E$.

If (i) holds, we can take $e_2\in E$, and we can find $f_1\in\mathrm{Ker}(v_1'-1)\cap\mathrm{Im}(v_1'-1)^{s-1}$, $f_1\notin\mathrm{Im}(v-1)^s$. Then for each $a\in k$ the subspace $E_a=ke_1+k(e_2+af_1)$ belongs to $\mathfrak{P}_{u,u'}$. Moreover (ii) holds for E_a if $a\neq 0$. For our purpose it is therefore sufficient to study $\mathfrak{P}^*_{u,u'}=\{E\in\mathfrak{P}_{u,u'} \mid E\not\subset\mathrm{Im}(v-1)^{h-2}\}$.

Let $E\in\mathfrak{P}^*_{u,u'}$. Choose $y\in E$ such that $(v-1)(y)=e_1$ and let $f_1=y-e_2\in\mathrm{Ker}(v-1)$. It is easily checked that $h=r$ and $f_1\in\mathrm{Im}(v-1)^{s-1}$, $f_1\notin\mathrm{Im}(v-1)^s$. Choose $f_2,\cdots,f_s$ such that $(v-1)(f_j)=f_{j-1}$ $(1<j\leqslant s)$. Let $V=V_1+\sum_{j=1}^s kf_j$, $V'=V^{\perp}$. Then $k^n=V\oplus V'$ and $E\subset V$. Let $\mathfrak{P}^V_{u,u'}=\{E'\in\mathfrak{P}_{u,u'} \mid E'\subset V\}$. We need only to check that $\mathfrak{P}^0_{u,u'}\cap\mathfrak{P}^V_{u,u'}$ is dense in $\mathfrak{P}^V_{u,u'}$. Notice that $\mathfrak{P}^V_{u,u'}$ is isomorphic to an open subset of $\mathbf{P}^1$, hence is irreducible.

Let $H=\{g\in G \mid g \text{ fixes } V' \text{ pointwise}\}$. Then $H=\mathrm{Spin}_{r+s}$. Let $y\in p^{-1}(E)$, and $Y^{y,H}_{u,u'}=\{h\in H \mid hy\in Y_{u,u'}\}=\{h\in H \mid h^{-1}uh\in {}^yu'\,{}^yU\}$. Looking at the restriction of π to $Y^{y,H}_{u,u'}$, we have reduced the problem to the case where λ has only two parts, and we have replaced $\mathfrak{P}_{u,u'}$ by $\mathfrak{P}^V_{u,u'}$. In this case however the result is already known. In particular the set $Y^{y,H,0}_{u,u'}$ of all $y' \in Y^{y,H}_{u,u'}$ such that $\pi^{-1}(\pi(y'))$ is contained in some irreducible component of $Y^{y,H}_{u,u'}$ has a dense image in $\mathfrak{P}^V_{u,u'}$. As we certainly have $Y^{y,H,0}_{u,u'} \subset Y^0_{u,u'}$, we find that $\mathfrak{P}^0_{u,u'} \cap \mathfrak{P}^V_{u,u'}$ is dense in $\mathfrak{P}^V_{u,u'}$, as required.

4.9. Corollary. *Let $n=d(2d-1)$, with $d\neq 0$. If $d>0$ let $\lambda \in X_n$ be the partition* $(1, 5, 9, 13, \cdots)$. *If $d<0$ let $\lambda \in X_n$ be the partition* $(3, 7, 11, 15, \cdots)$. *Let $(u, \varphi) \in \mathfrak{N}_\chi$ correspond to λ. Then (u, φ) is cuspidal.*

By 4.8 we have $\mathrm{Res}(\lambda)=\emptyset$. (This was conjectured in [4, 14.6].)

4.10. Corollary. *$d(\lambda)=\sum d(\lambda_i)$ for every $\lambda \in X_n$.*

If λ correspond to a cuspidal pair $(u, \varphi) \in \mathfrak{N}_\chi$, this follows from 4.9. By 4.8, if $\lambda \in X_n$ and $\lambda' \in \mathrm{Res}(\lambda)$, then $d(\lambda)=d(\lambda')$ and $\sum d(\lambda_i)=\sum d(\lambda'_i)$. The result follows.

4.11. Let $\lambda \in X_n$. Write it as $(\lambda_1, \lambda_2, \cdots, \lambda_m)$, with $\lambda_1\neq 0$. We define by induction on m a pair $\rho(\lambda)$ of finite sequences of integers.

$$\text{If } m=0, \qquad \rho(\lambda)=((0), (0)).$$

Suppose $m\neq 0$. If λ_m is odd, let $\mu=(\lambda_1, \cdots, \lambda_{m-1}) \in X_{n-\lambda_m}$. If λ_m is even, let $\mu=(\lambda_1, \cdots, \lambda_{m-2}) \in X_{n-2\lambda_m}$. We assume that $\rho(\mu)=(\gamma, \delta)$ is already known and we define $\rho(\lambda)=(\alpha, \beta)$.

If $\nu=(\nu_1, \cdots, \nu_h)$ is a finite sequence of integers and $r \in \mathbf{N}$, let (ν, r) denote the sequence $(\nu_1, \cdots, \nu_h, r)$.

(a) $d(\lambda_m)=0$. Set $r=[\frac{1}{4}(\lambda_m+2)]-d(\mu)$, $s=[\frac{1}{4}\lambda_m]+d(\mu)$. Then

a$_1$) If $d(\mu)>0$, $\alpha=(\gamma, r)$, $\beta=(\delta, s)$.

a$_2$) If $d(\mu)\leqslant 0$, $\alpha=(\gamma, s)$, $\beta=(\delta, r)$.

(b) $d(\lambda_m)=1$. Set $r=\frac{1}{4}(\lambda_m-1)-d(\mu)$. Then

b$_1$) If $d(\mu)>0$, $\alpha=(\gamma, r)$, $\beta=\delta$.

b$_2$) If $d(\mu)=0$, $\alpha=(\delta, r)$, $\beta=\gamma$.

b$_3$) If $d(\mu)<0$, $\alpha=\gamma$, $\beta=(\delta, r)$.

(c) $d(\lambda_m)=-1$. Set $r=\frac{1}{4}(\lambda_m-3)+d(\mu)$. Then

c$_1$) If $d(\mu)>1$, $\alpha=\gamma$, $\beta=(\delta, r)$.

c$_2$) If $d(\mu)=1$, $\alpha=(\delta, r)$, $\beta=\gamma$.

c$_3$) If $d(\mu)<1$, $\alpha=(\gamma, r)$, $\beta=\delta$.

Notice that $r\geqslant 0$ in all cases and $s\geqslant 0$ in case (a).

4.12. Lemma. *Let $\lambda=(\lambda_1, \cdots, \lambda_m) \in X_n$, define $(\alpha, \beta)=\rho(\lambda)$ as in 4.11 and let a (resp. b) be the last term of α (resp. β). Then*

(i) $\Sigma\alpha_i+\Sigma\beta_i=\frac{1}{4}(n-d(\lambda)(2d(\lambda)-1))$.

(ii) *If $d(\lambda)>0$, then*
$a\leqslant[\frac{1}{4}(\lambda_m+3)]-d(\lambda),\ b\leqslant[\frac{1}{4}(\lambda_m+1)]+d(\lambda)$.

(iii) *If $d(\lambda)\leqslant 0$, then*
$a\leqslant[\frac{1}{4}(\lambda_m+1)]+d(\lambda),\ b\leqslant[\frac{1}{4}(\lambda_m+3)]-d(\lambda)$

(iv) *If $d(\lambda)>0$ (resp. $d(\lambda)\leqslant 0$), let $m=b-a$ (resp. $m=a-b$). Then for the various cases of 4.11 we have*

$$\begin{array}{ll} m\leqslant 2d(\lambda)-2 & \textit{in case } (\mathrm{b}) \\ 2d(\lambda)-1\leqslant m\leqslant 2d(\lambda) & \textit{in case } (\mathrm{a}) \\ 2d(\lambda)+1\leqslant m & \textit{in case } (\mathrm{c}). \end{array}$$

(v) *α and β are partitions.*

We use induction. Suppose for example that we are in case (b_1) of 4.11. With the notation used there ,we have

$$\begin{aligned}\Sigma\alpha_i+\Sigma\beta_i&=r+\Sigma\gamma_i+\Sigma\delta_i\\ &=\tfrac{1}{4}(\lambda_m-1)-d(\mu)+\tfrac{1}{4}(n-\lambda_m-d(\mu)(2d(\mu)-1))\\ &=\tfrac{1}{4}(n-d(\lambda)(2d(\lambda)-1)).\end{aligned}$$

This proves (i) in this case. We have also

$$\begin{aligned}&a=r=\tfrac{1}{4}(\lambda_m+3)-d(\lambda),\\ &b\leqslant[\tfrac{1}{4}(\mu_{m-1}+1)]+d(\mu)\leqslant[\tfrac{1}{4}\lambda_m]+d(\lambda)-1=\tfrac{1}{4}(\lambda_m-5)+d(\lambda).\end{aligned}$$

This proves (ii) and (iv) in this case. For (v), it is clear that β is a partition. Let c be the last term of γ. Then α is a partition if $c\leqslant a$. But $c\leqslant[\frac{1}{4}(\mu_{m-1}+3)]-d(\mu)\leqslant[\frac{1}{4}(\lambda_m+2)]-d(\mu)=\frac{1}{4}(\lambda_m-1)-d(\mu)=a$, as required. Thus (v) holds in this case.

The remaining cases are dealt with in a similar way.

4.13. Corollary. *Let $d \in \mathbf{Z}$, $d\equiv n \pmod 4$, and let $m=\frac{1}{4}(n-d(2d-1))$. Then $\lambda\mapsto\rho(\lambda)$ defines a bijection from $\{\lambda \in X_n | d(\lambda)=d\}$ to $W_m^{\wedge}$.*

By 4.12(i), (v) we known that $\lambda\mapsto\rho(\lambda)$ gives a map from $\{\lambda \in X_n | d(\lambda)=d\}$ to $W_m^{\wedge}$. Let $\lambda \in X_n$, $d(\lambda)=d$, and let $(\alpha, \beta)=\rho(\lambda)$. By 4.12 (iv) the knowledge of (α, β) and $d(\lambda)$ determines which case of 4.11 applies to λ. Thus we can recover λ_1 (and λ_2 in case (a)), the pair $(\gamma, \delta)=\rho(\mu)$ and $d(\mu)$ from (α, β) and $d(\lambda)$. By induction we can assume that $\rho(\mu)$ and $d(\mu)$ determine μ, and this gives λ. Thus we have an injective map from $\{\lambda \in X_n | d(\lambda)=d\}$ to $W_m^{\wedge}$. It must be surjective since these two sets have the same number of elements [4] (the surjectivity can also be checked directly).

4.14. Theorem. *For $\lambda \in X_n$ let $\rho(\lambda)$ be defined as in* 4.11. *Then $\rho(\lambda) = \rho_n(\lambda)$, i.e. the map*

$$X_n \longrightarrow \bigcup_{\substack{d \in \mathbf{Z} \\ 4 | n-d}} W^{\wedge}_{(n-d(2d-1))/4}$$

$$\lambda \longmapsto \rho(\lambda)$$

represents the part pertaining to χ of the generalized Springer correspondence for $G = \mathrm{Spin}_n$.

We know that this map is bijective. It is easily checked that $\lambda \mapsto \rho(\lambda)$ is compatible with 0.4 (2) and 0.4 (3). As in the previous paragraphs, it is thus sufficient to verify that it is also compatible with the restriction formula 0.4 (4). For $\lambda \in X_n$ let $\mathrm{Res}'(\lambda)$ be the set of all $\lambda' \in X_{n-4}$ for which 4.5 (a) holds. Using 4.12, one can check that $\rho(\mathrm{Res}'(\lambda)) \subset \mathrm{Res}(\rho(\lambda))$. Thus we have

$$\sum_{\lambda \in X_n} |\mathrm{Res}'(\lambda)| \leqslant \sum_{\substack{d \in \mathbf{Z} \\ 4 | n-d}} \sum_{\theta \in W^{\wedge}_{(n-d(2d-1))/4}} |\mathrm{Res}(\theta)|.$$

But $\mathrm{Res}(\lambda) \subset \mathrm{Res}'(\lambda)$ by 4.8, and we certainly have

$$\sum_{\lambda \in X_n} |\mathrm{Res}(\lambda)| = \sum_{\substack{d \in \mathbf{Z} \\ 4 | n-d}} \sum_{\theta \in W^{\wedge}_{(n-d(2d-1))/4}} |\mathrm{Res}(\theta)|.$$

Therefore $\mathrm{Res}'(\lambda) = \mathrm{Res}(\lambda)$ and $\rho(\mathrm{Res}(\lambda)) = \mathrm{Res}(\rho(\lambda))$, as required.

4.15. In the proof of the theorem, we have also checked:

Corollary. *Let $\lambda \in X_n$, $\lambda' \in X_{n-4}$. Then $\lambda' \in \mathrm{Res}(\lambda)$ if and only if condition* (a) *of* 4.5 *holds.*

§ 5. Special linear groups

In this section $G = \mathrm{SL}_n(k)$. Let $p = \mathrm{char}(k)$ if $\mathrm{char}(k) > 0$, $p = 1$ if $\mathrm{char}(k) = 0$.

5.1. The unipotent classes of G are parametrized by partitions of n in the usual way, with the regular unipotent class corresponding to the partition (n), and $1 \in G$ to $(1, 1, \cdots, 1)$. We identify the Weyl group W of G with $\mathfrak{S}_n$ and parametrize $\mathfrak{S}_n^{\wedge}$ by the set of all partitions of n as usual, with (n) corresponding to the trivial representation and $(1, 1, \cdots, 1)$ to the sign representation.

The generalized Springer correspondence decomposes according to the one dimensional characters of the centre Z of G [4, 14.2]. In our case Z is cyclic of order n', where n' is the largest divisor of n which is prime to p. Let the unipotent element u of G correspond to the partition $\lambda=(\lambda_1, \lambda_2, \cdots)$. Let n'_λ be the greatest common divisor of n', $\lambda_1, \lambda_2, \cdots$. Then $A_G(u)$ is cyclic of order n'_λ. Moreover, if $\chi \in Z^\wedge$ is of order d, then

$$|A_G(u)^\wedge_\chi| = \begin{cases} 1 & \text{if } d \text{ divides each } \lambda_i \ (i \geqslant 1) \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [4, 10.3] that the part of the generalized Springer correspondence pertaining to $\chi \in Z^\wedge$ falls into a single family. More precisely, up to conjugacy there is a unique Levi factor L which has a cuspidal pair $(v, \psi) \in \mathfrak{N}_L$ corresponding to χ, and this cuspidal pair is also unique. If χ is of order d, then L is of type $A_{d-1}+\cdots+A_{d-1}$ (n/d factors) and v is a regular unipotent element of L. The group $N_G(L)/L$ can be identified with $\mathfrak{S}_{n/d}$, and $(N_G(L)/L)^\wedge$ is thus parametrized by the set of all partitions of n/d.

5.2. Proposition. *Let $\chi \in Z^\wedge$ have order d and let the unipotent element u of G correspond to the partition $\lambda=(\lambda_1, \lambda_2, \cdots)$. Assume that $d|\lambda_i$ $(i \geqslant 1)$, and let φ be the unique element of $A_G(u)^\wedge_\chi$. Then the partition of n/d corresponding to $\rho^G_{u,\varphi} \in (\mathfrak{S}_{n/d})^\wedge$ is $\lambda/d=(\lambda_1/d, \lambda_2/d, \cdots)$.*

Let $m=n/d$. For $m=1$ there is nothing to prove, and for $m=2$ the result follows from 0.4 (2), (3). For $m \geqslant 3$ we use induction on m and the restriction formula 0.4 (4) with M of type $A_{n-d-1}+A_{d-1}$. The group $N_M(L)/L$ gets identified with $\mathfrak{S}_{m-1} \subset \mathfrak{S}_m$. As $m \geqslant 3$, a representation $\rho \in \mathfrak{S}_m^\wedge$ is completely determined by its restriction $\operatorname{Res}(\rho)$ to $\mathfrak{S}_{m-1}$. It is therefore sufficient to check that the bijection described in the proposition is compatible with the special case of 0.4 (4) under consideration. If ρ corresponds to the partition μ of m, then the irreducible constituants of $\operatorname{Res}(\rho)$ correspond to the partitions of $m-1$ which are of the form

$$(\mu_1, \cdots, \mu_{i-1}, \mu_i-1, \mu_{i+1}, \cdots) \qquad (i \text{ such that } \mu_i > \mu_{i-1}).$$

Moreover $\operatorname{Res}(\rho)$ is multiplicity free. On the other hand, it follows from [8, II. 5.5] that in the restriction formula 0.4(4) the only unipotent elements u' which can give a non zero contribution are those which on the factor of M of type A_{n-d-1} correspond to a partition $\lambda'=(\lambda'_1, \lambda'_2, \cdots)$ with $\lambda'_i \leqslant \lambda_i$ for each i. But we are looking at the part of the generalized Springer correspondence pertaining to χ. Therefore if a pair $(u', \varphi') \in \mathfrak{N}_M$ gives a non zero contribution in 0.4 (4), u' must correspond on the factor of M of type A_{n-d-1} to a partition $\lambda'=(\lambda'_1, \lambda'_2, \cdots)$ with $d|\lambda'_i$ for each i, and on the other factor to the regular unipotent class; moreover φ' is determined by χ. Thus to get a non zero contribution, λ' must be of the form

$$(\lambda_1, \cdots, \lambda_{i-1}, \lambda_i-d, \lambda_{i+1}, \cdots) \qquad (i \text{ such that } \lambda_i \geqslant \lambda_{i-1}+d).$$

Let ρ_λ be the representation of $\mathfrak{S}_m$ corresponding to λ/d. The above and the induction hypothesis show that $\mathrm{Res}(\rho^G_{u,\varphi})$ is a subrepresentation of $\mathrm{Res}(\rho_\lambda)$. As this holds for all partitions λ of n with $d|\lambda_i$ for each i, we must have equality. This proves the proposition.

§ 6. Examples

6.1. Let char $(k)=2$. The unipotent classes in $\mathrm{Sp}_{2n}(k)$ are characterized by pairs (λ, ε) (2.1). In the tables below we give the partition λ with a subscript 0 to the parts of size m whenever $\varepsilon(m)=0$. If $(A, B) \in X_n^{2,2}$, we identify A and B with subsets of $\mathbf{N}$.

Elements of $X_2^{2,2}$	Conjugacy classes in $\mathrm{Sp}_4(k)$
$(\{2\}, \emptyset), (\emptyset, \{2\})$	4
$(\{0, 5\}, \{3\})$	$2+2$
$(\{0, 4\}, \{4\})$	$(2+2)_0$
$(\{1, 5\}, \{2\})$	$1+1+2$
$(\{0, 4, 8\}, \{3, 7\})$	$1+1+1+1$

Elements of $X_3^{2,2}$	Conjugacy classes in $\mathrm{Sp}_6(k)$
$(\{3\}, \emptyset), (\emptyset, \{3\})$	6
$(\{0, 6\}, \{3\})$	$2+4$
$(\{1, 6\}, \{2\}), (\{1\}, \{2, 6\})$	$1+1+4$
$(\{0, 5\}, \{4\}), (\{0, 4\}, \{5\})$	$3+3$
$(\{1, 5\}, \{3\})$	$2+2+2$
$(\{0, 4, 9\}, \{3, 7\})$	$1+1+2+2$
$(\{0, 4, 8\}, \{3, 8\})$	$1+1+(2+2)_0$
$(\{1, 5, 9\}, \{2, 6\})$	$1+1+1+1+2$
$(\{0, 4, 8, 12\}, \{3, 7, 11\})$	$1+1+1+1+1+1$

6.2. Let char $(k)=2$. To denote the unipotent classes in $\mathrm{SO}_{2n}(k)$ we use the same convention as in 6.1 for the unipotent classes of $\mathrm{Sp}_{2n}(k)$. If $\{A, B\} \in Y^4_{n,\,\mathrm{even}}$, we consider A and B as subsets of $\mathbf{N}$.

Elements of $Y^4_{4,\,\mathrm{even}}$	Conjugacy classes in $\mathrm{SO}_8(k)$
$\{\{0\}, \{4\}\}, \{\{0, 4\}, \emptyset\}$	$2+6$
$\{\{1\}, \{3\}\}$	$4+4$
$\{\{2\}, \{2\}\}$	$(4+4)_0$ (twice)
$\{\{0, 4\}, \{1, 7\}\}$	$1+1+2+4$
$\{\{0, 5\}, \{1, 6\}\}, \{\{1, 5\}, \{0, 6\}\}$	$1+1+3+3$
$\{\{0, 4\}, \{2, 6\}\}$	$2+2+2+2$
$\{\{1, 5\}, \{1, 5\}\}$	$(2+2+2+2)_0$ (twice)

$\{\{0, 4, 8\}, \{1, 5, 10\}\}$	$1+1+1+1+2+2$
$\{\{0, 4, 9\}, \{1, 5, 9\}\}$	$1+1+1+1+(2+2)_0$
$\{\{0, 4, 8, 12\}, \{1, 5, 9, 13\}\}$	$1+1+1+1+1+1+1+1$

6.3. Let char $(k) \neq 2$. We describe the part of the generalized Springer correspondence for $G = \mathrm{Spin}_n(k)$ $(12 \leqslant n \leqslant 15)$ pertaining to a one dimensional representation of the centre of G which is non trivial on the kernel of the natural isogeny $G \to \mathrm{SO}_n(k)$. For $n \in \{12, 13, 15\}$ this part of the generalized Springer correspondence forms a single family indexed by $W_3^\wedge$. For $n = 14$ there are two families indexed respectively by $W_1^\wedge$ and $W_2^\wedge$. We let $\emptyset$ denote the empty partition.

$n = 12$

$W_3^\wedge$	unipotent class
$(3, \emptyset)$	$1+11$
$(1+2, \emptyset)$	$1+2+2+7$
$(1+1+1, \emptyset)$	$1+2+2+2+2+3$
$(2, 1)$	$5+7$
$(1+1, 1)$	$1+3+4+4$
$(1, 2)$	$6+6$
$(1, 1+1)$	$2+2+4+4$
$(\emptyset, 3)$	$3+9$
$(\emptyset, 1+2)$	$2+2+3+5$
$(\emptyset, 1+1+1)$	$2+2+2+2+2+2$

$n = 13$

$W_3^\wedge$	unipotent class
$(3, \emptyset)$	13
$(1+2, \emptyset)$	$2+2+9$
$(1+1+1, \emptyset)$	$2+2+2+2+5$
$(2, 1)$	$1+3+9$
$(1+1, 1)$	$4+4+5$
$(1, 2)$	$1+6+6$
$(1, 1+1)$	$1+2+2+3+5$
$(\emptyset, 3)$	$1+5+7$
$(\emptyset, 1+2)$	$1+2+2+4+4$
$(\emptyset, 1+1+1)$	$1+2+2+2+2+2+2$

$n = 14$

$W_1^\wedge$	unipotent class
$(1, \emptyset)$	$3+11$
$(\emptyset, 1)$	$2+2+3+7$

$W_2^\wedge$	unipotent class
$(2, \emptyset)$	$1+13$
$(1+1, \emptyset)$	$5+9$
$(1, 1)$	$1+2+2+9$
$(\emptyset, 2)$	$1+4+4+5$
$(\emptyset, 1+1)$	$1+2+2+2+2+5$

$n = 15$

$W_3^\wedge$	unipotent class
$(3, \emptyset)$	15
$(1+2, \emptyset)$	$1+3+11$
$(1+1+1, \emptyset)$	$1+2+2+3+7$
$(2, 1)$	$2+2+11$
$(1+1, 1)$	$4+4+7$
$(1, 2)$	$3+5+7$
$(1, 1+1)$	$2+2+2+2+7$
$(\emptyset, 3)$	$3+6+6$
$(\emptyset, 1+2)$	$2+2+3+4+4$
$(\emptyset, 1+1+1)$	$2+2+2+2+2+2+3$

References

[1] Andrews, G. E., Partitions, q-functions and the Lusztig-Macdonald-Wall conjectures. Invent. Math., **41** (1977), 91–102.

[2] Bala, P. and Carter, R. W., Classes of unipotent elements in simple algebraic groups, Math. Proc. Cambridge Philos. Soc., **79** (1976), 401–425 and **80** (1976), 1–18.

[3] Hesselink, W. H., Nilpotency in classical groups over a field of characteristic 2. Math. Z., **166** (1979), 165–181.

[4] Lusztig, G., Intersection cohomology complexes on a reductive group, Invent. Math., **75** (1984), 205–272.

[5] ——, Irreducible representations of finite classical groups, Invent. Math., **43** (1977), 125–175.

[6] Lusztig, G. and Spaltenstein, N., Induced unipotent classes, J. London Math. Soc., **19** (1979), 41–52.

[7] Shoji, T., On the Springer representations of the Weyl groups of classical algebraic groups. Comm. Algebra, **7** (1979), 1713–1745. Correction, Comm. Algebra, **7** (1979), 2027–2033.

[8] Spaltenstein, N., Classes unipotentes et sous-groupes de Borel, Lecture Notes in Math., **946** Berlin-Heidelberg-New York: Springer Verlag 1982.

[9] ——, On the generalized Springer correspondence for exceptional groups, in this volume.

[10] Springer, T. A., Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math., **36** (1976), 173–207.

[11] Wall, G. E., On the conjugacy classes in the unitary, symplectic and orthogonal groups, J. Austral. Math. Soc., **3** (1963), 1–62.

G. Lusztig
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
USA

N. Spaltenstein
Mathematisches Institut der Universität Bern
Sidlerstrasse 5
3012 Bern
Switzerland