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# The Generalized Poincaré Series of a Principal Series Representation

**Akihiko Gyoja***

***Dedicated to Professor Hirosi Nagao on his 60th birthday***

## Introduction

In [2], we defined a matrix valued function $L(t, W, \rho)$ for a representation $\rho$ of the Hecke algebra $H_q(W)$ $(q>1)$ associated to a Coxeter group $W$. And we showed that this function is similar, in property, to the congruence zeta function of an algebraic variety, i.e.,

(1) matrix components of $L(t, W, \rho)$ are rational functions,

(2) under some assumptions on $W$, the function $L(t, W, \rho)$ satisfies a functional equation,

(3) the zeros of $\det L(t, W, \rho)$ are of the forms $\zeta q^{-a}$ with some roots of unity $\zeta$ and some rational numbers $0 \leq a \leq 1$ and

(4) if $W$ is finite, the zeros on the boundary of "the critical strip" can be described explicitly in terms of vertices of a $W$-graph affording $\rho$. (See [2, introduction] for "the critical strip.")

The purpose of this paper is to determine the denominator of $\det L(t, W, R_\lambda)$ explicitly for an affine Weyl group $W$ and the "generic principal series representation" $R_\lambda$. (See (4.5) for the "generic principal series representation.")

Let us describe our result more explicitly. Let $R$ be an irreducible root system, $\{\alpha_i \mid 1 \leq i \leq l\}$ a basis of $R$, $\{\omega_i \mid 1 \leq i \leq l\}$ the fundamental weights of $R^\vee$ (=the inverse root system of $R$), $Q(R^\vee)$ (resp. $P(R^\vee)$) the root lattice (resp. weight lattice) of $R^\vee$, $\Phi(R^\vee)$ the quotient group $P(R^\vee)/Q(R^\vee)$, $\Phi(R^\vee)^\vee = \mathrm{Hom}(\Phi(R^\vee), \mathbf{C}^\times)$, $\Omega_0$ the Weyl group of $R$, $\Omega = \Omega_0 \ltimes Q$ (=the affine Weyl group), and $R_i = \{\alpha \in R \mid \langle \alpha, \omega_i \rangle = 0\}$. Define the length function $l$ on $\Omega_0 \ltimes P$ as usual (cf. [5; 3.2.1]). Suppose that $R_i$ is a direct sum of irreducible root systems $R_{i,v}$ $(v=1, 2, \cdots)$. Let $f_i = \prod_v (\sharp\Phi(R^\vee_{i,v}))$. (For a

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set $X$, $\sharp X$ denotes its cardinality.) Let $\Omega_i$ be the stabilizer of $\omega_i$ in $\Omega_0$. We have

**Main Theorem.** *The denominator of* $\det L(t, W, R_\lambda)$ *is equal to*

$$\prod_{\substack{\omega_i \bmod \Omega_0 \\ \gamma \in \Omega_i \backslash \Omega_0 \\ \chi \in \Phi(R^\vee)^\vee}} (1-(\lambda\chi)(\omega_i\gamma)(q^{1/2}t)^{l(\omega_i)})^{\sharp\Omega_i/f_i}.$$

(Ssee (4.5) for $\lambda$.)

This paper consists of four sections. In the first section, we give the Taylor expansion of

$$(\sharp) \qquad \det(1+A_1t^{l(1)}+A_2t^{l(2)}+\cdots),$$

where $A_1, A_2, \cdots$ are square matrices of the same size and $\{l(i)\}$ is a sequence of positive integers such that every number appears only finitely many times in it. (See (1.5) for the exact form of the Taylor expansion of ($\sharp$).) In the second and third sections, we define the concepts of $S$-graphs and $S$-digraphs, and construct some special $S$-digraphs. (See the beginning of Section 2 for the definitions of $S$-graphs and $S$-digraphs.) We study these $S$-digraphs closely and get an equality (3.22) as a consequence. This equality, together with the Taylor expansion of ($\sharp$), proves our main theorem (Section 4).

**Notations.** For a set $X$, $\sharp X$ denotes its cardinality. For a Coxeter group $W$, $\leq$ denotes the usual Bruhat order.

## 1.

The purpose of this section is to prove the equality (1.5) below.

Let $e_n$ be the $n$-th elementary symmetric function in "infinitely many variables" $x_1, x_2, \cdots$. (See [4; Chap. 1, Section 2] for the justification of "infinitely many variables.") Put $p_n=\sum_{i=1}^{\infty} x_i^n$. For a partition $\mu=(\mu_1\geq \mu_2\geq\cdots\geq 0)$, define

$$\begin{aligned}
|\mu| &= \textstyle\sum_{i\geq 1}\mu_i \\
\varepsilon(\mu) &= \textstyle\prod_{i\geq 1}(-1)^{m_i(i-1)} \\
z(\mu) &= \textstyle\prod_{i\geq 1} i^{m_i}\cdot m_i! \\
p(\mu) &= p_{\mu_1}p_{\mu_2}\cdots,
\end{aligned}$$

where $m_i=m_i(\mu)$ is the number of parts of $\mu$ equal to $i$. Then we have

$$(1.1) \qquad e_n=\sum_{\substack{\mu \\ |\mu|=n}} \varepsilon(\mu)z(\mu)^{-1}p(\mu)$$

[4; Chap. 1, (2.14′)]. Let $A$ be a square matrix. We shall denote by $\mathrm{tr}^{(n)}A$ the $n$-th elementary symmetric function of the eigenvalues of $A$. As a consequence of (1.1), we get

$$(1.2)\qquad \mathrm{tr}^{(n)}A=\sum_{\substack{\mu\\|\mu|=n}}\varepsilon(\mu)z(\mu)^{-1}(\mathrm{tr}\,A)^{m_1}(\mathrm{tr}\,A^2)^{m_2}\cdots,$$

where $m_i=m_i(\mu)$. (Note that $\mathrm{tr}^{(0)}A=1$.)

Let $A_1, A_2, \cdots$ be a sequence of square matrices of the same size and $l(1), l(2), \cdots$ a sequence of positive integers such that any integer appears only finitely many times in it. Then, we have the following identity.

$$\begin{aligned}(1.3)\qquad &\det(1+A_1t^{l(1)}+A_2t^{l(2)}+\cdots)\\ &=\exp(\mathrm{tr}(\log(1+A_1t^{l(1)}+\cdots)))\\ &=\exp\left(\mathrm{tr}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(A_1t^{l(1)}+\cdots)^n\right)\right)\\ &=\prod_{n=1}^{\infty}\exp\left(\frac{(-1)^{n-1}}{n}\mathrm{tr}(A_1t^{l(1)}+\cdots)^n\right)\\ &=\prod_{n=1}^{\infty}\sum_{a_n=0}^{\infty}\frac{(-1)^{(n-1)a_n}}{a_n!\,n^{a_n}}(\mathrm{tr}(A_1t^{l(1)}+\cdots)^n)^{a_n}.\end{aligned}$$

Put $\mathbf{N}=\{1, 2, \cdots\}$. The automorphism $(i_1, \cdots, i_d)\to(i_d, i_1, \cdots, i_{d-1})$ of $\mathbf{N}^d$ generates a group $G(d)$ of automorphisms. An element $I=(i_1, \cdots, i_d)$ of $\mathbf{N}^d$ is said to be primitive if $\{g\in G(d)\,|\,gI=I\}=\{1\}$. We shall denote by $P(d)$ the set of primitive elements in $\mathbf{N}^d$. Put $P=\coprod_{d\geq 1}P(d)/G(d)$. For an element $I=(i_1, \cdots, i_d)$ of $\mathbf{N}^d/G(d)$, put

$$\begin{aligned}&\mathrm{tr}\,A_I=\mathrm{tr}(A_{i_1}A_{i_2}\cdots A_{i_d}),\\ &|I|=d,\\ &l(I)=l(i_1)+\cdots+l(i_d).\end{aligned}$$

Then

$$\begin{aligned}&(\mathrm{tr}(A_1t^{l(1)}+A_2t^{l(2)}+\cdots)^n)^{a_n}\\ &=\Big(\sum_{d|n}\sum_{I\in P(d)/G(d)}\mathrm{tr}(dA_I^{n/d}\,t^{(n/d)l(I)})\Big)^{a_n}\\ &=\sum_{f_n}\frac{a_n!}{\prod_I f_n(I)!}\prod_I(\mathrm{tr}(|I|A_I^{n/|I|}\,t^{(n/|I|)l(I)}))^{f_n(I)},\end{aligned}$$

the last summation being taken over the mappings $f_n\colon \coprod_{d|n}P(d)/G(d)\to \mathbf{N}\cup\{0\}$ such that $\sum_I f_n(I)=a_n$. Hence

$$\sum_{a_n=0}^{\infty}\frac{(-1)^{(n-1)a_n}}{a_n!\,n^{a_n}}(\mathrm{tr}\,(A_1t^{l(1)}+\cdots)^n)^{a_n}$$
$$=\sum_{f_n}\Big(\prod_I\frac{(-1)^{(n-1)f_n(I)}}{f_n(I)!\,(n/|I|)^{f_n(I)}}\Big)(\prod_I(\mathrm{tr}\,A_I^{n/|I|})^{f_n(I)})(\prod_I t^{(n/|I|)l(I)f_n(I)})$$
$$=\sum_{f_n}\Big(\prod_I\frac{(-1)^{((n/|I|)-1)f_n(I)}}{f_n(I)!\,(n/|I|)^{f_n(I)}}\Big)(\prod_I(-1)^{(n-n/|I|)f_n(I)})(\prod_I(\mathrm{tr}A_I^{n/|I|})^{f_n(I)})$$
$$\cdot(\prod_I t^{(n/|I|)l(I)f_n(I)}),$$

the second and the third summations being taken all over the mappings $f_n$: $\coprod_{d|n}P(d)/G(d)\to\mathbf{N}\cup\{0\}$ such that $\sum_I f_n(I)<\infty$. This equality, together with (1.3), implies

$$(1.4)\qquad \det\,(1+A_1t^{l(1)}+A_2t^{l(2)}+\cdots)$$
$$=\sum_{(f_1,f_2,\dots)}\Big(\prod_{\substack{I,n\\|I||n}}\frac{(-1)^{((n/|I|)-1)f_n(I)}}{f_n(I)!\,(n/|I|)^{f_n(I)}}\Big)$$
$$\cdot(\prod_{\substack{I,n\\|I||n}}(-1)^{(n-n/|I|)f_n(I)})(\prod_{\substack{I,n\\|I||n}}(\mathrm{tr}\,A_I^{n/|I|})^{f_n(I)})$$
$$\cdot(\prod_{\substack{I,n\\|I||n}}t^{(n/|I|)l(I)f_n(I)}).$$

Put $g_m(I)=f_{m|I|}(I)$ for $I\in P$ and $m\in\mathbf{N}$. Define a partition $\mu(I)$ by $\mu(I)=(1^{g_1(I)}2^{g_2(I)}\cdots)$. (See [4; Chap. 1] for this expression.) Let $\Phi$ be the set of mappings $\varphi$: $P\to\mathbf{N}\cup\{0\}$ such that $\varphi(I)=0$ except for finitely many $I$'s. Then (1.4) can be rewritten as

$$\det\,(1+A_1t^{l(1)}+A_2t^{l(2)}+\cdots)$$
$$=\sum_{(g_1,g_2,\dots)}\Big(\prod_{I,m}\frac{(-1)^{(m-1)g_m(I)}}{g_m(I)!\,m^{g_m(I)}}\Big)(\prod_{I,m}(-1)^{m(|I|-1)g_m(I)})$$
$$\cdot(\prod_{I,m}(\mathrm{tr}\,A_I^m)^{g_m(I)})(\prod_{I,m}t^{ml(I)g_m(I)})$$
$$=\sum_{\varphi\in\Phi}\sum_{\substack{(g_1,g_2,\dots)\\|\mu(I)|=\varphi(I)}}(\prod_I\varepsilon(\mu(I)z(\mu(I))^{-1})(\prod_I(-1)^{\varphi(I)(|I|-1)})$$
$$\cdot(\prod_{I,m}(\mathrm{tr}\,A_I^m)^{g_m(I)})(\prod_I t^{\varphi(I)l(I)}).$$

Then by (1.2), we get

$$(1.5)\qquad \det\,(1+A_1t^{l(1)}+A_2t^{l(2)}+\cdots)$$
$$=\sum_{\varphi\in\Phi}(-1)^{\Sigma\varphi(I)(|I|-1)}(\prod_I\mathrm{tr}^{(\varphi(I))}A_I)\,t^{\Sigma\varphi(I)l(I)}.$$

**1.6. Example.** Let $\det\,(1+A_1t+A_2t^2+\cdots)=1+a_1t+a_2t^2+\cdots.$

Then

$$a_1 = \mathrm{tr}\, A_1$$
$$a_2 = \mathrm{tr}\, A_2 + (\mathrm{tr}^{(2)} A_1 + (\mathrm{tr}\, A_1)^2)$$
$$a_3 = \mathrm{tr}\, A_3 + (-\mathrm{tr}\, A_2 A_1 + \mathrm{tr}\, A_2\, \mathrm{tr}\, A_1) + (\mathrm{tr}^{(3)} A_1 + \mathrm{tr}^{(2)} A_1\, \mathrm{tr}\, A_1 + (\mathrm{tr}\, A_1)^3)$$
$$\begin{aligned} a_4 = {} & \mathrm{tr}\, A_4 + (-\mathrm{tr}\, A_3 A_1 + \mathrm{tr}\, A_3\, \mathrm{tr}\, A_1) + (\mathrm{tr}^{(2)} A_2 + (\mathrm{tr}\, A_2)^2) \\ & + (\mathrm{tr}\, A_2 A_1^2 - \mathrm{tr}\, A_2 A_1\, \mathrm{tr}\, A_1 + \mathrm{tr}\, A_2\, \mathrm{tr}^{(2)} A_1 + \mathrm{tr}\, A_2\, (\mathrm{tr}\, A_1)^2) \\ & + (\mathrm{tr}^{(4)} A_1 + \mathrm{tr}^{(3)} A_1\, \mathrm{tr}\, A_1 + (\mathrm{tr}^{(2)} A_1)^2 + \mathrm{tr}^{(2)} A_1\, (\mathrm{tr}\, A_1)^2 + (\mathrm{tr}\, A_1)^4) \end{aligned}$$

etc.

## 2.

In this section, we define the notions of $S$-graphs and $S$-diagraphs, and study them.

Let $(W, S)$ be a Coxeter system. We define an $S$-graph to be a (pseudo-) graph together with the following datum: for each edge $x - y$, we are given an element $s$ of $S$. (We write $x \overset{s}{—} y$). This datum is subject to the following requirement. If

$$x_0 \overset{s_1}{——} x_1 \overset{s_1}{——} \cdots \overset{s_n}{——} x_n$$

is a path such that $s_1 s_2 \cdots s_n = 1$, then $x_0 = x_n$.

An $S$-digraph (=directed $S$-graph) $\Gamma$ is a directed (pseudo-) graph together with the following datum: for each directed edge $x \to y$, we are given an element $s$ of $S$. (We write $x \xrightarrow{s} y$.) This datum is subject to the following requirements.

(1) If we forget the directions of edges, $\Gamma$ becomes an $S$-graph, which is denoted by $f(\Gamma)$.

(2) If $x \underset{t}{\overset{s}{\rightrightarrows}} y$, then $s \neq t$.

A morphism between $S$-graphs (resp. $S$-digraphs) is a morphism $\varphi$ of graphs (resp. digraphs) such that $x \overset{s}{—} y$ implies $\varphi(x) \overset{s}{—} \varphi(y)$ (resp. $x \xrightarrow{s} y$ implies $\varphi(x) \xrightarrow{s} \varphi(y)$). Thus the totality of the $S$-graphs (resp. the $S$-digraphs) becomes a category. The automorphisms, the injections, etc. of $S$-graphs (resp. $S$-digraphs) can be defined as usual. (A morphism of $S$-digraphs is injective (resp. epimorphic) iff it induces an injection between vertices (resp. iff it induces an epimorphism between the connected components).)

An $S$-graph is said to be simply connected if for any closed path

$$x_0 \overset{s_1}{\text{———}} x_1 \overset{s_2}{\text{———}} \cdots \overset{s_n}{\text{———}} x_n = x_0,$$

we have $s_1 s_2 \cdots s_n = 1$.

If a morphism $\varphi$ of $S$-digraphs induces epimorphisms of vertices and edges, then $\varphi$ is called a covering map. Let $\Gamma_1, \Gamma_2$ be $S$-digraphs. If there exists a covering map $\Gamma_1 \to \Gamma_2$, $\Gamma_1$ is called a covering of $\Gamma_2$. If $\Gamma_1$ is a covering of $\Gamma_2$, $f(\Gamma_1)$ is connected and $f(\Gamma_2)$ is simply connected, then $\Gamma_1 \to \Gamma_2$ is an isomorphism.

**2.1.** An $S$-digraph $\Gamma$ is said to be complete if the following condition is satisfied. If $\Gamma$ has a path of the form

$$(\sharp) \qquad x_0 \xleftarrow{s(1)} x_1 \xleftarrow{s(2)} \cdots \xleftarrow{s(m)} x_m,$$

where

$$s(i) = \begin{cases} s, & \text{if } i \text{ is odd} \\ t, & \text{if } i \text{ is even,} \end{cases}$$

$$s, t \in S,$$

$$m = \operatorname{ord}(st),$$

$$1 < m < \infty,$$

then $\Gamma$ has also a path from $x_m$ to $x_0$ such that

$$x_0 \xleftarrow{s(0)} x'_1 \xleftarrow{s(1)} \cdots \longleftarrow x'_{m-1} \xleftarrow{s(m-1)} x_m.$$

(We call a path of the form ($\sharp$) a dihedral path.)

**2.2.** Let $\Gamma$ be an $S$-digraph. A pair $(\bar{\Gamma}, \iota)$ of a complete $S$-digraph $\bar{\Gamma}$ and an injection $\iota: \Gamma \to \bar{\Gamma}$ is, by definition, a completion of $\Gamma$, if the following condition is satisfied. If $\Gamma'$ is an arbitrary complete $S$-digraph and $\varphi$ is a morphism of $\Gamma$ into $\Gamma'$, then there exists a unique morphism $\bar{\varphi}$ such that the following diagram becomes commutative:

$$\begin{tikzcd} \Gamma' & \\ \Gamma \arrow[u, "\varphi"] \arrow[r, "\iota"'] & \bar{\Gamma} \arrow[ul, "\bar{\varphi}"'] \end{tikzcd}$$

**2.3. Lemma.** *For any $S$-digraph $\Gamma$, there exists a unique completion $(\bar{\Gamma}, \iota)$ up to isomorphism.*

*Proof.* It suffices to show the existence. Let

$$x_0 \xleftarrow{s(1)} x_1 \xleftarrow{s(2)} \cdots \xleftarrow{s(m)} x_m \qquad (m<\infty)$$

be a dihedral path. Assume that $\Gamma$ contains paths

$$x_0 \xleftarrow{s(0)} x_1' \xleftarrow{s(1)} \cdots \xleftarrow{s(k-1)} x_k',$$

and

$$x_l' \xleftarrow{s(l)} \cdots \xleftarrow{s(m-2)} x_{m-1}' \xleftarrow{s(m-1)} x_m,$$

and that $\Gamma$ does not contain edges of the form

$$x_k' \xleftarrow{s(k)} x_{k+1}' \quad \text{or} \quad x_{l-1}' \xleftarrow{s(l-1)} x_l'.$$

Then $k \leq l-1$. Construct an $S$-digraph $\Gamma^*$ by adding to $\Gamma$ new vertices $x_{k+1}', \cdots, x_{l-1}'$ and new edges

$$x_k' \xleftarrow{s(k)} x_{k+1}' \longleftarrow \cdots \xleftarrow{s(l-1)} x_l'.$$

Let $\varphi$ be a morphism of $\Gamma$ into a complete $S$-digraph $\Gamma'$. Since $\Gamma'$ is complete, there is a unique path of the form

$$(\varphi(x_0)=)\ y_0 \xleftarrow{s(0)} y_1 \xleftarrow{s(1)} \cdots \xleftarrow{s(m-1)} y_m \ (=\varphi(x_m)).$$

Hence $\varphi\colon \Gamma \to \Gamma'$ can be uniquely extended to a morphism $\varphi^*\colon \Gamma^* \to \Gamma'$. We make this operation to all the dihedral path of $\Gamma$ which satisfy our assumption and construct a new $S$-digraph $\Gamma_1$. Then $\varphi\colon \Gamma \to \Gamma'$ can be uniquely extended to $\varphi_1\colon \Gamma_1 \to \Gamma'$. In this way, we construct succesively $S$-digraphs $\Gamma_1, \Gamma_2, \cdots$ and put $\bar{\Gamma} = \varinjlim \Gamma_n$. Then this $\bar{\Gamma}$ with the natural inclusion map $\iota\colon \Gamma \to \bar{\Gamma}$ is a completion of $\Gamma$.

**2.4.** Let $\Gamma$ be an $S$-graph. Let $\Gamma^+$ be the two dimensional cell complex which is obtained by attaching one 2-cell for each subgraph of the form

$$\begin{tikzcd}
 & x_1 \arrow[r, dash, "s(2)\ \cdots\ s(m-1)"] & x_{m-1} \arrow[dr, dash, "s(m)"] & \\
x_0 \arrow[ur, dash, "s(1)"] \arrow[dr, dash, "s(0)"'] & & & x_m \\
 & y_1 \arrow[r, dash, "s(1)\ \cdots\ s(m-2)"'] & y_{m-1} \arrow[ur, dash, "s(m-1)"'] &
\end{tikzcd} \qquad (m<\infty)$$

or

$$x_1 \overset{s}{\underset{s}{=\!=\!=}} x_2,$$

where $s(i)$ and $m$ are defined as in (2.1). (Here we do not assume that $x_i$'s and $y_j$'s are all distinct.) Let

$$x_0=\omega(0)\overset{s_1}{\text{———}}\omega(1/n)\overset{s_2}{\text{———}}\cdots\text{———}\omega((n-1)/n)\overset{s_n}{\text{———}}\omega(1)=x_0$$

be a closed path of $\Gamma$ and $[\omega]$ its homotopy class of $\pi_1(\Gamma^+, x_0)$. Then the element $s_n \cdots s_2 s_1$ depends only on the homotopy class $[\omega]$. We denote this element by $\theta([\omega])$.

**2.5.** Let $w$ be an element of $W$ such that $l(w^k)=kl(w)$ $(k\geq 0)$. Let $w=s_n \cdots s_2 s_1$ $(s_i \in S)$ be a reduced expression of $w$. Consider the following $S$-digraph

$$\begin{tikzcd} \bar{1} \arrow[r, "s_2"'] & \bar{2} \arrow[r, "s_3"'] & \cdots \arrow[r, "s_n"'] & \bar{n} \arrow[lll, bend right, "s_1"'] \end{tikzcd}$$

where $\bar{i}=i \bmod n$. Denote this graph by $\Gamma(s_1, \cdots, s_n)$. We know that any reduced expression of $w$ can be obtained from one reduced expression by using the relation

$$sts\cdots=tst\cdots \qquad (m \text{ factors}),$$
$$s, t \in S, \qquad m=\operatorname{ord}(st).$$

(See [1; Chap. IV, § 1, Lemma 4]). Hence the completion $\bar{\Gamma}(s_1, \cdots, s_n)$ of $\Gamma(s_1, \cdots, s_n)$ does not depend on the choice of the reduced expression. (More precisely, let $w=s'_n \cdots s'_1$ be another reduced expression. There is a unique path $\Gamma'(s'_1, \cdots, s'_n)$ of the form

$$\begin{tikzcd} x_1 \arrow[r, "s'_2"'] & x_2 \arrow[r, "s'_3"'] & \cdots \arrow[r, "s'_n"'] & x_n=\bar{n} \arrow[lll, bend right, "s'_1"'] \end{tikzcd}$$

in $\bar{\Gamma}(s_1, \cdots, s_n)$. (Here $x_i \neq x_j$ if $i \neq j$.) And

$$\Gamma(s'_1, \cdots, s'_n) \xrightarrow{\sim} \Gamma'(s'_1, \cdots, s'_n) \hookrightarrow \bar{\Gamma}(s_1, \cdots, s_n)$$

is a completion. Note that the point $\bar{0}=\bar{n}$ is also independent of the choice of the reduced expression. We denote this completion by $\bar{\Gamma}(w)$.

Let

$$\omega(0) \overset{s_1'}{\text{———}} \omega(1/N) \overset{s_2'}{\text{———}} \omega(2/N) \text{———} \cdots \overset{s_N'}{\text{———}} \omega(1)$$

be a path of $f(\bar{\Gamma}(w))$. We count the edges contained in this path with alternating signs; an edge $\omega(i/N) - \omega(i+1/N)$ is counted with $+1$ if $\omega(i/N) \to \omega(i+1/N)$ in $\bar{\Gamma}(w)$ and is counted with $-1$ if $\omega(i/N) \leftarrow \omega(i+1/N)$ in $\bar{\Gamma}(w)$. The sum of these $\pm 1$ over all the edges contained in this path is denoted by $i([\omega])$. If $\omega(0) = \omega(1) = x_0$, this number $i([\omega])$ depends only on the homotopy class $[\omega] \in \pi_1((f\bar{\Gamma}(w))^+, x_0)$ of $\omega$ and defines an isomorphism

$$i\colon \pi_1((f\bar{\Gamma}(w))^+, x_0) \xrightarrow{\sim} n\mathbf{Z}.$$

Hence the local system $x_0 \mapsto \pi_1((f\bar{\Gamma}(w))^+, x_0)$ is trivial and there is a uniquely determined isomorphism

$$\pi_1((f\bar{\Gamma}(w))^+, x_0) \xrightarrow{\sim} \pi_1((f\bar{\Gamma}(w))^+, x_0'). \qquad (x_0, x_0' \in \bar{\Gamma}(w)).$$

This isomorphism is compatible with the isomorphism $i$. Let $\alpha(x_0)$ be the element of $\pi_1((f\bar{\Gamma}(w))^+, x_0)$ which corresponds to $n \in n\mathbf{Z}$ by the isomorphism $i$. Denote the element $\theta(\alpha(x_0))$ by $\theta(x_0)$. (See (2.4) for $\theta$.) Let

$$x_0 = y_0 \overset{s_1'}{\text{———}} y_1 \overset{s_2'}{\text{———}} \cdots \overset{s_m'}{\text{———}} y_m = x_0'$$

be a path of $f\bar{\Gamma}(w)$ connecting two vertices $x_0$ and $x_0'$. Put $\gamma = s_m' \cdots s_2' s_1'$. Then

$$\theta(x_0) = \gamma^{-1}\theta(x_0')\gamma. \tag{2.6}$$

**2.7.** Let $w$ be an element of $W$ as in (2.5). Let $S_0$ be a subset of $S$ such that $l(w) = l(sws)$ $(s \in S_0)$. Let $W_0$ be the parabolic subgroup generated by $S_0$. Let $\gamma \in W_0$ and $w^\gamma = \gamma^{-1}w\gamma$. If $l(sw^\gamma) = l(w^\gamma s)$ $(s \in S_0)$, $sw^\gamma = w^\gamma s$. (In fact, for any elements $s, t \in S$ and $w \in W$, "$l(swt) = l(w)$ and $l(sw)$ $l(wt)$" implies $sw = wt$.) Hence if $w^{\gamma s} \neq w^\gamma$, $l(sw^\gamma) > l(w^\gamma) > l(w^\gamma s)$ or $l(sw^\gamma) < l(w^\gamma) < l(w^\gamma s)$. Let $\Gamma_0(w)$ be the $S_0$-digraph whose vertices are $\{w^\gamma \mid \gamma \in W_0\}$ and such that two vertices $w^\gamma$ and $w^{\gamma s}$ $(s \in S_0)$ are connected in the following way. If $l(w^\gamma s) < l(w^\gamma)$, $w^{\gamma s} \overset{s}{\leftarrow} w^\gamma$. And we assume that $\Gamma_0(w)$ has no other edges. The $S_0$-graph $f\Gamma_0(w)$ is connected.

**2.8.** Let $\bar{\Gamma}_1(w)$ be the $S_0$-digraph which is obtained from $\bar{\Gamma}(w)$ by deleting all the edges corresponding to the elements in $S - S_0$. Let $\bar{\Gamma}_0(w)$ be the connected component of $\bar{\Gamma}_1(w)$ which contains $\bar{0}$.

**Lemma.** The $S_0$ -digraph $\bar{\Gamma}_0(w)$ is a covering of $\bar{\Gamma}_0(w)$. Especially, if $f\Gamma_0(w)$ is simply connected, the two $S_0$-digraphs $\Gamma_0(w)$ and $\bar{\Gamma}_0(w)$ are isomorphic.

*Proof.* In (2.5) we defined a mapping $\theta: \bar{\Gamma}(w) \to W$. Let us show that this mapping induces a covering map $\bar{\Gamma}_0(w) \to \Gamma_0(w)$. By (2.6), $\theta(\bar{\Gamma}_0(w))$ is contained in $\{w^{\gamma} \mid \gamma \in W_0\}$. Let $x \xrightarrow{s} y$ be an edge of $\bar{\Gamma}_0(w)$. Let

$$\begin{tikzcd} x=\omega(n-1/n) \arrow[rrr, bend left, "s'_n=s"] & \cdots \arrow[l, "s'_{n-1}"'] & \omega(1/n) \arrow[l, "s'_2"'] & \omega(0)=y \arrow[l, "s'_1"'] \end{tikzcd}$$

be a closed path of $\bar{\Gamma}(w)$ which contains $x \xrightarrow{s} y$ as an edge. Then

$$\theta(x) = s'_{n-1} \cdots s'_1 s,$$

$$\theta(y) = s s'_{n-1} \cdots s'_1.$$

By the assumption on $S_0$, $l(\theta(x)) = l(\theta(y)) = n$. Hence

$$\theta(x) = \theta(y)^s \xrightarrow{s} \theta(y).$$

Thus $\theta$ induces a morphism between $S$-digraphs.

Assume that $\theta(\bar{\Gamma}_0(w)) \subsetneq \{w^{\gamma} \mid \gamma \in W_0\}$. Then there exist $x \in \bar{\Gamma}_0(w)$, $y' \in \Gamma_0(w) - \theta(\bar{\Gamma}_0(w))$ such that $\theta(x) \xrightarrow{s} y'$ or $\theta(x) \xleftarrow{s} y'$ ($s \in S_0$). If $\theta(x) \xrightarrow{s} y'$, then $y' = \theta(x)^s$ and $l(\theta(x)s) < l(\theta(x))$ ($=n$). Hence there is a reduced expression of the form

$$\theta(x) = s'_n \cdots s'_2 s.$$

Hence there is a closed path of $\bar{\Gamma}(w)$ of the form

$$\begin{tikzcd} y \arrow[r, "s'_2"'] & \cdots \arrow[r, "s'_n"'] & x \arrow[ll, bend right, "s"'] \end{tikzcd}$$

But then $y \in \bar{\Gamma}_0(w)$ and $\theta(y) = y'$, which is absurd. The case $\theta(x) \xleftarrow{s} y'$ can be treated in the same way. Hence $\theta(\bar{\Gamma}_0(w)) = \{w^{\gamma} \mid \gamma \in W_0\}$. Moreover, it can be proved in the same way that every edge $x' \xrightarrow{s} y'$ of $\Gamma_0(w)$ comes from some edge $x \xrightarrow{s} y$ of $\bar{\Gamma}_0(w)$. Hence $\theta$ induces a covering map.

**2.9.** Let $\bar{\Gamma}'_0(w)$ be any connected component of $\bar{\Gamma}_1(w)$. If $\theta(\bar{\Gamma}'_0(w))$ is contained in $\{w^{\gamma} | \gamma \in W_0\}$, by the same argument as in (2.8), we can show that $\Gamma_0'(w)$ is a covering of $\Gamma_0(w)$.

## 3.

In this section we construct some $S$-digraphs and study them. The main purpose of this study is to get the equality (3.22), which will be used in the next section.

**3.1.** First of all, let us fix some notations relative to affine Weyl groups. The basic references are [1] and [3].

Let $R$ be a reduced, irreducible root system of rank $l \geq 1$ and $\{\alpha_1, \cdots, \alpha_l\}$ a set of simple roots. Let $\alpha_0$ be the highest root of $R$. Let $V$ be the vector space spanned by $R$, $V^*$ the dual space of $V$, $E$ the underlying affine space of $V^*$ and $R^\vee$ the inverse root system of $R$. For $\alpha \in R$ and $k \in \mathbf{Z}$, put

$$H_{\alpha, k} = \{x \in E | \langle \alpha, x \rangle = k\},$$

where $\langle\ ,\ \rangle$ is the natural pairing of $V$ and $V^*$. Let $\mathscr{F}$ be the totality of these hyperplanes. Each hyperplane $H \in \mathscr{F}$ defines an orthogonal reflection $e \rightarrow e\sigma_H$ in $E$ with fixed point set $H$. Let $\Omega$ be the group of affine motions generated by $\sigma_H$ ($H \in \mathscr{F}$). It is known that this group $\Omega$ satisfies the assumption in [3; 1.1], i.e., $\Omega$ is an infinite discrete subgroup of the group of affine motions of $E$, acting irreducibly on $V^*$ and leaving stable the set $\mathscr{F}$. For each special point $v$, we put

$$\mathscr{C}_v^+ = \{x \in E | 0 < \langle \alpha_i, x - v \rangle \ (1 \leq i \leq l)\}.$$

These cones $\mathscr{C}_v^+$ also satisfies the assumption in [3; 1.1], i.e., for any two special points $v$ and $v'$, $\mathscr{C}_{v'}^+$ is a translate of $\mathscr{C}_v^+$. Thus we may use the notations and definitions of [3; 1.1–1.4] without any change. For any unexplained notation, the reader is referred to [3; 1.1–1.4].

Let $\{\omega_1, \cdots, \omega_l\}$ be the vectors in $V^*$ such that $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$. These are the fundamental weights of $R^\vee$. Let $P = P(R^\vee)$ (resp. $Q = Q(R^\vee)$) be the lattice of $V^*$ generated by $\{\omega_1, \cdots, \omega_l\}$ (resp. $\{\alpha_1^\vee, \cdots, \alpha_l^\vee\}$).

Let $W$ be an affine Weyl group and $S$ its canonical generator ([3; 1.1]). This group $W$ acts on the set of alcoves from the left. For an element $\gamma \in \Omega$ (resp. $w \in W$), there is a unique element $\bar{\gamma} \in W$ (resp. $\bar{w} \in \Omega$) such that $\bar{\gamma} A_0^+ = A_0^+ \gamma$ (resp. $A_0^+ \bar{w} = w A_0^+$). For two elements $\gamma_1, \gamma_2 \in \Omega$, we have $\bar{\gamma}_1 \bar{\gamma}_2 A_0^+ = \bar{\gamma}_1 A_0^+ \gamma_2 = A_0^+ \gamma_1 \gamma_2 = \overline{\gamma_1 \gamma_2} A_0^+$. Hence $\gamma \rightarrow \bar{\gamma}$ is a homomorphism of $\Omega$ into $W$. The mapping $w \rightarrow \bar{w}$ is also a homomorphism of $W$ into $\Omega$. It is clear that $\bar{\bar{\gamma}} = \gamma$ ($\gamma \in \Omega$) and $\bar{\bar{w}} = w$ ($w \in W$). Especially $\Omega$ is isomorphic to $W$ and $(\Omega, \bar{S})$ is a Coxeter system.

**3.2.** Let $c_i$ be the smallest positive integer such that $c_i\omega_i \in Q$. For an element $\omega$ of $V^*$, $t(\omega)$ denotes the translation by $\omega$. Let $R_i$ be the intersection of $R$ with the subspace spanned by $\{\alpha_j | j \neq i\}$ and $R_i^+ = R_i \cap \{\alpha \in R | \alpha > 0\}$. Put

$$\beta_i = t(c_i\omega_i), \qquad l_i = l(\bar{\beta}_i),$$
$$K_i = \{x \in E \,|\, 0 < \langle \alpha, x \rangle < 1 \qquad (\alpha \in R_i^+)\}.$$

In the rest of this section, we fix $i$. So we write sometimes $l$ for $l_i$, if there is no fear of confusion.

We construct an $S$-digraph $\bar{\Gamma}_s^{\sim}$ as follows. The vertices are the alcoves contained in $K_i$. If $A$, $B$ are two alcoves contained in $K_i$ such that they have a common face of type $s\,(\in S)$ and $s \notin \mathscr{L}(A)$, then two vertices $A$, $B$ are connected in the following way.

$$A \xrightarrow{s} B$$

And assume that $\bar{\Gamma}_i^{\sim}$ has no other edges. Then $\bar{\Gamma}_i^{\sim}$ is an $S$-digraph and $f(\bar{\Gamma}_i^{\sim})$ is simply connected. Let $G_i$ be the group generated by $\beta_i$. Then $G_i$ acts on $\bar{\Gamma}_i^{\sim}$ as an automorphism group by

$$A \longmapsto A\gamma \qquad (\gamma \in G_i).$$

Hence we can naturally construct a new $S$-digraph $\bar{\Gamma}_i = \bar{\Gamma}_i^{\sim}/G_i$.

Let $\bar{\beta}_i = s_l \cdots s_2 s_1$ $(s_i \in S)$ be a reduced expression of $\bar{\beta}_i$. Then the set of alcoves

$$\begin{gathered} A_0^+\beta_i^n \\ s_1A_0^+\beta_i^n \\ \cdots \\ s_{l-1}\cdots s_1A_0^+\beta_i^n \qquad (n \in \mathbf{Z}) \end{gathered}$$

defines a full subgraph $\Gamma_i^{\sim}$ of $\bar{\Gamma}_i^{\sim}$, which becomes an $S$-digraph. Note that $\Gamma_i^{\sim}$ depends on the choice of the reduced expression. The action of $G_i$ preserves $\Gamma_i^{\sim}$. Hence we can construct another $S$-digraph $\Gamma_i = \Gamma_i^{\sim}/G_i$.

**3.3. Lemma** (1) *The $S$-digraph $\bar{\Gamma}_i^{\sim}$ is a completion of $\Gamma_i^{\sim}$.*
(2) *The $S$-digraph $\bar{\Gamma}_i$ is a completion of $\Gamma_i$.*

*Proof.* (1) Let $\Gamma$ be a complete $S$-digraph and $\varphi: \Gamma_i^{\sim} \to \Gamma$ be a morphism. Let $x$ be a vertex of $\bar{\Gamma}_i^{\sim}$. Then there is a path of $\bar{\Gamma}_i^{\sim}$ of the form

$$x_{-N-1} \xleftarrow{s_{-N}} \cdots \xleftarrow{s_{-1}} x_{-1} \xleftarrow{s_0} x_0 = x \xleftarrow{s_1} x_1 \xleftarrow{s_2} \cdots \xleftarrow{s_M} x_M,$$

$$x_{-N-1}, \qquad xM \in \Gamma_i^{\sim}$$

(Take alcoves $x_{-N-1}$, $x_M$ far enough from the alcove $x$. Take points $a_- \in x_{-N-1}$, $a_0 \in x$ and $a_+ \in x_M$ in general position. Since any face contained in $K_i$ is transversal to $\omega_i$, it is also transversal to the vectors $\overrightarrow{a_+a_0}$ and $\overrightarrow{a_0a^-}$. Let $x_M, \cdots, x_0$ (resp. $x_0, \cdots, x_{-N-1}$) be the alcoves which intersect the segment $\overrightarrow{a_+a_0}$ (resp. $\overrightarrow{a_0a_-}$). We may assume that these segments do not intersect with any facets of codimension greater than one and that $x_i$ and $x_{i+1}$ have a common face. Thus we get a path of the above form.) Then $\Gamma_i^{\sim}$ has a path connecting $x_{-N-1}$ and $x_M$

$$x_{-N-1}=y_{-N-1}\xleftarrow{s'_{-N}}\cdots\xleftarrow{s'_M}y_M=x_M.$$

Then $\Gamma$ has the path

$$\varphi(y_{-N-1})\xleftarrow{s'_{-N}}\cdots\xleftarrow{s'_M}\varphi(y_M).$$

Since $\Gamma$ is complete, $\Gamma$ has a path of the form

$$\varphi(y_{-N-1})=z_{-N-1}\xleftarrow{s_{-N}}\cdots\xleftarrow{s_0}z_0\xleftarrow{s_1}\cdots\xleftarrow{s_M}z_M=\varphi(y_M).$$

Put $\bar{\varphi}(x)=z_0$. Since $\bar{\Gamma}_i^{\sim}$ is simply connected, this is well defined and an extension of $\varphi$. Hence $\bar{\Gamma}_i^{\sim}$ is a completion of $\Gamma_i^{\sim}$.

(2) Let $\Gamma$ be a complete $S$-digraph and $\varphi$: $\Gamma_i \to \Gamma$ be a morphism. Then there is a uniquely determined morphism $\psi$: $\bar{\Gamma}_i^{\sim} \to \Gamma$ such that the following diagram becomes commutative.

$$\begin{tikzcd}
\Gamma & \\
\Gamma_i \arrow[u, "\varphi"] & \\
\Gamma_i^{\sim} \arrow[u] \arrow[r, hook] & \bar{\Gamma}_i^{\sim} \arrow[uul, "\psi"']
\end{tikzcd}$$

Since $\psi$ is uniquely determined, $\psi$ is $G_i$-invariant and induces a morphism

$$\bar{\varphi}: \bar{\Gamma}_i \to \Gamma,$$

which is an extension of $\varphi$.

**3.4.** The element $\bar{\beta}_i$ satisfies the assumption of (2.5), i.e., $l(\bar{\beta}_i^k)=kl(\bar{\beta}_i)$ $(k \geq 0)$. (See [5; 3.2.3]). Hence we can use the results of (2.5). If

$\bar{\beta}_i = s_l \cdots s_2 s_1$ ($s_i \in S$) is the reduced expression used to construct $\Gamma_i^{\sim}$, then $\Gamma_i$ is isomorphic to $\Gamma(s_1, \cdots, s_l)$. Hence $\bar{\Gamma}_i$ is isomorphic to $\bar{\Gamma}(\bar{\beta}_i)$. (See (2.5) for the definition of $\bar{\Gamma}(\bar{\beta}_i)$.)

For an alcove $A$, define an element $\theta(A)$ of $W$ by

$$\theta(A)A = A\beta_i.$$

Then $\theta$: $\bar{\Gamma}_i^{\sim} \to W$ is $G_i$-invariant and induces $\theta$: $\bar{\Gamma}_i \to W$. Then the diagram

$$\begin{tikzcd} \bar{\Gamma}_i \arrow[r, "\theta"] \arrow[d, "\wr"'] & W \\ \bar{\Gamma}(\bar{\beta}_i) \arrow[ur, "\theta"'] & \end{tikzcd}$$

is commutative. (See (2.5) for the definition of $\theta$.) Any alcove can be expressed uniquely as $A = w^{-1}A_0^+ t(p)$ ($w \in W_0, p \in Q$). Since $w^{-1}A_0^+ t(p)\beta_i = w^{-1}\bar{\beta}_i w.\ w^{-1}A_0^+ t(p)$, we have $\theta(A) = \bar{\beta}_i^w$. Hence

(3.5) $$\theta(\bar{\Gamma}(\bar{\beta}_i)) = \{\bar{\beta}_i^w \mid w \in W_0\}.$$

**3.6.** Let $\Omega_i$ be the stabilizer of $\omega_i$ in $\Omega_0$, where $\Omega_0$ is the stabilizer of 0 in $\Omega$. For a natural number $f$ and an element $w$ of $W$, put

$$N(f, w) = \{I = (w_1, \cdots, w_f) \mid w_j \in W - \{1\},\ \textstyle\sum_{j=1}^f l(w_j) = l(w),$$
$$w_f \cdots w_2 w_1 = w\}.$$

Let $G(f)$ be the group generated by the automorphism

$$(w_1, \cdots, w_f) \longmapsto (w_f, w_1, \cdots, w_{f-1})$$

of $W^f$. Put

$$N(f) = \textstyle\coprod_{\gamma \in \Omega_i \backslash \Omega_0} N(f, \overline{\beta_i^\gamma}),$$
$$N = \textstyle\coprod_{f \leq l} N(f)/G(f).$$

A subgraph of $\bar{\Gamma}(\bar{\beta}_i)$ of the form

(3.7) $$\begin{tikzcd} x_l & \cdots \arrow[l, "s'_{l-1}"] & x_1 \arrow[l, "s'_1"] \arrow[from=1-1, to=1-3, bend left, "s'_l"] \end{tikzcd}$$

is called a global section. Let $J$ be a set of vertices of $\bar{\Gamma}(\bar{\beta}_i)$ which is contained in some global section. Let $M$ be the totality of such a set $J$. Put $M(f) = \{J \in M \mid \# J = f\}$. Assume that $J(\in M(f), \neq \phi)$ is contained in the global section (3.7) and put $J=\{x_{i_1}, \cdots, x_{i_f}\}$ $(i_1<\cdots<i_f)$. Put

$$w=s'_{i_2-1}\cdots s'_{i_1},\ w_2=s'_{i_3-1}\cdots s'_{i_2},\ \cdots,\ w_f=s'_{i_1-1}\cdots s'_1 s'_l\cdots s'_{i_f}.$$

Then $(w_1, \cdots. w_f)$ defines an element of $N(f)/G(f)$. Let $\mathrm{Aut}_i$ be the automorphism group of $\bar{\Gamma}(\bar{\beta}_i)$. Then the mapping $M(f)\to N(f)/G(f)$ is $\mathrm{Aut}_i$-invariant and induces a mapping

$$\xi\colon M(f)/\mathrm{Aut}_i \to N(f)/G(f).$$

Assume that two elements $J$, $J'$ of $M(f)$ correspond to the same element of $N(f)/G(f)$. Let $\Gamma$(resp. $\Gamma'$) be a global section containing $J$ (resp. $J'$). By the assumption, we may assume that $\Gamma$ is isomorphic to $\Gamma'$. Moreover we may assume that there is an isomorphism $f\colon \Gamma\to\Gamma'$ such that $f(J)=J'$. As is easily verified, $\bar{\Gamma}(\bar{\beta}_i)$ is a completion of any global section. Hence $f$ can be extended to an automorphism of $\bar{\Gamma}(\bar{\beta}_i)$. Hence $\xi$ is injective. By (2.8), for any $\gamma\in\Omega_0$, there is a global section of the form (3.7) such that $s'_l\cdots s'_1=\bar{\beta}_i^{\gamma}$. Assume that $(w_1, \cdots, w_f)$ is an element of $N(f, \bar{\beta}_i^{\gamma})$ and that $w_1=s''_{i_2-1}\cdots s''_1$, $w_2=s''_{i_3-1}\cdots s''_{i_2}$, $\cdots$ be reduced expressions. Since $\bar{\Gamma}(\bar{\beta}_i)$ is complete, there is a global section of the form

$$\begin{tikzcd} x'_l \arrow[rr, bend left, "s''_l"] & \cdots \arrow[l, "s''_{l-1}"'] & x'_1 = x_1. \arrow[l, "s''_1"'] \end{tikzcd}$$

Put $J=\{x_1, x_{i_2}, \cdots, x_{i_f}\}$. Then $\xi(J)$ is the class of $(w_1, \cdots, w_f)$. Hence

(3.8) $$\xi\colon M(f)/\mathrm{Aut}_i \xrightarrow{\sim} N(f)/G(f).$$

Put $M'=M-\{\phi\}$. Then

(3.9) $$\xi\colon M'/\mathrm{Aut}_i \xrightarrow{\sim} \textstyle\coprod_{f\le l} N(f)/G(f).$$

Let $J=\{x_{i_1}, \cdots, x_{i_f}\}$ be an element of $M(f)$ which is contained in the global section (3.7). Define an element $I=(w_1, \cdots, w_f)$ of $N(f)$ as before. Let $\sigma$ be an element of $\mathrm{Aut}_i$ such that $\sigma(J)=J$. Put $y_j=x_{i_j}$. Here we consider the index $j$ as an element of $\mathbf{Z}/f\mathbf{Z}$. Then $\sigma(y_j)=y_{j+\tau}$ with some $\tau\in\mathbf{Z}/f\mathbf{Z}$. Define an element $\sigma'$ of $G(f)$ by $\sigma'(w'_1, \cdots, w'_f)=(w'_{1+\tau}, \cdots, w'_{f+\tau})$. Here also we consider the index $j$ of $w'_j$ as an element of $\mathbf{Z}/f\mathbf{Z}$. Then $\sigma'$ is an element of the stabilizer $G(I)$ of $I$ in $G(f)$. Conversely, assume that $\sigma'\colon (w'_1, \cdots, w'_f)\mapsto(w'_{1+\tau}, \cdots, w'_{f+\tau})$ stabilizes the element $I$. Then $J$ is contained in a global section which admits an automorphism $\sigma$ such that $\sigma(y_j)=y_{j+\tau}$. This $\sigma$ can be extended to an automorphism of $\bar{\Gamma}(\bar{\beta}_i)$, which we shall denote by the same letter $\sigma$. Then $\sigma$ is an element of the stabilizer

$G(J)$ of $J$ in $\mathrm{Aut}_l$. Thus we get

$$G(I)\cong G(J). \tag{3.10}$$

From (3.9) and (3.10), we have the following equality.

$$\begin{aligned}\sum_{\substack{f\leq l \\ I\in N(f)/G(f)}}(-1)^{f-1}/\# G(I) &= \sum_{J\in M'/\mathrm{Aut}_l}(-1)^{|J|-1}/\# G(J)\\ &= (\#\mathrm{Aut}_l)^{-1}\sum_{J\in M'}(-1)^{|J|-1}.\end{aligned} \tag{3.11}$$

**3.12.** Let $\Gamma_0$ $(n\geqq 2)$ be the graph of the form

$$\begin{array}{c}\overline{\;/\qquad\qquad\qquad\qquad\qquad\qquad\backslash\;}\\ x_1 \text{———} x_2 \text{———} \cdots \text{———} x_n.\end{array}$$

Let $\Gamma$ be a finite graph and $p$: $\Gamma\to\Gamma_0$ a morphism. We define the admissibility of such a pair $(\Gamma, p)$ as follows:

(3.12.1) $(\Gamma_0, \mathrm{id})$ is admissible.

(3.12.2) Assume that $(\Gamma, p)$ is admissible. Take two vertices $x_k''$ and $x_l''$ of $\Gamma$ such that $p(x_k'')=x_k$ and $p(x_l'')=x_l$. Construct a graph $\Gamma'$ by adding to $\Gamma$ new vertices $x_{k+1}'', \cdots, x_{l-1}''$ and new edges

$$x_k'' \text{———} x_{k+1}'' \text{———} \cdots \text{———} x_{l-1}'' \text{———} x_l''.$$

Define an extension $p'$: $\Gamma'\to\Gamma_0$ of $p$: $\Gamma\to\Gamma_0$ naturally. Then $(\Gamma', p')$ is admissible.

(3.12.3) A pair $(\Gamma, p)$ is admissible iff it can be obtained in this way.

Assume that $(\Gamma, p)$ is admissible. A subgraph $C$ of $\Gamma$ is called a global section if $p|_C$: $C\to\Gamma_0$ is an isomorphism. Let $J$ be a set of vertices of $\Gamma$ which is contained in some global section. Let $M=M(\Gamma)$ be the totality of such a set $J$. Let $|M|$ be the simplicial complex whose vertices are the vertices of $\Gamma$ and whose simplices are the nonempty set belonging to $M$.

Let us show that $|M|$ is contractible. If $(\Gamma, p)=(\Gamma_0, \mathrm{id})$, $|M|$ is a simplex, hence contractible. Assume that $|M(\Gamma)|$ is contractible and that $(\Gamma', p')$ is obtained from $(\Gamma, p)$ by the procedure (3.12.2). Let $\{C_i\}$ be the totality of the global sections of $(\Gamma', p')$ which contains $\{x_{k+1}'', \cdots, x_{l-1}''\}$. Let $|C_i|$ be the simplex of $|M(\Gamma')|$ corresponding to $C_i$. Then $|M(\Gamma')|=\cup_i|C_i|\cup|M(\Gamma)|$. Since each simplex $|C_i|$ contains the vertices $\{x_k'', \cdots, x_l''\}$, $\cup_i|C_i|$ is contractible. Since each $|C_i|\cap|M(\Gamma)|$ is a simplex and contains the vertices $x''_k$ and $x''_l$, $(\bigcup_i |C_i|) \cap |M(\Gamma)|$ is contractible. Since, by the induction hypothesis, $|M(\Gamma)|$ is also contractible, $|M(\Gamma')|$ is contractible.

Thus we have shown that $|M|$ is contractible. Especially the Euler characteristic of $|M|$ is equal to one, in another word,

$$\sum_{J \in M(\Gamma)} (-1)^{|J|} = 0.$$

**3.13.** By (3.11) and (3.12), we get

$$\sum_{\substack{f \le l \\ I \in N(f)/G(f)}} (-1)^{f-1} / \# G(I) = (\# \mathrm{Aut}_i)^{-1}.$$

Let us give an explicit formula for $\# \mathrm{Aut}_i$. Since an element $\sigma$ of $\mathrm{Aut}_i$ is determined by $\sigma(\bar{0})$, it suffices to determine the cardinality of $\mathrm{Aut}_i$-orbit of $\bar{0}$. (See (2.5) for $\bar{0}$.)

**3.14.** Let $S_0 = S \cap W_0$. Then $S_0$ satisfies the assumption of (2.7) with $w = \bar{\beta}_i$, i.e., we have $l(\bar{\beta}_i) = l(s \bar{\beta}_i s)$ for $s \in S_0$. Thus we can define the $S_0$-digraph $\Gamma_0(\bar{\beta}_i)$.

**Lemma.** *The $S_0$-graph $f\Gamma_0(\bar{\beta}_i)$ is simply connected.* (*It follows that the two $S_0$-digraphs $\Gamma_0(\bar{\beta}_i)$ and $\bar{\Gamma}_0(\bar{\beta}_i)$ are isomorphic. See* (2.8).)

*Proof.* Asssume that

$$(3.15) \qquad (\overline{\beta_i^\gamma})^s \xleftarrow{\ s\ } \overline{\beta_i^\gamma} \qquad (\gamma \in \Omega_0,\ s \in S_0).$$

Let $H_{\alpha,0}$ $(\alpha > 0)$ be the fixed point set of the reflection $\bar{s}$. Then (3.15) is equivalent to

$$(3.16) \qquad \langle \alpha, \omega_i^\gamma \rangle < 0.$$

Since $\bar{W} = \Omega$, $\Omega$ is a Coxeter group. Let $\gamma_0$ be the minimal element in the coset $\Omega_i \gamma$. Then (3.16) is equivalent to

$$(3.17) \qquad \alpha^{\gamma_0^{-1}} < 0.$$

In fact (3.16)$\Rightarrow$(3.17) is trivial. Assume that $\alpha^{\gamma_0^{-1}} < 0$ and $\langle \alpha, \omega_i^\gamma \rangle \ge 0$. Then $\langle \alpha^{\gamma_0^{-1}}, \omega_i \rangle = 0$. Hence $\alpha^{\gamma_0^{-1}}$ can be expressed as

$$\alpha^{\gamma_0^{-1}} = \sum_{j \ne i} c_j \alpha_j.$$

Since $\alpha^{\gamma_0^{-1}} < 0$, $c_j \le 0$. Since $\gamma_0$ is the minimal element of $\Omega_i \gamma$, $\alpha_j^{\gamma_0} > 0$ $(j \ne i)$. Hence

$$\alpha = \sum_{j \ne i} c_j \alpha_j^{\gamma_0} < 0.$$

This is absurd. Hence (3.16)$\Leftarrow$(3.17). It is easy to see that (3.17) is equivalent to

$$\gamma_0\sigma<\gamma_0, \tag{3.18}$$

where $\sigma=\bar{s}$ (=the reflection with respect to $H_{\alpha,0}$). Let $\sigma_j$ be the reflection with respect to $H_{\alpha_j,0}$, $\Sigma_0=\{\sigma_j\,|\,1\leq j\leq l\}$ and $\Sigma_i=\{\sigma_j\in\Sigma_0\,|\,j\neq i\}$. Let $\Gamma_i$ be the $S_0$-digraph whose vertices are the $(\Sigma_i, \phi)$-reduced element of $\Omega_0$ (see [1; Chap. IV, § 1, Ex. 3]) and two vertices are connected in the following way. Let $\gamma$ be a vertex of $\Gamma_i$ and $\sigma$ an element of $\Sigma_0$ such that $\gamma\sigma<\gamma$. Then $\gamma\sigma$ is also a vertex of $\Gamma_i$ and we set

$$\gamma\sigma\xleftarrow{\ \bar{\sigma}\ }\gamma.$$

Since $\gamma\mapsto\overline{\beta_i^\gamma}$ defines a bijection between the vertices of $\Gamma_i$ and $\Gamma_0(\bar{\beta}_i)$ and (3.15) is equivalent to (3.18), these two $S_0$-digraphs $\Gamma_i$ and $\Gamma_0(\bar{\beta}_i)$ are isomorphic.

Let

$$\gamma_0\overset{s_1}{\text{———}}\gamma_1\overset{s_2}{\text{———}}\cdots\overset{s_N}{\text{———}}\gamma_N$$

be a path of $f\Gamma_i$. Then

$$\bar{\gamma}_0 s_1 s_2\cdots s_N=\bar{\gamma}_N.$$

Hence $f\Gamma_i$ is simply connected and $f\Gamma_0(\bar{\beta}_i)$ is also simply connected.

**3.19.** As is noted in the proof of the above lemma, (3.15) is equivalent to (3.16). Hence every edge goes in at $\overline{\beta_i^\gamma}$ iff $\omega_i^\gamma$ is dominant, i.e., $\beta_i^\gamma=\beta_i$. Since $\bar{\Gamma}_0(\bar{\beta}_i)$ is isomorphic to $\Gamma_0(\bar{\beta}_i)$ and the vertex $\bar{0}$ corresponds to $\bar{\beta}_i$, $\bar{0}$ is the unique vertex of $\bar{\Gamma}_0(\bar{\beta}_i)$ at which every edge goes in. By (2.9) and (3.5), every connected component of $\bar{\Gamma}_1(\bar{\beta}_i)$ is also isomorphic to $\Gamma_0(\bar{\beta}_i)$. Hence the cardinality of $\pi_0(\bar{\Gamma}_1(\bar{\beta}_i))$ is equal to the cardinality of the set $V_0$ of the vertices at which every edge goes in. Since $\mathrm{Aut}_i$-orbit of $\bar{0}$ is contained in $V_0$,

$$\#\,\mathrm{Aut}_i=\#\,\mathrm{Aut}_i(\bar{0})\leq\#\,V_0=\#\,\pi_0(\bar{\Gamma}_1(\bar{\beta}_i)).$$

**3.20.** Suppose that $R_i$ is a direct sum of irreducible root systems $R_{i,\nu}$ $(\nu=1, 2, \cdots)$. Let $R_{i,\nu}^+=R_{i,\nu}\cap R_i^+$ and $\tilde{\alpha}_\nu$ the highest root of $R_{i,\nu}$. Then

$$K_i=\{x\in E\,|\,0<\langle\alpha_j, x\rangle\ (j\neq i),\ \langle\tilde{\alpha}_\nu, x\rangle<1\ (\nu=1, 2, \cdots)\}.$$

Put $\tilde{\alpha}_\nu=\sum n_{\nu,j}\alpha_j$, $J_\nu=\{j\,|\,n_{\nu,j}=1\}$ and $J'_\nu=\{j\,|\,\alpha_j\in R_{i,\nu}\}$. Let $J$ be a subset of $\cup_\nu J_\nu$ such that $\sharp(J\cap J_\nu)\leq 1$ for every $\nu$. For a subset $I$ of $\{j|1\leq j\leq l\}$, let $\Omega$ $(I)$ be the group generated by $\{\sigma_j|j\in I\}$ and $\gamma(I)$ the longest element of $\Omega(I)$. Put

$$\gamma(J,\nu)=\gamma(J'_\nu)\gamma(J'_\nu-J).$$

**Lemma.** *We have*

$$K_i\,(\textstyle\prod_\nu\gamma(J,\nu))t(\sum_{j\in J}\omega_j+r\omega_i)=K_i \qquad (r\in\mathbf{Z}).$$

*Proof.* Let $x$ be an element of $K_i$. For $k\in J\cap J_\nu$,

$$\langle\alpha_k,\,x(\textstyle\prod_\nu\gamma(J,\nu))+\sum_{j\in J}\omega_j+r\omega_i\rangle=1-\langle-\alpha_k\gamma(J'_\nu-J)\gamma(J'_\nu),\,x\rangle>0.$$

For $k\in J'_\nu-J$,

$$\langle\alpha_k,\,x(\textstyle\prod_\nu\gamma(J,\nu))+\sum_{j\in J}\omega_j+r\omega_i\rangle=\langle\alpha_k\gamma(J'_\nu-J)\gamma(J'_\nu),\,x\rangle>0.$$

Finally, for $\nu=1,2,\cdots$

$$\langle\tilde{\alpha}_\nu,\,x(\textstyle\prod_\nu(J,\nu))+\sum_{j\in J}\omega_j+r\omega_i\rangle=\langle\tilde{\alpha}_\nu\gamma(J'_\nu-J)\gamma(J'_\nu),\,x\rangle+\sharp(J\cap J_\nu)<1.$$

**3.21.** Put

$$\mathscr{A}=\{J|J\subset\cup J_\nu,\ \sharp(J\cap J_\nu)\leq 1\}.$$

Since $\bar{\Gamma}_i$ is isomorphic to $\bar{\Gamma}(\bar{\beta}_i)$ (see (3.4)) and

$$Q\cap\bar{K}_i=Q\cap\{\textstyle\sum_{j\in J}\omega_j+r\omega_i|J\in\mathscr{A},\quad r\in\mathbf{Z}\},$$

(3.20) implies that $\mathrm{Aut}_i$ acts on $\pi_0(\bar{\Gamma}_1(\bar{\beta}_1))$ transitively. (Note that, if one deletes all the faces corresponding to $S-S_0$ from $\bar{K}_i$ and denotes it by $K'$, then there is a one-to-one correspondence $\pi_0(K')\rightleftarrows Q\cap\bar{K}_i$.) Hence

$$\sharp\,\mathrm{Aut}_i=\sharp\,\mathrm{Aut}_i\,(\bar{0})=\sharp\,V_0=\sharp\,\pi_0(\bar{\Gamma}_1(\bar{\beta}_1))=\sharp\,Q_i,$$

where

$$Q_i=(Q\cap\{\textstyle\sum_{j\in J}\omega_j+r\omega_i|J\in\mathscr{A},\ r\in\mathbf{Z}\})/\mathbf{Z}\,c_i\omega_i.$$

Then the equality in (3.13) can be rewritten as follows

$$\text{(3.22)}\qquad \sum_{\substack{f\leq l\\ i\in N(f)/G(f)}}(-1)^{f-1}/\sharp\,G(I)=(\sharp\,Q_i)^{-1}.$$

## 4.

The purpose of this section is to prove the main theorem. (See introduction.)

Let us introduce some notations. Put

$$\Pi^{\sim}=\{x \in E \mid 0<\langle\alpha_i, x\rangle<c_i \ (1\leq i\leq l)\}.$$

Let $Q^{++}$ be the set $\{\sum_{i=1}^{l} a_i c_i \omega_i \mid a_i \in \mathbf{Z},\ a_i\geq 0\}$. For a subset $E'$ of $E$, put

$$W(E')=\{w \in W \mid wA_0^+\subset E'\}.$$

**4.1.** The following statement is easily verified. Every element $w$ of $W$ can be expressed uniquely as $w=w_1w_2$ ($w_1 \in W(\mathscr{C}_0^+)$, $w_2 \in W_0$). And, then, $l(w)=l(w_1)+l(w_2)$.

**4.2.** For $w \in W(\mathscr{C}_0^+)$, we have $l(w)=d(A_0^+, wA_0^+)$. Hence for $w_1 \in W(\tilde{\Pi})$ and $p \in Q^{++}$, we have

$$\begin{aligned} l(w_1\overline{t(p)})&=d(A_0^+, w_1\overline{t(p)}A_0^+)=d(A_0^+, A_0^+t(p))+d(A_0^+t(p), w_1A_0^+t(p)) \\ &=l(\overline{t(p)})+l(w_1). \end{aligned}$$

**4.3.** Let $K$ be the quotient field of the group ring $\mathbf{C}[P]$. Let $q$ be a positive real number. The Hecke algebra $H_q(W)$ is the associative $K$-algebra which has basis element $T(w)$ (one for each $w \in W$) and multiplication defined by the rules

$$\begin{aligned} &(T(s)+1)(T(s)-q)=0 \qquad (s \in S), \\ &T(w)T(w')=T(ww'), \qquad \text{if } l(ww')=l(w)+l(w'). \end{aligned}$$

For a representation $R$ of $H_q(W)$, put

$$L(t, R)=\sum_{w\in W} R(T(w))t^{l(w)}.$$

(See [2] for its properties.)

**4.4.** As a consequence of (4.1) and (4.2), we get the following identities.

$$\begin{aligned} \sum_{w\in W} T(w)t^{l(w)} &=(\sum_{w\in W(\Pi^{\sim})}T(w)t^{l(w)})(\sum_{p\in Q^{++}}T(\overline{t(p)})\,t^{l}(t^{l(\overline{t(p)})})) \\ &\quad\cdot(\sum_{w\in W_0}T(w)t^{l(w)}) \\ &=(\sum_{w\in W(\Pi^{\sim})} T(w)t^{l(w)})\prod_{i=1}^{l}(1-T(\overline{\beta}_i)t^{l_i})^{-1} \\ &\quad\cdot(\sum_{w\in W_0} T(w)t^{l(w)}). \end{aligned}$$

Hene, for a representation $R$ of $H_q(W)$, the denominator of det $L(t, R)$ divides

$$\prod_{i=1}^{l} \det(1-R(T(\overline{\beta}_i))t^{l_i}).$$

**4.5.** Let $X$ be the set of alcoves and $M$ be the $K$-vector space with basis $X$. There is a unique $H_q(W)$-module structure on $M$ such that, for $A \in X$ and $s \in S$, we have

$$T(s)A = sA, \qquad \text{if } s \notin \mathscr{L}(A).$$

Let $\lambda$ be the natural inclusion map $P \subsetneq K$. Let $M_\lambda$ be the $K$-vector space spanned by all the infinite formal linear combinations

$$F(A) = \sum_{p \in Q} \lambda(-p) q^{d(At(p),A)/2} At(p) \qquad (A \in X).$$

The Hecke algebra $H_q(W)$ acts naturally on $M_\lambda$. This $H_q(W)$-module is called a "generic principal series representation." The set $\{F(wA_0^+) | w \in W_0\}$ is a basis of $M_\lambda$. This basis gives a matrix representation $R_\lambda$. For $p \in Q$, we have

$$F(At(p)) = (\lambda\rho)(p)F(A).$$

Here $\rho(p) = q^{d(A,At(p))/2}$, which is independent of the choice of the alcove $A$.

**4.6.** It is known that the eigenvalues of $R_\lambda(T(\bar{\omega}))$ $(\omega \in Q^{++})$ are $\{(\gamma\lambda)(\omega) q^{l(t(\omega))/2} | \gamma \in \Omega_0\}$ ([5; (4.3.3)]). Hence, by (4.4), the denominator of $\det L(t, R_\lambda)$ divides

$$\prod_{i=1}^{l} \prod_{\gamma \in \Omega_0} (1 - (\gamma\lambda)(c_i\omega_i)(q^{1/2}t)^{l_i}).$$

We normalize the denominator so that its constant term equals 1. It is also known that $M_{\gamma\lambda}$ $(\gamma \in \Omega_0)$ is isomorphic to $M_\lambda$ ([5; (4.3.3)]). Hence the denominator of $\det L(t, R_\lambda)$ is invariant under the change $\lambda \to \gamma\lambda$ $(\gamma \in \Omega_0)$. Since each $c_i\omega_i$ is not divisible in $Q$, the polynomial $1 - (\gamma\lambda)(c_i\omega_i)(q^{1/2}t)^{l_i}$ is irreducible. Hence the denominator of $\det L(t, R_\lambda)$ is a product of the factors $1 - (\gamma\lambda)(c_i\omega_i)(q^{1/2}t)^{l_i}$ $(1 \le i \le l, \gamma \in \Omega_0)$. Hence it is of the form

$$\prod_{\omega_i \bmod \Omega_0} \prod_{\gamma \in \Omega_i \backslash \Omega_0} (1 - (\gamma\lambda)(c_i\omega_i)(q^{1/2}t)^{l_i})^{n_i}, \tag{4.7}$$

with some non-negative integers $n_i$. Here $\prod_{\omega_i \bmod \Omega_0}$ means that $\omega_i$ runs over a representative of $\Omega_0$-conjugacy classes of the fundamental weights.

**4.8.** For an element $a = \sum_{p \in Q} a(p)\lambda(p)$ $(a(p) \in \mathbf{C})$, put

$$[a : p] = a(p).$$

If the numerator of $\det L(t, R_\lambda)$ is of the form $1 + a_1't + a_2't^2 + \cdots$, then $[a_j' : p] = 0$ $(p \in Q - \{0\})$. (See [2; 1.14.2].) Hence, if $\det L(t, R_\lambda) = 1 + a_1 t + a_2 t^2 + \cdots$, then

(4.9) $$[a_{l_i}\colon c_i\omega_i]=n_i q^{l_i/2}.$$

Thus to get an explicit formula of the denominator of det $L(t, R_\lambda)$, it suffices to calculate the value of $[a_{l_i}\colon c_i\omega_i]$ using (1.5).

For $x \in W$ and $\gamma \in \Omega_0$, we have

$$T(x)F(A_0^+\gamma)=\sum_{x'\leq x} a(x', \gamma)F(x' A_0^+\gamma)$$

with some $a(x', \gamma) \in \mathbf{Z}[q]$ ($\subset \mathbf{C}$). For each $x' \in W$, there are uniquely determined $p \in Q$ and $\gamma' \in \Omega_0$ such that $x'A_0^+=A_0^+t(p)\gamma'$. Then

$$\begin{aligned}T(x)F(A_0^+\gamma)&=\sum_{x'\leq x} a(x', \gamma)F(A_0^+t(p)\gamma'\gamma)\\&=\sum_{x'\leq x} a(x', \gamma)(\lambda\rho)(p\gamma'\gamma)F(A_0^+\gamma'\gamma).\end{aligned}$$

Hence

$$\operatorname{tr} R_\lambda(T(x))=\sum_{\substack{\overline{t(p)}\leq x\\ \gamma\in\Omega_0}} a(\overline{t(p)}, \gamma)(\lambda\rho)(p\gamma).$$

Let $x_j$ $(1\leq j\leq n)$ be elements in $W$ such that $\sum_{j=1}^n l(x_j)=l_i$. Then

$$\prod_{j=1}^n \operatorname{tr} R_\lambda(T(x_j))=\sum_{\substack{\overline{t(p_j)}\leq x_j\\ \gamma_j\in\Omega_0}} \left(\prod_{j=1}^n a(\overline{t(p_j)}, \gamma_j)\right)(\lambda\rho)\left(\sum_{j=1}^n p_j\gamma_j\right).$$

Hence if $[\prod_{j=1}^n \operatorname{tr} R_\lambda(T(x_j))\colon c_i\omega_i]\neq 0$, there exist $p_j \in Q$, $\gamma_j \in \Omega_0$ $(1\leq j\leq n)$ such that $\sum_{j=1}^n p_j\gamma_j=c_i\omega_i$ and $\overline{t(p_j)}\leq x_j$. But then

$$\begin{aligned}l_i=l(\bar\beta_i)=l\left(\prod_{j=1}^n \overline{t(p_j\gamma_j)}\right)&\leq\sum_{j=1}^n l(\overline{t(p_j\gamma_j)})\\&=\sum_{j=1}^n l(\overline{t(p_j)})\leq\sum_{j=1}^n l(x_j)=l_i.\end{aligned}$$

Hence $\overline{t(p_j)}=x_j$ and each $p_j\gamma_j$ is contained in the segment joining $c_i\omega_i$ and the origin of $V^*$. But since $p_j\gamma_j \in Q$, $n$ must be equal to 1 and $p_1\gamma_1=c_i\omega_i$. Hence, by (1.2), we get

(4.10) $$\begin{aligned}[a_{l_i}\colon c_i\omega_i]&=\sum_{I\in P}(-1)^{(d-1)(l_i/n)}[\operatorname{tr}^{(l_i/n)} R_\lambda(T(w_1\cdots w_d))\colon c_i\omega_i]\\&=\sum_{I\in P}(-1)^{(l_i/n)d-1}(n/l_i)[\operatorname{tr} R_\lambda(T(w_1\cdots w_d)^{l_i/n})\colon c_i\omega_i],\end{aligned}$$

where

$$\begin{gathered}P=\coprod_{d\leq n,\, n|l_i} P(d, n)/G(d),\\ P(d, n)=\{I=(w_1, \cdots, w_d)\,|\, w_j \in W-\{1\},\\ \sum_{j=1}^d l(w_j)=n,\\ (w_1\cdots w_d)^{l_i/n} \in \{\overline{\beta_i^\gamma}\,|\,\gamma \in \Omega_0\},\\ (w_1, \cdots, w_d)\colon \text{primitive}\}\end{gathered}$$

and $G(d)$ is the group generated by the automorphism $(w_1, \cdots, w_d) \to (w_d, w_1, \cdots, w_{d-1})$. Here $I=(w_1, \cdots, w_d)$ is called primitive iff $G(I)=\{1\}$, where $G(I)$ is the stabilizer of $I$ in $G(d)$.

By [5; 4.3.3], we get tr $R_\lambda(T(\overline{\beta_i^\gamma}))=$ tr $R_\lambda(T(\overline{\beta}_i))$ $(\gamma \in \Omega_0)$ and

$$[\mathrm{tr}\, R_\lambda(T(\overline{\beta_i^\gamma})) : c_i\omega_i]=[\mathrm{tr}\, R_\lambda(T(\overline{\beta}_i)) : c_i\omega_i]=(\# \Omega_i)\times q^{l_i/2}.$$

Hence (4.9) and (4.10) imply

$$n_i=(\textstyle\sum_{I\in P}(-1)^{(l_i/n)d-1}(n/l_i))\times(\# \Omega_i). \tag{4.11}$$

Using the notations in (3.6), the above equality can be rewritten as

$$n_i=(\textstyle\sum_{\substack{f\leq l_i \\ I\in N(f)/G(f)}}(-1)^{f-1}(\# G(I))^{-1})\times(\# \Omega_i),$$

where $G(I)$ is the stabilizer of $I$ in $G(f)$. Using (3.22), we get

$$n_i=\# \Omega_i/\# Q_i. \tag{4.12}$$

**4.13. Lemma.** *Let $G$ be a finite commutative group and $g$ its element of order $n$. Then*

$$\textstyle\prod_{\chi\in G^\vee}(1-\chi(g)x)=(1-x^n)^{\# G/n},$$

*where* $G^\vee=\mathrm{Hom}\,(G, \mathbf{C}^\times)$.

By using (4.12) and (4.13), (4.7) can be rewritten as follows:

$$\textstyle\prod_{\substack{\omega_i \bmod \Omega_0 \\ \gamma\in\Omega_i\backslash\Omega_0 \\ \chi\in\Phi(R^\vee)^\vee}}(1-(\lambda\chi)(\omega_i\gamma)(q^{1/2}t)^{l(\omega_i)})^{(c_i/f)(\# Q_i/\#\Omega_i)}, \tag{4.14}$$

where $\Phi(R^\vee)=P(R^\vee)/Q(R^\vee)$ and $f=\#\Phi(R^\vee)$. Since we can check (case by case) that

$$f\cdot \# Q_i/c_i=f_i, \tag{4.15}$$

(see Introduction for $f_i$), we have proved the main theorem.

**4.16. Remark.** It is well known that $\Phi=\Phi(R^\vee)$ can be regarded as an automorphism group of the Coxeter system $(W, S)$. Consider the semidirect product $\tilde{W}=W\rtimes\Phi$. We can define the length function $l$ on $\tilde{W}$, the Hecke algebra $H_q(\tilde{W})$, the action $\tilde{R}_\lambda$ of $H_q(\tilde{W})$ on $M_\lambda$ which is an extension of $R_\lambda$ etc. as usual. Put

$$L(t, \tilde{W}, \tilde{R}_\lambda)=f^{-1}\textstyle\sum_{w\in\tilde{W}}\tilde{R}_\lambda(T(w))t^{l(w)}$$

and

$$e=f^{-1}\sum_{x\in\Phi}\tilde{R}_\lambda(T(x)).$$

Then $L(t, \tilde{W}, \tilde{R}_\lambda)$ stabilizes the subspace $eM_\lambda$ of $M_\lambda$ and the denominator of $\det(L(t, \tilde{W}, \tilde{R}_\lambda)|_{eM_\lambda})$ is equal to

$$\prod_{\substack{\omega_i \bmod \Omega_0 \\ \gamma\in\Omega_i\backslash\Omega_0}}(1-\lambda(\omega_i\gamma)(q^{1/2}t)^{l(\omega_i)})^{\#\Omega_i/f_i}.$$

(*Sketch of the proof.* We can prove that the denominator of $\det(L(t, \tilde{W}, \tilde{R}_\lambda)|_{eM_\lambda})$ is of the form

$$\prod_{\substack{\omega_i \bmod \Omega_0 \\ \gamma\in\Omega_i\backslash\Omega_0}}(1-\lambda(\omega_i\gamma)(q^{1/2}t)^{l(\omega_i)})^{m_i}$$

with some non-negative integers $m_i$ (cf. (4.6)) and that the numerator is "independent of $\lambda$" by the same argument as in [2]. Since

$$\det L(t, W, R_\lambda)=\prod_{\chi\in\Phi^\vee}\det(L(t, \tilde{W}, \tilde{R}_{\lambda\otimes\chi})|_{eM_{\lambda\otimes\chi}}),$$

this fact and our main theorem imply the above statement.)

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*Department of Mathematics*
*Osaka University*
*Toyonaka, 560 Japan*