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Some Generalization of Asai's Result for Classical Groups

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Introduction

Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$, $F:G\to G$ be the corresponding Frobenius map and for each positive integer m, G^{F^m} be the group of F^m-fixed points in G. Let $G^{F^m}/\sim_F$ be the set of F-twisted conjugacy classes of G^{F^m}. In the case where $m=1$, we simply express it as $G^F/\sim$. A bijection $N_{F^m/F}$: $G^F/\sim\to G^{F^m}/\sim_F$ is defined by attaching $x=F^m(a)a^{-1}$ to $\hat{x}=a^{-1}F(a)$, where $x\in G^F$, $\hat{x}\in G^{F^m}$ and $a\in G$. We denote by $C(G^{F^m}/\sim_F)$ the space of $\overline{\mathbf{Q}}_l$-valued functions on the set $G^{F^m}/\sim_F$. Then we get the induced map $N^*_{F^m/F}$: $C(G^{F^m}/\sim_F)\to C(G^F/\sim)$.

Let $\tilde{G}^{F^m}$ be the semidirect product of G^{F^m} with the cyclic group of order m with generator σ, where σ acts on G^{F^m} by $\sigma g\sigma^{-1}=F(g)$. For each representation $\tilde{\rho}$ of $\tilde{G}^{F^m}$, we denote by $[\tilde{\rho}]$ the restriction on $G^{F^m}\sigma$ of the character of $\tilde{\rho}$, which we regard as an element of $C(G^{F^m}/\sim_F)$ under the natural bijection $G^{F^m}/\sim_F\simeq G^{F^m}\sigma/\sim$ ($\sim$ means the conjugation under $\tilde{G}^{F^m}$).

Assume that the center of G is connected. By Lusztig [11], the set $\mathscr{E}(G^{F^m})$ of isomorphism classes of irreducible representations of G^{F^m} over $\overline{\mathbf{Q}}_l$ is partitioned into the disjoint union of subsets $\mathscr{E}(G^{F^m},(s))$ where (s) runs over all F^m-stable semisimple conjugacy classes in the dual group G^* of G. Moreover, by [11], taking $s\in G^{*F^m}$, we have a canonical bijection

$$\mathscr{E}(G^{F^m},(s))\simeq\mathscr{E}(Z_{G^*}(s)^{*F^m},(1)). \tag{0.1}$$

F acts naturally on $\mathscr{E}(G^{F^m})$ and for each F-stable class (s), F stabilizes $\mathscr{E}(G^{F^m},(s))$. Let $\mathscr{E}(G^{F^m},(s))^F$ be the set of F-stable representations in $\mathscr{E}(G^{F^m},(s))$. We denote by $C^{(s)}(G^{F^m}/\sim_F)$ the subspace of $C(G^{F^m}/\sim_F)$ generated by $[\tilde{\rho}]$, where $\tilde{\rho}$ runs over all the irreducible representations of $\tilde{G}^{F^m}$ whose restriction to G^{F^m} lies in $\mathscr{E}(G^{F^m},(s))^F$. Thus, if $m=1$, $C^{(s)}(G^F/\sim)$ is the subspace of $C(G^F/\sim)$ generated by various elements in $\mathscr{E}(G^F,(s))$.

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The purpose of this paper is to investigate the map $N^*_{F^m/F}$ in the case of classical groups.

If $m=1$, the map $N^*_{F/F}$ becomes an automorphism on the space of class functions of G^F and in the case of classical groups of split type, Asai [2], [3] has shown using the lifting theory of Kawanaka [8], that $N^*_{F/F}$ leaves $C^{(1)}(G^F/\sim)$ invariant and that $N^*_{F/F}$ restricted to $C^{(1)}(G^F/\sim)$ is closely related with the "Fourier transform" (or rather almost characters in the sense of [11, § 4]) of unipotent characters. (He also obtained the similar result ([4]) in the case of exceptional groups using the twisted operator instead of $N^*_{F/F}$).

In this paper, we shall treat the case where G is a classical group with connected center and m is sufficiently divisible, i.e., $\mathbf{F}_{q^m}$ contains some fixed sufficiently large extension of $\mathbf{F}_q$. Then $\mathscr{E}(G^{F^m}, (s))^F$ is parametrized by $X(W_s, \gamma_s)$ (see 2.1 for the definition) independently of m, and for each $x \in X(W_s, \gamma_s)$ an almost character $R_x \in C^{(s)}(G^F/\sim)$ can be defined by [11]. By this correspondence, we can attach to each $\rho \in \mathscr{E}(G^{F^m}, (s))^F$ corresponding to $x_\rho \in X(W_s, \gamma_s)$, an almost character R_{x_ρ} up to a root of unity multiple. Then our main result is Theorem 2.2, which asserts that under the above assumptions, $N^*_{F^m/F}$ maps $C^{(s)}(G^{F^m}/\sim_F)$ onto $C^{(s)}(G^F/\sim)$ and that $N^*_{F^m/F}([\mu_{\tilde\rho}\tilde\rho])=R_{x_\rho}$, where $\tilde\rho$ is an extension of ρ to $\tilde G^{F^m}$ and $\mu_{\tilde\rho}$ is a root of unity depending on the choice of $\tilde\rho$ and m. In particular, $N^*_{F^m/F}$ is compatible with the map (0.1).

In the case where $s=1$, our result is already contained in [2], [3]. Hence, Theorem 2.2 can be regarded as a generalization of Asai's result to arbitrary s, although his result itself (which is concerned with $N^*_{F/F}$) is not extended to the general case by our argument.

As a corollary (Corollary 2.19), we can decompose $R^G_{M\subset P}(\pi)$ into irreducible constituents, where M is an F-stable Levi subgroup of (not necessarily F-stable) parabolic subgroup P of G and π is an irreducible representation of M^F.

As regards the proof, Asai's method can be applied to our case, essentially. However, it should be noticed that, as we are dealing the case where m is sufficiently large, Kawanaka's theory cannot be applied to our case. Instead, using the argument of Lusztig ([11]), we can show that $N^*_{F^m/F}([\tilde\rho])$ gives the same element in $C(G^F/\sim)$, up to a root of unity multiple, for infinitely many m. This enables us to apply the specialization argument to our situation, and once this is done, Asai's method works as well to ours by making use of results of Lusztig [11].

The author understands that B. Srinivasan obtained independently the similar result as Corollary 2.19.

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§ 1. The maps $\mathbf{R}^{(m)}_{M(\dot{w})}$ and a_{Fw}

1.1. Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$, with Frobenius map F. We may assume further that G has a split $\mathbf{F}_p$-structure with Frobenius map F_0 such that $F_0F=FF_0$ and that some power of F_0 is equal to some power of F, where $\mathbf{F}_p$ is a prime field contained in $\mathbf{F}_q$. We shall fix an F_0-stable Borel subgroup B, an F_0-stable maximal torus T contained in B, and denote by W the Weyl group of G relative to T. We assume further $F(B)=B$ and $F(T)=T$. Let Σ be the set of roots of G with respect to T and $\Pi\subset\Sigma$ be the set of simple roots with respect to (B, T). Then any parabolic subgroup containing B is expressed as $P_J=M_JU_J$ for some $J\subset\Pi$, where M_J is a Levi subgroup of P_J containing T and U_J is the unipotent radical of P_J. Put $M=M_J$. Take $w\in W$ such that $Fw(J)=J$, and let $\dot{w}$ be a representative of w in $N_G(T)^{F_0}$. Then $F\dot{w}$: $g\to F(\dot{w}g\dot{w}^{-1})$ may be considered as a Frobenius map of M commuting with F_0 with respect to some $\mathbf{F}_q$-structure. Consider the variety $S=\{g\in G|g^{-1}F(g)\in F(\dot{w}U_J)\}$ and put $\bar{S}=S/U_J\cap F(\dot{w}U_J\dot{w}^{-1})$. Then $G^F\times M^{F\dot{w}}$ acts on $H^i_c(\bar{S}, \overline{\mathbf{Q}}_l)$. According to [9], [2], we associate a virtual G^F-module $R^G_{M(\dot{w})}(\pi)$ to an irreducible $M^{F\dot{w}}$-module π as follows.

$$R(^G_{M(\dot{w})}(\pi)=\sum_{i\geq 0}(-1)^i(H^i_c(\bar{S}, \overline{\mathbf{Q}}_l)\otimes\pi)^{M^{F\dot{w}}}.$$

Thus, extending linearly, we get a homomorphism $R^G_{M(\dot{w})}$: $\mathscr{R}(M^{F\dot{w}})\to\mathscr{R}(G^F)$, where $\mathscr{R}(\ )$ denotes the Grothendieck group of representations of a finite group over $\overline{\mathbf{Q}}_l$. (Note our definition of $R^G_{M(\dot{w})}$ here is slightly different from that of [2], where he uses $\dot{w}F$ instead of $F\dot{w}$).

1.2. We recall here some related notations of [11]. For each $w\in W$, we define $X_w=\{gB\in G/B|g^{-1}F(g)\in BwB\}$ and for each representative $\dot{w}\in N_G(T)^{F_0}$, we define $\tilde{X}_{\dot{w}}=\{g\in G|g^{-1}F(g)\in\dot{w}U\}/U\cap\dot{w}U\dot{w}^{-1}$, where U is the unipotent radical of B. Put $T_w=\{t\in T|w(F(t))=t\}$. Then $G^F\times T_w$ acts on $\tilde{X}_{\dot{w}}$ by $x\to gxt^{-1}$ and induces the isomorphism $\tilde{X}_{\dot{w}}/T_w\simeq X_w$, which is G^F-equivariant with respect to the action of G^F by left multiplication on X_w. We denote by $\mathscr{F}_\theta$ the locally constant G^F-equivariant $\overline{\mathbf{Q}}_l$-sheaf of rank 1 over X_w corresponding to $\theta\in T^\wedge_w$. Then $H^i_c(X_w, \mathscr{F}_\theta)$ becomes a G^F-module and in fact,

$$R^G_{T(F^{-1}(\dot{w}))}(\theta)=\sum_{i\geq 0}(-1)^iH^i_c(X_w, \mathscr{F}_\theta).$$

Let $\bar{X}_w$ be the Zariski closure of X_w in G/B. Then $\bar{X}_w$ is the disjoint union of $X_{w'}$ $(w'\leq w)$. We shall consider, following [11, §2], the cohomology sheaves $\mathscr{H}^i(\bar{X}_w, \mathscr{F}_\theta)$ of the intersection cohomology complex $\mathrm{IC}(\bar{X}_w, \mathscr{F}_\theta)$ and its hypercohomology group $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_\theta)$, which becomes a G^F-module.

1.3. Let G^* be the dual group of G defined over $\mathbf{F}_q$ and T^* be an F-stable maximal torus of G^* which is dual to T over $\mathbf{F}_q$.

From now on, throughout this section, we assume that the center of G is connected.

According to [9, §7], $\theta \in T_w^{\wedge}$ determines an F-stable semisimple class (s) of G^*. Then, by [11], for each F-stable class $(s) \subset G^*$, the set $\mathscr{E}(G^F, (s))$ consists of $\rho \in \mathscr{E}(G^F)$ such that ρ appears as a constituent in a G^F-module $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_\theta)$ for some i and w under the condition that θ corresponds to (s).

Fix an F-stable class (s) in G^*. Let s be an element of (s) contained in T^* and d be the smallest integer such that $F_0^d(s)=s$. Then F_0^d acts on X_w, and since θ is F_0^d-stable, $\mathscr{F}_\theta$ is endowed with an F_0^d-structure. So, F_0^d acts naturally on $H_c^i(X_w, \mathscr{F}_\theta)$ and $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_\theta)$. However, this F_0^d-structure depends on the choice of a representative $\dot{w}$ of w, we shall write $\mathscr{F}_\theta$ as $\mathscr{F}_{\dot{w},\theta}$ (as G^F-equivariant sheaf, $\mathscr{F}_{\dot{w},\theta}$ are mutually isomorphic). Hence, from now on, according to [11, 1.23], we shall fix a suitable representative $\dot{w} \in N_G(T)^{F_0}$ for each $w \in W$.

Let b be the smallest integer such that F_0^{db} is an integral power of F. In the following, for (G^F, F_0^{db})-module H, we denote by H_ρ the ρ-isotypic subspace of H and by $H_{\rho,\mu}$ the generalized μ-eigenspace with respect to F_0^{db} of H_ρ. The following lemma, which is a usual cohomology version of [11, Proposition 2.20], is due to G. Lusztig. The author is very grateful to him for communicating this.

1.4. Lemma. *Assume we are in the setting of* 1.3. *Let $G^F\langle\vartheta\rangle$ be the semidirect product of G^F with the cyclic group of order b with generator ϑ, where ϑ acts on G^F by $\vartheta g \vartheta^{-1}=F_0^d(g)$. Then each representation ρ in $\mathscr{E}(G^F, (s))$ is F_0^d-stable. Moreover, for each $\rho \in \mathscr{E}(G^F, (s))$, there exists an extension $\tilde{\rho}$ to $G^F\langle\vartheta\rangle$ and a root of unity $\lambda'_\rho \in \overline{\mathbf{Q}}_l^*$ such that the following holds.*

(i) *Put $\lambda_\rho=(\lambda'_\rho)^b$. Then the eigenvalues of F_0^{db} on $H_c^i(X_w, \mathscr{F}_{\dot{w},\theta})_\rho$ are λ_ρ times integral powers of $p^{db/2}$.*

(ii) *Put $\mu=\lambda_\rho p^{dbk/2}$ be an eigenvalue of F_0^{db} as given in* (i). *Then $H_c^i(X_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu}$ is F_0^d-stable and admits a (G^F, F_0^d)-stable filtration each of whose successive quotients is isomorphic as a $G^F\langle\vartheta\rangle$-module (with ϑ acting as $(\lambda'_\rho)^{-1}p^{-dk/2}F_0^d$) to $\tilde{\rho}$.*

Proof. All the statements are certainly true for $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})$ in view of [11, Proposition 2.20, Theorem 3.8]. Hence the first statement follows. We shall show (i). Take $\rho \in \mathscr{E}(G^F, (s))$. They by [loc. cit.], the eigenvalues of F_0^{db} on $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})_\rho$ are of the form $\lambda_\rho p^{idb/2}$, where λ_ρ is a root of unity independent of i and w. Suppose the lemma does not hold and let w be a minimal element with respect to the Coxeter order where the lemma fails. Hence there exists i and $\mu \in \bar{\mathbf{Q}}_l^*$, not of the form λ_ρ times integral power of $p^{db/2}$, such that $H_c^i(X_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu} \neq 0$. The spectral sequence of G^F-modules

$$H^i(\bar{X}_w, \mathscr{H}^j(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})) \Longrightarrow \mathbf{H}^{i+j}(\bar{X}_w, \mathscr{F}_{\dot{w},\theta}),$$

which is F_0^d-equivariant, implies

$$(1.4.1) \qquad H^i(\bar{X}_w, \mathscr{H}^j(\bar{X}_w, \mathscr{F}_{\dot{w},\theta}))_{\rho,\mu} \Longrightarrow \mathbf{H}^{i+j}(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu}.$$

But, by [11, Theorem 2.4], for each $w' \leq w$, the restriction of $\mathscr{H}^j(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})$ to $X_{w'}$ has a filtration of G^F-equivariant sheaves defined over $\mathbf{F}_{p^d}$ if it is non-zero, where each successive quotient is isomorphic to $\mathscr{F}_{\dot{w}',\theta'}(-j/2)$ (Tate twist) for some $\theta' \in T_{w'}^\wedge$ corresponding to (s). Moreover when $w' = w$, this restriction is isomorphic to $\mathscr{F}_{\dot{w},\theta}$ if $j=0$ and 0 otherwise. Hence, by assumption on w, the left hand side of (1.4.1) vanishes except when $j=0$. Thus we have

$$H_c^i(X_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu} \simeq \mathbf{H}^i(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu}.$$

This is a contradiction since $\mathbf{H}^i(\bar{X}_w, \mathscr{F}_{\dot{w},\theta})_{\rho,\mu} = 0$. Thus (i) is proved. (ii) follows from Proposition 2.20 of [11] using the similar argument as in (i) if we notice that (1.4.1) turns out to be the spectral sequence of $G^F\langle\vartheta\rangle$-modules. Thus the lemma is proved.

1.5. Let $w \in W$ be such that $Fw(J)=J$. We shall choose a positive integer m such that F^m is a power of F_0 and that $(F\dot{w})^m = F^m$ on $M=M_J$. Then F^m acts on S and so acts on $H_c^i(\bar{S}, \bar{\mathbf{Q}}_l) \otimes \pi$ commuting with the action of $M^{F\dot{w}}$ (with trivial action on π). Hence we get a natural action of F^m on the virtual G^F-module $R_{M(\dot{w})}^G(\pi)$.

The following proposition describes the eigenvalues of F^m on $R_{M(\dot{w})}^G(\pi)$ in the case where m is sufficiently large.

1.6. Proposition. *Let $w \in W$ be as in* 1.5. *There exists an integer $m_1 > 0$ such that for any integer $m > 0$ divisible by m_1, the eigenvalues of F^m on $(H_c^i(\bar{S}, \bar{\mathbf{Q}}_l) \otimes \pi)^{M^{F\dot{w}}}$ are integral powers of $q^{m/2}$.*

Proof. Take m as in 1.5. Then for each $\pi \in \mathscr{E}(M^{F\dot{w}})$, there exists $X_{w',M}$ (the similar variety as X_w defined replacing (G, F) by $(M, F\dot{w})$), $\theta' \in T_{w'}^\wedge$ and F^m-stable subspace V_π of $H_c^j(X_{w',M}, \mathscr{F}_{\dot{w}',\theta'})$ isomorphic to π as $M^{F\dot{w}}$-module. Then by the similar argument as in [11, 3.5], [2, 1.1], there exists $w'' \in W$ and $\theta'' \in T^{\wedge}_{w''}$ such that

$$(H^i_c(\bar{S}, \bar{\mathbf{Q}}_l) \otimes V_\pi)^{M^{F\dot{w}}} \hookrightarrow H^{i+j}_c(X_{w''}, \mathcal{F}_{\dot{w}'', \theta''}).$$

The inclusion is F^m-equivariant as m is taken as in 1.5. Hence the proposition follows from Lemma 1.4.

1.7. We fix a parabolic subgroup $P=P_J$. Taking m such that F^m is a power of F_0, consider an irreducible representation π: $M^{F^m} \to \mathrm{GL}(V)$. π is naturally extended to a representation of P^{F^m}, which we also denote by π. Let $\mathscr{P}_m$ be the space of all functions f: $G^{F^m} \to V$. It is a G^{F^m}-module by $(gf)(x)=f(xg)$, g, $x \in G^{F^m}$, $f \in \mathscr{P}_m$. Let us define a subspace of $\mathscr{P}_m$ by

$$\mathscr{P}(M, \pi)=\{f \in \mathscr{P}_m | f(pg)=\pi(p)f(g) \text{ for } p \in P^{F^m},\ g \in G^{F^m}\}.$$

Then $\mathscr{P}_\pi=\mathscr{P}(M, \pi)$ is a G^{F^m}-submodule of $\mathscr{P}_m$ isomorphic to $\mathrm{Ind}^{G^{F^m}}_{P^{F^m}}(\pi)$. For each $w \in W$ such that $wJ \subset \Pi$, choose a representative $\dot{w} \in N_G(T)^{F_0}$ and define a linear map $\tau_{\pi, \dot{w}}$: $\mathscr{P}_m \to \mathscr{P}_m$ by

$$\tau_{\pi,\dot{w}}(f)(x)=\frac{1}{|U^{F^m}_{wJ}|}\sum_{y \in U^{F^m}_{wJ}} f(\dot{w}^{-1}yx). \tag{1.7.1}$$

Then $\tau_{\pi,\dot{w}}$ is G^{F^m}-equivariant and we have

$$\tau_{\pi,\dot{w}}\colon \mathscr{P}(M, \pi) \longrightarrow \mathscr{P}(wMw^{-1}, {}^{\dot{w}}\pi), \tag{1.7.2}$$

where ${}^{\dot{w}}\pi$ is a representation of $(wMw^{-1})^{F^m}$ given by ${}^{\dot{w}}\pi(x)=\pi(\dot{w}^{-1}x\dot{w})$. We also define F: $\mathscr{P}_m \to \mathscr{P}_m$ by $F(f)(x)=f(F^{-1}(x))$.

Now, assume given $w \in W$ and m as in 1.5. We assume further that π is $F\dot{w}$-stable. Then since $Fw(J)=J$, $\tau_{\pi,\dot{w}}$ can be defined. Let $\sigma\dot{w}$ be the restriction of $F\dot{w}$ to M^{F^m}. Since $F^m=(F\dot{w})^m$, we can define $\tilde{M}^{F^m}$ as the semidirect product of M^{F^m} with the cyclic group of order m generated by $\sigma\dot{w}$. Let $\tilde{\pi}$ be an extension of π to $\tilde{M}^{F^m}$. Then $\tilde{\pi}(\sigma\dot{w})$: $V \to V$ gives a map $\mathscr{P}(M, {}^{F\dot{w}}\pi) \to \mathscr{P}(M, \pi)$ by $f \to \tilde{\pi}(\sigma\dot{w}) \circ f$, which we denote also by $\tilde{\pi}(\sigma\dot{w})$. Hence, we get a map

$$\tilde{\pi}(\sigma\dot{w})F\tau_{\pi,\dot{w}}\colon \mathscr{P}(M, \pi) \longrightarrow \mathscr{P}(M, \pi), \tag{1.7.3}$$

which is independent of the choice of representatives $\dot{w}$ of w. Note that $\tilde{\pi}(\sigma\dot{w})F\tau_{\pi,\dot{w}}$ is nothing but $a_{F(w)F}$ in Asai's notation up to a constant multiple ([2, 1.3]).

1.8. Let $C(G^F/\sim)$ and $C(G^{F^m}/\sim_F)$ be as in Introduction. We define the similar objects with respect to M with Frobenius map $F\dot{w}$. (Note $(F\dot{w})^m=F^m$ by assumption). Following [2, 1.4], we shall define a linear map $a_{Fw}\colon C(M^{F^m}/\sim_{F\dot{w}})\to C(G^{F^m}/\sim_F)$ by putting

$$(1.8.1)\qquad (a_{Fw}([\tilde{\pi}]))(\hat{x}\sigma)=q^{md'}\,\mathrm{Tr}\,(\hat{x}\tilde{\pi}(\sigma\dot{w})F\tau_{\pi,\dot{w}},\mathscr{P}_\pi)$$

for each $\tilde{\pi}$ which is an extension to $\tilde{M}^{F^m}$ of $\pi\in\mathscr{E}(M^{F^m})^{F\dot{w}}$, and extending linearly to $C(M^{F^m}/\sim_{F\dot{w}})$. Here $d'=\dim(U_J\cap\dot{w}^{-1}U^-\dot{w})$. ($U^-$ is the unipotent radical of the opposite Borel subgroup of B with respect to T).

Nextly, we define a linear map $R^{(m)}_{M(\dot{w})}\colon C(M^{F\dot{w}}/\sim)\to C(G^F/\sim)$ by putting

$$R^{(m)}_{M(\dot{w})}(\pi)(x)=\sum_{i\geq 0}(-1)^i\,\mathrm{Tr}\,((x^{-1}F^m)^*,(H^i_c(\bar{S},\bar{\mathbf{Q}}_l)\otimes\pi)^{M^{F\dot{w}}})$$

for each $\pi\in\mathscr{E}(M^{F\dot{w}})$ and extending linearly to $C(M^{F\dot{w}}/\sim)$. Note our definition of $R^{(m)}_{M(\dot{w})}$ is slightly different from that of [2, 1.4], Now, using the same argument as in [2, 1.4], [11, 2.10], we have

1.9. Proposition. *Let w and m be as in* 1.5. *Then the following diagram is commutative.*

$$(1.9.1)\qquad \begin{array}{ccc} C(G^F/\sim) & \xrightarrow{N^*_{F^m/F}} & C(G^{F^m}/\sim_F) \\ {\scriptstyle R^{(m)}_{M(\dot{w})}}\uparrow & & \uparrow{\scriptstyle a_{Fw}} \\ C(M^{F\dot{w}}/\sim) & \xrightarrow[N^*_{F^m/F}]{} & C(M^{F^m}/\sim_{F\dot{w}}) \end{array}$$

1.10. As in [2, 2.4], [11, 3.6], we shall express the map a_{Fw} more explicitly using Hecke algebras. Let δ be an irreducible cuspidal representation of M^{F^m}. Put $W_\delta=\{w\in W\mid wJ=J,\ {}^w\delta\simeq\delta\}$, where $M=M_J$ as before. Then by the result of Howlett and Lehrer [6] and [11, § 8], W_δ is a reflection group on the orthogonal complement of $\langle J\rangle$ in $X(T)\otimes R$. ($X(T)$ is the group of characters of T). Moreover there exists a "root system" $\Gamma\subset\Sigma$ and the set of "positive roots" $\Gamma^+=\Gamma\cap\Sigma^+$ (actually the projection on $\langle J\rangle^\perp$ is a root system in the usual sense). Now, δ can be extended to a representation on $N_G(M)^{F^m}$ by means of (6.4) of [6] since W_δ is generated by reflections. We denote by $\tilde{\delta}$ an extension of δ to $N_G(M)^{F^m}$. Let $S_\delta\subset W_\delta$ be the set of simple reflections with respect to Γ^+. Following [6, 4.11], we shall define for each $y\in W_\delta$, $T_y\colon\mathscr{P}_\delta\to\mathscr{P}_\delta$ by

$$(1.10.1)\qquad T_y=\varepsilon^{(m)}_y(q_y)^{m/2}q^{l(y)m/2}\tilde{\delta}(\dot{y})\tau_{\delta,\dot{y}}$$

where $y\mapsto\varepsilon^{(m)}_y=\pm 1$ is a linear character of W_δ and $q_y=\prod_s q^{\lambda(s)}$, s runs through the elements in a reduced expression of y in W_δ and $\lambda\colon S_\delta\to\mathbf{Z}^+$ is a function which takes constant value under W_δ-conjugate (cf. [11, Theorem 8.6]). Note that T_y is independent of the choice of representatives $\dot{y}$ of y.

Then T_y ($y \in W_\delta$) gives a basis of $\mathrm{End}_{G^{F^m}} \mathrm{Ind}_{P^{F^m}}^{G^{F^m}}(\delta)$. Moreover, by [6], [11, Theorem 8.6], T_y ($y \in W_\delta$) gives rise to a basis of the Hecke algebra $H(q^m)$ over $\overline{\mathbf{Q}}_l$ with relations

$$T_w T_{w'} = T_{ww'}, \qquad \text{if } \tilde{l}(ww') = \tilde{l}(w) + \tilde{l}(w')$$
$$(T_s + 1)(T_s - q^{m\lambda(s)}) = 0, \qquad s \in S_\delta,$$

where $\tilde{l}$ is the length function of W_δ and λ: $S_\delta \to \mathbf{Z}^+$ is as above.

We define the set $Z_\delta = \{w \in W \mid Fw(J) = J,\ {}^{Fw}\delta \simeq \delta\}$. Then Z_δ can be written as wW_δ for some $w \in Z_\delta$. Since $F(wW_\delta w^{-1}) = W_\delta$ and Fw stabilizes $\langle J \rangle^\perp$, there exists $w_1 \in Z_\delta$ such that $Fw_1(\Gamma^+) \subset \Sigma^+$ by [6, Lemma 2.2]. Then $w_1(\Gamma^+) \subset \Sigma^+$ and w_1 is uniquely determined by this property. In the following, let us fix suitable representatives of Z_δ in $N_G(T)^{F_0}$ (a coherent lifting of Z_δ in the sense of [11, 1.23]). Now, $\tilde{\delta}$ can be extended to $N_G(M)^{F^m}\langle \sigma \dot{w}_1 \rangle$ (semidirect product), which we denote also by $\tilde{\tilde{\delta}}$.

We now want to show analogous formulae of (3.5.1), (3.5.2) and (3.5.3) in [11]. In order to do this, we need the following lemma, which is a variant of [6, Lemma 4.2] and can be proved by the same way.

1.11. Lemma. *Let v, $w \in W$. Assume one of the following conditions holds.*

(i) $v \in W_\delta$, $wJ \subset \Pi$ and $w\Gamma^+ \subset \Sigma^+$.

(ii) $vJ = J' \subset \Pi$, $wJ' \subset \Pi$ and $v\Gamma^+ \subset \Sigma^+$.

Then we have

$$\tau_{\dot{v}\delta, \dot{w}} \tau_{\delta, \dot{v}} = q^{m/2(l(wv) - l(w) - l(v))} \tau_{\delta, \dot{w}\dot{v}}.$$

1.12. Put $\gamma = \gamma_\delta$ and $\tau_{\dot{w}_1} = \tau_{\delta, \dot{w}_1}$. The linear map $\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1}$: $\mathscr{P}_\delta \to \mathscr{P}_\delta$ has the following properties:

(1.12.1) $\quad (\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1}) g = F(g) (\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1}) \qquad$ for $g \in G^{F^m}$,

(1.12.2) $\quad T_{\gamma(y)} (\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1}) = (\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1}) T_y \qquad$ for $y \in W_\delta$,

(1.12.3) $$(\tilde{\tilde{\delta}}(\sigma \dot{w}_1) F \tau_{\dot{w}_1})^i = q^{1/2(l(F^{-i+1}(w_1)F^{-i+2}(w_1)\cdots F^{-1}(w_1)w_1) - il(w_1))m} \times \tilde{\tilde{\delta}}(\sigma \dot{w}_1)^i F^i \tau_{\delta, F^{-i+1}(\dot{w}_1)F^{-i+2}(\dot{w}_1)\cdots F^{-1}(\dot{w}_1)\dot{w}_1}.$$

In fact, (1.12.1) is obvious. We shall prove (1.12.2). Since γ is an automorphism of the Coxeter group (W_δ, S_δ), we have $\varepsilon_y^{(m)} = \varepsilon_{\gamma(y)}^{(m)}$ and $q_y^{(m)} = q_{\gamma(y)}^{(m)}$. Then (1.12.2) is equivalent to

(1.12.4) $$\tau_{F\dot{w}_1\delta, \gamma(\dot{y})} F \tau_{\delta, \dot{w}_1} = F \tau_{\dot{y}\delta, \dot{w}_1} \tau_{\delta, \dot{y}} q^{1/2(l(y) - l(\gamma(y)))m}.$$

Lemma 1.11, (i) can be applied to the right hand side of (1.12.4) since

$y \in W_\delta$, $w_1 J \subset \Pi$ and $w_1 \Gamma^+ \subset \Sigma^+$. Hence,

$$\tau_{\dot{y}\delta,\dot{w}_1} \tau_{\delta,\dot{y}} = q^{1/2(l(w_1 y) - l(w_1) - l(y))m} \tau_{\delta,\dot{w}_1\dot{y}}.$$

While, for the left hand side,

$$\begin{aligned}\tau_{F\dot{w}_1\delta,\gamma(\dot{y})} F\tau_{\delta,\dot{w}_1} &= F\tau_{\dot{w}_1\delta,\dot{w}_1\dot{y}\dot{w}_1^{-1}} \tau_{\delta,\dot{w}_1} \\ &= q^{1/2(l(w_1 y) - l(w_1) - l(w_1 y w_1^{-1}))m} F\tau_{\delta,\dot{w}_1\dot{y}}.\end{aligned}$$

The last equality follows from Lemma 1.11, (ii). Since $l(w_1 y w_1^{-1}) = l(F(w_1 y w_1^{-1})) = l(\gamma(y))$, (1.12.4) follows.

Next, we show (1.12.3). The left hand side of (1.12.3) is equal to

$$\tilde{\delta}(Fw_1)^i F^i \tau_{F^{-i+2}(\dot{w}_1)\cdots F^{-1}(\dot{w}_1)\dot{w}_1\delta, F^{-i+i}(\dot{w}_1)} \cdots \tau_{\dot{w}_1\delta, F^{-1}(\dot{w}_1)} \tau_{\delta,\dot{w}_1}.$$

We want to apply Lemma 1.11, (ii) successively from the left. For this, we have only to verify that for each $j \geqq 1$,

(i) $F^{-i+j}(w_1)F^{-i+j+1}(w_1)\cdots F^{-1}(w_1)w_1 J \subset \Pi$,

(ii) $F^{-i+j}(w_1)F^{-i+j+1}(w_1)\cdots F^{-1}(w_1)w_1 \Gamma^+ \subset \Sigma^+$.

But these are obvious since $Fw_1 J = J$ and $Fw_1 \Gamma^+ = \Gamma^+$.

1.13. Let $\tilde{W}_\delta = W_\delta \langle \gamma_\delta \rangle$ be the semidirect product of W_δ with the cyclic group generated by γ_δ. The Hecke algebra $H(q^m)$ can be extended to an algebra $\tilde{H}(q^m)$ with basis T_w ($w \in \tilde{W}_\delta$) as in [11, 3.3]. Let us denote by $(W_\delta)^\wedge_{\text{ex}}$ the set of isomorphism classes of irreducible W_δ-modules over $\mathbf{Q}$ which is extendable to a $\tilde{W}_\delta$-module over $\bar{\mathbf{Q}}$. Let $E(q^m)$ be an irreducible $H(q^m)$-module corresponding to $E \in W_\delta^\wedge$. If $E \in (W_\delta)^\wedge_{\text{ex}}$, there exists exactly two extensions to $\tilde{W}_\delta$ over $\mathbf{Q}$. Let $\tilde{E} \in \tilde{W}_\delta^\wedge$ be one of them. Then, corresponding to $\tilde{E}$, $E(q^m)$ can be extended to an $\tilde{H}(q^m)$-module, which we denote by $\tilde{E}(q^m)$.

Now let us take m sufficiently large so that

(1.13.1) F^m is a power of F_o and $F^{-m+1}(\dot{w}_1)F^{-m+2}(\dot{w}_1)\cdots F^{-1}(\dot{w}_1)\dot{w}_1 = 1$.

Then from (1.12.3), $(\tilde{\delta}(\sigma\dot{w}_1)F\tau_{\dot{w}_1})^m = q^{-1/2l(w_1)m^2}$ id. on $\mathscr{P}_\delta$. Thus, by the same argument as in [11, 3.6], we have

(1.13.2)
$$\begin{aligned}&\mathrm{Tr}\,(\hat{x}(\tilde{\delta}(\sigma\dot{w}_1)F\tau_{\dot{w}_1})T_y, \mathscr{P}_\delta) \\ &= \sum_{E \in (W_\delta)^\wedge_{\text{ex}}} q^{-1/2l(w_1)m}\,\mathrm{Tr}\,(\hat{x}\sigma, \tilde{\rho}_E)\,\mathrm{Tr}\,(T_{\gamma y}, \tilde{E}(q^m)),\end{aligned}$$

for each $\hat{x} \in G^{F^m}$. Here $\tilde{\delta}_E$ is an extension of the irreducible G^{F^m}-module ρ_E corresponding to $E \in (W_\delta)^\wedge_{\text{ex}}$ and this extension is uniquely determined by the choice of an extension $\tilde{\delta}$ of δ and by the choice of $\tilde{E}$ of E.

1.14. Following [2, 2.3, 2.4], [3, 1.3], we shall extend the formula (1.13.2) to $\mathscr{P}_\pi$ where π is not necessarily cuspidal. Let π be an irreducible representation of $M_K^{F^m}$ $(K\subset\Pi)$, where $Fw(K)=K$ and ${}^{F\dot{w}}\pi\simeq\pi$. Let W_K be the Weyl subgroup of W with respect to K. There exists an irreducible cuspidal representation δ of $M_J^{F^m}$ $(J\subset K)$ such that π can be written as $\pi_{E'}$ for $E'\in(W_K)_\delta^\wedge$. We assume here that (*) $Fw(J)=J$. Then as $\pi_{E'}$ is $F\dot{w}$-stable, there exists $w'\in W_K$ such that ${}^{F\dot{w}\dot{w}'}\delta\simeq\delta$. Hence $ww'\in Z_\delta$ and we can write $ww'=w_1y'$, $y'\in W_\delta$. Moreover, $w'\in Z'_\delta$ (the subset of W_K with respect to $W'_\delta=(W_K)_\delta$ and $F\dot{w}$) and we have $w'=w'_1y''$, where $y''\in W'_\delta$ and w'_1 is the similar element of Z'_δ as w_1 in Z_δ. Hence there exists $y\in W_\delta$ such that $w=w_1yw_1'^{-1}$.

Let γ'_δ be the automorphism of W'_δ defined by $(Fw)w'_1$ similar to γ_δ for W_δ, and $\tilde{W}'_\delta$ be the semidirect product of W'_δ with $\langle\gamma'_\delta\rangle$. We denote by $H'(q^m)$ the subalgebra of $H(q^m)$ generated by T_z $(z\in W'_\delta)$ which corresponds to $\mathrm{Ind}_{P_J^{F^m}}^{P_K^{F^m}}(\delta)$. Let $\tilde{H}'(q^m)$ be the extended algebra corresponding to $\tilde{W}'_\delta$, and we denote by $T_{\gamma'_\delta}$ the element of $\tilde{H}'(q^m)$ corresponding to γ'_δ. In the following, for each $E\in(W_\delta)^\wedge_{\mathrm{ex}}$ and $E'\in(W'_\delta)^\wedge$, we denote by $\tilde{E}(q^m)_{E'}$ the $E'(q^m)$-isotypic subspace of $H'(q^m)$-module $\tilde{E}(q^m)$. On the other hand, as $\pi_{E'}$ is Fww'_1-stable, E' is γ'_δ-stable. Hence the extension $\tilde{\pi}_{E'}$ of $\pi_{E'}$ to $\tilde{M}_K^{F^m}$ is determined canonically as in 1.13 from $\tilde{\delta}$. Then we have

1.15. Lemma. *Let* $\pi_{E'}\in\mathscr{E}(M_K^{F^m})^{F\dot{w}}$ *and* $w=w_1yw_1'^{-1}$ *as in* 1.14. *Put* $\gamma=\gamma_\delta$, $\gamma'=\gamma'_\delta$. *Then*

$$a_{Fw}([\tilde{\pi}_{E'}])(\hat{x}\sigma)=\frac{1}{\dim E'}\varepsilon_y^{(m)}q^{md'}(q_y)^{-m/2}q^{-1/2(l(w_1)+l(y)-l(w_1'))m}$$
$$\times\sum_{E\in(W_\delta)^\wedge_{\mathrm{ex}}}\mathrm{Tr}\,(\hat{x}\sigma,\tilde{\rho}_E)\,\mathrm{Tr}\,(T_{\gamma y}T_{\gamma'}^{-1},\tilde{E}(q^m)_{E'}),$$

where $d'=\dim(U_{wK}\cap w^{-1}U^-w)$.

Proof. Let

$$\mathscr{P}=\{f\colon P_K^{F^m}\to V_1\,|\,f(px)=\delta(p)f(x)\text{ for }p\in P_J^{F^m},\ x\in P_K^{F^m}\}$$

be a realization of $\mathrm{Ind}_{P_J^{F^m}}^{P_K^{F^m}}(\delta)$, where V_1 is a representation space of δ. We denote by $\mathscr{P}_{E'}$ the $E'(q^m)$-isotypic subspace of $\mathscr{P}$ and $p_{E'}$ be the representation of $P_K^{F^m}$ on $\mathscr{P}_{E'}$. Hence $p_{E'}$ is isomorphic to $\pi_{E'}\otimes E'(q^m)$ as $P_K^{F^m}\times H'(q^m)$-module. Moreover the map $\phi\colon\mathscr{P}(M_K,p_{E'})\to\mathscr{P}(M_J,\delta)$ given by $\phi(f)(x)=f(x)(1)$ (evaluation of $f(x)\in\mathscr{P}_{E'}$ at $1\in P_K^{F^m}$) induces an isomorphism of $G^F \times H'(q^m)$-modules $\mathscr{P}(M_K, p_{E'}) \simeq \mathscr{P}(M_J, \delta)_{E'}$, which becomes an isomorphism of $\tilde{H}(q^m)$-modules. Here $\mathscr{P}(M_J, \delta)_{E'}$ denotes the $E'(q^m)$-isotypic subspace of $\mathscr{P}(M_J, \delta)$.

Let $\tilde{\delta}(\sigma\dot{w}\dot{w}_1')F\tau^K_{\delta,\dot{w}_1'}: \mathscr{P} \to \mathscr{P}$ be the map defined for $P_K^{F^m}$ with respect to δ and $w_1' \in Z_\delta'$ similar to G^{F^m}, and we denote by $b_{\dot{w}_1'}$ its restriction on $\mathscr{P}_{E'}$. Thus, by 1.13, $b_{\dot{w}_1'}$ acts on $\pi_{E'} \otimes E'(q^m)$ as $q^{-1/2l(w_1')m} F\dot{w} \otimes T_{\gamma'}$. Since $\mathscr{P}(M_K, p_{E'}) \simeq \mathscr{P}(M_K, \pi_{E'}) \otimes E'(q^m)$, $b_{\dot{w}_1'}$ induces a map $\mathscr{P}(M_K, {}^{F\dot{w}}p_{E'}) \to \mathscr{P}(M_K, p_{E'})$, which we denote also by $b_{\dot{w}_1'}$. Hence we can define a map

$$b_{\dot{w}_1'} F\tau p_{E', \dot{w}}: \mathscr{P}(M_k, p_{E'}) \to \mathscr{P}(M_K, p_{E'}).$$

Now by assumption, $Fw(J) = J$ and $Fw(K) = K$. Thus $U_{wJ} = U^K_{wJ} U_{wK}$ and $w^{-1} U^K_{wJ} w = U^K_J$, where $U^K_I = U_I \cap M_K$ for any $I \subset \Pi$. From this, we see easily that, under the isomorphism ϕ, $b_{\dot{w}_1'} F\tau_{p_{E'}, \dot{w}}$ turns out to be the map $\tilde{\delta}(\sigma\dot{w}\dot{w}_1')F\tau_{\delta, \dot{w}\dot{w}_1'}: \mathscr{P}(M_J, \delta)_{E'} \to \mathscr{P}(M_J, \delta)_{E'}$, which is nothing but the map $\tilde{\delta}(\sigma\dot{w}_1\dot{y})F\tau_{\delta, \dot{w}_1\dot{y}}$.

On the other hand, using $\mathscr{P}(M_K, p_{E'}) \simeq \mathscr{P}(M_K, \pi_{E'}) \otimes E'(q^m)$, we have

$$\begin{aligned} &\operatorname{Tr}(\hat{x} b_{w_1'} F\tau_{p_{E'}, \dot{w}} T_{\gamma'}^{-1}, \mathscr{P}(M_K, p_{E'})) \\ &= (\dim E') q^{-1/2l(w_1')m} \operatorname{Tr}(\hat{x} \tilde{\pi}_{E'}(\sigma\dot{w}) F\tau_{\pi_{E'}, \dot{w}}, \mathscr{P}(M_K, \pi_{E'})). \end{aligned}$$

This implies the lemma in view of (1.13.2).

§ 2. The main result

2.1. In this section, we assume that G is a connected classical group with connected center. Let (s) be an F-stable semisimple class in the dual group G^* of G. Taking $s \in (s) \cap T^*$, define $W_s = \{w \in W \mid w(s) = s\}$. Since (s) is F-stable, there exists $w \in W$ such that $Fw(s) = s$. Then Fw stabilizes W_s and we may take $w_0 \in W$ such that $Fw_0(s) = s$ and that Fw_0 induces a graph automorphism $\gamma_s: W_s \to W_s$. According to [11, §4], the set $\bar{X}(W_s, \gamma_s)$, $X(W_s, \gamma_s)$ and a pairing $\{,\}: \bar{X}(W_s, \gamma_s) \times X(W_s, \gamma_s) \to \bar{\mathbf{Q}}_l$ is defined. Moreover, a finite group M_c acts freely on $X(W_s, \gamma_s)$, where c is the order of γ_s and $M_c = \{\alpha \in \bar{\mathbf{Q}}_l^* \mid \alpha^c = 1\}$. In our case, W_s is isomorphic to a product of various W_I and γ_s stabilizes each W_I, where W_I is either an irreducible Weyl group of type C_l or D_l, or $W_I \simeq \prod_{i \in I} W_i$ where W_i is an irreducible Weyl group of type A_l for various l and γ_s permutes transitively each component W_i. If we denote by γ_I the restriction of γ_s to W_I, $\bar{X}(W_s, \gamma_s)$ (resp. $X(W_s, \gamma_s)$) is defined as the product set of $\bar{X}(W_I, \gamma_I)$ (resp. $X(W_I, \gamma_I)$), and the pairing $\{\ ,\ \}$ is defined as the product of each pairing.

If $W_I \simeq \prod_{i \in I} W_i$, ($W_i$: type A_l), we may assume $I = \mathbf{Z}/r\mathbf{Z}$ and $\gamma_I(W_i) = W_{i+1}$ for $i \in I$. Then $\gamma_I^r(W_1) = W_1$. Let c be the order of γ_I^r on W_1.

Then the order of γ_I is equal to rc. Now, $\bar{X}(W_I, \gamma_I) \simeq \bar{X}(W_1, \gamma_I^r) \simeq W_1^\wedge$, and $X(W_I, \gamma_I) \simeq W_1^\wedge \times M_{rc}$. The pairing $\{\ ,\ \}$: $\bar{X}(W_I, \gamma_I) \times X(W_I, \gamma_I) \to \bar{\mathbf{Q}}_l$ is given by $\{\lambda, (\lambda', \alpha)\} = \delta_{\lambda,\lambda'}\alpha^{-1}(\lambda, \lambda' \in W_1^\wedge, \alpha \in M_{rc})$.

If W_I is a Weyl group of type C_l, γ_I is identity. Then $\bar{X}(W_I, \gamma_I) \simeq X(W_I, \gamma_I) = \Phi_l$: the set of symbol classes of rank l and odd defects ([10, §3], [11, 4.5]).

If W_I is a Weyl group of type D_l, $\bar{X}(W_I, \gamma_I) = \Phi_l^\pm$ according as γ_I is trivial or not, where Φ_l^+ (resp. Φ_l^-) is the set of symbol classes of rank l and defect $\equiv 0$ (mod 4), with reduced symbol (S, S) counted twice, (resp. defect $\equiv 2$ (mod 4)), ([11, 4.6]). If γ_I is trivial, $X(W_I, \gamma_I) = \bar{X}(W_I, \gamma_I)$. While if γ_I is non-trivial, $X(W_I, \gamma_I) = \Psi_l$: the set of ordered symbol classes (S, T) such that $S \neq T$, of rank l and defect $\equiv 0$ (mod 4). $M_2 \cong \mathbf{Z}/2\mathbf{Z}$ acts on Ψ_l by $(S, T) \leftrightarrow (T, S)$, ([11, 4.18]). For each of above cases, the pairing is given in terms of symbols, ([11, 4.5, 4.6, 4.18]).

It is known by Theorem 4.23 of [11], that $\mathscr{E}(G^F, (s)) \cong \bar{X}(W_s, \gamma_s)$. We express this correspondence by $\rho \leftrightarrow \bar{x}_\rho$. Take m large enough so that $s \in T^{*F^m}$ and that F^m is a power of F_0. Then there exists a surjection from $X(W_s, \gamma_s)$ to $\mathscr{E}(G^{F^m}, (s))^F$ each of whose fibre is just an M_c-orbit. Hence $\mathscr{E}(G^{F^m}, (s))^F \simeq X(W_s, \gamma_s)/M_c$.

For each $x \in X(W_s, \gamma_s)$, we shall define, following [11, (4.24.1)], an almost character associated to x,

$$R_x = (-1)^{l(w_0)} \sum_{\rho \in \mathscr{E}(G^F, (s))} \{\bar{x}_\rho, x\}\rho \in \mathscr{R}(G^F) \otimes \bar{\mathbf{Q}}_l. \tag{2.1.1}$$

The action of M_c on $X(W_s, \gamma_s)$ gives rise to the scalar multiplication by elemets of M_c on R_x. Hence, for a given ρ in $\mathscr{E}(G^{F^m}, (s))^F$, an element $x = x_\rho$ in $X(W_s, \gamma_s)$ is determined up to the M_c-orbit, and we can attach $R_{x_\rho} \in \mathscr{R}(G^F) \otimes \bar{\mathbf{Q}}_l$ to ρ up to a c-th root of unity multiple.

We note here that by our assumption on m, a root of unity λ_ρ (in Lemma 4.1) is associated to each $\rho \in \mathscr{E}(G^{F^m}, (s))$. We can now state our main result.

2.2. Theorem. *Let G be a classical group with connected center. Then there exists an integer $m_0 = m_0(G^F)$ satisfying the following properties:*

Let ρ be a representation in $\mathscr{E}(G^{F^m}, (s))^F$ and $\tilde{\rho}$ an extension to $\tilde{G}^{F^m}$. If m is divisible by m_0, there exists $\mu_{\tilde{\rho}}$ (depending on m, $\tilde{\rho}$ and the choice of x_ρ) such that

$$N^*_{F^m/F}([\mu_{\tilde{\rho}}\tilde{\rho}]) = R_{x_\rho}.$$

Here $\mu_{\tilde{\rho}}$ is a root of unity satisfying $(\mu_{\tilde{\rho}})^m = \lambda_\rho^{-1}$.

2.3. Remark. The definition of λ_ρ in [11, Proposition 2.20] depends on the choice of a coherent lifting ([11, 1.23]). However, our theorem implies that, at least in our setting, i.e., m is sufficiently divisible and ρ is F-stable, λ_ρ is independent from that choice since $(\mu_{\tilde{\rho}})^m$ is uniquely determined by ρ.

2.4. The remainder part of this paper is devoted to the proof of the theorem.

If G is of type A_n, the lifting always exists by [12], [7] and the theorem is proved easily from this. Hence we assume that $G=G_n$ is of type B_n, C_n or D_n. Using induction on n, we shall assume that the theorem is valid for $G_{n'}$ $(n'<n)$.

Let $M=M_J$ be a proper Levi subgroup of G and $F\dot{w}$ be a Frobenius map on M (i.e., $Fw(J)=J$). Since the Coxeter diagram of M is a direct sum of diagrams of classical type, using the argument in [1, § 2], we may assume that the theorem is valid for M.

2.5. Let M and $F\dot{w}$ be as in 2.4 and $(s)\subset M^*$ be an $F\dot{w}$-stable semisimple class. We assume that $\mathscr{E}(M^{Fm}, (s))^{F\dot{w}}$ contains a cuspidal representation δ, which is unique in $\mathscr{E}(M^{Fm}, (s))$. By induction hypothesis, for each m divisible by $m_0(M^{F\dot{w}})$, we can attach a root of unity $\mu_{\tilde{\delta}}$ such that $N^*_{Fm/F,M}([\mu_{\tilde{\delta}}\tilde{\delta}])$ is independent of m. Let $\rho_E \in \mathscr{E}(G^{Fm}, (s))^F$ be the representation corresponding to $E \in (W_\delta)^{\wedge}_{\text{ex}}$ and $\tilde{\rho}_E$ be as in 1.13.

Following [11, § 3], we shall show that $N^*_{Fm/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])$ takes the same value for infinitely many m.

Let $H(q^m)$ be the Hecke algebra corresponding to $\mathrm{Ind}_{P_J^{Fm}}^{G^{Fm}}(\delta)$. Then, since $W_\delta=\{w \in W_s \mid w(J)=J\}$ by [1], $H(q^m)$ is a tensor product of various Hecke algebras of classical type. Hence by [11, § 3] and Benson and Curtis [4], we see that, for each $E \in (W_\delta)^{\wedge}_{\text{ex}}$

$$\mathrm{Tr}\,(T_{\gamma y}, \tilde{E}(q^m)) \in \mathbf{Q}[q^m]. \tag{2.5.1}$$

Let $m_1(G^F)$ be the smallest integer such that $m_1(G^F)$ is divisible by both of $m_0(M^{F\dot{w}})$ and m_1 in Proposition 1.6 for various M and $F\dot{w}$, and that $m_1(G^F)$ satisfies (1.13.1) for various $\dot{w}_1$. We denote by $\mathscr{M}'$ the set of positive integers m divisible by $m_1(G^F)$. Then, in particular, $N^*_{Fm/F,M}([\mu_{\tilde{\delta}}\tilde{\delta}])$ $=R_{x_\delta}$ for $m \in \mathscr{M}'$. Put $\alpha_{\psi,E}(m)=\langle\psi, N^*_{Fm/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])\rangle_{G^F}$ for each $\psi \in \mathscr{E}(G^F)$. Now using the orthogonality relations of Hecke algebra $\tilde{H}(q^m)$, we see, by Proposition 1.9 together with (1.13.2), that $N^*_{Fm/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])$ is contained in $C^{(s)}(G^F/\sim)$. Moreover, by virtue of Proposition 1.6, we see that $\alpha_{\psi,E}(m)$ is contained in a fixed algebraic number field in $\overline{\mathbf{Q}}_l$. On the other hand, $\alpha_{\psi,E}(m)$ are cyclotomic integers divided by $|G^F|$, and have absolute value ≤ 1. The last property follows from the Cauchy-Schwarz inequality, (cf.

[11, 3.8]). Hence there are only finitely many $\alpha_{\psi,E}(m)$ for $m \in \mathcal{M}'$. Therefore we can divide $\mathcal{M}'$ into a finite number of sets $\mathcal{M}_i$ $(i=1, \cdots, r)$ such that $\alpha_{\psi,E}$ takes constant value on $\mathcal{M}_i$ for each pair (ψ, E).

Let $\mathcal{M}$ be one of the $\mathcal{M}_i$ such that $|\mathcal{M}|=\infty$. Then $N^*_{F^m/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])$ is independent of m for $m \in \mathcal{M}$. Hence, by Lemma 1.15 applied to the case $J=K$, we see that $\varepsilon_y^{(m)}$ is independent of $m \in \mathcal{M}$ for each $y \in W_\delta$. We denote by ε_y this constant value $\varepsilon_y^{(m)}$, (the assumption (*) in 1.14 is trivial in this case).

2.6. Let $\mathcal{U}^{(s)}(G, F)$ be an Euclidean space over $\overline{\mathbf{Q}}_l$ with inner product $\langle\ ,\ \rangle$ generated by f_x, $(x \in X(W_s, \gamma_s))$ with relations

$$f_{\zeta x}=\zeta f_x \qquad \text{for each } \zeta \in M_c,$$
$$\langle f_x, f_y\rangle=\begin{cases}1 & \text{if } x=y\\ 0 & \text{if } y \notin M_c x.\end{cases}$$

Moreover, let $\mathcal{V}^{(s)}(G, F)$ be an Euclidean space over $\overline{\mathbf{Q}}_l$ with inner product $\langle\ ,\ \rangle$ and with orthonormal basis $e_{\bar{x}}$ $(\bar{x} \in \bar{X}(W_s, \gamma_s))$. As in 2.1, $\mathcal{E}(G^{F^m}, (s))^F$ is bijective with $X(W_s, \gamma_s)/M_c$. We fix a representative $x=x_\rho$ in $X(W_s, \gamma_s)$ for each $\rho \in \mathcal{E}(G^{F^m}, (s))^F$. Let $C^{(s)}(G^{F^m}/\sim_F)'$ be the subspace of $C^{(s)}(G^{F^m}/\sim_F)$ generated by $[\tilde{\rho}_E]$ for various $(M, F\dot{w})$ with $M\neq G$. Also we denote by $\mathcal{U}^{(s)}(G, F)'$ the subspace of $\mathcal{U}^{(s)}(G, F)$ generated by f_x for x corresponding to ρ_E as above. Then we may identify $\mathcal{U}^{(s)}(G, F)'$ with $C^{(s)}(G^{F^m}/\sim_F)'$ by associating $x=x_{\rho_E}$ to $[\mu_{\tilde{\delta}}\tilde{\rho}_E]$. We consider also the similar spaces $\mathcal{U}^{(s)}(M, F\dot{w})$ and $\mathcal{V}^{(s)}(M, F\dot{w})$. We may identify $\mathcal{U}^{(s)}(M, F\dot{w})$ with $C^{(s)}(M^{F^m}/\sim_F)$ by associating $x=x_\rho$ to $\mu_{\tilde{\rho}}\tilde{\rho}$, where $\rho \in \mathcal{E}(M^{F^m}, (s))^{F\dot{w}}$ and $\mu_{\tilde{\delta}}$ is given as in the theorem. Then a_{Fw} (resp. $R^{(m)}_{M(\dot{w})}$) induces the map a_{Fw}: $\mathcal{U}^{(s)}(M, F\dot{w})\to\mathcal{U}^{(s)}(G, F)'$ (resp. $R^{(m)}_{M(\dot{w})}$: $\mathcal{V}^{(s)}(M, F\dot{w})\to\mathcal{V}^{(s)}(G, F)$) by above identifications.

2.7 Let us define Δ_M: $\mathcal{U}^{(s)}(M, F\dot{w})\to\mathcal{V}^{(s)}(M, F\dot{w})$ by

$$\Delta_M : f_x \longmapsto \hat{e}_x=(-1)^{l(w_0')} \sum_{\bar{y} \in \bar{X}((w_J)_s, \gamma_s')} \{\bar{y}, x\}\bar{y},$$

where w_0' is the corresponding element in W_J of w_0 in W. Hence Δ_M coincides with $N^*_{F^m/F,M}$ under our identifications. Moreover, we define Δ_G: $\mathcal{U}^{(s)}(G, F)'\to\mathcal{V}^{(s)}(G, F)$ by associating $x=x_{\rho_E}$ to the element corresponding to $N^*_{F^m/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])$ which is independent of $m \in \mathcal{M}$ by 2.5. Then Δ_M and Δ_G becomes isometries between two spaces and (1.9.1) turns out to be the following commutative diagram.

(2.7.1)
$$\begin{array}{ccc}
\mathcal{V}^{(s)}(G, F) & \xleftarrow{\ \Delta_G\ } & \mathcal{U}^{(s)}(G, F)' \\
R^{(m)}_{M(\dot{w})}\uparrow & & \uparrow a_{Fw} \\
\mathcal{V}^{(s)}(M, F\dot{w}) & \xleftarrow{\ \Delta_M\ } & \mathcal{U}^{(s)}(M, F\dot{w})
\end{array}$$

In (2.7.1), each spaces and Δ_G, Δ_M are independent of m, while $R^{(m)}_{M(\dot{w})}$ is the map whose coefficients are given by (laurent) polynomials in $q^{m/2}$ by Proposition 1.6. We show that a_{Fw} is also the map whose coefficients are given by polynomials in $q^{m/2}$. In view of Lemma 1.15 and (2.5.1), we have only to show that the assumption (*) in 1.14 is satisfied. Thus we shall show that for each $\pi_{E'} \in \mathscr{E}(M_K^{F^m})^{F\dot{w}}$, there exists an $F\dot{w}$-stable Levi subgroup M_J and a cuspidal representation $\delta \in \mathscr{E}(M_J^{F^m})$ to which $\pi_{E'}$ belongs. Since we are dealing with classical groups, this is reduced to the case where K is of type A_l and $\sigma\dot{w}$ is a non-trivial automorphism of K. But in this case, by the existence of the lifting ([7]), (*) is transferred to the similar problem in $M_K^{F\dot{w}}$. Hence (*) holds in this case.

Now, by specializing $q^m \to 1$, we get the following diagram.

(2.7.2)
$$\begin{array}{ccc} \mathscr{V}^{(s)}(G, F) & \xleftarrow{\ \Delta_G\ } & \mathscr{U}^{(s)}(G, F)' \\ R_{(\dot{w})}\uparrow & & \uparrow a_w \\ \mathscr{V}^{(s)}(M, F\dot{w}) & \xleftarrow{\ \Delta_M\ } & \mathscr{U}^{(s)}(M, F\dot{w}) \end{array}$$

The map a_w is given for each $x_{E'} = x_{\pi_{E'}}$ ($\pi_{E'}$ as in 1.15),

(2.7.3)
$$a_w(f_{x_{E'}}) = \frac{1}{\dim E'}\varepsilon_y \sum_{E \in (W_\delta)^\wedge_{\mathrm{ex}}} \mathrm{Tr}\,(\gamma_\delta y \gamma_\delta'^{-1}, \tilde{E}_{E'}) f_{x_E}$$

where $f_{x_E} \in \mathscr{U}^{(s)}(G, F)'$ is the element corresponding to $\rho_E \in \mathscr{E}(G^{F^m}, (s))^F$ and $w = w_1 y w_1'^{-1}$ is as in 1.15. $\tilde{E}_{E'}$ is the E'-isotypic subspace of W'_δ-module $\tilde{E}$. $R_{(\dot{w})}$ is nothing but $R^G_{M(\dot{w})}$ by our identifications.

The following transitivity of $R_{(\dot{w})}$ is known ([9], [2, 1.1.3]).

(2.7.4)
$$\begin{array}{ccc} \mathscr{V}^{(s)}(G, F) & & \\ \uparrow & \nwarrow{\scriptstyle R_{(\dot{w})}} & \\ R_{(\dot{w}\dot{w}')} & & \mathscr{V}^{(s)}(M_K, F\dot{w}) \\ & \nearrow{\scriptstyle R_{(\dot{w}')}} & \\ \mathscr{V}^{(s)}(M_J, F\dot{w}\dot{w}') & & \end{array}$$

where $w \in W$ and $w' \in W_K$. (s) is a class in M_J which is Fww'-stable and is extended to the classes in M_K and in G.

The following transitivity of a_w also follows easily from (2.7.3), (cf. [2, Lemma 2.7.7]). Under the same setting as above,

(2.7.5)
$$\begin{array}{ccc} \mathcal{U}^{(s)}(G,F) & & \\ \uparrow a_{ww'} & \nwarrow a_w & \\ & & \mathcal{U}^{(s)}(M_K, F\dot{w}) \\ & \nearrow a_{w'} & \\ \mathcal{U}^{(s)}(M_J, F\dot{w}\dot{w}') & & \end{array}$$

2.8. We now show that the proof of the theorem is reduced to the special case where the centralizer $Z_{G^*}(s)^*$ has the same semisimple rank as G. Assume that the semisimple rank of $Z_{G^*}(s)^*$ is less than that of G. Then there exists some $M \neq G$ with Frobenius map $F\dot{w}$ such that $Z_{G^*}(s)$ is contained in M^*. In this case, W_s is contained in W_K (here we put $M = M_K$) and $\bar{X}(W_s, \gamma_s)$ for M coincides with the one for G. By [9, 8, 10], $R_{M(\dot{w})}$ becomes the scalar multiplication $(-1)^{\sigma(G)-\sigma(M)}$ under our identification $\mathcal{V}^{(s)}(M, F\dot{w}) = \mathcal{V}^{(s)}(G, F)$, where $\sigma(G)$ (resp. $\sigma(M)$) is the $\mathbf{F}_q$-split rank of G (resp. M) with respect to F (resp. $F\dot{w}$), respectively. Hence

$$(-1)^{\sigma(G)-\sigma(M)} = (-1)^{l(w)}.$$

On the other hand, since $W_\delta = W'_\delta$ for each cuspidal representation δ of M_J $(J \subset K)$, we have $ww'_1 = w_1$. Hence $\gamma_\delta = \gamma'_\delta$ and $y = 1$, and a_w turns out to be the identity map on $\mathcal{U}^{(s)}(M, F\dot{w}) = \mathcal{U}^{(s)}(G, F)'$ $(= \mathcal{U}^{(s)}(G, F))$. Now our assertion follows from the fact that the element w_0 in W with respect to (W_s, γ_s) in $\mathcal{E}(G^F, (s))$ is equal to w, while w'_0 in $\mathcal{E}(M^{F\dot{w}}, (s))$ is equal to 1.

2.9. In view of 2.8, we may assume $Z_{G^*}(s)$ has the same semisimple rank as G^*. Then W_s has the form $W_1 \times W_2$, where W_i $(i = 1, 2)$ is a Weyl group of type C_k or D_k. We may take $s \in T^{*F}$ in this case and therefore $w_0 = 1$.

Let us define a linear map $\tilde{\Delta} = \tilde{\Delta}_G \colon \mathcal{U}^{(s)}(G, F) \to \mathcal{V}^{(s)}(G, F)$ by associating f_x $(x \in X(W_s, \gamma_s))$ to $\hat{e}_x = \Sigma\{\bar{y}, x\}\bar{y}$, where $\bar{y}$ runs over the elements in $\bar{X}(W_s, \gamma_s)$. We want to show that $\Delta = \tilde{\Delta}$ on $\mathcal{U}^{(s)}(G, F)'$. Let $M_r = M_{J_r}$ $(r \geq 0)$ be the Levi subgroup of G whose Coxeter diagram has the same type as G with rank r $(r \neq 1, 2$ if G is of type $D_n)$. It is clear that $\mathcal{U}^{(s)}(G, F)'$ is generated by the images of a_w from $\mathcal{U}^{(s')}(M_r, F\dot{w})$ ((s') is a class in M_r such that $(s') \subset (s)$) for various M_r, w and (s'). So, it is enough to show that $\Delta = \tilde{\Delta}$ on $a_w(\mathcal{U}^{(s')}(M_r, F\dot{w}))$ for each triple $(M_r, w, (s'))$. We note here that

$$\Delta = \tilde{\Delta} \quad \text{on} \quad a_w(\mathcal{U}^{(s')}(T, F\dot{w})). \tag{2.9.1}$$

In fact, since $l(w_0) = 1$, this follows immediately from Corollary 4.24 of [11].

Assume $r>0$ (resp. $r\geq 4$) for G of type B_n, C_n (resp. D_n) Put $W^r=\{w \in W \mid w(J_r)=J_r\}$. Then W^r is isomorphic to a Weyl group of type C_{n-r}, and an element $w \in W^r$ can be expressed as a product of positive cycles and negative cycles. Hence, from the transitivity of $R_{(\dot w)}$ and a_w ((2.7.4), (2.7.5)), the verification of $\Delta=\tilde{\Delta}$ on $\mathcal{U}^{(s)}(G, F)'$ is reduced to showing that $\Delta=\tilde{\Delta}$ on $a_w(\mathcal{U}^{(s)}(M_r, F\dot w))$ where w is a positive or negative cycle of length $n-r$.

2.10. Lemma. *Assume that $w \in W^r$ is a positive cycle of length $n-r$. Then $\Delta=\tilde{\Delta}$ on $a_w(\mathcal{U}^{(s)}(M_r, F\dot w))$.*

Proof. Let M be the Levi subgroup of G whose Coxeter diagram is a direct sum of A_{n-r-1} and the diagram of M_r. Then using the transitivity (2.7.4), (2.7.5) to $M_r^{F\dot w}$, M^F and G^F, we see that to prove the lemma it is enough to show the commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{V}^{(s)}(G, F) & \xleftarrow{\;\tilde{\Delta}\;} & \mathcal{U}^{(s)}(G, F)' \\ R_{(1)}\Big\uparrow & & \Big\uparrow a_1 \\ \mathcal{V}^{(s)}(M, F) & \xleftarrow{\;\Delta\;} & \mathcal{U}^{(s)}(M, F) \end{array}$$

As $R_{(1)}$ is nothing but the induction from P^F to G^F, all the maps are explicitly computable. Hence using the similar computation as in [2, Lemma 2.8.3], we get the lemma.

2.11. Next we consider the case where $w \in W^r$ is a negative cycle of length $n-r$. In order to apply (2.7.3) to this case, we shall determine γ_δ, γ_δ' and others. Assume δ is a cuspidal representation of M_t^{Fm} $(t\leq r)$, where J_t is Fw-stable. Then, since $W_s\simeq W_1\times W_2$, we can express $(W_{J_r})_s\simeq W_1'\times W_2'$ and $(W_{J_t})_s\simeq W_1''\times W_2''$ with $W_i''\subset W_i'\subset W_i$ $(i=1, 2)$. In our setting, we may assume $W_2=W_2'$. Put $W_\delta'=W_\delta\cap W_{J_r}$. Since $W_\delta\simeq W^t\cap W_s$, we can express W_δ and W_δ' as $W_\delta\simeq(W_\delta)_1\times(W_\delta)_2$, $W_\delta'\simeq(W_\delta')_1\times(W_\delta')_2$. Let γ_i^s: $W_i\rightarrow W_i$, γ_i: $(W_\delta)_i\rightarrow(W_\delta)_i$ and γ_i': $(W_\delta')_i\rightarrow(W_\delta')_i$ be the maps on the i-th factor $(i=1, 2)$ induced from γ_s: $W_s\rightarrow W_s$, γ_δ: $W_\delta\rightarrow W_\delta$ and γ_δ': $W_\delta'\rightarrow W_\delta'$, respectively. Moreover we put γ_i^r: $W_i'\rightarrow W_i'$ the map induced on the i-th factor from γ_s': $(W_{J_r})_s\rightarrow(W_{J_r})_s$.

First consider the case where W_1 is of type C_k. In this case, $(W_\delta)_1$ and $(W_\delta')_1$ are also of type C. Hence, $\gamma_1^s=\gamma_1=\gamma_1'=\gamma_1^r=$trivial. Moreover, since $w \in (W_\delta)_1$, we have $w_1=w_1'=1$.

Next consider the case where W_1 is of type D_k. If $W_1''=\{1\}$, then $(W_\delta)_1=W_1$, $(W_\delta')_1=W_1'$ and both of these are of type D. In this case, since F stablilizes $(W_\delta)_i$ and $(W_\delta')_i$, w_1 stabilizes $(W_\delta)_i$ and w_1' stabilizes $(W_\delta')_i$ $(i=1, 2)$. From this, considering the possibility of w_1 and w_1', we see that $y=w_1^{-1}ww_1' \in (W_\delta)_1$ and that exactly one of w_1 and w_1' is equal to 1. Thus, $\gamma_1^s=\gamma_1$, $\gamma_1'=\gamma_1^\tau=-\gamma_1$. (Here we regarded γ_1^s, γ_1, etc. as elements in $M_2=\{1, -1\}$). Moreover $\gamma_1 y\gamma_1'^{-1}$ coincides with w in $(\tilde{W}_\delta)_1$. If $W_1''\neq\{1\}$, $(W_\delta)_1$ and $(W_\delta')_1$ has type C, and w is contained in $(W_\delta)_1$. Hence $\gamma_1=\gamma_1'=$ trivial and $w=y$. Moreover, as w acts non-trivially on W_1', we have $\gamma_1^s=-\gamma_1^\tau$. Throughout the above cases $\gamma_2=\gamma_2'$ and the contribution of $\gamma_\delta y\gamma_\delta'^{-1}$ on$(W_\delta)_2$ is trivial.

2.12. Before proceeding further, we note here about ε_y in (2.7.3). This is described as follows. Let $y \in (W_\delta)_1$ as in 2.12. Then by [11, § 5], [1], there exists $\varepsilon_\delta'=\pm 1$ such that

$$\varepsilon_y=\begin{cases} 1 & \text{if } (W_\delta)_1\text{: type } D, \\ (\varepsilon_\delta')^{l'(y)} & \text{if } (W_\delta)_1\text{: type } C, \end{cases} \tag{2.12.1}$$

where $l'(y)$ is the number of reflections corresponding to long roots (in C) appearing in the reduced expression of y in $(W_\delta)_1$.

2.13. Lemma. *Let $w \in W^\tau$ be a negative cycle of length $n-r$. Then $\Delta=\tilde{\Delta}$ on $a_w(\mathcal{U}^{(s)}(M_r, F\dot{w}))$.*

Proof. We shall show the lemma, following [2], only in the case where W_1 is of type D_k. The case W_1 is of type C_k is dealt similarly (cf. [3]), (see also Remark 2.14).

Let $\mathcal{U}_k^\varepsilon$ (resp. $\mathcal{V}_k^\varepsilon$) be the space corresponding to $X(W_1, \gamma_1^s)$ (resp. $\bar{X}(W_1, \gamma_1^s)$) as in 2.6, where $\varepsilon=\pm 1$ according as γ_1 is trivial or not. Thus, as in 2.1, $\mathcal{U}_k^\varepsilon$ and $\mathcal{V}_k^\varepsilon$ are described by symbols. For each symbol Λ in $\Phi_k^\pm$ or Ψ_k, we denote by f_Λ or e_Λ the element corresponding to f_x or $e_{\bar{x}}$. We may identify $\mathcal{U}^{(s)}(G, F)$ (resp. $\mathcal{V}^{(s)}(G, F)$) with $\mathcal{U}_k^\varepsilon\otimes\mathcal{U}'$ (resp. $\mathcal{V}_k^\varepsilon\otimes\mathcal{V}'$) and also $\mathcal{U}^{(s)}(M, F\dot{w})$ (resp, $\mathcal{V}^{(s)}(M, F\dot{w})$) with $\mathcal{U}_l^\varepsilon\otimes\mathcal{U}'$ (resp. $\mathcal{V}_l^\varepsilon\otimes\mathcal{V}'$), respectively. Here $\mathcal{U}'$ (resp. $\mathcal{V}'$) denotes the space corresponding to $X(W_2, \gamma_2^s)$ (resp. $\bar{X}(W_2, \gamma_2^s)$).

Following [2, 2.8], for positive integer v, linear maps $I_{(v)}^-: \mathcal{U}_l^\varepsilon\to\mathcal{U}_k^{-\varepsilon}$ and $J_{(v)}: \mathcal{V}_l^\varepsilon\to\mathcal{V}_k^{-\varepsilon}$, $(k=l+v)$ are defined. Since $\tilde{\Delta}: \mathcal{U}_k^\varepsilon\otimes\mathcal{U}'\to\mathcal{V}_k^\varepsilon\otimes\mathcal{V}'$ can be decomposed as $\tilde{\Delta}=\tilde{\Delta}_k\otimes\tilde{\Delta}'$, where $\tilde{\Delta}_k$, $\tilde{\Delta}'$ is the corresponding map on $\mathcal{U}_k^\varepsilon$, $\mathcal{U}'$, respectively, we see that the following diagrams turns out to be commutative by [2, Lemma 2.8.3].

$$\begin{array}{ccc} \mathcal{V}_k^{-\varepsilon}\otimes\mathcal{V}' & \xleftarrow{\ \tilde{\Delta}\ } & \mathcal{U}_k^{-\varepsilon}\otimes\mathcal{U}' \\ {\scriptstyle J_{(v)}\otimes 1}\uparrow & & \uparrow{\scriptstyle I_{(v)}^-\otimes 1} \\ \mathcal{V}_l^\varepsilon\otimes\mathcal{V}' & \xleftarrow{\ \Delta\ } & \mathcal{U}_l^\varepsilon\otimes\mathcal{U}' \end{array} \tag{2.13.1}$$

Using the definition of a_w (2.7.3) together with 2.11, we see by [2, Lemma 2.8.2] that $I^-_{(v)}\otimes 1$ coincides with a_w for a negative cycle w of length v. Note, in this case, under the identification of $\mathcal{U}^{(s)}(G, F)$ with $X(W_s, \gamma_s)/M_c$, retaking representatives of M_c-orbit if necessary, we may regard $\varepsilon_y=1$ when comparing a_w with $I^-_{(v)}\otimes 1$, (cf. [2]).

Take $e_\Lambda\otimes e_{\bar{x}}\in\mathcal{V}^\varepsilon_l\otimes\mathcal{V}'$. Then by (2.7.2), (2.13.1), we have

$$\Delta\tilde{\Delta}^{-1}(J_{(v)}e_\Lambda\otimes e_{\bar{x}})=R_{(\dot{w})}(e_\Lambda\otimes e_{\bar{x}})$$

Hence $\Delta\tilde{\Delta}^{-1}(J_{(v)}e_\Lambda\otimes e_{\bar{x}})$ is an integral linear combination of various $e_{\Lambda'}\otimes e_{\bar{y}}\in\mathcal{V}^\varepsilon_k\otimes\mathcal{V}'$. Now $\tilde{\Delta}$ is an isometry and Δ is also an isometry where it is defined, and moreover we know already $\Delta=\tilde{\Delta}$ on $\mathcal{U}^\varepsilon_{k,0}\otimes\mathcal{U}'_0$ by (2.9.1), where $\mathcal{U}^\varepsilon_{k,0}$ is a space generated by symbols of defect 0 and $\mathcal{U}'_0$ is the similar subspace in $\mathcal{U}'$. Hence entirely similar proof as in Lemma 2.8.10 of [2] shows that, if we put $\tilde{f}=J_{(v)}e_\Lambda\otimes e_{\bar{x}}$, then

$$\tilde{f}-\Delta\tilde{\Delta}^{-1}\tilde{f}=f_1\otimes e_{\bar{x}},$$

where $f_1\in\mathcal{V}^{-\varepsilon}_k$ is written as in the form (II) of Lemma 2.8.10 in [2] with $f=J_{(v)}e_\Lambda$. Furthermore, $\langle f_1, \hat{e}_{\Lambda'}\rangle=0$ for any $e_{\Lambda'}$ of defect 0. Hence by the argument in Lemma 2.8.7, Lemma 2.8.8 in [2], we have $f_1=0$. This shows that $\Delta=\tilde{\Delta}$ on $I^-_{(v)}(\mathcal{U}^\varepsilon_l)\otimes\mathcal{U}'$. Hence the lemma is proved.

2.14. Remark. The case where W_1 is of type C_k is dealt similarly according to [2]. In this case, as in [3], we encounter the problem to determine ε_δ explicitly on the way to the proof. This is done similarly as in [3] and we have the following. Let $x=(x_1, x_2)$ be the element in $X(W_s, \gamma_s)\simeq X(W_1, \gamma^s_1)\times X(W_2, \gamma^s_2)$ corresponding to δ. Now $X(W_1, \gamma^s_1)$ is identified with symbols of odd defect. If x_1 corresponds to a symbol of defect d, then we have

$$\varepsilon'_\delta=(-1)^{(d-1)/2}.$$

Now, in view of 2.9, we have

2.15. Proposition. *In the setting of* 2.9, $\Delta=\tilde{\Delta}$ *on* $\mathcal{U}^{(s)}(G, F)'$.

2.16. We keep the assumption on s as in 2.9. Then, as is easily seen, $\mathcal{U}^{(s)}(G, F)'$ coincides with the space generated by the elements corresponding to non-cuspidal representations. Moreover, in the case of classical groups, $\mathcal{E}(G^{F^m}, (s))$ contains at most one cuspidal representation for each $(s)\subset G^*$. Thus, in view of Proposition 2.15, to prove the theorem, it is enough to show the following lemma. (Note our result does not depend on the choice of $\mathcal{M}$.)

2.17. Lemma. *Let $s \in G^{*F}$ be as before and $m \in \mathcal{M}$. Assume $\mathscr{E}(G^{F^m}, (s))^F$ contains a cuspidal representation ρ_0. Then for each extension $\tilde{\rho}_0$ to $\tilde{G}^{F^m}$, there exists a root of unity $\mu_{\tilde{\rho}_0}$ such that*

$$N^*_{F^m/F}([\mu_{\tilde{\rho}_0}\tilde{\rho}_0]) = R_{x_0},$$

where $x_0 = x_{\rho_0}$. Moreover $(\mu_{\tilde{\rho}_0})^m = \lambda_{\rho_0}^{-1}$, where λ_{ρ_0} is a root of unity associated to ρ_0 (see Lemma 1.4).

Proof. Let us take $w \in W_s^F$ (the group of F-fixed points of W_s). Then F acts on T^{wF^m} and we can find $\theta \in \hat{T}^{wF^m}$ corresponding to $s \in T^{*wF^m}$ such that θ is F-stable. We denote by $\theta_0 \in \hat{T}^F$ the character obtained as the image of the map $N^*_{wF^m/F}$: $C(T^{wF^m}/\sim_F) \to C(T^F/\sim)$.

Let $X_w^{(m)}$ be the variety as in 1.2 with Frobenius map F^m, and $\mathscr{F}_{\dot{w},\theta}$ be the corresponding sheaf on $X_w^{(m)}$. Since w is F-stable, F acts naturally on $X_w^{(m)}$ and we get the induced action of F on $H_c^i(X_w^{(m)}, \mathscr{F}_{\dot{w},\theta})$ as θ is F-stable. Then using the similar argument as in the proof of Proposition 1.9, ([2, 1.4], [11, 2.10]), but with inverse setting, we have

$$\sum_i (-1)^i \operatorname{Tr}(F^*\hat{x}^*, H_c^i(X_w^{(m)}, \mathscr{F}_{\dot{w},\theta})) = \operatorname{Tr}(x^{-1}\tau_{\theta_0,\dot{w}}, \operatorname{Ind}_{B^F}^{G^F}(\theta_0)), \tag{2.17.1}$$

where $\hat{x} \in G^{F^m}$ and $x \in G^F$ are as in Introduction.

From Lemma 1.4, for each $\rho \in \mathscr{E}(G^{F^m}, (s))^F$, there exists a root of unity λ_ρ such that the eigenvalues of F^m on $H_c^i(X_w^{(m)}, \mathscr{F}_{\dot{w},\theta})$ are of the form $\lambda_\rho q^{jm/2}$ for some integer j. Let us fix an m-th root λ'_ρ of λ_ρ. For an eigenvalue $\mu = \lambda_\rho q^{jm/2}$, put $H^i_{\dot{w},\mu}$ be the generalized eigenspace of F^m with eigenvalue μ of $H_c^i(X_w^{(m)}, \mathscr{F}_{\dot{w},\theta})_\rho$. Then $H^i_{\dot{w},\mu}$ is a G^{F^m}-module on which F acts. There exists a filtration of G^F-modules, stable by F, whose successive quotient is isomorphic to ρ as a G^{F^m}-module. If we define the action of σ on this filtration by $\lambda'_\rho q^{j/2} F^{*-1}$, each successive quotient becomes a $\tilde{G}^{F^m}$-module. However, if we consider the action of F^2 instead of F, this filtration gives rise to an F^2-stable filtration and each successive quotient turns out to be a $G^{F^m}\langle\sigma^2\rangle$-module. Then, by Lemma 1.4, these $G^{F^m}\langle\sigma^2\rangle$-modules are mutually isomorphic for various filtration and various i and w. Hence, as $\tilde{G}^{F^m}$-modules, there are at most two possibilities, if we denote one by $\tilde{\rho}$, the other one is obtained by acting σ as $-\sigma$ on $\tilde{\rho}$, which we denote by $-\tilde{\rho}$. Since,

$$\operatorname{Tr}(F^*\hat{x}^*, H^i_{\dot{w},\mu}) = A\lambda'_\rho q^{j/2} \operatorname{Tr}((\hat{x}\sigma)^{-1}, \tilde{\rho}),$$

where $A = \#\{\tilde{\rho}\text{-factors in } H^i_{\dot{w},\mu}\} - \#\{-\tilde{\rho}\text{-factors in } H^i_{\dot{w},\mu}\}$, the left hand side of (2.17.1) can be expressed as

$$\sum_\rho c_{\dot{w},\rho} \lambda'_\rho \operatorname{Tr}((\hat{x}\sigma)^{-1}, \tilde{\rho}), \tag{2.17.2}$$

where ρ runs over all the representations in $\mathscr{E}(G^{F^m}, (s))^F$ and $c_{\dot{w},\rho}$ is a real number.

On the other hand, the right hand side of (2.17.1) becomes

$$C_w(q)\sum_E \mathrm{Tr}\,(x^{-1}, \rho_E)\,\mathrm{Tr}\,(T_w, E(q)) \tag{2.17.3}$$

where $C_w(q)$ is an integral power of q and E runs over all the irreducible representations of W_s^F. Moreover T_w is a standard basis of the Hecke algebra $H(q)$ corresponding to a Coxeter group W_s^F. Since the set of the dual representation of $\mathscr{E}(G^{F^m}, (s))^F$ coincides with $\mathscr{E}(G^{F^m}, (s^{-1}))^F$ and the dual of the cuspidal representation is again cuspidal, we may replace ρ by the dual ρ^* of ρ. Then (2.17.2) and (2.17.3) implies that

$$N^*_{F^m/F}(\sum_\rho c_{\dot{w},\rho^*}\lambda'_{\rho^*}[\tilde{\rho}]) = C_w(q)\sum_E \mathrm{Tr}\,(T_w, E(q))\rho_E \tag{2.17.4}$$

for each $w \in W_s^F$.

Let $C(W_s^F)$ be the subspace of $C^{(s)}(G^{F^m}/\sim_F)$ generated by $\Sigma c_{\dot{w},\rho^*}\lambda'_{\rho^*}[\tilde{\rho}]$ for various $w \in W_s^F$. Then (2.17.4) shows, by the orthogonality relations of Hecke algebra $H(q)$, that the image of $C(W_s^F)$ by $N^*_{F^m/F}$ coincides with the subspace of $C^{(s)}(G^F/\sim)$ generated by ρ_E ($E \in (W_s^F)^\wedge$). Let ρ_0 be the cuspidal representation in $\mathscr{E}(G^{F^m}, (s))^F$ and let x_0 the corresponding element in $X(W_s, \gamma_s)$. Then $\langle R_{x_0}, \rho_E\rangle_{G^F} \neq 0$ for some E, and in particular, $N^*_{F^m/F}(C(W_s^F))$ is not contained in the subspace of $C^{(s)}(G^F/\sim)$ generated by R_x with $x \neq x_0$, $x \in X(W_s, \gamma_s)$. This implies that $N^*_{F^m/F}([\tilde{\rho}_0])$ is contained in $C^{(s)}(G^F/\sim)$ since we know already $N^*_{F^m/F}(\mu_{\tilde{\rho}}[\tilde{\rho}]) = R_{x_\rho}$ for each $x_\rho \neq x_0$. Since $N^*_{F^m/F}$ is an isometry, we have

$$N^*_{F^m/F}(\lambda'_{\rho_0^*}[\tilde{\rho}_0]) = \alpha_0 R_{x_0} \tag{2.17.5}$$

for some $\alpha_0 \in \overline{\mathbf{Q}}_l$ of absolute value 1.

Let us take $w \in W_s^F$ such that $c_{\dot{w},\rho^*} \neq 0$, (such a w exists). Then (2.17.4) implies that the image of $\sum c_{\dot{w},\rho^*}\lambda'_{\rho^*}[\tilde{\rho}]$ by $N^*_{F^m/F}$ is written as a linear combination of R_x ($x \in X(W_s, \gamma_s)$) with coefficients in $\mathbf{R}$. Hence, in particular, $N^*_{F^m/F}(\lambda'_{\rho_0^*}[\tilde{\rho}_0])$ coincides with R_{x_0} up to a real number multiple. This shows, by (2.17.5),

$$N^*_{F^m/F}(\lambda'_{\rho_0^*}[\tilde{\rho}_0]) = \pm R_{x_0}.$$

Now, $(\pm\lambda'_{\rho_0^*})^m = \lambda_{\rho_0^*}$ and λ_{ρ_0} coincides with $\lambda_{\rho_0^*}^{-1}$ by the Poincaré duality. This proves the lemma.

2.18. Using Theorem 2.2, we can describe the map

$$R_{M(\dot{w})}\colon C^{(s)}(M^{F\dot{w}}/\sim) \longrightarrow C^{(s)}(G^F/\sim)$$

for $M=M_K$. If we choose a set X_1 of representatives of M_c^- orbits in $X((W_K)_s, \gamma_s')$, almost characters R_x $(x' \in X_1)$ give a basis of $C^{(s)}(M^{F\dot{w}}/\sim)$. For each $x' \in X_1$, there exists a Levi subgroup M_J contained in M_K and a cuspidal representation δ of $M_J^{F^m}$ (m: as in the theorem) such that x' can be expressed as $x'=x_{\rho_{E'}}$, where $E' \in W_\delta'^{\wedge}$ and ρ_E is an irreducible representation of $M_K^{F^m}$ corresponding to E'. As mentioned earlier, W_s is a product of various Weyl groups of classical type. Hence W_δ and the linear character $y \to \varepsilon_y$ $(y \in W_\delta)$ is decomposed according to it. We denote by $\eta(y)$ the part of ε_y corresponding to the component of type C in W_s. Hence $\eta(y)$ is explicitly known by Remark 2.14. Now, in view of (2.7.3), together with Theorem 2.2, we have the following corollary.

2.19. Corollary. *Let* $w=w_1yw_1'^{-1}$, γ_δ: $W_\delta \to W_\delta$ *and* γ_δ': $W_\delta' \to W_\delta'$ *be as in* (2.7.3). *Then*

$$R_{M(\dot{w})}(R_{x_{E'}})=\frac{1}{\dim E'}\eta(y)\sum_{E\in(W_\delta)^{\wedge}_{\mathrm{ex}}}\mathrm{Tr}(\gamma_\delta y\gamma_\delta'^{-1}, \tilde{E}_{E'})R_{x_E}.$$

2.20. Remark. It is likely that similar results hold for exceptional groups, in view of [4]. But more generally for arbitrary connected algebraic groups, we can consider the map $N^*_{F^m/F}$: $C(G^{F^m}/\sim_F) \to C(G^F/\sim)$ in a similar manner, and the number of F-stable irreducible representations of G^{F^m} is independent of m. Hence our result suggests the following conjecture.

Conjecture. Let G be a connected algebraic group defined over $\mathbf{F}_q$. There exists a good parametrization of the set $\mathscr{E}(G^{F^m})^F$ of F-stable irreducible representations of G^{F^m}, say $X(G)$ by $\rho_x \leftrightarrow x$ such that $N^*_{F^m/F}([\tilde{\rho}_x])$ $\in C(G^F/\sim)$ is independent of m (for sufficiently divisible m) up to a root of unity multiple.

Added in Proof. Recently Asai extended his result to the case of non-split orthogonal groups.
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