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Generalized Gelfand-Graev Representations and Ennola Duality*

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Dedicated to Professor Hisaaki Yoshizawa on his sixtieth birthday

Introduction

In 1963, V. Ennola [6] conjectured: the complex irreducible characters of the finite unitary group $U_n(\mathbf{F}_q)$ could be obtained from those of the finite general linear group $GL_n(\mathbf{F}_q)$ "*by simple formal change that q is everywhere replaced by $-q$.*" (See [6] or Section 4.1 for the precise formulation.) This has been verified for small n (Ennola [6], S. Nozawa [22]) and for the characters corresponding to a Coxeter torus (G. Lusztig [17]). Moreover, according to R. Hotta and T. A. Springer [11], the conjecture is true if the characteristic p of $\mathbf{F}_q$ is large compared with n. One of the purpose of the present paper is to give a proof of Ennola conjecture which works without any restriction. More precisely, we show that the conjecture, if suitably strengthened, is equivalent to a formula (3.2.10) on duality operation [1, 13] and Green polynomials [9, 21]. Since this formula can be considered as a system of equalities between polynomials in q with coefficients independent of p, the truthness of Ennola conjecture follows from the result of Hotta and Springer via "analytic continuation".

Another purpose of the paper is to begin the study of *generalized Gelfand-Graev representations*. An original Gelfand-Graev representation (see, e.g., [5, 7, 8, 24, 33]) of a finite (or real or p-adic) reductive group G is, in a sense, associated to a regular nilpotent $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}=\mathrm{Lie}(G)$. Using Dynkin-Kostant-Springer-Steinberg's theory on nilpotent $\mathrm{Ad}(G)$-orbits, one can generalize this construction and attach to every nilpotent orbit O an induced representation Γ_O (see (1.3.4)). If $G=GL_n(\mathbf{F}_q)$ or $U_n(\mathbf{F}_q)$, the nilpotent orbits are prametrized by the set $\mathscr{P}_n$ of partitions of n. We denote by 1_{O_μ} the characteristic function of the nilpotent orbit O_μ corresponding to $\mu\in\mathscr{P}_n$, and by γ_μ the character of Γ_{O_μ}. When applied

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to this special case, our main result (3.1.1) on generalized Gelfand-Graev characters gives the following "$q\leftrightarrow q^{-1}$" duality:

Theorem. *Let* $G=\mathrm{GL}_n(\mathbf{F}_q)$ *or* $\mathrm{U}_n(\mathbf{F}_q)$. *Let* $\mu, \nu \in \mathscr{P}_n$. *Then there exist polynomials* $H_{G,\mu,\nu}(t)$, $H^{\vee}_{G,\mu,\nu}(t) \in \mathbf{Z}[t]$ *with the following properties.*

(i) *Their coefficients are independent of* q.

(ii) $$\langle \gamma_\mu, \gamma_\nu\rangle_G = H_{G,\mu,\nu}(q)$$

and

$$|G||O_\mu|^{-1}\mathscr{F}(1_{O_\mu})(A_\nu)=H^{\vee}_{G,\mu,\nu}(q),$$

where $A_\mu \in O_\mu$ *and* $\mathscr{F}(\cdot)$ *is the Fourier transforms defined in* [13].

(iii) $H^{\vee}_{G,\mu,\nu}(t)=(-1)^{s(G)}t^{n(\mu,\nu)}H_{G,\mu,\nu}(t^{-1})$

for some positive integers $s(G)$ *and* $n(\mu, \nu)$.

This combined with the author's previous result [13] and the result of Hotta and Springer [11] gives an explicit formula (3.2.14) for the values of γ_μ in terms of Green polynomials. In particular, one can see that an Ennola-type duality ("$q\leftrightarrow -q$" duality) between $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{U}_n(\mathbf{F}_q)$ holds for these character values. This is the key-step towards the proof of Ennola conjecture.

The paper is organized as follows. In Sections 1.1–1.3, after some preliminaries, we give the definition of the generalized Gelfand-Graev representations of a finite reductive group. In 2.1–2.3, we derive some consequences from Dynkin-Kostant-Springer-Steinberg theory. In 3.1 we prove a relation (*a* "$q\leftrightarrow q^{-1}$" duality) between inner products of generalized Gelfand-Graev characters and values of Fourier transforms of nilpotently supported invariant functions. A detailed study of the generalized Gelfand-Graev characters of $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{U}_n(\mathbf{F}_q)$ is given in 3.2. In 3.3, some conjectures concerning generalized Gelfand-Graev characters of a general finite reductive group are stated. Finally, in 4.1–4.2, we prove Ennola conjecture.

The author is very grateful to A. Gyoja for showing his experimental results on induced characters of $\mathrm{GL}_n(\mathbf{F}_q)$ (see (4.2.9) (ii)), by which the author came to notice the importance of generalizing Gelfand-Graev representations. He is also very thankful to T. Asai for explaining his arguments in [2, 3]. This has been used to simplify the original proof of the equality (4.2.22). Some part of this work was done during the author's stay at the Mathematics Institute of the University of Warwick in 1977, and some part of this paper was written during the visit at the Department of Pure Mathematics of the University of Sydney in 1982. So the author would also like to express his hearty thanks to the both institutes for their hospitality.

General Notations. For a set X, $|X|$ denotes its cardinality. If σ is a transformation of X, X_σ denotes the set of σ-fixed elements of X. Let Y be a subset of X. Then $1_Y = 1_{Y\subset X}$ is the characteristic function of Y on X. If ϕ is a, say, complex valued function on X, $\phi \mid Y$ is the restriction of ϕ to Y and $\mathrm{supp}(\phi) = \{x \in X; \phi(x) \neq 0\}$. Let G be a group and let $x, y \in G$. Then $x^y = y^{-1}xy$. Similarly, if $\boldsymbol{g}$ is the Lie algebra of an algebraic group $\mathfrak{G}$, then $X^y = \mathrm{Ad}(y^{-1})X$, where $X \in \boldsymbol{g}$, $y \in \mathfrak{G}$ and Ad is the adjoint action of $\mathfrak{G}$ on $\boldsymbol{g}$. If X is an element of $\boldsymbol{g}$ or $\mathfrak{G}$, $Z_{\mathfrak{G}}(X) = \{g \in \mathfrak{G}; X^g = X\}$. Some of more specific notations are explained in 1.1.

§ 1. Gelfand-Graev representations

1.1. Notations and assumptions

Let K be an algebraically closed field containing a finite field $\mathbf{F}_q$ of q elements. Let $\mathfrak{G}$ be a connected reductive linear algebraic group over K, with a fixed $\mathbf{F}_q$-rational structure. The Frobenius morphism will be denoted by σ. We use the notations in [13; 1.1]. In particular, $\mathfrak{B}$ (resp. $\mathfrak{T}$) is a σ-stable Borel subgroup (resp. a σ-stable maximal torus contained in $\mathfrak{B}$), Σ is the root system of $\mathfrak{G}$ with respect to $\mathfrak{T}$ and Σ^+ (resp. π) is the positive (resp. simple) system of Σ corresponding to $\mathfrak{B}$. For a subset r of the standard generator system R of the Weyl group $W = (N_G(\mathfrak{T})/\mathfrak{T})_\sigma$, $\mathfrak{P}_r(\supset \mathfrak{B})$ denotes the corresponding parabolic subgroup of $\mathfrak{G}$. In general, algebraic subgroups (resp. Lie algebras of algebraic subgroups) of $\mathfrak{G}$ will be denoted by large (resp. small) Gothic letters, and their σ-fixed points set will be denoted by the corresponding large Roman letters (resp. small Roman scripts). For example, $G = \mathfrak{G}_\sigma$, $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})$ and $\boldsymbol{b} = \mathrm{Lie}(\mathfrak{B})_\sigma$. From now on, we need the following:

(1.1.1) **Assumption.** (0) $\mathfrak{G}$ is a direct product of connected reductive groups $\mathfrak{H}_j$ with the root system Σ_j, $1 \leqq j \leqq s$.

(i) The commutator groups $[\mathfrak{H}_j, \mathfrak{H}_j]$ $(1 \leqq j \leqq s)$ are simple.

(ii) p $(=\mathrm{char}(K))$ is good ([32; I, 4.3]) for Σ_j, $1 \leqq j \leqq s$.

(iii) If some Σ_{j_0} $(1 \leqq j_0 \leqq s)$ is of type A_l and p divides $l+1$, then $[\mathfrak{H}_{j_0}, \mathfrak{H}_{j_0}] \cong \mathrm{SL}_{l+1}$.

(iv) If some Σ_{j_0} $(1 \leqq j_0 \leqq s)$ is of type E_6, E_7, E_8, F_4 or G_2, then $p \geqq 4m_{j_0} + 3$, where m_{j_0} is the height of the highest root of $\Sigma^+_{j_0}$.

Under this assumption, we can use Dynkin-Kostant-Springer-Steinberg's theory ([32]; cf. also [13; 1.4]) on nilpotent $\mathrm{Ad}(\mathfrak{G})$-orbits in $\mathfrak{g}$. Thus, given a nilpotent element A of $\boldsymbol{g} = \mathfrak{g}_\sigma$, one can define a σ-stable $\mathbf{Z}$-grading

(1.1.2) $$\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)_A$$

of $\mathfrak{g}$ with the following properties (i)–(v).

(i) $\mathfrak{g}(2)_A \ni A$.

(ii) $\mathfrak{p}_A=\bigoplus_{i\geqq 0}\mathfrak{g}(i)_A$ is the Lie algebra of a σ-stable parabolic subgroup $\mathfrak{P}_A$ of $\mathfrak{G}$, and $\mathfrak{l}_A=\mathfrak{g}(0)_A$ is the Lie algebra of a σ-stable Levi subgroup $\mathfrak{L}_A$ of $\mathfrak{P}_A$.

(iii) $\mathfrak{g}(2)_A$ is $\mathrm{Ad}(\mathfrak{L}_A)$-stable and $\mathrm{Ad}(\mathfrak{L}_A)A$ is dense in $\mathfrak{g}(2)_A$.

(iv) For $i\geqq 1$, $\mathfrak{u}_{i,A}=\bigoplus_{j\geqq i}\mathfrak{g}(i)_A$ is the Lie algebra of a σ-stable connected normal unipotent subgroup $\mathfrak{U}_{i,A}$ of $\mathfrak{P}_A$. In particular, $\mathfrak{U}_{1,A}$ is the unipotent radical of $\mathfrak{P}_A$.

(v) By taking a suitable G-conjugate of A instead of A, if necessary, we can assume that $\mathfrak{P}_A\supset\mathfrak{B}\supset\mathfrak{T}\subset\mathfrak{L}_A$. Then there exists a unique σ-invariant $\mathbf{Z}$-valued function h_A on Σ such that:

(a) $h_A(\alpha)=0$, 1 or 2 for $\alpha\in\pi$;

(b) $h_A(\beta)=\sum_{\alpha\in\pi}n_\alpha h_A(\alpha)$, if $\beta=\sum_{\alpha\in\pi}n_\alpha\alpha\in\Sigma$ with $n_\alpha\in\mathbf{Z}$;

(c) $\mathfrak{g}(i)_A=\bigoplus_{h_A(\alpha)=i}\mathfrak{u}_\alpha$.

We denote by $H(\mathfrak{G})$ the set of all $\mathbf{Z}$-valued functions h (or "weighted Dynkin diagrams") which can be realized as h_A for some nilpotent $A\in\boldsymbol{g}$. Then the σ-stable nilpotent $\mathrm{Ad}(\mathfrak{G})$-orbits in $\mathfrak{g}$ are parametrized by $H(\mathfrak{G})$. For $h\in H(\mathfrak{G})$, we put

$$\mathfrak{P}_h=\mathfrak{P}_A$$

if $h=h_A$. $\mathfrak{L}_h$, $\mathfrak{g}(i)_h$, $\mathfrak{u}_{i,h}$, $\cdots$ are defined similarly.

1.2. Non-degeneracy of a skew symmetric bilinear form

Let f be a Springer's morphism [32; III, 3.12], i.e. a bijective $\mathbf{F}_q$-morphism from the unipotent variety $\mathfrak{G}_1(\subset\mathfrak{G})$ onto the nilpotent variety $\mathfrak{g}_0(\subset\mathfrak{g})$. For our later purpose, it is convenient to choose an f explicitly as follows. (Here one can assume that Σ is irreducible.)

(i) If Σ is of classical type, we can assume that $\mathfrak{G}=\mathrm{SL}_n$, Sp_{2n} or $(\mathrm{P})\mathrm{SO}_n$ for some n. If $\mathfrak{G}=\mathrm{SL}_n$ and σ is an untwisted (resp. twisted) Frobenius, we can take for f the map $x\to x-1$ (resp. $x\to(x-1)(\eta^q x+\eta)^{-1}$ with $\eta\in\mathbf{F}_{q^2}$ such that $\eta+\eta^q=1$). If $\mathfrak{G}=\mathrm{Sp}_{2n}$ or $(\mathrm{P})\mathrm{SO}_n$, we define f to be the Cayley map $x\to(x-1)(x+1)^{-1}$. (If σ is twisted, it can be written as $\sigma=j\circ\sigma_0$ with an untwisted Frobenius σ_0 and a non-trivial graph automorphism j of $\mathfrak{G}$ commuting with σ_0. The fact that f is defined over $\mathbf{F}_q$ can be checked using an explicit realization of j (see, e.g., [34]).

(ii) If Σ is of exceptional type, we define f to be the logarithm map. (Recall that, in this case, we are assuming $p\geqq 4m+3$ (see (1.1.1) (iv)), although this condition seems to be too restrictive than is actually necessary.)

Let A be a nilpotent element of $\boldsymbol{g}=\mathfrak{g}_\sigma$, and let $\mathfrak{g}(i)_A$, $\mathfrak{U}_{j,A}$, $\cdots$ be as in 1.1. Since we fix A throughout in this and next subsections, we shall use the simplified notations $\mathfrak{g}(i)$, $\mathfrak{U}_j$, $\cdots$ instead of $\mathfrak{g}(i)_A$, $\mathfrak{U}_{j,A}$, $\cdots$. By virtue of the explicit choice of f, we have the following:

(1.2.1) **Lemma.** (i) *$f(\mathfrak{U}_i)=\mathfrak{u}_i$ for any $i\geqq 1$.*
(ii) *Let $u\in\mathfrak{U}_i$ and $v\in\mathfrak{U}_j$ with $i, j\geqq 1$. Then we have*

$$f(uv)-f(u)-f(v)\in\mathfrak{u}_{i+j}.$$

(iii) *Notations being as in* (ii), *we have*

$$f(uvu^{-1}v^{-1})-c[f(u), f(v)]\in\mathfrak{u}_{i+j+1}$$

with some $c\in\mathbf{F}_q\backslash\{0\}$ independent of u and v.
(iv) *Let u be as above and let $X\in\mathfrak{g}(j)$ for some integer j. Then*

$$\mathrm{Ad}(u)X-\{X+d[f(u), X]\}\in\bigoplus_{l\geqq 2i+j}\mathfrak{g}(l)$$

with some $d\in\mathbf{F}_q\backslash\{0\}$ independent of u, A and X.
(v) *Let $u\in\mathfrak{G}_1$. If $[f(u), A]=0$, then $u\in Z_{\mathfrak{G}}(A)$.*

Let $\kappa(\cdot,\cdot)$ be a fixed $\mathrm{Ad}(\mathfrak{G})$-invariant symmetric bilinear form on $\mathfrak{g}$ defined over $\mathbf{F}_q$ with the following properties:

$$\kappa(X_1, [X_2, X_3])=\kappa([X_1, X_2], X_3),\quad X_i\in\mathfrak{g}; \tag{1.2.2}$$

$$\mathfrak{u}_\alpha^\perp=\mathfrak{t}\oplus\sum_{\beta\in\Sigma\backslash\{\alpha\}}\mathfrak{u}_{-\beta},\quad \alpha\in\Sigma, \tag{1.2.3}$$

where $\mathfrak{u}_\alpha^\perp=\{X\in\mathfrak{g};\kappa(X,\mathfrak{u}_\alpha)=0\}$. (Such $\kappa(\cdot,\cdot)$ exists by the proof of [32; I, 5.3].) Let $X\to X^*(X\in\mathfrak{g})$ be an opposition $\mathbf{F}_q$-automorphism [13; (3.1.4)] of $\mathfrak{g}$. Then, for $i\geqq 1$, $(\mathfrak{u}_i)^*=\mathfrak{u}_{-i}$ can be considered as the dual space of $\mathfrak{u}_i$ by

$$X^*: Y\longrightarrow\kappa(X^*, Y),\qquad X, Y\in\mathfrak{u}_i.$$

(1.2.4) **Lemma.** (i) *If $u\in\mathfrak{U}_1$ and $[f(u), A^*]=0$, then $u=e$.*
(ii) *The $\mathfrak{U}_1$-orbit of A^* under the coadjoint action is $A^*+\mathfrak{g}(-1)$.*
(iii) *The skew symmetric bilinear form*

$$(X, Y)\longrightarrow\kappa(A^*, [X, Y])$$

on $\mathfrak{g}(1)$ (or on $\mathbf{g}(1)=\mathfrak{g}(1)_\sigma$) is non-degenerate. In particular, $\dim\mathfrak{g}(1)$ *is even.*

Proof. (i) It is known (see, e.g., [13; (1.4.3) (vi)]) that $Z_{\mathfrak{G}}(A^*)\cap\mathfrak{U}_1=\{e\}$. Hence the statement follows from (1.2.1) (v).
(ii) By (i) and (1.2.1) (iv),

$$\mathrm{Ad}^*(\mathfrak{U}_1)A^*\subset A^*+\mathfrak{g}(-1) \tag{1.2.5}$$

and

$$\{x \in \mathfrak{U}_1;\ \mathrm{Ad}^*(x)A^* = A^*\} = \mathfrak{U}_2. \tag{1.2.6}$$

By a theorem of Rosenlicht, the left hand side of (1.2.5) is closed. Hence, to prove (ii), it is enough to show that the both sides of (1.2.5) have the same dimension. But, by (1.2.6),

$$\dim \mathrm{Ad}^*(\mathfrak{U}_1)A^* = \dim \mathfrak{U}_1 - \dim \mathfrak{U}_2 = \dim \mathfrak{g}(1) = \dim (A^* + \mathfrak{g}(-1)).$$

Hence (ii) is proved.

(iii) Let $X \in \mathfrak{g}(1)$. By (i), if $[A^*, X]=0$, then $f^{-1}(X)=e$, i.e., $X=0$. The required non-degeneracy follows from this fact, (1.2.2) and (1.2.3).

1.3. Generalized Gelfand-Graev representations

We fix a complex non-trivial additive character χ of $\mathbf{F}_q$. Let A be as in 1.2. We define a $\mathbf{C}$-valued function ξ_A on $\mathrm{U}_2 = \mathrm{U}_{2,A}$ by

$$\xi_A(u) = \chi(\kappa(A^*, f(u))), \quad u \in \mathrm{U}_2.$$

By (1.2.1) (ii) and (1.2.3), this is a linear character of the group U_2. By (1.2.4) (iii), there exists a linear subspace $\boldsymbol{s}$ of $\boldsymbol{g}(1)$ of dimension $(\dim \boldsymbol{g}(1))/2$ such that

$$\kappa(A^*, [X, Y]) = 0, \quad X, Y \in \boldsymbol{s}. \tag{1.3.1}$$

(1.3.2) **Lemma.** *Notations being as above, let*

$$\mathrm{U}_{1.5} = f^{-1}(\boldsymbol{s} + \boldsymbol{u}_2).$$

Then we have:

(i) $\mathrm{U}_{1.5}$ *is a subgroup of* G;
(ii) $\mathrm{U}_1 \supset \mathrm{U}_{1.5} \supset \mathrm{U}_2$ *and* $[\mathrm{U}_1 : \mathrm{U}_{1.5}] = [\mathrm{U}_{1.5} : \mathrm{U}_2]$;
(iii) ξ_A *is extendable to a linear character* $\xi_A^{\sim}$ *of* $\mathrm{U}_{1.5}$.

Proof. By (1.2.1) (i) (ii), we have (i). (ii) is trivial. To check (iii), it is enough to show that $\mathrm{U}_{1.5}/\mathrm{Ker}(\xi_A)$ is abelian, i.e., that

$$\xi_A(uvu^{-1}v^{-1}) = 1, \quad u, v \in \mathrm{U}_{1.5}. \tag{1.3.3}$$

(Note that $uvu^{-1}v^{-1} \in \mathrm{U}_2$.) But the left hand side of (1.3.3) is equal to

$$\chi(\kappa(A^*, f(uvu^{-1}v^{-1}))) = \chi(\kappa(A^*, c[f(u)f(v)]))$$

by (1.2.1) (iii). Hence (1.3.3) follows from (1.3.1) and (1.2.3).

(1.3.4) Definition. The representation Γ_A of G induced from the linear character $\xi_A^{\sim}$ of $U_{1.5}$ is called the *generalized Gelfand-Graev representation* of G associated with A. (See also (1.3.6) and (1.3.8) (ii).) The character of Γ_A will be denoted by γ_A.

(1.3.5) **Remark.** (i) If A is regular nilpotent [32], then Γ_A is a Gelfand-Graev representation as defined in [33] (See also [5, 7, 8, 24].)

(ii) As is evident from the above construction, one can also define generalized Gelfand-Graev representations of real, complex or $\mathfrak{p}$-adic reductive algebraic groups.

(1.3.6) **Lemma.** (i) *If $A' \in \boldsymbol{g}$ is G-conjugate to A, then $\gamma_A=\gamma_{A'}$.*

(ii) $$\gamma_A=q^{-m(A)}\operatorname{ind}_{U_2}^{G}(\xi_A),$$

where $m(A)=(\dim \boldsymbol{g}(1)_A)/2$. *In particular, γ_A does not depend on the choice of $\boldsymbol{s}$ and $\xi_A^{\sim}$ in* (1.3.2).

Proof. (i) Obvious.

(ii) It is enough to show:

$$\operatorname{ind}_{U_{1.5}}^{U_1}(\xi_A^{\sim})(u)=\begin{cases} q^{m(A)}\xi_A(u) & \text{if } u\in U_2, \\ 0 & \text{if } u \in U_1\backslash U_2. \end{cases} \tag{1.3.7}$$

By (1.2.1) (i) (ii), we have

$$\xi_A^{\sim}(f^{-1}(X+Y))=\xi_A^{\sim}(f^{-1}(X))\xi_A(f^{-1}(Y))$$

for any $X\in \boldsymbol{g}(1)$ and $Y\in \boldsymbol{u}_2$. Hence, if one put $u=f^{-1}(X+Y)$, the left hand side of (1.3.7) is equal to

$$|U_{1.5}|^{-1}\sum_{v\in U_1}\xi_A^{\sim}(u)\xi_A(f^{-1}(\mathrm{Ad}(v)X-X))$$

by (1.2.1) (iv). If $X=0$, i.e., $u\in U_2$, then this is evidently equal to $q^{m(A)}\xi_A(u)$. If $X\neq 0$, then

$$\sum_{v\in U_1}\xi_A(f^{-1}(\mathrm{Ad}(v)X-X))=\sum_{v\in U_1}\chi(\kappa(\mathrm{Ad}(v)A^*, X))=0$$

by (1.2.3) and (1.2.4) (ii). This proves (1.3.7).

(1.3.8) **Remark.** (i) By (1.3.7), $\operatorname{ind}_{U_{1.5}}^{U_1}(\xi_A^{\sim})$ is an irreducible representation of U_1. In fact, this is the irreducible representation of U_1 associated with the coadjoint U_1-orbit of $A^*\in \boldsymbol{u}_1^*$ in the sense of A. A. Kirillov [16] (and D. A. Kazhdan [15]).

(ii) Let $\mathfrak{G}'$ be a connected reductive group defined over $\mathbf{F}_q$ whose root system is isomorphic to that of $\mathfrak{G}$. Then, by (i), it is easy to see that one can define the generalized Gelfand-Graev representation Γ_o of G' associated with a unipotent class o of G'.

In Section 4, we need a slight generalization of Γ_A:

(1.3.9) **Definition.** Let $Z(\mathfrak{G})$ be the center of G, and let ϕ be a character of $Z(\mathfrak{G})_\sigma$. Using the notations in (1.3.4), we denote by $\Gamma_{A,\phi}$ the representation of G induced from the linear character $\phi\otimes\tilde{\xi}_A$ of $Z(\mathfrak{G})_\sigma \times \mathrm{U}_{1.5}$, and call it the *generalized Gelfand-Graev representation* associated to (A, ϕ). The character of $\Gamma_{A,\phi}$ will be denoted by $\gamma_{A,\phi}$.

§ 2. On nilpotently supported invariant functions

2.1. Induced invariant functions on $\boldsymbol{p}_h$ supported by $\boldsymbol{u}_{2,h}$

In this subsection, we fix an element h of $H(\mathfrak{G})$ (see 1.1). Hence, almost always, we will omit the letter h in $\mathfrak{P}_h$, P_h, $\mathfrak{p}_h$, $\mathfrak{u}_{i,h}$, $\mathfrak{g}(i)_h$, $\cdots$ and denote them simply by $\mathfrak{P}$, P, $\mathfrak{p}$, $\mathfrak{u}_i$, $\mathfrak{g}(i)$, $\cdots$.

As in [13; 2.2], we denote by $\mathrm{Inv}(\boldsymbol{p})$ the space of complex valued $\mathrm{Ad}(P)$-invariant functions on $\boldsymbol{p}$ with the standard hermitian inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{p}}$. We also consider the subspace $\mathrm{Inv}(\boldsymbol{p}; u_2/u_3)=\mathrm{Inv}(\boldsymbol{p}_h; u_{2,h}/u_{3,h})$ of $\mathrm{Inv}(\boldsymbol{p})$ which consists of all elements $\phi\in\mathrm{Inv}(\boldsymbol{p})$ such that $\mathrm{supp}(\phi)\subset\boldsymbol{u}_2$ and that $\phi(X+Y)=\phi(X)$ for any $X\in\boldsymbol{p}$ and $Y\in\boldsymbol{u}_3$. A σ-stable subset I of Σ^+ is called a σ-ideal [13; (1.3.1)] if $I\ni\alpha$ implies $I\supset(\alpha+\Sigma^+)\cap\Sigma^+$. We denote by $\mathscr{I}$ the set of σ-ideals of Σ^+. Then, for any positive integer i,

$$\Sigma(\geqq i)=\Sigma(\geqq i)_h=\{\alpha\in\Sigma;\ h(\alpha)\geqq i\}$$

is an element of $\mathscr{I}$. Put

$$\mathscr{I}[h]=\{I\in\mathscr{I};\ \Sigma(\geqq 2)\supset I\supset\Sigma(\geqq 3)\}$$

and let $\mathrm{Inv}(\boldsymbol{p};\boldsymbol{u}_2/\boldsymbol{u}_3)'=\mathrm{Inv}(\boldsymbol{p}_h;\boldsymbol{u}_{2,h}/\boldsymbol{u}_{3,h})'$ be the subspace of $\mathrm{Inv}(\boldsymbol{p},\boldsymbol{u}_2/\boldsymbol{u}_3)$ spanned by $\{\mathrm{ind}_{\boldsymbol{b}}^{\boldsymbol{p}}(1_{\boldsymbol{n}(I)});\ I\in\mathscr{I}[h]\}$, where

$$\boldsymbol{n}(I)=\mathfrak{n}(I)_\sigma=(\sum_{\alpha\in I}\mathfrak{u}_\alpha)_\sigma.$$

In [13; 3.2] we studied the subspace $\mathrm{Inv}(\boldsymbol{g}_0)'$ of $\mathrm{Inv}(\boldsymbol{g})$ spanned by $\{\mathrm{ind}_{\boldsymbol{b}}^{\boldsymbol{g}}(1_{\boldsymbol{n}(I)});\ I\in\mathscr{I}\}$. From the proof of [13; (3.2.2)], one can see that $\mathrm{Inv}(\boldsymbol{g}_0)'$ is spanned by

$$\bigcup_h \mathrm{ind}_{\boldsymbol{p}_h}^{\boldsymbol{g}}(\mathrm{Inv}(\boldsymbol{p}_h;\boldsymbol{u}_{2,h}/\boldsymbol{u}_{3,h})')$$

where the union is taken over $H(\mathfrak{G})$. (See also the conjecture [13; (3.2.9)] for the space $\mathrm{Inv}(\boldsymbol{g}_0)'$.) Thus the following proposition can be considered as a bit stronger version of [13; (3.2.2)].

(2.1.1) **Proposition.** *Let ϕ be an element of $\mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$ and let $\mathfrak{O}=\mathfrak{O}_h$ be the nilpotent $\mathrm{Ad}(\mathfrak{G})$-orbit corresponding to h. Define $\phi_O \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)$ by*

$$\phi_O(X)=\begin{cases}\phi(X) & \text{if } X \in O \cap \boldsymbol{u}_2;\\ 0 & \text{if } X \in \boldsymbol{p}\backslash(O \cap \boldsymbol{u}_2),\end{cases}$$

where $O=\mathfrak{O}_\sigma$.

Then $\phi_O \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$. In particular, we have

$$1_{O\cap \boldsymbol{u}_2}=(1_{\boldsymbol{u}_2})_O \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'.$$

For a proof of (2.1.1) we need the following:

(2.1.2) **Lemma.** *Let $\psi \in \mathrm{Inv}(\boldsymbol{g}_0)'$. Then there exists an element $\psi_{\boldsymbol{p}}$ of $\mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$ such that*

$$\langle \alpha, \psi|\boldsymbol{p}\rangle_{\boldsymbol{p}}=\langle \alpha, \psi_{\boldsymbol{p}}\rangle_{\boldsymbol{p}} \tag{2.1.3}$$

for any $\alpha \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)$.

Proof. It is enough to prove this when $\psi=\mathrm{ind}_{\boldsymbol{b}}^{\boldsymbol{g}}(1_{\boldsymbol{n}(I)})$ for some $I \in \mathscr{I}$. Let $R(h)$ be the subset of R defined by

$$\mathfrak{P}_{R(h)}=\mathfrak{P}_h. \tag{2.1.4}$$

Then, by Mackey's formula and Frobenius reciprocity, the left hand side of (2.1.3) is equal to

$$\sum_w \{|\boldsymbol{b}^w \cap \boldsymbol{p}|^{-1} \sum_{X \in \boldsymbol{n}(I)^w \cap \boldsymbol{b}} \alpha(X)\}, \tag{2.1.5}$$

where the outer sum is taken over the set $W(\phi, R(h))$ of $(\phi, R(h))$-reduced element of W. By [13; (3.2.7)], $(\boldsymbol{n}(I)^w \cap \boldsymbol{g}(2))\oplus \boldsymbol{u}_3$ can be written as $\boldsymbol{n}(J(w, I))$ for some $J(w, I) \in \mathscr{I}[h]$. Hence, by the definition of $\mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)$, the term in (2.1.5) corresponding to w is

$$\begin{aligned}|\boldsymbol{b}^w \cap \boldsymbol{p}|^{-1}|\boldsymbol{n}(J(w, I))|^{-1}|\boldsymbol{n}(I)^w \cap \boldsymbol{u}_2| &\sum_{X \in \boldsymbol{n}(J(w,I))} \alpha(X)\\ &=c(w, I)\langle \alpha, \mathrm{ind}_{\boldsymbol{b}}^{\boldsymbol{p}}(1_{\boldsymbol{n}(J(w,I))})\rangle_{\boldsymbol{p}},\end{aligned}$$

where $c(w, I)=|\boldsymbol{b}^w \cap \boldsymbol{p}|^{-1}|\boldsymbol{n}(J(w, I)|^{-1}|\boldsymbol{n}(I)^w \cap \boldsymbol{u}_2||\boldsymbol{b}|$. Hence

$$\psi_p = \sum_w c(w, I)\, \mathrm{ind}_b^p(1_{n(J(w,I))}) \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$$

satisfies (2.1.3).

Proof of Proposition (2.1.1). From [13; (1.4.7), (3.2.2)], we see that

(2.1.6) $$\phi_O \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3),$$

(2.1.7) $$\phi_O \equiv \mathrm{ind}_p^g(\phi_O) \quad \text{on } \boldsymbol{p}$$

and

(2.1.8) $$\mathrm{ind}_p^g(\phi_O) \in \mathrm{Inv}(\boldsymbol{g}_0)'.$$

By (2.1.7),

(2.1.9) $$\langle \alpha, \phi_O \rangle_p = \langle \alpha, \mathrm{ind}_p^g(\phi_O) \,|\, \boldsymbol{p} \rangle_p$$

for any $\alpha \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)$. On the other hand, by (2.1.2) and (2.1.8), there exists a function $(\mathrm{ind}_p^g(\phi_O))_p \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$ such that

(2.1.10) $$\langle \alpha, \mathrm{ind}_p^g(\phi_O) \,|\, \boldsymbol{p} \rangle_p = \langle \alpha, (\mathrm{ind}_p^g(\phi_O))_p \rangle_p$$

for any $\alpha \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)$. Hence, from (2.1.6), (2.1.9) and (2.1.10), we get

$$\phi_O = (\mathrm{ind}_p^g(\phi_O))_p \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)',$$

as desired.

Put

$$\mathscr{J}[h] = \{J \in \mathscr{I}[h];\ \boldsymbol{n}(J) \cap O_h = \phi\}.$$

(2.1.11) **Lemma.** *Let the notations be as in* (2.1.1). *Then* $\mathrm{supp}(\phi) \subset O$ *if and only if*

(2.1.12) $$\langle \phi, \mathrm{ind}_b^p(1_{n(J)}) \rangle_p = 0$$

for any $J \in \mathscr{J}[h]$.

Proof. The "only if" part is trivial. Assume that $\phi \in \mathrm{Inv}(\boldsymbol{p}; \boldsymbol{u}_2/\boldsymbol{u}_3)'$ satisfies (2.1.12). From the proof of (2.1.1), one sees that $\phi - \phi_O$ can be written as a linear combinations of $\{\mathrm{ind}_b^p(1_{n(J)});\ J \in \mathscr{J}[h]\}$. Hence, by (2.1.12) and the definition of ϕ_O, we have

$$\langle \phi, \phi - \phi_O \rangle_p = 0$$

and

$$\langle \phi_O, \phi - \phi_O \rangle_p = 0.$$

Hence we have $\phi=\phi_O$, i.e, supp $(\phi)\subset O$. This proves the "if" part.

For $I \in \mathscr{I}[h]$, we put $\mathfrak{p}(I)=\mathfrak{p}_h(I)=\mathrm{Lie}(\mathfrak{P}(I))$, where $\mathfrak{P}(I)=\mathfrak{P}_h(I)$ is the normalizer of $\mathfrak{n}(I)$ in $\mathfrak{P}_h$. (Hence $\mathfrak{P}\supset\mathfrak{P}(I)\supset\mathfrak{B}$.) The following lemma is easy to verify.

(2.1.13) **Lemma.** *Let w_h denote the longest element of $W_{R(h)}=\langle R(h)\rangle$, where $R(h)\subset R$ is defined by* (2.1.4). *For $I \in \mathscr{I}[h]$, let*

$$\delta(I)=\delta_h(I)=w_h(\Sigma(\geqq 2)_h\backslash I)\cup\Sigma(\geqq 3)_h. \tag{2.1.14}$$

Then $\delta(I)\in\mathscr{I}[h]$, and $w_h\mathfrak{P}(\delta(I))w_h^{-1}$ and $\mathfrak{P}(I)$ have a Levi subgroup in common.

2.2. Considering q as a variable

For a reduced root system Σ and a graph automorphism τ of Σ, we consider an infinite set S of prime powers and a family

$$\mathfrak{G}(S)=\{\mathfrak{G}(q);\, q\in S\} \quad (\text{resp. } \mathfrak{g}(S)=\{\mathfrak{g}(q);\, q\in S\})$$

of algebraic groups (resp. Lie algebras) which satisfies the following:

(2.2.1) **Assumption.** (i) For each $q\in S$, $\mathfrak{G}(q)$ is a connected reductive linear algebraic group over $\overline{\mathbf{F}}_q$ with a fixed $\mathbf{F}_q$-rational structure. (The corresponding Frobenius endomorphism is denoted by σ_q.)

(ii) Let $\mathfrak{B}(q)$ be a σ_q-stable Borel subgroup of $\mathfrak{G}(q)$, and let $\mathfrak{T}(q)$ be a σ_q-stable maximal torus contained in $\mathfrak{B}(q)$. Then the root system of $\mathfrak{G}(q)$ with respect to $\mathfrak{T}(q)$ can be identified with Σ in such a way that the automorphism of Σ induced from σ_q coincides with τ.

(iii) For each $q\in S$, the algebraic group $\mathfrak{G}=\mathfrak{G}(q)$ satisfies Assumption (1.1.1).

(iv) The rank and the $\mathbf{F}_q$-split rank of $\mathfrak{G}(q)$ are both independent of q (and will be denoted by $r(\mathfrak{G}(S))$ and $s(\mathfrak{G}(S))$ respectively).

(v) For each $q\in S$, $\mathfrak{g}(q)=\mathrm{Lie}\,\mathfrak{G}(q)$.

Then, for a given system $(S, \mathfrak{G}(S), \tau)$, the σ_q-stable nilpotent $\mathrm{Ad}(\mathfrak{G}(q))$-orbits in $\mathfrak{g}(q)$ can be parametrized, independently of $q\in S$, by a set $H(\mathfrak{G}(S))=H(\mathfrak{G}(S),\tau)$ of weighted Dynkin diagrams (see 1.1). We denote by $\mathfrak{O}_h(q)$ (resp. $O_h(q)$) the orbit corresponding to $h\in H(\mathfrak{G}(S))$ (resp. the set $(\mathfrak{O}_h(q))_{\sigma_q}$). Let $\mathfrak{P}_h(q)$, $\mathfrak{p}_h(q)$, $\boldsymbol{p}_h(q)$, $\boldsymbol{p}_h(I)(q)$ $(I\in\mathscr{I}[h])$, $\cdots$ be as in 1.1–2.1. When there is no fear of confusion, we will omit "q" and use the notations $\mathfrak{P}_h$, $\mathfrak{p}_h$, $\cdots$ (as we did in 1.1–2.1).

2.3. A "$q\leftrightarrow q^{-1}$" transformation Δ_h

Let $(S, \mathfrak{G}(S), \tau)$ be as in 2.2. In this subsection we fix an element h of $H(\mathfrak{G}(S))$, and usually denote $\mathfrak{P}_h(q)$, $\mathfrak{p}_h(q)$, $\mathfrak{O}_h(q)$, $\cdots$ simply by $\mathfrak{P}(q)$,

$\mathfrak{p}(q), \mathfrak{O}(q), \cdots$.

Let $\mathbf{C}[t]$ be the polynomial ring over the complex number field $\mathbf{C}$ in an indeterminate t. Put

$$Q=Q_S=\{f \in \mathbf{C}[t]; f(q^{\pm 1})=0 \text{ for some } q \in S\}.$$

Then

$$\mathbf{C}[t]_Q=\{f_1/f_2; f_1 \in \mathbf{C}[t], f_2 \in \mathbf{C}[t]\backslash Q\}$$

is a subring of the rational function field $\mathbf{C}(t)$. Let $\mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$ be the free $\mathbf{C}[t]_Q$-module with basis $\mathscr{I}[h]$. We define a $\mathbf{C}$-linear transformation $\Delta=\Delta_h$ of $\mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$ by

(2.3.1) $$\Delta(d(t)I)=d(t^{-1})t^{-c(I)}\delta(I)$$

for $d \in \mathbf{C}[t]_Q$ and $I \in \mathscr{I}[h]$, where $c(I)=c_h(I)=\dim \boldsymbol{p}_h(q)/\boldsymbol{p}_h(I)(q)$ (which is independent of $q \in S$) and $\delta(I)$ is an element of $\mathscr{I}[h]$ defined by (2.1.14). From (2.1.13) and (2.2.1), we have the following:

(2.3.2) **Lemma.** *The transformation Δ is involutory, i.e., $\Delta^2=$ identity.*

Now let $\alpha_q=\alpha_{h,q}$ ($q \in S$) be the $\mathbf{C}$-linear map

$$\mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle \longrightarrow \mathrm{Inv}\,(\boldsymbol{p}(q); \boldsymbol{u}_2(q)/\boldsymbol{u}_3(q))'$$

defined by

(2.3.3) $$\alpha_q(d(t)I)=d(q)\ \mathrm{ind}_{\boldsymbol{p}(I)(q)}^{\boldsymbol{p}(q)}(1_{\boldsymbol{n}(I)(q)})$$

for $d \in \mathbf{C}[t]_Q$ and $I \in \mathscr{I}[h]$. From the proof of (2.1.1) we have the following:

(2.3.4) **Lemma.** *Let $\Phi \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$. Then there exists an element $\Phi_h \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$ such that*

$$\alpha_q(\Phi_h)=\alpha_q(\Phi)_{O(q)}$$

for any $q \in S$. (See (2.1.1) for the definition of $\alpha_q(\Phi)_{O(q)} \in \mathrm{Inv}\,(\boldsymbol{p}(q), \boldsymbol{u}_2(q)/\boldsymbol{u}_3(q)')$.) In particular, there exists an element $\mathbf{1}_h$ of $\mathbf{C}[t]_Q\langle\mathscr{I}[\mathrm{h}]\rangle$ such that

$$\alpha_q(\mathbf{1}_h)=1_{O(q)\cap \boldsymbol{u}_2(q)}$$

for any $q \in S$.

(2.3.5) **Remark.** Note that Φ_h in (2.3.4) is not uniquely determined by Φ in general. Of course, it is unique modulo $\bigcap_{q\in S}\operatorname{Ker}\alpha_q$.

(2.3.6) **Proposition.** (i) *Let* $\Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle$ *and let* $I \in \mathscr{I}[h]$. *Then there exists an* $F_{\Phi,I}(t) \in \mathbf{C}[t]_Q$ *such that*

$$|P(q)|\langle \alpha_q(\Phi), \alpha_q(I)\rangle_{P(q)} = F_{\Phi,I}(q) \tag{2.3.7}$$

for any $q \in S$. *Moreover, we have*

$$F_{\Delta(\Phi),I}(t) = t^{a(I)}F_{\Phi,I}(t^{-1}) \tag{2.3.8}$$

where

$$a(I) = \dim \boldsymbol{p}(q)/\boldsymbol{p}(I)(q) + \dim \boldsymbol{n}(I)(q) + \dim \boldsymbol{u}_3(q)$$

(*which is independent of* $q \in S$).

(ii) *The transformation* Δ *stabilizes the linear subspace* $\bigcap_{q\in S}\operatorname{Ker}\alpha_q$ *of* $\mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle$.

(iii) *The linear transformation* Δ *stabilizes*

$$\mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle_h = \{\Phi_h ; \Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle\}. \tag{2.3.9}$$

(*See* (2.3.4) *and* (2.3.5) *for the definition of* Φ_h.)

Proof. (i) It is enough to prove this when $\Phi = J \in \mathscr{I}[h]$. Then, by Mackey's formula, the left hand side of (2.3.7) is equal to

$$\begin{aligned} &|B|^2|P||P(I)|^{-1}|P(J)|^{-1}\langle \operatorname{ind}_b^p(1_{\boldsymbol{n}(J)}), \operatorname{ind}_b^p(1_{\boldsymbol{n}(I)})\rangle_{\boldsymbol{p}} \\ &\quad = (|B||P(I)|^{-1})(|P||P(J)|^{-1}) \sum_{w\in W_{R(h)}} q^{L(w)}|\boldsymbol{n}(J)^w \cap \boldsymbol{n}(I)|, \end{aligned} \tag{2.3.10}$$

where $L(w) = \dim \mathfrak{B}/(\mathfrak{B}^w \cap \mathfrak{B})$. Hence the existence of $F_{\Phi,I}$ follows. Next we calculate

$$F_{\Delta(J),I}(q) = q^{-c(J)}|P|\langle \alpha_q(\delta(J)), \alpha_q(I)\rangle.$$

By (2.3.10) and (2.1.13), this is equal to

$$\begin{aligned} &(|B||P(I)|^{-1})(|P||P(J)|^{-1}q^{-c(J)}) \sum_{w\in W_{R(h)}} q^{L(w)}|\boldsymbol{n}(I)||\boldsymbol{u}_3||\boldsymbol{n}(J)^{w_h w} \cap \boldsymbol{n}(I)|^{-1} \\ &\quad = (|B||P(I)|^{-1})(|P||P(J)|^{-1}q^{-c(J)}) \sum_{w\in W_{R(h)}} q^{L(w_h w)}|\boldsymbol{n}(I)||\boldsymbol{u}_3||\boldsymbol{n}(J)^{w} \cap \boldsymbol{n}(I)|^{-1}. \end{aligned}$$

Comparing this with (2.3.10) we get

$$F_{\Delta(J),I}(t) = t^{a(I)}F_{J,I}(t^{-1}),$$

where

$$a(I)=\dim \mathfrak{B}-\dim \mathfrak{P}(I)+L(w_h)+\dim \boldsymbol{n}(I)+\dim \boldsymbol{u}_3$$
$$=\dim \boldsymbol{p}/\boldsymbol{p}(I)+\dim \boldsymbol{n}(I)+\dim \boldsymbol{u}_3.$$

This proves the part (i).

(ii), (iii) Let $\Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle$. Then $\Phi \in \bigcap_q \operatorname{Ker}(\alpha_q)$ if and only if

(2.3.11) $$\langle \alpha_q(\Phi), \alpha_q(I)\rangle_{\boldsymbol{p}(q)}=0$$

for any $I \in \mathscr{I}[h]$ and any $q \in S$. Analogously, by (2.1.11), $\Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle_h$ if and only if (2.3.11) holds for any $I \in \mathscr{I}[h]$ and any $q \in S$. Hence (ii) and (iii) follow from (i).

§ 3. Generalized Gelfand-Graev characters and Fourier transforms

3.1. *A "$q\leftrightarrow q^{-1}$" duality*

We use the notations in 2.2. For $q \in S$ and a nilpotently supported $\operatorname{Ad}(G(q))$-invariant function ϕ on $\boldsymbol{g}(q)$, the Fourier transform [13] $\mathscr{F}(\phi)=\mathscr{F}_q(\phi)$ of ϕ is defined by

$$\mathscr{F}_q(\phi)(X)=\begin{cases} q^{-N}\sum_{Y\in \boldsymbol{g}_0(q)} \chi(\kappa(X^*, Y))\phi(Y) & \text{if } X \in \boldsymbol{g}_0(q) \\ 0 & \text{if } X \in \boldsymbol{g}(q)\backslash \boldsymbol{g}_0(q), \end{cases}$$

where $N=|\Sigma^+|$, $\boldsymbol{g}_0(q)$ is the set of nilpotent elements of $\boldsymbol{g}(q)$, and χ, $\kappa(\cdot, \cdot)$ and $*$ are as in 1.2. Let h be an element of $H(\mathfrak{S}(S))$. For $q \in S$ and $\Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle$, we put

$$\gamma_{\Phi,q}=\sum_{C\in \boldsymbol{g}(2)_h(q)} \alpha_{h,q}(\Phi)(C)\gamma_{C,q},$$

where $\gamma_{C,q}$ is the generalized Gelfand-Graev character of $G(q)$ associated with a nilpotent element C of $\boldsymbol{g}(q)$. Our main result in this subsection is:

(3.1.1) **Theorem.** *Let $h, k \in H(\mathfrak{S}(S))$, and let $\Phi \in \mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle_h$, $\Psi \in \mathbf{C}[t]_Q\langle \mathscr{I}[k]\rangle_k$ (see* (2.3.9) *for the definition of* $\mathbf{C}[t]_Q\langle \mathscr{I}[h]\rangle_h$). *Then there exist $E_{\Phi,\Psi}$, $E^{\vee}_{\Phi,\Psi} \in \mathbf{C}[t]_Q$ such that*

$$\langle \gamma_{\Phi,q}, \gamma_{\Psi,q}\rangle_{G(q)}=E_{\Phi,\Psi}(q)$$

and

$$|P_h(q)||P_k(q)|\langle \mathscr{F}_q(\Phi^{\sim}_q), \Psi^{\sim}_q\rangle_{\boldsymbol{g}(q)}=E^{\vee}_{\Phi,\Psi}(q)$$

hold for $q \in S$, where the nilpotently supported $\operatorname{Ad}(G(q))$*-invariant functions*

$\Phi_q^{\sim}$, $\Psi_q^{\sim}$ *on* $\mathbf{g}(q)$ *are defined by*

$$\Phi_q^{\sim}(X)=\alpha_{h,q}(\Phi)(X) \quad \text{if } X \in \mathbf{p}_h(q);$$
$$\Psi_q^{\sim}(X)=\alpha_{k,q}(\Psi)(X) \quad \text{if } X \in \mathbf{p}_k(q).$$

Moreover, one has

$$E^{\vee}_{\Phi,\Psi}(t)=(-1)^{s(\mathfrak{G}(S))}t^{m(h,k)}E_{\Delta(\Phi),\Delta(\Psi)}(t^{-1}),$$

where $m(h, k)=2N+r(\mathfrak{G}(S))-(\dim \mathbf{g}(1)_h+\dim \mathbf{g}(1)_k)/2$, *and* Δ *is the involutory operation on* $\mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle_h$ *(or on* $\mathbf{C}[t]_Q\langle\mathscr{I}[k]\rangle_k$*) defined in* (2.3.6) (iii).

For $\Phi \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$, put

(3.1.2) $$\gamma'_{\Phi,q}=\sum_{C\in\mathbf{g}(2)_h}\alpha_{h,q}(\Phi)(C)\,\mathrm{ind}\,_{U_{2,h}(q)}^{G(q)}(\eta_C),$$

where $\eta_C(u)=\chi(\kappa(C^*,f(u)))$ for $u \in U_{2,h}(q)$. We also put

$$\mathrm{ind}_{h,q}(\Phi)=\mathrm{ind}\,_{\mathbf{p}_h(q)}^{\mathbf{g}(q)}(\alpha_{h,q}(\Phi)).$$

Assume that $\Phi \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle_h$. Then we have $\Delta(\Phi) \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle_h$ by (2.3.6) (iii). Hence, by (1.3.6) (ii),

$$\gamma'_{\Delta(\Phi)}=q^{m(h)}\gamma_{\Delta(\Phi)},$$

where $m(h)=(\dim \mathbf{g}(1)_h)/2$. By [13; (1.4.7) (i) (ii)], we also have

(3.1.3) $$\Phi_q^{\sim}=\mathrm{ind}_{h,q}(\Phi).$$

Hence, to prove (3.1.1), it is enough to show the following more general

(3.1.4) **Lemma.** *Let* $h, k \in H(\mathfrak{G}(S))$, *and let* $\Phi \in \mathbf{C}[t]_Q\langle\mathscr{I}[h]\rangle$, $\Psi \in \mathbf{C}[t]_Q\langle\mathscr{I}[k]\rangle$. *Then there exist* $E'_{\Phi,\Psi}$, $E'^{\vee}_{\Phi,\Psi} \in \mathbf{C}[t]_Q$ *such that*

$$\langle\gamma'_{\Phi,q},\gamma'_{\Psi,q}\rangle_{G(q)}=E'_{\Phi,\Psi}(q)$$

and

$$|P_h(q)||P_k(q)|\langle\mathscr{F}_q(\mathrm{ind}_{h,q}(\Phi)),\,\mathrm{ind}_{k,q}(\Psi)\rangle_{\mathbf{g}(q)}=E'^{\vee}_{\Phi,\Psi}(q)$$

for any $q \in S$. *Moreover, one has*

$$E'^{\vee}_{\Phi,\Psi}(t)=(-1)^{s(\mathfrak{G}(S))}t^{2N+r(\mathfrak{G}(S))}E'_{\Delta(\Phi),\Delta(\Psi)}(t^{-1}).$$

By (3.1.1), (3.1.4) and [13; (4.2.1)], we have

(3.1.5) **Corollary.** *In the statements of* (3.1.1) *and* (3.1.4), *one can replace the Fourier transformation* $\mathscr{F}_q$ *with the duality operation* $D=D_q$ *on* $\mathrm{Inv}(\mathbf{g}_0(q))$ (*defined in* [13; 2.2]).

Before proving (3.1.4) we prepare several lemmas.

(3.1.6) **Lemma.** *Let* $I \in \mathscr{I}[h]$ *and* $J \in \mathscr{I}[k]$. *For a nilpotent* $X \in \mathbf{g}(q)$, *we have*

$$\mathscr{F}_q(\mathrm{ind}_{h,q}(I))(X)=q^{-N}|\boldsymbol{n}(I)| \sum_{w \in W(R(I),R(J))} |(P_h(I)^*)^w \cap P_k(J)|^{-1} \times |\{y \in P_k(J);\ X \in (\boldsymbol{n}(I)^\circ)^{wy}\}|,$$

where $P_h(I)^*$ *is the* $\mathbf{F}_q$*-rational points of the algebraic subgroup of* $\mathfrak{G}$ *whose Lie algebra is* $\mathfrak{p}_h(I)^*$, $R(I)=R_h(I)\subset R$ *is defined by*

$$P_{R(I)}=P_h(I),$$

$W(R(I), R(J))$ *is the set of* $(R(I), R(J))$*-reduced elements of* W *and*

$$(\boldsymbol{n}(I))^\circ=\{X \in \mathbf{g};\ \kappa(X, \boldsymbol{n}(I)^*)=0\}.$$

This can be proved by an analogous way as [13; (4.1.5)].

(3.1.7) **Lemma.** *Notations being as in* (3.1.6), *let* $A_w(t)$ *and* $A_w^\vee(t)$ $(w \in W)$ *be the polynomials in* t *such that*

$$|P_h(I)(q)^w \cap P_k(J)(q)|=A_w(q)$$

and

$$|(P_h(I)(q)^*)^w \cap P_k(J)(q)|=A_w^\vee(q)$$

for $q \in S$ *and* $w \in W$. *Then*

$$A_w^\vee(t)=(-1)^{s(\mathfrak{G}(S))}t^{a(I,J)}A_w(t^{-1}),$$

where

$$a(I, J)=\dim \mathfrak{P}_h(I)+\dim \mathfrak{P}_k(J)-\dim \mathfrak{B}.$$

This can be proved easily by noting that $\mathfrak{P}_h(I)(q)$ and $\mathfrak{P}_h(I)(q)^*$ have a Levi subgroup in common. The next lemma follows from (3.1.2), (1.2.3) and (2.1.13).

(3.1.8) **Lemma.** *Let* $I \in \mathscr{I}[h]$. *Then*

$$\gamma'_I=|\boldsymbol{n}(\delta(I))|^{-1}|P_h|\,\mathrm{ind}^{g}_{\mathfrak{p}_h(\delta(I))}(1_{\boldsymbol{n}(\delta(I))})\circ f \quad \text{on } U_{2,h}.$$

Proof of (3.1.4). It is enough to prove this when $\Phi=I \in \mathscr{I}[h]$ and $\Psi=J \in \mathscr{I}[k]$. The existence of $E'_{I,J}(t)$ and $E'^{\vee}_{I,J}(t)$ follows from (3.1.5) and (3.1.7). By (3.1.6), $\langle \mathscr{F}_q(\mathrm{ind}_{h,q}(I)), \mathrm{ind}_{k,q}(J)\rangle_g$ is equal to

$$(3.1.9) \qquad \begin{aligned} &q^{-N}|\boldsymbol{n}(I)| \sum_w |(P_h(I)^*)^w \cap P_k(J)|^{-1}|(\boldsymbol{n}(I)^\circ)^w \cap \boldsymbol{n}(J)| \\ &= q^{-N}|\boldsymbol{n}(I)||\boldsymbol{n}(J)| \sum_w |(P_h(I)^*)^w \cap P_k(J)|^{-1}|\boldsymbol{n}(I)^w \cap \boldsymbol{n}(J)|^{-1}, \end{aligned}$$

where the sums are both taken over $W(R(I), R(J))$. On the other hand, by (3.1.8), $|P_h|^{-1}|P_k|^{-1}\langle \gamma'_{\Delta(I)}, \gamma'_{\Delta(J)}\rangle_G$ is equal to

$$(3.1.10) \qquad q^{-c_h(I)-c_k(J)}|\boldsymbol{n}(I)|^{-1}|\boldsymbol{n}(J)^{-1} \sum_w |P_h(I)^w \cap P_k(J)|^{-1}|\boldsymbol{n}(I)^w \cap \boldsymbol{n}(J)|,$$

where the sum is again over $W(R(I), R(J))$. From (3.1.7), (3.1.9) and (3.1.10), we see that

$$E'^{\vee}_{I,J}(t) = (-1)^{s(\mathfrak{G}(S))} t^{2N+r(\mathfrak{G}(S))} E'_{\Delta(I),\Delta(J)}(t^{-1}).$$

This proves (3.1.4).

3.2. The generalized Gelfand-Graev characters of $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{U}_n(\mathbf{F}_q)$

Let S be the set of all prime powers. For $q \in S$, let $G(q)=\mathrm{GL}_n(\overline{\mathbf{F}}_q)$ and let $\sigma_q=\sigma_{q,\varepsilon}$ ($\varepsilon=\pm 1$) be defined by

$$(x_{ij})^{\sigma_{q,1}}=(x_{ij}^q), \qquad (x_{ij})^{\sigma_{q,-1}}=(x_{ji}^q)^{-1}$$

for $(x_{ij}) \in \mathfrak{G}(q)$. (Hence $G(q)=\mathfrak{G}(q)_{\sigma_q}$ is either isomorphic to $\mathrm{GL}_n(\mathbf{F}_q)$ or to $\mathrm{U}_n(\mathbf{F}_q)$.) For this system $(S, \mathfrak{G}(S)_\varepsilon)=(S, \{\mathfrak{G}(q), \sigma_{q,\varepsilon}\}_{q \in S})$, the set $H(\mathfrak{G}(S)_\varepsilon)$ (see 2.2) can naturally be identified with the set

$$\mathscr{P}_n=\{(m_1, m_2, \cdots, m_r);\ m_i \in \mathbf{Z},\ m_1 \geqq m_2 \geqq \cdots \geqq m_r > 0,\ n=\sum_i m_i\}$$

of partitions of n. Hence, for $\mu \in \mathscr{P}_n$, $\mathfrak{P}_\mu$, $\mathfrak{O}_\mu$, $\mathrm{U}_{2,\mu}$, $\cdots$ will mean $\mathfrak{P}_h$, $\mathfrak{O}_h$, $\mathrm{U}_{2,h}$, $\cdots$ respectively, if μ corresponds to $h \in H(\mathfrak{G}(S)_\varepsilon)$. Then $O_\mu=(\mathfrak{O}_\mu)_\sigma$ is an $\mathrm{Ad}(G(q))$-orbit in $\boldsymbol{g}(q)$ by [32; III, 3.22]. We denote by $\gamma_\mu=\gamma_{\mu,q}$ the generalized Gelfand-Graev character of $G(q)$ associated with an element of O_μ.

(3.2.1) **Theorem.** *Let* $(S, \mathfrak{G}(S)_\varepsilon)$ ($\varepsilon=\pm 1$) *be as above. Let* $\mu, \nu \in \mathscr{P}_n$. *There exist elements* ${}^\varepsilon H_{\mu,\nu}(t)$ *and* ${}^\varepsilon H^{\vee}_{\mu,\nu}(t)$ *of* $\mathbf{C}[t]_Q$ *(in fact, of* $\mathbf{Z}[t]$; *see* (3.2.18) (i)) *such that*

$$(3.2.2) \qquad \langle \gamma_{\mu,q}, \gamma_{\nu,q}\rangle_{G(q)} = {}^\varepsilon H_{\mu,\nu}(q)$$

and

$$|G(q)||O_\mu(q)|^{-1}D_q(1_{O_\mu(q)})(A_\nu)={}^\varepsilon H^\vee_{\mu,\nu}(q),\ A_\nu\in O_\nu(q) \tag{3.2.3}$$

hold for $q\in S$, *where* D_q *is the duality operation* [1, 13]. *Moreover,*

$${}^\varepsilon H^\vee_{\mu,\nu}(t)=(-1)^{s(\varepsilon)}t^{n+n(\mu)+n(\nu)}\,{}^\varepsilon H_{\mu,\nu}(t^{-1}), \tag{3.2.4}$$

where

$$s(\varepsilon)=s(\mathfrak{G}(S)_\varepsilon)=\begin{cases} n & \text{if } \varepsilon=1;\\ [n/2] & \text{if } \varepsilon=-1\end{cases}$$

and

$$n(\mu)=\sum_i (i-1)\mu_i$$

if $\mu=(\mu_1,\mu_2,\cdots,\mu_r)\in\mathscr{P}_n$.

Proof. For $\Phi\in\mathbf{C}[t]_Q\langle\mathscr{I}[\mu]\rangle$, we define $|\Phi|(t)\in\mathbf{C}[t]_Q$ by

$$|\Phi|(q)=|P_\mu(q)|\langle\alpha_q(\Phi),1_{u_{2,\mu}}\rangle_{P_\mu(q)},\quad q\in S.$$

By (2.3.6), we have

$$|\Phi|(t)=t^{a_\mu}|\varDelta(\Phi)|(t^{-1}), \tag{3.2.5}$$

where $a_\mu=\dim\boldsymbol{u}_{2,\mu}+\dim\boldsymbol{u}_{3,\mu}$. Let $\mathbf{1}_\mu$ be an element of $\mathbf{C}[t]_Q\langle\mathscr{I}[\mu]\rangle_\mu$ defined in (2.3.4). Note that $|\mathbf{1}_\mu|(q)=|O_\mu(q)\cap\boldsymbol{u}_{2,\mu}(q)|$. Hence, by [13; (1.4.7) (i) (ii)],

$$|G(q)||O_\mu(q)|^{-1}=|P_\mu(q)||\mathbf{1}_\mu|(q)^{-1}.$$

Hence, by (3.1.4), the left hand side of (3.2.3) is equal to

$$\begin{aligned}|P_\mu||P_\nu||\mathbf{1}_\mu|(q)^{-1}|\mathbf{1}_\nu|(q)^{-1}\langle D(1_{O_\mu}),1_{O_\nu}\rangle_q\\ =|\mathbf{1}_\mu|(q)^{-1}|\mathbf{1}_\nu|(q)^{-1}E^\vee_{\mathbf{1}_\mu,\mathbf{1}_\nu}(q)\end{aligned} \tag{3.2.6}$$

in the notation of (3.1.1). On the other hand, by (1.3.6) (i), [13; (1.4.7) (iii)] and the fact that $\varDelta(\mathbf{1}_\mu)=d(t)\mathbf{1}_\mu$ for some $d(t)\in\mathbf{C}[t]_Q$, the left hand side of (3.2.2) is equal to

$$|\boldsymbol{u}_{3,\mu}||\boldsymbol{u}_{3,\nu}||\varDelta(\mathbf{1}_\mu)|(q)^{-1}|\varDelta(\mathbf{1}_\nu)|(q)^{-1}E_{\varDelta(\mathbf{1}_\mu,\varDelta(\mathbf{1}_\nu)}(q). \tag{3.2.7}$$

Hence we see the existence of ${}^\varepsilon H_{\mu,\nu}(t)$, ${}^\varepsilon H^\vee_{\mu,\nu}(t)\in\mathbf{C}[t]_Q$ satisfying (3.2.2) and (3.2.3). Comparing (3.2.6) and (3.2.7), and using (3.1.1) and (3.2.5), we have

$$ {}^{\varepsilon}H^{\vee}_{\mu,\nu}(t)=(-1)^{s(\varepsilon)}t^{n(\mu,\nu)\varepsilon}H_{\mu,\nu}(t^{-1}), $$

where

$$ n(\mu,\nu)=2N+r(\mathfrak{G}(S))-(\dim \mathbf{g}(1)_{\mu}+\dim \mathbf{g}(1)_{\nu})/2-(\dim \boldsymbol{u}_{2,\mu}+\dim \boldsymbol{u}_{2,\nu}). $$

But, using [32; IV, 1.13], one can check that

$$ n(\mu)=N-\dim \mathbf{g}(1)_{\mu}/2-\dim \boldsymbol{u}_{2,\mu}. $$

Hence (3.2.4) follows.

Let $Q^{\lambda}_{\rho}(t)\in \mathbf{Z}[t]$ $(\rho,\lambda\in\mathscr{P}_n)$ be the Green polynomials [9, 21]. We define a nilpotently supported invariant function $Q_{\rho}(\rho\in\mathscr{P}_n)$ on $\mathbf{g}(q)$ by

(3.2.8) $$ Q_{\rho}(A_{\lambda})=Q^{\lambda}_{\rho}(\varepsilon q),\qquad A_{\lambda}\in O_{\lambda}(q). $$

(3.2.9) **Lemma.** *Let D be the duality operation* [13; 2.2] *on the space of invariant functions on $\mathbf{g}(q)$. For $\rho\in\mathscr{P}_n$, we put*

$$ \mathrm{sgn}_{\varepsilon}(\rho)=\varepsilon^{[n/2]}(-1)^{n+r(\rho)} $$

where $r(\rho)$ is the number of parts of ρ. Then we have

(3.2.10) $$ D(Q_{\rho})=\mathrm{sgn}_{\varepsilon}(\rho)Q_{\rho},\qquad \rho\in\mathscr{P}_n. $$

Proof. Assume, for the moment, that p $(=\mathrm{char}\,(\mathbf{F}_q))$ is large. Then, by Springer [29] and Hotta-Springer [11], the values of Q_{ρ} can be expressed using Springer's trigonometric sums [30] associated with a strongly regular element in Lie $(\mathfrak{T}_{\rho})_{\sigma}$. Hence, if $\mathfrak{T}_{\rho}$ is not contained in any proper parabolic $\mathbf{F}_q$-subgroup of $\mathfrak{G}$, Q_{ρ} is precuspidal in the sense of [13; 2.2] (see [29; 1.7 (iii)]). Hence (3.2.10) follows from [30; 5.5] and [13; (2.1.7)], provided that p is large. But (3.2.10) is equivalent to

$$ \sum_{\mu} Q^{\mu}_{\rho}(\varepsilon q)|G(q)||O_{\nu}(q)|^{-1}\langle D(1_{O_{\mu}}),1_{O_{\nu}}\rangle_{\mathbf{g}(q)}=\mathrm{sgn}_{\varepsilon}(\rho)Q^{\nu}_{\rho}(\varepsilon q),\quad \rho,\nu\in\mathscr{P}_n. $$

By the definition of D and (3.1.3) (with $\Phi=\mathbf{1}_h$), we see that this is, in fact, a finite number of identities between polynomials in q whose coefficients are independent of p. Hence it must be true for all p. (It is possible that a more elementary proof exists, since (3.2.10) is much weaker than the result of Hotta and Springer.)

Let $\mathfrak{T}_1=\mathfrak{T}_1(q)$ be the σ-stable maximal torus of $\mathfrak{G}$ which consists of the diagonal elements of $\mathfrak{G}=GL_n(\bar{\mathbf{F}}_q)$. Then $N_{\mathfrak{G}}(\mathfrak{T}_1)/\mathfrak{T}_1$ can naturally be identified with the n-th symmetric group S_n. For $s\in S_n$, take a representative $\dot{s}$ of s in $\mathfrak{G}$. By Lang's theorem, there exists an elements $g(s)\in\mathfrak{G}$ such that $(g(s)^{\sigma})^{-1}g(s)=\dot{s}$. The torus $g(s)\mathfrak{T}_1g(s)^{-1}$ is clearly σ-stable, and the correspondence $s \to g(s)\mathfrak{T}_1 g(s)^{-1}$ induces a well-defined bijection between the conjugacy classes of S_n and the $\mathfrak{G}_\sigma$-conjugacy classes of σ-stable maximal torus of $\mathfrak{G}$ (see [32; I, § 2]). Hence the latters can be parametrized by $\mathscr{P}_n$. For $\rho \in \mathscr{P}_n$, we shall denote by $\mathfrak{T}_\rho$ (resp. T_ρ) a σ-stable maximal torus corresponding to ρ (resp. the finite group $(\mathfrak{T}_\rho)_\sigma$). Then, by [32; II, 1.7],

$$|T_\rho| = q^n e_\rho((\varepsilon q)^{-1})$$

where

$$e_\rho(t) = \prod_i (1 - t^{\rho_i})$$

if $\rho = (\rho_1, \rho_2, \cdots)$. We also put

$$W_\rho = (N_{\mathfrak{G}}(\mathfrak{T}_\rho)/\mathfrak{T}_\rho)_\sigma,$$

which, as an abstract group, is independent of ε and is isomorphic to $Z_{S_n}(s_\rho)$, where s_ρ is an element of S_n contained in the conjugacy class corresponding to $\rho \in \mathscr{P}_n$.

(3.2.11) **Theorem.** *Let* $\mu, \nu \in \mathscr{P}$.

(i) *Let* ${}^\varepsilon H_{\mu,\nu}(t)$, ${}^\varepsilon H^\vee_{\mu,\nu}(t) \in \mathbf{C}[t]_Q$ *be as in* (3.2.1). *Then we have*

(3.2.12) $$ {}^\varepsilon H^\vee_{\mu,\nu}(t) = \sum_\rho |W_\rho|^{-1} \operatorname{sgn}_\varepsilon(\rho) t^n e_\rho((\varepsilon t)^{-1}) Q^\mu_\rho(\varepsilon t) Q^\nu_\rho(\varepsilon t)$$

and

(3.2.13) $${}^\varepsilon H_{\mu,\nu}(t) = \varepsilon^{n(\mu)+n(\nu)} \sum_\rho |W_\rho|^{-1} t^n e_\rho((\varepsilon t)^{-1}) X^\mu_\rho(\varepsilon t) X^\nu_\rho(\varepsilon t),$$

where $X^\mu_\rho(t) \in \mathbf{Z}[t]$ $(\rho, \mu \in \mathscr{P}_n)$ *is defined by*

$$X^\mu_\rho(t) = t^{n(\mu)} Q^\mu_\rho(t^{-1})$$

(*see* [21; III, 7] *and* [31], *where combinatorial and geometrical descriptions of* X^μ_ρ*'s are given*).

(ii) *Let* a_ν *be a unipotent element of* $G(q)$ *contained in the inverse image* $f^{-1}(O_\nu)$ *of the Springer map* f (*see* 1.2). *Then*

(3.2.14) $$\gamma_\mu(a_\nu) = \varepsilon^{n(\mu)} \sum_\rho |W_\rho|^{-1} \operatorname{sgn}_\varepsilon(\rho) q^n e_\rho((\varepsilon q)^{-1}) X^\mu_\rho(\varepsilon q) Q^\nu_\rho(\varepsilon q).$$

Proof. (i) By the orthogonality relations [21; III, (7.9)] of the Green polynomials, we have

$$|G||O_\mu|^{-1}1_{O_\mu}=\sum_\rho |W_\rho|^{-1}q^n e_\rho((\varepsilon q)^{-1})Q^\mu_\rho(\varepsilon q)Q_\rho.$$

Hence (3.2.12) and (3.2.13) follow from (3.2.10), (3.2.4) and (3.2.8).

(ii) Let γ'_μ be the unipotently supported class function on $G(q)$ whose value at a_ν is given by the right hand side of (3.2.14). Then, by the orthogonality relations [21; III, (7.10)] and (3.2.13), we see that

(3.2.15) $$\langle \gamma'_\mu, \gamma'_\nu\rangle = {}^\varepsilon H_{\mu,\nu}(q).$$

Using [21; III, (7.11)] and the orthogonal relations, we also have

(3.2.16) $$\langle \gamma'_\mu, 1_G\rangle_G=\begin{cases}1 & \text{if } \mu=\{1^n\},\\ 0 & \text{otherwise,}\end{cases}$$

and

(3.2.17) $$\gamma'_\mu(a_\nu)=0 \quad \text{unless } \nu\leqq\mu,$$

where "$\leqq$" is the natural ordering [21; I, 1] on $\mathscr{P}_n$. We note that the conditions (3.2.15)–(3.2.17) determine the class function γ'_μ on G uniquely. Since γ_μ also satisfy the same conditions, we have $\gamma'_\mu=\gamma_\mu$. This proves (ii).

(3.2.18) **Corollary.** (i) ${}^\varepsilon H_{\mu,\nu}(t)$, ${}^\varepsilon H^\vee_{\mu,\nu}(t) \in \mathbf{Z}[t]$.

(ii) *Let $\varepsilon=1$. (See Remark (3.2.24) (i) below for the case $\varepsilon=-1$.) Let η be an irreducible character of G. Since the values of η at unipotent elements can be written as polynomials in q* [9], *one can define a polynomial $M_\mu(\eta)[t]$ (with coefficients in* $\mathbf{Z}$*) by*

$$\langle \gamma_\mu, \eta\rangle_G=M_\mu(\eta)[q].$$

Then

$$D(\eta)(a_\mu)=\pm q^{n(\mu)}M_\mu(\eta)(q^{-1}).$$

(iii) *Let $\varepsilon=1$ (see Remark (3.2.24) (i)). For any irreducible character η of G, there exists a partition $\mu(\eta)$ of n such that*

$$\langle \gamma_{\mu(\eta)}, \eta\rangle_G=1.$$

The correspondence $\eta\rightarrow\mu(\eta)$ is well defined if

(a) *q is large,*

or if

(b) *η is a unipotent character.*

Moreover, under this correspondence, the set of unipotent irreducible characters is sent bijectively to the set $\mathscr{P}_n$.

Proof. (i) This follows from (3.2.12) and (3.2.13) using orthogonality relations of the irreducible characters of S_n

(ii) This is a consequence of (3.2.10), (3.2.14) and the result of J.A. Green [9].

(iii) In Ohmori [23], it is shown that, for any η, there exists a $\mu(\eta)^* \in \mathscr{P}_n$ such that

$$\eta(a_{\mu(\eta)^*}) = \pm q^{n(\mu(\eta)^*)}.$$

This, together with (ii), implies the existence of $\mu(\eta)$. The uniqueness of $\mu(\eta)$ in the case (b) and the last assertion follow from (ii) and the formula [21; III, (6.5)] for the values of unipotent characters at the unipotent elements. By [9] (cf. also [23]), to prove the uniqueness of $\mu(\eta)$ under the condition (a), it is enough to show:

(3.2.19) **Lemma.** *Let m and d be positive integers. For $\nu \in \mathscr{P}_m$, let χ^ν_ρ ($\rho \in \mathscr{P}_m$) be the values of the irreducible character χ^ν of S_m, and $d\cdot\nu = (d\cdot\nu_1, d\cdot\nu_2, \cdots) \in \mathscr{P}_{d\cdot m}$ if $\nu = (\nu_1, \nu_2, \cdots)$. Then, for $\lambda \in \mathscr{P}_{d\cdot m}$,*

(3.2.20) $$\sum_{\rho \in \mathscr{P}_m} |W_\rho|^{-1} \chi^\nu_\rho X^\lambda_{d\cdot\rho}(t) = 1$$

if and only if $\lambda = d\cdot\nu$.

Proof. By using almost the same argument as that in [23; (2.8)], one can show:

$$\sum_{\rho \in \mathscr{P}_m} |W_\rho|^{-1} \chi^\nu_\rho \chi^\mu_{d\cdot\rho} = \begin{cases} 1 & \text{if } \mu = d\cdot\nu, \\ 0 & \text{unless } \mu \leqq d\cdot\nu \end{cases}$$

for $\mu \in \mathscr{P}_{d\cdot m}$. (Notice that our "$\leqq$" is the natural ordering, whereas the author of [23] uses lexicographical one.) By this formula and [21; III, (6.5), (7.6')], we see that the left hand side of (3.2.20) is monic of degree $n(\lambda) - n(d\cdot\nu)$ if $\lambda \leqq d\cdot\nu$ and is equal to 0 otherwise. Hence we get (3.2.19).

(3.2.21) **Theorem.** *Let $\varepsilon = 1$. (See Remark (3.2.24) (i).) Let* $\mathrm{Char}(G)_{\mathrm{unip.}}$ *be the **Z**-module of generalized characters of G supported by the set of unipotent elements.*

(i) *A unipotently supported class function ζ on G is an element of* $\mathrm{Char}(G)_{\mathrm{unip.}}$ *if and only if it can be written as*

(3.2.22) $$\zeta = \sum_\rho |W_\rho|^{-1} |T_\rho| \alpha(s_\rho) Q_\rho,$$

where α is a generalized charactor of $S_n = W_{\{1^n\}}$ and s_ρ is an element of S_n contained in the conjugacy class corresponding to $\rho \in \mathscr{P}_n$.

(ii) $\{\gamma_\mu;\ \mu \in \mathscr{P}_n\}$ *is a* **Z**-*basis of* $\mathrm{Char}(G)_{\mathrm{unip.}}$

Proof. (i) ("only-if" part) By the orthogonalities [21; III, (7.10)] of Q_ρ^μ's, $\{Q_\rho;\ \rho \in \mathscr{P}_n\}$ is a **C**-basis of $\mathbf{C}\otimes_{\mathbf{Z}} \mathrm{Char}(G)_{\mathrm{unip.}}$. Hence ζ can be written in the form (3.2.22) with some complex numbers $\{\alpha(s_\rho)\}$. Since ζ is a generalized character, we have

$$\langle \zeta, \eta^\mu \rangle_G \in \mathbf{Z}, \qquad \mu \in \mathscr{P}_n, \tag{3.2.23}$$

where η^μ is the unipotent irreducible character of G which corresponds to $\mu \in \mathscr{P}_n$ under the mapping in (3.2.18) (iii). Recall that

$$\eta^\mu(a_\nu) = \sum_\rho |W_\rho|^{-1} \chi_\rho^{\mu^*} Q_\rho^\nu(q), \qquad \nu \in \mathscr{P}_n,$$

where μ^* is the dual partition of μ. Hence by (3.2.23) and the orthogonalities of Q_ρ^ν's, we have

$$\sum_\rho |W_\rho|^{-1} \alpha(s_\rho) \chi_\rho^{\mu^*} \in \mathbf{Z},$$

which implies that $\{\alpha(s_\rho)\}$ are the values of a generalized character of S_n.

(i) ("if" part), (ii) These can be shown simultaneously, if one proves: a class function ζ on G which has the form (3.2.22) with $\alpha = \chi^\nu$ for some $\nu \in \mathscr{P}_n$ can be written as a **Z**-linear combination of $\{\gamma_\mu;\ \mu \in \mathscr{P}_n\}$. But this follows from (3.2.14) and [21; III, (7.6')] by induction on ν.

(3.2.24) **Remark.** (i) The results (3.2.18) (ii), (iii) and (3.2.21) which were proved only for $\varepsilon = 1$ in the above are, in fact, true also in the case $\varepsilon = -1$, by the Ennola duality proved in Section 4.

(ii) It might be interesting to know the values of γ_μ in a form more explicit than (3.2.14). For example, it can be shown:

$$\gamma_{\{n\}}(a_\lambda) = (-1)^{s(\varepsilon)} \prod_{i=1}^{r(\lambda)} (1-(\varepsilon q)^i);$$

$$\gamma_{\{n-1,1\}}(a_\lambda) = (-1)^{s(\varepsilon)} q \left\{ \prod_{i=1}^{r(\lambda)-1} (1-(\varepsilon q)^i) \right\} \times \{(1-(\varepsilon q)^{r(\lambda)}) - (\varepsilon q)^{r(\lambda)-1}(1-(\varepsilon q)^{r_1(\lambda)})\},$$

where $r_1(\lambda)$ is the number of parts $\neq 1$ of λ;

$$\gamma_\mu(a_\lambda) = (-1)^{s(\varepsilon)} q^{n(\mu)} \prod_{i=1}^{v} (1-(\varepsilon q)^i) \prod_{j=s-u+1}^{s} (1-(\varepsilon q)^j)$$

where $\mu = (2^s 1^t)$ and $\lambda = (2^u t^v)$. But the author does not know such a formula valid for any γ_μ.

3.3. Conjectures for a general finite reductive group

Here we collect some conjectural statements concerning generalized Gelfand-Graev characters of a general finite reductive group $G=\mathfrak{G}_\sigma$ over $\mathbf{F}_q$. The first one generalizes (3.2.21) (ii).

(3.3.1) **Conjecture.** The set $\{\gamma_O\}$ of generalized Gelfand-Graev characters (indexed by the set of nilpotent Ad(G)-orbits O) is a $\mathbf{Z}$-basis of the space of unipotently supported generalized characters of G.

To state the next conjecture, we need terminologies and results of T. Shoji [26], [27] and W.M. Beynon and N. Spaltenstein [4]. For simplicity, we assume that $\mathfrak{G}$ is split over $\mathbf{F}_q$. Let $A \in \mathbf{g}=\mathrm{Lie}(\mathfrak{G})_\sigma$ be a distinguished (or $\mathbf{F}_q$-split) nilpotent element in the sence of [27, 4]. Let $C(A)^\wedge$ be the set of irreducible characters of the component group $C(A) =Z_{\mathfrak{G}}(A)/Z_{\mathfrak{G}}(A)^0$ of $Z_{\mathfrak{G}}(A)$. For $\phi \in C(A)^\wedge$ we put

$$\gamma_A[\phi]=|C(A)|^{-1}\sum_{c\in C(A)}\overline{\phi(c)}\gamma_{A_c},$$

where A_c ($c \in C(A)$) are representatives of G-orbits of $(\mathrm{Ad}(\mathfrak{G})A)_\sigma$. Let $\mathfrak{T}_w$ be a σ-stable maximal torus of G corresponding to $w \in W$, and, for nilpotent $C \in \mathbf{g}$, let $Q_{\mathfrak{T}_w}(C)(t)$ be the Green polynomial [4, 26, 27] of G. We define another polynomial $X_{\mathfrak{T}_w}(C)(t)$ by

$$X_{\mathfrak{T}_w}(C)(t)=t^{n(C)}Q_{\mathfrak{T}_w}(C)(t^{-1}),$$

where $n(C)$ is the dimension of the variety of Borel subgroups $\mathfrak{B}'$ such that $\mathrm{Lie}(\mathfrak{B}') \ni C$. For $\phi \in C(A)^\wedge$, we put

$$X_{\mathfrak{T}_w}(A)[\phi]=|C(A)|^{-1}\sum_{c\in C(A)}\overline{\phi(c)}X_{\mathfrak{T}_w}(A_c).$$

We also put

$$C(A)_0^\wedge=\{\phi \in C(A)^\wedge;\ X_{\mathfrak{T}_w}(A)[\phi]\neq 0 \text{ for some } w \in W\}.$$

Then, generalizing (3.2.14), we expect the following:

(3.3.2) **Conjecture.** (i) Let $\phi \in C(A)_0^\wedge$. Then

$$\gamma_A[\phi](u)=|W|^{-1}\sum_{w\in W}(-1)^{s(\mathfrak{G})-s(\mathfrak{T}_w)}|T_w|X_{\mathfrak{T}_w}(A)[\phi](q)Q_{\mathfrak{T}_w}(f(u))$$

for any unipotent element u of G, where f is a Springer map (see 1.2).

(ii) Let $\phi \in C(A)^\wedge\backslash C(A)_0^\wedge$. Then $\gamma_A[\phi]$ is orthogonal to the space of uniform functions on G. The next one generalizes (3.2.18) (iii).

(3.3.3) **Conjecture.** For any irreducible character χ of $G(q)$, there exists a unique element h_χ of $H(\mathfrak{G}(S))$ (see 2.2) such that the multiplicity

$$\langle \chi, \gamma_{A,q} \rangle_{G(q)}$$

for some $A \in (\mathfrak{O}_{h_\chi})_\sigma$ is non-zero and "independent of q". (Rigorously, this must be defined using suitable parametrizations of A's and χ's.) Moreover, the correspondence $\chi \to h_{D(\chi)}$ would be essentially the one appearing in a work of Lustzig [19; 13. 4]. In particular, the set

$$\{h_\chi;\ \chi \text{ is a unipotent irreducible character}\}$$

parametrizes the special unipotent classes [19; 13.1] of $\mathfrak{G}$, and two unipotent characters χ_1 and χ_2 would be in the same family [19] if and only if $h_{\chi_1} = h_{\chi_2}$.

Notice that the idea of using Gelfand-Graev representations for a classification of irreducible representations goes back to a paper [7] of Gelfand and Graev themselves.

§ 4. Ennola duality between $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{U}_n(\mathbf{F}_q)$

4.1. Preliminaries

We use the notations of 3.2. For $\rho \in \mathscr{P}_n$, we denote by $\mathscr{H}_\rho$ the set of all σ-stable subgroups $\mathfrak{H} \subset \mathfrak{G}$ of the form:

$$\mathfrak{H} = Z_{\mathfrak{G}}(t), \qquad t \in \mathfrak{T}_\rho.$$

For $H \in \mathscr{H}_\rho$ and a character θ of T_ρ, let $r^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]$ be the character of the Deligne-Lusztig virtual representation $R^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]$ (see [5]) of $H = \mathfrak{H}_\sigma$. The restriction of $r^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]$ to the set of unipotent elements is independent of θ and is denoted by $Q^{\mathfrak{H}}_{\mathfrak{T}_\rho}$. Let $x = su$ be the Jordan decomposition of $x \in H$. Then, by [5],

$$r^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta](x) = |Z(s)_\sigma|^{-1} \sum_h \theta(s^h) Q^{Z(s)}_{h\mathfrak{T}_\rho h^{-1}}(u), \tag{4.1.1}$$

where $Z(s) = Z_{\mathfrak{H}}(s)$ and the summation is taken over the set $\{h \in H;\ h\mathfrak{T}_\rho h^{-1} \subset Z(s)\}$. In [20], G. Lusztig and B. Srinivasan have shown that the irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{U}_n(\mathbf{F}_q)$ can be written, in a unified way, as explicit $\mathbf{Q}$-linear combinations of the $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]$'s. This result implies that, for a proof of Ennola conjecture [6], it is enough to show:

(4.1.2) **Theorem.** *For $\lambda \in \mathscr{P}_n$,*

$$Q^{\mathfrak{G}}_{\mathfrak{T}_\rho}(u_\lambda) = Q^\lambda_\rho(\varepsilon q).$$

Note that, if $\varepsilon = 1$, this is well known and follows, e.g., from [5] and [28;

I, 5.3] or [17; 40]. Let H and θ be as above. Following D. Kazhdan [15] (see also Lusztig [18; 2.14]), we define a generalized character $k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]$ of $H=\mathfrak{H}_\sigma$ inductively by

$$r^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]=\sum_{\mathfrak{M}} \mathrm{ind}_M^H\,(k^{\mathfrak{M}}_{\mathfrak{T}_\rho}[\theta]), \tag{4.1.3}$$

where the summation is taken over $\{\mathfrak{M}\in\mathscr{H}_\rho;\ \mathfrak{M}\subset\mathfrak{H}\}$. If $x=su$ is the Jordan decomposition of $x\in H=\mathfrak{H}_\sigma$, then

$$k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta](su)=\begin{cases}\theta(s)k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[1](u) & \text{if } s\in Z(H);\\ 0 & \text{otherwise,}\end{cases} \tag{4.1.4}$$

where $Z(H)$ is the center of H. Note that, in particular, $\mathrm{supp}\,(k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta])$ is contained in $Z(H)\times\{$unipotent elements of $H\}$.

In 4.2 we also need the following:

(4.1.5) **Lemma.** *Let G be a finite group and $\langle\tau\rangle$ a finite cyclic group which acts on G. We assume that $(|\langle\tau\rangle|, |G_\tau|)=1$. Let H be a τ-stable subgroup of G and let Ψ be a class function on the semi-direct product $\langle\tau\rangle H$. Define a class function ψ on H_τ by*

$$\Psi(\tau x)=\psi(x),\quad x\in H_\tau.$$

Then

$$\mathrm{ind}\,_{\langle\tau\rangle H}^{\langle\tau\rangle G}\Psi(\tau g)=\mathrm{ind}_{H_\tau}^{G_\tau}(\psi)(g),\quad g\in G_\tau.$$

This is a version of T. Shintani [25; Lemma 2.8], and can be proved easily using the fact that two elements of the coset τG_τ are $\langle\tau\rangle G$-conjugate if and only if they are G_τ-conjugate (see the proof of [12; (13.6)]; cf. also [14; II]).

4.2. Proof of Theorem (4.1.2)

For $\mu\in\mathscr{P}_n$ and a character ψ of $Z(\mathfrak{G})_\sigma=Z(G(q))$, let $\gamma_{\mu,\psi,q}$ be the generalized Gelfand-Graev character of $G(q)$ associated with (A_μ, ψ) (see (1.3.9)). If $x=su$ is the Jordan decomposition of $x\in G$, we have

$$\gamma_{\mu,\psi,q}(x)=\begin{cases}(q-\varepsilon)^{-1}\psi(s)\gamma_{\mu,q}(u) & \text{if } s\in Z(G(q));\\ 0 & \text{otherwise.}\end{cases} \tag{4.2.1}$$

(4.2.2) **Lemma.** *Let $\varepsilon=1$. (See (4.2.9) for the case $\varepsilon=-1$.) For $\rho\in\mathscr{P}_n$ and a character θ of T_ρ, let $k^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q$ be the Kazhdan's generalized character (see 4.1) of $G(q)$. Then there exists integers $c^\mu_\rho(\rho, \mu\in\mathscr{P}_n)$ independent of θ and $q\in S$ such that*

(4.2.3) $$k^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q=\sum_\mu c^\mu_\rho \gamma_{\mu,\theta|Z,q},$$

where $Z=Z(G(q))$.

Proof. We already know, by (4.1.4), (4.2.1) and (a slight generalization of) (3.2.21), that there exist integers $c^\mu_\rho=c^\mu_\rho(q)$ which are independent of θ and satisfy (4.2.3). Next we prove that $c^\mu_\rho(q)$'s are independent of q. By [35], we can assume, by induction, that the corresponding statements for $k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]_q$ ($\mathfrak{H}\in\mathscr{H}_\rho$, $\mathfrak{H}\neq\mathfrak{G}$) are true. Let $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q$ be the Deligne-Lusztig generalized character of $G(q)$. Then, by (4.1.3) and the induction assumption,

(4.2.4) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q=\sum_\mu c^\mu_\rho(q)\gamma_{\mu,\theta|Z,q}+\sum_{\substack{\mathfrak{H}(q)\in\mathscr{H}_\rho(q)\\ \mathfrak{H}(q)\neq\mathfrak{G}(q)}}\operatorname{ind}^{G(q)}_{H(q)}(\gamma_{H,q}),$$

where $\gamma_{H,q}$ is a $\mathbf{Z}$-linear combination of generalized Gelfand-Graev characters of $H(q)$ with coefficients independent of q. Comparing the values of the both hand sides of (4.2.4) at unipotent elements and noting that $c^\mu_\rho(q)\in\mathbf{Z}$ for any $q\in S$, we have

(4.2.5) $$c^\mu_\rho(q)\in\mathbf{Z}_D[q],\quad \mu,\rho\in\mathscr{P}_n,$$

where D is a multiplicatively closed subset of $\mathbf{Z}$ generated by a finite number of primes. Hence, to prove that these are actually independent of q, we can assume that q is large. Then there exists a θ such that $\{w\in W_\rho;\,\theta=\theta^w\}=\{1\}$. We fix such q and θ. Then, by [5], $\alpha\cdot r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]$ is irreducible for some $\alpha=\alpha_\rho=\pm1$. Consider the set E_q of prime numbers e which does not divide $|G(q)|$. For $e\in E_q$, we can assume that $\mathfrak{T}_\rho(q^e)=\mathfrak{T}_\rho(q)$ and that $\sigma_{q^e}=\sigma^e$, where $\sigma=\sigma_{q,1}$. Let θ_e be the unique character of $T_\rho(q^e)$ satisfying

$$\theta_e\,|\,T_\rho(q)=\theta$$

and

$$\theta_e(t^\sigma)=\theta_e(t),\quad t\in T_\rho(q^e).$$

Consider a cyclic group $\langle\tau_e\rangle$ of order e which acts on $G(q^e)$ by

$$x^{\tau_e}=x^\sigma,\quad x\in G(q^e).$$

Put $Z_e=Z(G(q^e))$. Then, by (4.1.5), there exist natural extensions

$$\gamma^\sim_{\mu,\theta_e|Z_e,q^e}\quad(\text{resp. }\operatorname{ind}^{G(q^e)}_{H(q^e)}(\gamma_{H,q^e})^\sim(\mathfrak{H}\in\mathscr{H}_\rho(q^e)=\mathscr{H}_\rho(q)))$$

of

$$\gamma_{\mu,\theta_e|Z_e,q^e}\quad(\text{resp. }\operatorname{ind}^{G(q^e)}_{H(q^e)}(\gamma_{H,q^e}))$$

to characters of the semi-direct product $\langle\tau_e\rangle G(q^e)$ such that

$$(4.2.6)\qquad \begin{aligned}&\gamma^{\sim}_{\mu,\theta_e|Z_e,q^e}(\tau_e g)=\gamma_{\mu,\theta|Z,q}(g)\\ &\qquad(\text{resp. } \mathrm{ind}_{H(q^e)}^{G(q^e)}(\gamma_{H,q^e})^{\sim}(\tau_e g)=\mathrm{ind}_{H(q)}^{G(q)}(\gamma_{H,q})(g))\end{aligned}$$

for $g\in G(q)$. Using these characters, one can define the generalized character $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]^{\sim}_{q^e}$ of $\langle\tau_e\rangle G(q^e)$ by

$$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]^{\sim}_{q^e}=\sum_{\mu} c^{\mu}_{\rho}(q^e)\gamma^{\sim}_{\mu,\theta_e|Z_e,q^e}+\sum_{\mathfrak{H}\in\mathscr{H}_\rho(q)}\mathrm{ind}_{H(q^e)}^{G(q^e)}(\gamma_{H,q^e})^{\sim}.$$

Since $\mathscr{H}_\rho(q)=\mathscr{H}_\rho(q^e)$, we see, from (4.2.4), that $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]^{\sim}_{q^e}\,|\,G(q^e)=r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]_{q^e}$. Moreover, by (4.2.6),

$$(4.2.7)\qquad r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]^{\sim}_{q^e}(\tau_e g)=\sum_{\mu} c^{\mu}_{\rho}(q^e)\gamma_{\mu,\theta|Z,q}(g)+\sum_{\mathfrak{H}\in\mathscr{H}_\rho(q)}\mathrm{ind}_{H(q)}^{G(q)}(\gamma_{H,q})(g)$$

for $g\in G(q)$. By a theorem of G. Glauberman (see [12; (13.6), (13.14)]), (4.1.1) and (4.1.2) for $\varepsilon=1$, we have, for $g\in G(q)$,

$$(4.2.8)\qquad \alpha\cdot r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta_e]^{\sim}_{q^e}(\tau_e g)=\zeta_{2e}\alpha\cdot r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q(g)$$

where ζ_{2e} is a $2e$-th root of unity. But, by (4.2.4), (4.2.5), (4.2.7) and (4.2.8),

$$(1-\zeta_{2e})\alpha\cdot r^{\mathfrak{G}}_{\mathfrak{H}_\rho}[\theta]_q(1)\in e\mathbf{Z}_D,$$

if $e\in E_q\backslash D$. Since e does not divide $|G(q)|$, this implies that $\zeta_{2e}=1$. Hence, by (4.2.4), (4.2.7) and (4.2.8), we have

$$c^{\mu}_{\rho}(q^e)=c^{\mu}_{\rho}(q),\quad e\in E_q\backslash D.$$

This, together with (4.2.5), implies that $c^{\mu}_{\rho}(q)$'s are independent of q. This proves the lemma.

(4.2.9) **Remark.** (i) Lemma (4.2.2) is also true in the case $\varepsilon=-1$, because of the Ennola duality.

(ii) It is likely that an analogue of (4.2.2) holds for a general reductive group. A weaker result of this type was proved by A. Gyoja [10]. For GL_n, $n\leqq 4$, he also verified a version of (4.2.2) by direct calculations. This result was one of the main motivations of the present work.

From now on, unless otherwise stated, we consider exclusively the case $\varepsilon=-1$. Let c^{μ}_{ρ} ($\rho,\mu\in\mathscr{P}_n$) be as in (4.2.2), and let θ be the character of $T_\rho(q)\subset \mathrm{U}_n(\mathbf{F}_q)$. Using c^{μ}_{ρ}'s appearing in (4.2.3) (where $G(q)=\mathrm{GL}_n(\mathbf{F}_q)$), we define a generalized character $k^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]^{\#}_q$ of $G(q)=\mathrm{U}_n(\mathbf{F}_q)$ by

(4.2.10) $$k^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]^\#_q = \sum_\mu c^\mu_\rho (-1)^{n+[n/2]+n(\mu)} \gamma_{\mu,\theta|Z(G(q)),q}.$$

Analogously we can also define a generalized character $k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]^\#_q$ of $H = \mathfrak{H}_\sigma$ for any $\mathfrak{H} \in \mathscr{H}_\rho$. Thus we can define

(4.2.11) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]^\#_q = \sum_{\mathfrak{H} \in \mathscr{H}_\rho} \mathrm{ind}^{G(q)}_{H(q)} (k^{\mathfrak{H}}_{\mathfrak{T}_\rho}[\theta]^\#_q).$$

By [35], (4.1.3), (4.2.1), (4.2.10) and (3.2.14), this is just the class function $B^\rho(h^\rho)$ defined by Ennola [5]. (Thus we have proved that Ennola's $B^\rho(h^\rho)$'s are actually generalized characters of $\mathrm{U}_n(\mathbf{F}_q)$. Hence Theorem (4.1.2) (for $\varepsilon = -1$) will follow if one shows:

(4.2.12) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]^\#_q = r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]_q.$$

But the generalized characters $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]^\#_q$ and $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[\theta]_q$ of $\mathrm{U}_n(\mathbf{F}_q)$ satisfy the same-type character formula (see (4.1.1)) and the same-type orthogonality relations (see [5; Th. 6.8]). Hence, by Kazhdan's argument in the proof of [15; Th. 3], we see that (4.2.12) is true if q is sufficiently large (for any characteristic p). We now fix an arbitrary q. Let e be a prime number such that $(e, |G(q)|) = 1$ and that (4.2.12) is true if q is replaced by q^e. By [20], there exist integers $d^\mu_\rho (\rho, \mu \in \mathscr{P}_n)$ independent of q such that

(4.2.13) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]_{q^e} = \sum_\mu d^\mu_\rho \xi^\mu_{q^e}$$

and

(4.2.14) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]_q = \sum_\mu d^\mu_\rho \xi^\mu_q,$$

where $\xi^\mu_{q^e}$'s (resp. ξ^μ_q's) are the unipotent irreducible characters of $G(q^e)$ (resp. $G(q)$). As in the proof of (4.2.2), we consider a cyclic group $\langle \tau_e \rangle$ which acts on $G(q^e)$ by

$$x^{\tau_e} = x^{\sigma_q, -1}, \quad x \in G(q^e)$$

and consider the natural extension $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]^\sim_{q^e}$ of $r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]^\#_{q^e} = r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]_{q^e}$ to a generalized character of $\langle \tau_e \rangle G(q^e)$. Then

(4.2.15) $$r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]^\sim_{q^e}(\tau_e g) = r^{\mathfrak{G}}_{\mathfrak{T}_\rho}[1]^\#_q(q)$$

for $g \in G(q)$ (cf. (4.2.7)). For $\mu \in \mathscr{P}_n$, let $\xi^{\mu\#}_q$ be the irreducible character of $G(q)$ which corresponds to $\xi^\mu_{q^e}$ under the Glauberman correspondence (see [12; Ch. 13] and [14; II]), in other words, we have

(4.2.16) $$\xi^{\mu\sim}_{q^e}(\tau_e g) = \zeta_{2e} \xi^{\mu\#}_q(g)$$

for $g \in G(q)$, where $\xi_{q^e}^{\mu\sim}$ is an extension of $\xi_{q^e}^{\mu}$ to an irreducible character of $\langle\tau_e\rangle G(q^e)$ and ζ_{2e} is a $2e$-th root of unity.

(4.2.17) **Lemma.** *Let* $d_\rho^\mu(\rho, \mu \in \mathscr{P}_n)$ *be as in* (4.2.14). *Then we have*

$$r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^\# = \sum_\mu d_\rho^\mu \xi_q^{\mu\#}.$$

Proof. By (4.2.13), (4.2.15), (4.2.16) and the fact that $r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q$ is a generalized character, we have

(4.2.18)
$$r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^\# = \sum_\mu d_\rho^{\mu\#} \xi_q^{\mu\#}$$

with $d_\rho^{\mu\#} \in \mathbf{Z}$ ($\mu \in \mathscr{P}_n$) such that

(4.2.19)
$$|d_\rho^{\mu\#}| \leqq |d_\rho^\mu|.$$

But, since

$$\langle r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1], r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]\rangle_{G(q)} = \langle r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]^\#, r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]^\#\rangle_{G(q)} = |W_\rho|,$$

we have, from (4.2.14), (4.2.18) and (4.2.19),

(4.2.20)
$$|d_\rho^{\mu\#}| = |d_\rho^\mu|.$$

On the other hand, by (4.2.11), (4.2.10), (4.2.1) and (3.2.14),

$$r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_{q^e}^\#(g) - r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^\#(g) \in e\mathbf{Z}$$

for any $g \in G(q)$. Hence, by [12; (13.14)], (4.2.13), (4.2.18) and the validity of (4.2.12) with q replaced by q^e, we have

$$d_\rho^{\mu\#} \equiv d_\rho^\mu \pmod{e}$$

for any $\rho, \mu \in \mathscr{P}_n$. This, together with (4.2.20) implies that

(4.2.21)
$$d_\rho^{\mu\#} = d_\rho^\mu, \quad \rho, \mu \in \mathscr{P}_n$$

at least when e is sufficiently large. But, since d_ρ^μ's are independent of e, (4.2.21) always holds. This proves the lemma.

By (4.2.14), for a proof of (4.2.12) (and, hence, of Ennola conjecture), it is enough to show:

(4.2.22)
$$r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^\# = \sum_\mu d_\rho^\mu \xi_q^\mu.$$

But, by the argument of T. Asai [2, 3], one has

$$\xi_q^{\mu\#}(g) = \xi_q^{\mu}(g^e), \quad g \in G(q).$$

Hence (4.2.22) follows from (4.2.17) and the fact that

$$r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^{\#}(g) = r_{\mathfrak{T}_\rho}^{\mathfrak{G}}[1]_q^{\#}(g^e), \quad g \in G(q).$$

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