Advanced Studies in Pure Mathematics 6, 1985
Algebraic Groups and Related Topics
pp. 155–174

# The Dimensions of the Spaces of All Class Functions Invariant under the Twisting Operators of Finite Classical Groups

**Teruaki Asai***)

## § 1. Introduction

In [1]–[3], the author has determined the action of the twisting operator on the unipotent class functions of finite split reductive groups in terms of the Fourier transform of Lusztig (cf. [9]). Although the close relation between the twisting operator and the Lusztig Fourier transform should be expected for the non-unipotent class functions, there seem to be some difficulties for the direct approach. So, we do not take up the problem straightforwardly in this paper; instead we are to determine the dimensions of the spaces of all class functions invariant under the twisting operator of finite classical groups. The formulae are presented so as to give us the distinct features concerning the twisting operators and also to be a possible key for the future investigations.

Now, let us introduce several notations to show our results. Let $H$ be a connected reductive group over a finite field $\mathbf{F}_q$ with $q$-elements and $F$ be the Frobenius mapping. Let $t_1^*$ be the twisting operator on the space $C(H^F)$ of all class functions of $H^F$ associated with the twisting $t_1$ of the conjugacy classes of $H^F$ (cf. [3, Introduction]),

$$t_1\colon\ H^F/\sim\,\to H^F/\sim.$$

The twisting $t_1$ maps unipotent conjugacy classes of $H^F$ to unipotent ones, and thus decomposes unipotent conjugacy classes $H^F_{\mathrm{unip}}/\sim$ into equivalence classes:

$$(H^F_{\mathrm{unip}}/\sim)/\underset{t_1}{\sim}.$$

Now, the adjoint group $H_{\mathrm{ad}}$ of $H$ is written as a direct product

*) This paper has been supported in part by National Science Foundation grant MCS-8108814 (AO2).

Received December 27, 1983.

$$(*) \qquad H_{\mathrm{ad}}=\prod_{i=1}^{r} R_{\mathbf{F}_{q^{m_i}}/\mathbf{F}_q}(H_i)$$

over $\mathbf{F}_q$, where each $H_i$ is an absolutely simple adjoint group over $\mathbf{F}_{q^{m_i}}$ with the Frobenius mapping $F_i$ and $R_{\mathbf{F}_{q^{m_i}}/\mathbf{F}_q}$ is the restriction of the field of definition (cf. [11]). We say that $H$ is of classical type if the Dynkin graph of each $H_i$ is of type $A_{n_i}$, $B_{n_i}$, $C_{n_i}$ or $D_{n_i}$ and is not of type ${}^3D_4$ if the action of the Frobenius mapping is considered.

Let $B(H^F)$ be the number of unipotent representations (up to equivalence) of $H^F$. We define related non-negative integers $B^+(H^F)$ and $B^-(H^F)$ for any connected reductive group $H$ of classical type by the following rules:

(i) $B(H^F)=B^+(H^F)+B^-(H^F)$,

(ii) if we write $H_{\mathrm{ad}}$ as in (*), then

$$B^+(H^F)=\sum_{\substack{\varepsilon_i=1 \text{ or } -1 \\ \text{st. } \operatorname{sgn}(\varepsilon_1\cdots\varepsilon_r)=+}} B^{\operatorname{sgn}(\varepsilon_1)}(H_1^{F_1})\cdots B^{\operatorname{sgn}(\varepsilon_r)}(H_r^{F_r}),$$

where sgn $(a)$ means the signature of $a$,

(iii) if the Dynkin graph of $H$ is of type $A_n$, then we simply set

$$B^+(H^F)=B(H^F),$$

(iv) if $H=\mathrm{Sp}_{2n}$, $\mathrm{SO}_{2n+1}$ or $\mathrm{SO}_{2n}^{\pm}$ with the characteristic of $\mathbf{F}_q=2$, then we set $B^+(H^F)$ to be the number of unipotent conjugacy classes $H^F_{\mathrm{unip}}/\sim$ in $H^F$ divided by the equivalence relation defined by the twisting $t_1$.

(v) $B^+(H^F)$ depends only on the Dynkin graph of $H$ and the action of $F$ on the graph.

Our main result is as follows.

**Theorem A.** *Let $G$ be one of the classical groups in* [8], i.e.

(i) $\mathrm{GL}_n$ *or* $\mathrm{U}_n$,

(ii) $\mathrm{Sp}_{2n}$ *or* $\mathrm{SO}_{2n}^{\pm}$ *with* char $\mathbf{F}_q=2$,

(iii) $\mathrm{CSp}_{2n}$, $\mathrm{CO}_{2n}^{\pm,0}$ *or* $\mathrm{SO}_{2n+1}$ *with* char $\mathbf{F}_q\neq 2$.

*Let $C(G^F)^{t_1^*}$ be the space of all $t_1^*$-invariant class functions (over an algebraically closed field of characteristic* 0). *Then*

$$\dim C(G^F)^{t_1^*}=\sum_s B^+(Z_{G^*}(s)^{*F})$$

*where $s$ runs through all the $F$-stable semisimple conjugacy classes in the dual group $G^*$ and $Z_{G^*}(s)^*$ is the dual group of the centralizer of $s$ in $G^*$, which is a connected reductive group of classical type.*

As for $B^+(G)$, we shall prove

**Theorem B.** *Assume* char $\mathbf{F}_q=2$. *Let* $c_n(-1)$, $c_n^+(-1)$ *and* $c_n^-(-1)$ *be the numbers of the* $t_1$*-invariant unipotent conjugacy classes of* $\mathrm{Sp}_{2n}^F$, $\mathrm{SO}_{2n}^{+,F}$ *and* $\mathrm{SO}_{2n}^{-,F}$ *respectively, and* $c_n(1)$, $c_n^+(1)$ *and* $c_n^-(1)$ *be the numbers of unipotent conjugacy classes of* $\mathrm{Sp}_{2n}^F$, $\mathrm{SO}_{2n}^{+,F}$ *and* $\mathrm{SO}_{2n}^{-,F}$ *respectively. Then:*

(a) $B^+(\mathrm{Sp}_{2n}^F)=\frac{1}{2}(c_n(-1)+c_n(1))$,
$B^+(\mathrm{SO}_{2n}^{+,F})=\frac{1}{2}(c_n^+(-1)+c_n^+(1))$,
$B^+(\mathrm{SO}_{2n}^{-,F})=\frac{1}{2}(c_n^-(-1)+c_n^-(1))$.

(b) *For* $\eta=1$ *and* $-1$,

(i)
$$1+\sum_{n\geq 1} c_n(\eta)t^n=\frac{\sum_{n\geq 0}\eta^{n(n+1)/2}t^{n(n+1)}}{\prod_{n\geq 1}(1-t^n)^2},$$

(ii)
$$\sum_{n\geq 1} c_n^-(\eta)t^n=\frac{\sum_{n\geq 1}(\eta^n-1)t^{(2n)^2}+\sum_{n\geq 1}t^{(2n-1)^2}}{\prod_{n\geq 1}(1-t^n)^2},$$

(iii)
$$1+\sum_{n\geq 1} c_n^+(\eta)t^n=\frac{\sum_{n\geq 1}\eta^n t^{(2n)^2}}{\prod_{n\geq 1}(1-t^n)^2}+D(t),$$

*where* $D(t)$ *is the generating function of the numbers of the conjugacy classes of Weyl groups of type* $D_n$*'s, i.e.*

$$D(t)=2^{-1}\Big(\prod_{n\geq 1}(1-t^n)^{-2}+3\prod_{n\geq 1}(1-t^{2n})^{-1}-2\Big).$$

The statement (a) is obvious and the statement (b) for $\eta=1$ is the result of G.E. Andrews [5] (cf. 8, 6]). Therefore only (b) for $\eta=-1$ is the new result. Let us mention some open problems suggested by our theorems. By Lusztig [8],

$$(**)\qquad C(G^F)\cong\bigoplus_s C^1(Z_{G^*}(s)^{*F})$$

where $s$ runs through all the $F$-stable semisimple conjugacies in $G^*$ and $C^1(Z_{G^*}(s)^{*F})$ is the space of all unipotent class functions on $Z_{G^*}(s)^{*F}$. By Theorem B and [1]–[3], if $H$ is a split simple classical group, then $B^+(H^F)$ is the dimension of the space of all $t_1$-invariant unipotent class functions on $H^F$. Therefore, Theorem A together with the isomorphism (*) suggests that $B^+(H^F)$ might be the dimension of the space of all $t$-invariant unipotent functions on $H^F$ for any connected reductive group $H$ of classical type and moreover that the isomorphism (**) might commute with the twisting operator $t_1^*$. At any rate, these are open questions to be left for future investigations.

As for proofs, we prove Theorem B first. Since the unipotent conjugacy classes in the finite classical groups are completely determined by G.E. Wall [12], we may obtain the inductive relations between numbers of $t_1$-invariant unipotent conjugacy classes as a matter of course in the first place. Therefore, our main concern in proving Theorem B is to obtain the generating functions from the inductive relations. As we have remarked, (b) of Theorem B for $\eta=1$ is the result of G.E. Andrews [5]. His method applied in [loc. cit.] is as follows. If we are given the inductive relations between the numbers in series, then consider a more general series which is in double variables, and next find the functional equations which determine the double series together with the initial conditions. To go further, construct some new double series in an explicit manner. If this double series has the same property as the old one, we may conclude that they are identical. Then by reducing to the 1-variable situation, we get the desired expression of the generating function. This method of Andrews works quite efficiently in proving (b) of Theorem B for $\eta=-1$ as well. For this, we are concerned in the next sections. Once Theorem B is proved, Theorem A follows immediately. This will be treated in Section 4.

Finally, the author thanks the Institute for Advanced Study, Princeton, for the hospitality which he received during the preparation of this paper in 1983.

## § 2. Unipotent conjugacy classes in finite classical groups in characterictic 2

Let $m$ be a positive integer. We consider the set $\mathrm{U(O)}_m$ of all the sequences of integers $(a_1, \cdots, a_m)$ with the following conditions:

(i) if $i$ is odd, then $a_i=0$ or $-2$,

(ii) if $i$ is even, then $a_i=0, 1, 2, -1$ or $-2$,

(iii) if $i$ is odd and $a_i \neq 0$, then $a_{i-1}=a_{i+1}=0$ (where we set $a_j=0$ if $j \leq 0$ or $j \geq m+1$ for convenience),

(iv) if $i$ is even and $a_i<0$, then $a_{i-2}=0$.

For $A=(a_1, \cdots, a_m) \in \mathrm{U(O)}_m$, a subsequence $(a_{2i}, a_{2i+2}, a_{2i+4}, \cdots, a_{2i+2s})$ $(0<2i \leq 2i+2s \leq m)$ of $A$ is called an even continuous block in $A$ if the following conditions are satisfied:

$$a_{2i-2}=0,\ a_{2i+2s+2}=0 \quad \text{and} \quad a_{2j} \neq 0$$

for any $i \leq j \leq i+s$.

Let $B=(a_{2i}, a_{2i+2}, \cdots, a_{2i+2s})$ be an even continuous block in $A$. We define

$$\operatorname{rank} B = \sum_{i \leq j \leq i+s} |a_{2j}|.$$

Let $\bar{\mathrm{U}}(\mathrm{O})_m$ (resp. $\bar{\mathrm{U}}(\mathrm{Sp})_m$) be the subset of $\mathrm{U}(\mathrm{O})_m$ (resp. $\mathrm{U}(\mathrm{Sp})_m$) consisting of all $A \in \mathrm{U}(\mathrm{O})_m$ (resp. $\mathrm{U}(\mathrm{Sp})_m$) with the additional condition:

(iv) for any even continuous block $B$ in $A$, rank $B$ is an even integer (resp. rank $B$ is an even integer provided that the first entry of $B$ is not the second entry of $A$).

For $A=(a_1, \cdots, a_m) \in \mathrm{U}(\mathrm{O})_m$, define

$$\deg A = \sum_{1 \leq i \leq m} i|a_i|.$$

If $m<m'$, then by the mapping

$$(a_1, \cdots, a_m) \longmapsto (a_1, \cdots, a_m, \underbrace{0, \cdots, 0}_{(m'-m)-\text{times}}),$$

we have the following canonical inclusions:

$$\begin{array}{ll} \mathrm{U}(\mathrm{Sp})_m \hookrightarrow \mathrm{U}(\mathrm{Sp})_{m'}, & \bar{\mathrm{U}}(\mathrm{Sp})_m \hookrightarrow \bar{\mathrm{U}}(\mathrm{Sp})_{m'}, \\ \mathrm{U}(\mathrm{O})_m \hookrightarrow \mathrm{U}(\mathrm{O})_{m'}, & \bar{\mathrm{U}}(\mathrm{O})_m \hookrightarrow \bar{\mathrm{U}}(\mathrm{O})_{m'}. \end{array}$$

Let

$$\begin{array}{ll} \mathrm{U}(\mathrm{Sp}) = \bigcup_{m \geq 1} \mathrm{U}(\mathrm{Sp})_m, & \bar{\mathrm{U}}(\mathrm{Sp}) = \bigcup_{m \geq 1} \bar{\mathrm{U}}(\mathrm{Sp})_m, \\ \mathrm{U}(\mathrm{O}) = \bigcup_{m \geq 1} \mathrm{U}(\mathrm{O})_m, & \bar{\mathrm{U}}(\mathrm{O}) = \bigcup_{m \geq 1} \bar{\mathrm{U}}(\mathrm{O})_m. \end{array}$$

For $A=(a_1, \cdots, a_m) \in \mathrm{U}(\mathrm{O})$, let

$$\varepsilon(A) = +(\text{resp. } -)$$

if $\sum_{1 \leq i \leq m} a_i \equiv 0 \bmod 2$ (resp. $1 \bmod 2$), and for $\varepsilon = +$ or $-$ we put

$$\mathrm{U}(\mathrm{O}^\varepsilon) = \{A \in \mathrm{U}(\mathrm{O});\ \varepsilon(A) = \varepsilon\}.$$

Define the subset $\mathrm{U}(\mathrm{SO}^\varepsilon)$ of $\mathrm{U}(\mathrm{O}^\varepsilon)$ as follows. Let $A \in \mathrm{U}(\mathrm{O}^\varepsilon)$ and $\{B_1, \cdots, B_r\}$ be the set of all even continuous blocks in $A$. Then $A \in \mathrm{U}(\mathrm{SO}^\varepsilon)$ if and only if

$$\sum_{1 \leq i \leq r} \operatorname{rank} B_i = \text{even}.$$

We put

$$\bar{\mathrm{U}}(\mathrm{SO}^\varepsilon) = \mathrm{U}(\mathrm{SO}^\varepsilon) \cap \bar{\mathrm{U}}(\mathrm{O}).$$

Note that $U(SO^+)\cup U(SO^-)\supset \bar{U}(O)$.

The following theorem describes unipotent conjugacy classes of finite classical groups with its defining field of characteristic 2.

**Theorem 2.1** (G.E. Wall [12]). *Assume* char $\mathbf{F}_q=2$.

(I) *The unipotent conjugacy classes in* $Sp_{2n}^F$ *are parametrized by the set of all pairs* $(A, P)$ *where* $A\in U(Sp)$ *with* $n\geq \deg A$ *and* $P$ *is an even partition of* $n-\deg A$. *Here by an even partition we mean a partition consisting only of even integers. If* $\{u\}$ *is the unipotent conjugacy class in* $Sp_{2n}^F$ *corresponding with* $(A, P)$, *then* $u$ *belongs to the identity component of its centralizer in* $Sp_{2n}$ *if and only if* $A\in \bar{U}(Sp)$.

(II) *The unipotent conjugacy classes in* $O_{2n}^{\varepsilon,F}$ *are parametrized by the set of pairs* $(A, P)$ *where* $A\in U(O^{\varepsilon})$ *with* $\deg A\leq n$ *and* $P$ *is an even partition of* $n-\deg A$.

(a) *Let* $\{u\}$ *be the unipotent conjugacy class in* $O_{2n}^{\varepsilon,F}$ *corresponding with* $(A, P)$. *Then* $\{u\}\cap SO_{2n}^{\varepsilon,F}\neq\phi$ *if and only if* $A\in U(SO^{\varepsilon})$.

(b) *If* $\{u\}\cap SO_{2n}^{\varepsilon,F}\neq\phi$, *then the number of* $SO_{2n}^{\varepsilon,F}$*-conjugacy classes in* $\{u\}\cap SO_{2n}^{\varepsilon,F}$ *is* 2 (*resp.* 1) *if* $A\neq(0,\cdots,0)$ (*resp.* $A=(0,\cdots,0)$).

(c) *Assume* $u\in SO_{2n}^{\varepsilon,F}$. *Then* $u$ *belongs to the identity component of its centralizer in* $SO_{2n}^{\varepsilon}$ *if and only if* $A\in \bar{U}(SO_{2n}^{\varepsilon})$.

*Proof.* Our correspondence is as follows. Take a unipotent $x\in Sp_{2n}^F$. If we consider the Jordan blocks of $x$, we get a partition $P_0$ of 2n. The unipotent conjugacy classes in $Sp_{2n}^F$ yielding the same partition $P_0$ are parametrized by attaching signatures on some blocks of $P_0$. Moreover, [12] shows that the partition $P_0$ is regarded as a sum of two partitions $P_1$ and $P$ so that only on the blocks of the partitions $P_1$, signatures $\pm$ are allowed to be attached, and that the partition $P$ is necessarily an even partition. For the essential partition $P_1$, we may correspond a sequence of integers

$$(a_1,\cdots,a_m)\qquad (a_i\geq 0)$$

so that

$$P_1=(\underbrace{1,\cdots,1}_{a_1-\text{times}},\cdots,\underbrace{m,\cdots,m}_{a_m-\text{times}}).$$

For each block $(\underbrace{i,\cdots,i}_{a_i-\text{times}})$, if this block has a negative signature, then replace $a_i$ with $-a_i$. Finally, we get a certain sequence of integers $(a_1,\cdots,a_m)$ which is verified to belong to $U(Sp)$ by [12]. The correspondence $x\mapsto((a_1,\cdots,a_m), P)$ is that of (I). The bijectivity is the result of [12], and the second part of (I) is checked easily. (II) is similar.

Let $t$ be an indeterminate and define the polynomials $X_m^{(so)}(t)$ and $X_m^{(sp)}(t)$ as follows.

$$X_m^{(so)}(t^2)=1+\sum_{i\geq 1}\#\{A\in\bar{\mathrm{U}}(\mathrm{O})_m;\ \deg A=2i\}t^{2i},$$
$$X_m^{(sp)}(t^2)=1+\sum_{i\geq 1}\#\{A\in\bar{\mathrm{U}}(\mathrm{Sp})_m;\ \deg A=2i\}t^{2i}.$$

We may define the formal power series $X^{(so)}(t)$ and $X^{(sp)}(t)$ by

$$X^{(so)}(t)=\lim_{m\to\infty}X_m^{(so)}(t),$$
$$X^{(sp)}(t)=\lim_{m\to\infty}X_m^{(sp)}(t).$$

**Lemma 2.2.** *Assume* char $\mathbf{F}_q=2$.

(i) *Let $c_n$ be the number of unipotent conjugacy classes of* $\mathrm{Sp}_{2n}^F$ *invariant under the twisting $t_1$. Then*

$$1+\sum_{n\geq 1}c_nt^{2n}=X^{(sp)}(t^2)\prod_{n\geq 1}(1-t^{2n})^{-1}.$$

(ii) *For $\varepsilon=+$ and $-$, let $c_n^\varepsilon$ be the number of unipotent conjugacy classes of* $\mathrm{SO}_{2n}^{\varepsilon,F}$ *invariant under the twisting $t_1$. Then*

$$1+\sum_{n\geq 1}(c_n^++c_n^-)t^{2n}=X^{(so)}(t^2)\left(\prod_{n\geq 1}(1-t^{2n})^{-1}+\prod_{n\geq 1}(1-t^{4n})^{-1}-1\right),$$
$$1+\sum_{n\geq 1}(c_n^+-c_n^-)t^{2n}=1+2\left(\prod_{n\geq 1}(1-t^{4n})^{-1}-1\right).$$

*Proof.* Let $G$ be a connected reductive group over $\mathbf{F}_q$ with the Frobenius mapping $F$. For $x\in G^F$, the conjugacy class of $x$ in $G^F$ is $t_1$-invariant if and only if $x$ belongs to the identity component of its centralizer in $G$. Therefore our lemma is a direct consequence of Theorem 2.1.

**Definition 2.3.** Let $z$ be an indeterminate. For non-negative integers $a$, $b$ and for any integer $m$, we define the polynomials $X_m=X_m(a,b,t;z)$ by the following.

$$X_{-1}=a,\quad X_0=b,\quad X_{-i}=0\quad\text{for } i\geq 2,$$

and for $m\geq 0$,

$$X_{2m+1}-X_{2m}=zt^{2m+1}X_{2m-1}$$

$$X_{2m+2}-X_{2m+1}=zt^{2m+1}(t+1)X_{2m}+zt^{2m+1}(t-1)X_{2m-1}$$
$$+(zt^{2m+1}-z^2t^{4m+1})(X_{2m-2}+X_{2m-3}).$$

We define

$$X(a, b, t; z)=\lim_{m\to\infty} X_m(a, b, t; z)$$

as a formal power series in $t$ and $z$. Note that $X(a, b, t; z)$ is analytic in $t$ and $z$ for fixed $a, b$ if $|t|<1$. See [5, p. 93].

**Lemma 2.4.** *For any integer $m\geq 1$,*

$$X_m^{(sp)}=X_m(0, 1, t; 1), \qquad X_m^{(so)}=X_m(1, 1, t; 1).$$

*Therefore*

$$X^{(sp)}=X(0, 1, t; 1), \qquad X^{(so)}=X(1, 1, t; 1).$$

*Proof.* Let $X_{-1}^{(sp)}(t)=0$, $X_0^{(sp)}(t)=1$, $X_{-1}^{(so)}(t)=1$ and $X_0^{(so)}(t)=1$. For any integer $m\geq -1$, we put

$$X_m^{(*)}(t)=X_m^{(sp)}(t) \text{ (resp. } X_m^{(so)}(t)).$$

Then we can check

(i) if $m\geq 0$, $X_{2m+1}^{(*)}(t)-X_{2m}^{(*)}(t)=t^{2m+1}X_{2m-1}^{(*)}(t)$,

(ii) if $m\geq 1$, $X_{2m}^{(*)}(t)-X_{2m-1}^{(*)}(t)=t^{2m}(X_{2m-2}^{(*)}(t)+X_{2m-3}^{(*)}(t))$
$$+\sum_{1\leq i\leq m-2} t^{m^2-i^2}(X_{2i-2}^{(*)}(t)+X_{2i-3}^{(*)}(t))+f_m(t),$$

where $f_m(t)=t^{m^2}$ (resp. $f_m(t)=0$). From these relations, the lemma follows.

## § 3. Explicit expressions of $X^{(so)}(t)$ and $X^{(sp)}(t)$

Quite identically with [5, p. 93] we can express $X_{2m+2}(0, 1, t; z)$ as the determinant of the $(2m+2)\times(2m+2)$-matrix as follows.

$$X_{2m+2}(0, 1, t; z)$$
$$=\left[\begin{array}{ccccccccc}
1 & zt(t+1) & 0 & zt^3-z^2t^5 & \cdots & & & & \\
-1 & 1 & zt^3 & zt^3(t-1) & \cdots & & & & \vdots \\
0 & -1 & 1 & zt^3(t+1) & \cdots & & & & 0 \\
0 & 0 & -1 & 1 & \cdots & & & \vdots & zt^{2m+1}-z^2t^{4m+1} \\
\cdot & \cdot & & & \cdots & & & 0 & zt^{2m+1}-z^2t^{4m+1} \\
 & & & & & -1 & 1 & zt^{2m+1} & zt^{2m+1}(t-1) \\
 & & & & & & -1 & 1 & zt^{2m+1}(t-1) \\
 & & & & & & & -1 & 1
\end{array}\right]$$

$X_{2m+1}(0, 1, t; z)$ is the determinant of the $(2m+1)\times(2m+1)$-matrix obtained by deleting the final row and the final column of the preceding matrix. $X_{2m+2}(1, 1, t; z)$ also can be expressed as the determinant of the $(2m+3)\times(2m+3)$-matrix as follows.

$$X_{2m+2}(1, 1, t; z)$$

$$=\begin{vmatrix} 1 & zt & zt(t-1) & 0\cdots & & & \\ -1 & 1 & zt(t+1) & 0\cdots & & & \vdots \\ 0 & -1 & 1 & zt^3\cdots & & & 0 \\ 0 & 0 & -1 & 1\cdots & \vdots & & zt^{2m+1}-z^2t^{4m+1} \\ & & & \cdots & 0 & & zt^{2m+1}-z^2t^{4m+1} \\ & & & & 1 & zt^{2m+1} & zt^{2m+1}(t-1) \\ & & & & -1 & 1 & zt^{2m+1}(t+1) \\ & & & & & -1 & 1 \end{vmatrix}$$

$X_{2m+1}(1, 1, t; z)$ is the determinant of the $(2m+2)\times(2m+2)$-matrix obtained by deleting the final row and the final column of the preceding matrix.

Now, by expanding along the first rows of the preceding determinants, we obtain the relations:

$$\begin{aligned} X_n(0, 1, t; z) &= X_{n-2}(1, 1, t; zt^2)+zt(t+1)X_{n-2}(0, 1, t; zt^2) \\ &\quad +(zt^3-z^2t^5)X_{n-4}(0, 1, t; zt^4), \\ X_n(1, 1, t; z) &= X_n(0, 1, t; z)+ztX_{n-2}(1, 1, t; zt^2) \\ &\quad +zt(t-1)X_{n-2}(0, 1, t; zt^2) \\ &\quad \times(zt^3-z^2t^5)X_{n-4}(0, 1, t; zt^4). \end{aligned}$$

Hence

$$\begin{aligned} X(0, 1, t; z) &= X(1, 1, t; zt^2)+zt(t+1)X(0, 1, t; zt^2) \\ &\quad +(zt^3-z^2t^5)X(0, 1, t; zt^4) \\ X(1, 1, t; z) &= X(0, 1, t; z)+ztX(1, 1, t; zt^2)+zt(t-1)X(0, 1, t; zt^2) \\ &\quad +(zt^3-z^2t^5)X(0, 1, t; zt^4). \end{aligned}$$

By these relations the next lemma follows.

**Lemma 3.1.** *We have the following functional equations.*

(i) $X(1, 1, t; z)$

$$=(1+zt)X(0, 1, t; z)+(zt(t-1)-(zt)^2(t+1))X(0, 1, t; zt^2)$$
$$+(zt^3-z^2t^5)(1-tz)X(0, 1, t; zt^4)$$

(ii) $X(0, 1, t; z)$

$$=(1+zt+zt^2+zt^3)X(0, 1, t; zt^2)+(zt^4-z^2t^5-z^2t^6-z^2t^7)$$
$$\times X(0, 1, t; zt^4)+(zt^5-z^2t^8-z^2t^9-z^3t^{12})X(0, 1, t; zt^6).$$

**Lemma 3.2.** *Let* $P_n(t)=\prod_{1\le i\le n}(1-t^n)$ *and* $P_n^+(t)=\prod_{1\le i\le n}(1+t^n)$. *Define* $Y(z)=\sum_{n,m\ge 0}(P_{2n}^+(t)/(P_n(t^2)P_m(t^2)))t^{(n+m)^2+n}z^{n+m}$. *Then*

$$Y(z)=(1+zt+zt^2+zt^3)Y(zt^2)+(zt^4-z^2t^5-z^2t^6-z^2t^7)Y(zt^4)$$
$$+(zt^5-z^2t^8-z^2t^9+z^3t^{12})Y(zt^6).$$

*Proof. Step* 1. Let $\{f_n(t); n\ge 0\}$ be the set of rational functions in $t$. Define

$$L(z)=\sum_{n,m\ge 0}\frac{1}{P_m(t^2)}f_n(t)t^{(n+m)^2+n}z^{n+m}.$$

By shifting $m\mapsto m-1$ in the summation,

$$L(z)=ztL(zt^2)+\sum_{n,m\ge 0}\frac{t^{2n}}{P_m(t^2)}f_n(t)t^{(n+m)^2+n}z^{n+m}.$$

Define

$$M(z)=\sum_{n,m\ge 0}f_m(t)\frac{P_{2n}^+(t)}{P_n(t^2)}t^{(n+m)^2+n}z^{n+m}.$$

By shifting $n\mapsto n-1$, we get

$$zt^2M(zt^2)=\sum_{n,m\ge 0}f_m(t)\frac{P_{2n}^+(t)}{P_n(t^2)}\frac{1-t^{2n}}{(1+t^{2n-1})(1+t^{2n})}t^{(n+m)^2+n}z^{n+m}.$$

*Step* 2. Let

$$F(z)=\sum_{n,m\ge 0}\frac{P_{2n}^+(t)}{P_m(t^2)P_n(t^2)}\frac{t^{2n}}{(1+t^{2n-1})(1+t^{2n})}t^{(n+m)^2+n}z^{n+m},$$

$$H(z)=\sum_{n,m\ge 0}\frac{P_{2n}^+(t)}{P_m(t^2)P_n(t^2)}\frac{1}{(1+t^{2n-1})(1+t^{2n})}t^{(n+m)^2+n}z^{n+m}.$$

Then by Step 1,

(1) $H(z)-F(z)=zt^2Y(zt^2)$

(2) $F(z)-ztF(zt^2)=H(zt^2)$

(1) and (2) show

(3) $H(z)-(zt+1)H(zt^2)=zt^2Y(zt^2)-z^2t^5Y(zt^4)$.

*Step* 3. Let

$$A(z)=\sum_{n,m\geq 0}\frac{P_{2n}^{+}(t)}{P_m(t^2)P_n(t^2)}\frac{t^{4n}}{(1+t^{2n})(1+t^{2n-1})}t^{(n+m)^2+n}z^{n+m}.$$

Then by Step 1,

(4) $A(z)-ztA(zt^2)=F(zt^2)$.

Since $t^{4n-1}+t^{2n}+t^{2n-1}+1=(t^{2n}+1)(t^{2n-1}+1)$, we have

$$t^{-1}A(z)+(1+t^{-1})F(z)+H(z)=Y(z).$$

Thus

$$\begin{aligned}
&Y(z)-ztY(zt^2)\\
&=t^{-1}(A(z)-ztA(zt^2))+(1+t^{-1})(F(z)-ztF(zt^2))+H(z)-ztH(zt^2)\\
&=t^{-1}F(zt^2)+(1+t^{-1})H(zt^2)+H(z)-ztH(zt^2) \text{ (by (2) and (4))}\\
&=H(z)+(1+2t^{-1}-zt)H(zt^2)-zt^3Y(zt^4) \text{ (by (1))}.
\end{aligned}$$

Hence

(5) $H(z)+(1+2t^{-1}-zt)H(zt^2)=Y(z)-ztY(zt^2)+zt^3Y(zt^4)$.

*Step* 4. By the relations (3) and (5), both $H(z)$ and $H(zt^2)$ are expressed as linear combinations of $Y(z)$, $Y(zt^2)$ and $Y(zt^4)$. Replacing $z$ with $zt^2$ in the expression of $H(z)$, we get an expression of $H(zt^2)$ as a linear combination of $Y(zt^2)$, $Y(zt^4)$, and $Y(zt^6)$. As a result, $H(zt^2)$ is expressed in two ways as linear combinations of $Y(z)$, $Y(zt^2)$, $Y(zt^4)$ and $Y(zt^6)$ and therefore it yields a relation consisting only of $Y(z)$, $Y(zt^2)$, $Y(zt^4)$ and $Y(zt^6)$, which turns out to be the desired one in our lemma.

**Proposition 3.3.** *We have the following equations.*

(i) $X(0, 1, t; 1)=\dfrac{\sum_{m\geq 0}(-1)^{m(m+1)/2}t^{m(m+1)}}{\prod_{m\geq 1}(1-t^m)}$

(ii) $$X(1, 1, t; 1)=\frac{\sum_{-\infty<m<\infty}(-1)^m t^{(2m)^2}}{\prod_{m\geq 1}(1-t^m)}.$$

*Proof.* The power series $X(0, 1, t; z)$ is uniquely determined by the functional equation in Lemma 3.1, (ii) and the initial conditions:

$$X(0, 1, 0; z)=X(0, 1, t; 0)=1.$$

Therefore we have

$$X(0, 1, t; z)=Y(1)$$

by Lemma 3.2. Now,

$$\begin{aligned}Y(1)&=\sum_{n,m\geq 0}\frac{(1+t)(1+t^2)\cdots(1+t^{2n})}{((1-t^2)\cdots(1-t^{2m}))((1-t^2)\cdots(1-t^{2n}))}t^{(n+m)^2+n}\\&=\sum_{n\geq 0}\left(\sum_{m\geq 0}\frac{t^{m^2+2mn}}{(1-t^2)\cdots(1-t^{2m})}\right)\frac{(1+t)(1+t^2)\cdots(1+t^{2n})}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}t^{n^2+n}.\end{aligned}$$

By [4; p. 19, Corollary 2.2],

$$1+\sum_{m\geq 1}\frac{c^m t^{m(m-1)/2}}{(1-t)\cdots(1-t^m)}=\prod_{m\geq 0}(1+ct^m).$$

By replacing $t$ with $t^2$ and $c$ with $t^{1+2n}$,

$$\sum_{m\geq 0}\frac{t^{m^2+2mn}}{(1-t^2)\cdots(1-t^{2m})}=\sum_{m\geq 0}(1+t^{1+2n+2m}).$$

Hence

$$\begin{aligned}Y(1)&=\sum_{n\geq 0}(\prod_{m\geq 0}(1+t^{1+2n+2m}))\frac{(1+t)(1+t^2)\cdots(1+t^{2n})}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}t^{n^2+n}\\&=(\prod_{m\geq 0}(1+t^{2m+1}))\prod_{n\geq 0}\frac{(1+t^2)\cdots(1+t^{2n})}{(1-t^2)\cdots(1-t^{2n})}t^{n^2+n}\\&=(\prod_{m\geq 0}(1+t^{2m+1}))\prod_{m\geq 1}(1+(t^2)^{2m})(1+t^{2m}).\end{aligned}$$

Here the last equality is by [4; p. 21, Corollary 2.7]. Therefore

$$Y(1)=\prod_{m\geq 1}(1+t^m)(1+t^{4m}).$$

Now,

$$\frac{\sum_{m\geq 0}(-1)^{m(\pi+1)/2}t^{m(m+1)}}{\prod_{m\geq 1}(1-t^m)}$$
$$=\frac{1}{\prod_{m\geq 1}(1-t^m)}\prod_{m\geq 1}\frac{1-t^{4m}}{1+t^{4m-2}} \qquad \text{(by [4, Corollary 2.10])}$$
$$=(\prod_{m\geq 1}(1+t^{4m}))(\prod_{m\geq 1}(1+t^{4m-2}))(\prod_{m\geq 1}(1+t^m))(\prod_{m\geq 1}(1+t^{4m-2}))^{-1}$$
$$=(\prod_{m\geq 1}(1+t^{4m}))(\prod_{m\geq 1}(1+t^m)).$$

Hence we have (i) of the proposition. The relation (ii) of the proposition is obtained by Lemma 3.1, (ii) in a similar manner. We omit the detail.

*Proof of Theorem* B. What we have to prove are the relations in (b) for $\eta=-1$, which are obtained by Lemma 2.2, Lemma 2.4, and Proposition 3.3.

## § 4. Proof of Theorem A

The statement is obvious if $G$ is one of the groups in (i) or (ii). For any group $G$ in (iii), we have

$$\begin{aligned}\dim C(G^F)^{t_1^*} &= \#(G^F/\sim)/_{\tilde{t}_1}\\ &=\#\{\{x\}\in G^F/\sim\,;\ x\in Z_G(x)^0\}+\tfrac{1}{2}\#\{\{x\}\in G^F/\sim\,;\ x\notin Z_G(x)^0\},\end{aligned}$$

where $Z_G(x)^0$ means the identity component of the centralizer of $x$ in $G$. If one is familiar with the parametrizations of conjugacy classes in the groups in (iii), then one can check our statement in a straightforward manner. The conjugacy classes in $\mathrm{SO}_{2n+1}$ are explicitly described in [10] or [13]. The parametrizations of the conjugacy classes in $\mathrm{CSp}_{2n}^F$ and $(\mathrm{CO}_{2n}^{\varepsilon,0})^F$ with $\mathrm{char}\,\mathbf{F}_q\neq 2$ are easily derived from [10] or [13] and was necessary in [8, § 6] in obtaining generating functions of the conjugacy classes, however without being mentioned. There is an article [12, 1] by Shinoda which describes explicitly the conjugacy classes in $\mathrm{CSp}_{2n}^F$ and $(\mathrm{CO}_{2n}^{\varepsilon})^F$, but not $(\mathrm{CO}_{2n}^{\varepsilon,0})^F$. So for the completeness of our proof, let me introduce here the parametrizations of the conjugacy classes in $(\mathrm{CO}_{2n}^{\varepsilon,0})^F$ and $\mathrm{CSp}_{2n}^F$ from our standpoint, which are obtained from [10], [12] or [13].

Consider a partition

$$P=(\underbrace{1,\cdots,1}_{a_1\text{-times}},\underbrace{2,\cdots,2}_{a_2\text{-times}},\cdots,\underbrace{r,\cdots,r}_{a_r\text{-times}})$$

of some non-negative integer $d$. $d$ is called the degree of $P$ and is denoted by $d(P)$. If $d(P)=0$, then $P$ is the empty partition. We call the partition

$$P_i=(\underbrace{i,\ \cdots,\ i}_{a_i\text{-times}})$$

the $i$-part of $P$. If $i$ is an odd (resp. even) integer, then $P_i$ may be called an odd (resp. even) part of $P$ and $a_i$ may be called the rank of the $i$-part. We define a signatured partition $P^{\mathrm{sig}}$ of orthogonal (resp. symplectic) type, which is a partition $P$ attached with a set of signatures to some non-empty parts of $P$ subject to the following conditions (i) and (ii).

(i) The rank of any even (resp. odd) part of $P_i$ is an even integer,

(ii) to each non-empty odd (resp. even) part $P_i$ of $P$, a signature $\varepsilon_i$ ($+$ or $-$) is attached.

If the signatures $\varepsilon_i$ are changed to its negative for all non-empty odd (resp. even) parts of odd rank, then we may also obtain a signatured partition of orthogonal (resp. symplectic) type, which is denoted by $-P^{\mathrm{sig}}$. The degree and the rank of the $i$-part of $P^{\mathrm{sig}}$ are defined to be those of $P$. If $P^{\mathrm{sig}}$ is a signatured partition of orthogonal type, then we define the signature of $P^{\mathrm{sig}}$ by

$$\mathrm{sig}\,(P^{\mathrm{sig}})=\prod_i \varepsilon_i$$

where the signature $+$ (resp. $-$) is identified with 1 (resp. $-1$) and the product is over all the signatures of all non-empty odd parts of $P$.

Let $V$ be a $2n$-dimensional linear space and $\psi$ a non-degenerate symplectic or symmetric bilinear form. If $\psi$ is a non-degenerate symplectic form, or if the quadratic form $Q$ associated with $\psi$ is non-split (resp. split), then the group of all similitudes with respect to $\psi$ is $\mathrm{CSp}_{2n}^F$ or $(\mathrm{CO}_{2n}^+)^F$ (resp. $(\mathrm{CO}_{2n}^-)^F$).

Let $x$ be a similitude with respect to $\psi$ and $\lambda$ be the multiplicator of $x$, i.e.

$$\psi(xu,\ xv)=\lambda\psi(u,\ v)$$

for any $u, v \in V$. For a monic polynomial

$$m(X)=X^d+a_{d-1}X^{d-1}+\cdots+a_0$$

of degree $d$ with $a_0\neq 0$, we define the $\lambda$-dual $m^*(X)$ of $m(X)$ by

$$m^*(X)=m(\lambda/X)X^d a_0^{-1}.$$

Assume that $m(X)$ is the minimal polynomial of $x$. Then $m(X)$ is necessarily self $\lambda$-dual:

$$m^*(X)=m(X)$$

**Lemma 4.1.** *Let $x \in (CO^{\varepsilon}_{2n})^F$ (resp. $CSp^F_{2n}$) and $\lambda$ be the multiplicator of $x$. Then we can associate the following pairs:*

(i) $(m_i(X), P_i)$ $1 \leq i \leq r+s$

*where $P_i$ is a partition of some positive integer $(1 \leq i \leq r+s)$; $m_1(X), \cdots, m_{r+s}(X)$ are monic self $\lambda$-dual polynomials prime to the polynomial $X(X^2-\lambda)$ and are mutually prime to each other; for $1 \leq i \leq r$, $m_i(X)$ is irreducible, and for $r+1 \leq i \leq r+s$, $m_i(X) = h_i(X)h_i^*(X)$ with $h_i(X)$ a monic irreducible polynomial different from its $\lambda$-dual $h_i^*(X)$. If $\lambda$ is a square (resp. non-square) in $\mathbf{F}_q$, then we can further associated the following pairs in* (ii) *(resp.* (ii)′*).*

(ii) $(m_i(X), P_i^{\mathrm{sig}})$ $\quad i = r+s+1, r+s+2$

*where $m_{r+s+1}(X) = X - \lambda^{1/2}$, $m_{r+s+1}(X) = X + \lambda^{1/2}$ with $\lambda^{1/2}$ being a fixed square root of $\lambda$; $P^{\mathrm{sig}}_{r+s+1}$ and $P^{\mathrm{sig}}_{r+s+2}$ are signatured partitions of orthogonal (resp. symplectic) types, which are determined up to the operation:*

$$(P^{\mathrm{sig}}_{r+s+1}, P^{\mathrm{sig}}_{r+s+2}) \longrightarrow (-P^{\mathrm{sig}}_{r+s+1}, -P^{\mathrm{sig}}_{r+s+2})$$

(ii)′ $(m_{r+s+1}(X), P^{\mathrm{sig}}_{r+s+1})$

*where $m_{r+s+1}(X) = X^2 - \lambda$ and $P^{\mathrm{sig}}_{r+s+1}$ is a signatured partition of orthogonal (resp. symplectic) type.*

*The degrees of partitions in* (i) *and the signatured partitions in* (ii) *(or* (ii)′*) are subject to the following condition.*

$$\text{(iii, a)} \quad 2n = \sum_{1 \leq i \leq r+s} \deg m_i(X) d(P_i) + \begin{cases} \sum_{i=r+s+1, r+s+2} \deg m_i(X)\, d(P_i^{\mathrm{sig}}) & \text{if } \lambda \text{ is a square,} \\ \deg m_{r+s+1}(X)\, d(P^{\mathrm{sig}}_{r+s+1}) & \text{if } \lambda \text{ is a non-square.} \end{cases}$$

*If $x \in (CO^{\varepsilon}_{2n})^F$, then the signatures of the signatured partitions in* (ii) *or* (ii)′ *are subject to the following condition:*

$$\text{(iii, b)} \quad \begin{array}{ll} (-1)^{\delta} \operatorname{sig}(P^{\mathrm{sig}}_{r+s+1}) \operatorname{sig}(P^{\mathrm{sig}}_{r+s+2}) = \varepsilon & \text{if } \lambda \text{ is a square,} \\ (-1)^{\delta} \operatorname{sig}(P^{\mathrm{sig}}_{r+s+1}) = \varepsilon & \text{if } \lambda \text{ is a non-square,} \end{array}$$

*where $\delta = \sum_{1 \leq i \leq r} \delta_i$ with $\delta_i$ the sum of all ranks of odd parts of the partition $P_i$ $(1 \leq i \leq r)$.*

*The pairs in* (i) *and* (ii) *(if $\lambda$ is a square) or* (ii)′ *(if $\lambda$ is a non-square) with the conditions* (iii, a) (iii, b) *(resp. the condition* (iii, a)*) are uniquely determined by the conjugacy classes of $x$ in $(CO^{\varepsilon,F}_{2n})$ (resp. $CSp^F_{2n}$), and conversely if we are given $\lambda \neq 0 \in \mathbf{F}_q$ and pairs in* (i) *and* (ii) *(if $\lambda$ is a square)* *or* (ii)′ (*if* $\lambda$ *is a non-square*) *with the conditions* (iii, a) *and* (iii, b) (*resp.* (iii, a)), *then a conjugacy in* $(CO_{2n}^{\varepsilon})^F$ (*resp.* $CSp_{2n}^F$) *is uniquely associated. This parametrization affords the following criterions*:

(C1) *Assume that* $x \in (CO_{2n}^{\varepsilon})^F$ *and that the multiplicator* $\lambda$ *of* $x$ *is a square* (*resp. non-square*). *Then* $x \in (CO_{2n}^{\varepsilon,0})^F$ *if and only if the degree of the signatured partition* $P_{r+s+1}^{\mathrm{sig}}$ *is an even integer.*

(C2) *Assume that* $x \in (CO_{2n}^{\varepsilon,0})^F$ *and* $\lambda$ *is a square. Then the conjugacy class of* $x$ *in* $(CO_{2n}^{\varepsilon,0})^F$ *splits into two conjugacy classes in* $(CO_{2n}^{\varepsilon,0})^F$ if *and only if there does not exist an odd part in the signatured partitions* $P_{r+s+1}^{\mathrm{sig}}$ *or* $P_{r+s+2}^{\mathrm{sig}}$. *Assume that* $x \in (CO_{2n}^{\varepsilon,0})^F$ *and* $\lambda$ *is a non-square. Then the conjugacy class of* $x$ *in* $(CO_{2n}^{\varepsilon})^F$ *always splits into two conjugacy classes in* $(CO_{2n}^{\varepsilon,0})^F$.

(C3) *Assume that* $G = CO_{2n}^{\varepsilon,0}$ *and* $x \in G^F$. *If the multiplicator of* $x$ *is a square, then* $x$ *does not belong to identity component* $Z_G(x)^0$ *of the centralizer of* $x$ *in* $G$ *if and only if there exists an odd part of odd rank in the signatured partition* $P_{r+s+1}^{\mathrm{sig}}$ *and also in* $P_{r+s+2}^{\mathrm{sig}}$. *If* $\lambda$ *is a non-square, then* $x \notin Z_G(x)^0$ *if and only if there exists an odd part of odd rank in the signatured partition* $P_{r+s+1}^{\mathrm{sig}}$.

(C4) *Assume that* $G = CSp_{2n}$ *and* $x \in G^F$. *If the multiplicator* $\lambda$ *of* $x$ *is a square, then* $x \notin Z_G(x)^0$ *if and only if there exists and even part of odd rank in the signatured partition* $P_{r+s+1}^{\mathrm{sig}}$ *and also in* $P_{r+s+2}^{\mathrm{sig}}$. *If* $\lambda$ *is a non-square, then* $x \notin Z_G(x)^0$ *if and only if there exists an even part of odd rank in the signatured partition* $P_{r+s+1}^{\mathrm{sig}}$.

As our lemma is derived from [10], [12] or [13], the proof is omitted. We may be content in remarking that the minimal polynomial $m(X)$ of $x$ is

$$m(X) = \prod_{1 \le i \le r+s} m_i(X)^{d_i} \times \begin{cases} \prod_{i=r+s+1,r+s+2} m_i(X)^{d_i} & \text{if } \lambda \text{ is a square,} \\ m_{r+s+1}(X)^{d_{r+s+1}} & \text{if } \lambda \text{ is a non-square.} \end{cases}$$

where $d_i$ is the degree of the partition $P_i$ for $1 \le i \le r+s$ and the degree of the signatured partition $P_i^{\mathrm{sig}}$ for $i > r+s$.

Let $a_n$ be the number of the signatured partitions of orthogonal type with the following property:

$$d(P^{\mathrm{sig}}) = 2n,$$

*there exists an odd part of odd rank in* $P^{\mathrm{sig}}$.

Then we can check

$$1+\sum_{n\geq 1} a_n t^{2n} = \left(\left[\prod_{n\geq 1}\frac{1+t^{2n-1}}{1-t^{2n-1}}\right]_{\text{even}} - \prod_{n\geq 1}\frac{1+t^{4n-2}}{1-t^{4n-2}}\right)\frac{1}{\prod_{n\geq 1}(1-t^{4n})}$$

where for a formal power series $g(t)=\sum_{n\geq 0} b_n t^n$ in $t$, we define $g(t)_{\text{even}}=\sum_{n\geq 0} b_{2n} t^{2n}$. We set

$$f(t)=1+\sum_{n\geq 1} a_n t^{2n}$$

for simplicity. Let $\lambda\neq 0\in \mathbf{F}_q$ and define a formal power series $h_\lambda(t)$ in $t$ by

$$h_\lambda(t)=\prod_{j=1}^{\infty}\prod_{i=0}^{\infty} b(i)t^{ij}$$

where $b(i)$ is the number of monic polynomials $m(X)$ which are prime to $X(X^2-\lambda)$, self $\lambda$-dual and of degree $i$. If $\lambda$ is a square, then by [13, p. 37],

$$h_\lambda(t)=\prod_{n\geq 1}\frac{(1-t^{2n})^2}{1-qt^{2n}}.$$

If $\lambda$ is a non-square, then by a similar method as [loc. cit.] we can check

$$h_\lambda(t)=\prod_{n\geq 1}\frac{1-t^{4n}}{1-qt^{2n}}$$

$h_\lambda(t)$ is the generating function of the sum of the numbers of the conjugacy classes $\{x\}$ in $(\mathrm{CO}_{2n}^{+})^F$ and $(\mathrm{CO}_{2n}^{-})^F$ such that the multiplicator of $x$ is $\lambda$ and the minimal polynomial of $x$ is prime to $X^2-\lambda$.

For $\varepsilon=+$ or $-$, let

$$c_n^{\varepsilon}=\#(\mathrm{CO}_{2n}^{\varepsilon,0})^F/\sim,$$
$$\bar{c}_n^{\varepsilon}=\#((\mathrm{CO}_{2n}^{\varepsilon,0})^F/\sim)/\tilde{t}_1$$

and let $c_n'^{\varepsilon}$ be the number of the conjugacy classes $\{x\}$ in $(\mathrm{CO}_{2n}^{\varepsilon,0})^F$ such that $x$ does not belong to the identity component of its centralizer. $c_n^{\varepsilon}$, $\bar{c}_n^{\varepsilon}$ and $c_n'^{\varepsilon}$ are related by

$$c_n^{\varepsilon}=\bar{c}_n^{\varepsilon}+\tfrac{1}{2}c_n'^{\varepsilon}.$$

Then

$$\begin{aligned}&(1+\sum_{n\geq 1}(c_n^{+}+c_n^{-})t^{2n})-(1+\sum_{n\geq 1}(\bar{c}_n^{+}+\bar{c}_n^{-})t^{2n}\\&=\tfrac{1}{2}\sum_{n\geq 1}(c_n'^{+}+c_n'^{-})t^{2n}\end{aligned}$$

$$=\frac{q-1}{2}\,\frac{1}{2}\left(\frac{1}{2}h_{\lambda_0}(t)f(t)^2+2h_{\lambda_1}(t)f(t^2)\right)$$

by Lemma 4.1, where $\lambda_0$ is a fixed square in $\mathbf{F}_q$ (say 1) and $\lambda_1$ is a fixed non-square. Therefore

$$(1)\qquad (1+\sum_{n\geq 1}(c_n^+ + c_n^-)t^{2n})-(1+\sum_{n\geq 1}(\bar{c}_n^+ + \bar{c}_n^-)t^{2n})$$
$$=\frac{q-1}{2}\times\frac{1}{2}\times\frac{1}{2}\left(\sum_{n\geq 1}\frac{(1-t^{2n})^2}{1-qt^{2n}}\right)f(t)^2$$
$$+\frac{q-1}{2}\times\frac{1}{2}\times 2\left(\prod_{n\geq 1}\frac{1-t^{4n}}{1-qt^{2n}}\right)f(t^2).$$

Since $c_n'^+=c_n'^-$, we get

$$(2)\qquad (1+\sum_{n\geq 1}(c_n^+ - c_n^-)t^{2n})-(1+\sum_{n\geq 1}(\bar{c}_n^+ - \bar{c}_n^-)t^{2n})=0.$$

For $\varepsilon=+$ or $-$, let

$$d_n^\varepsilon=\sum_{s\in G^{*F}_{s.s.}/\sim} B(Z_{G^*}(s)^{*F}),$$
$$\bar{d}_n^\varepsilon=\sum_{s\in G^{*F}_{s.s.}/\sim} B^+(Z_{G^*}(s)^{*F}),$$

where $G=\mathrm{CO}_{2n}^{\varepsilon,0}$. Let

$$\mathrm{U}(t)=1+\sum_{n\geq 1}c_n^+(1)t^n+\sum_{n\geq 1}c_n^-(1)t^n,$$
$$\bar{\mathrm{U}}(t)=1+\sum_{n\geq 1}c_n^+(-1)t^n+\sum_{n\geq 1}c_n^-(-1)t^n$$

where $c_n(1)$ and $c_n(-1)$ are as in Theorem B. Let

$$V(t)=\mathrm{U}(t)-\bar{\mathrm{U}}(t).$$

Then

$$(3)\quad (1+\sum_{n\geq 1}(d_n^+ + d_n^-)t^n)-(1+\sum_{n\geq 1}(\bar{d}_n^+ + \bar{d}_n^-)t^n)$$
$$=(q-1)2\left(\prod_{n\geq 1}\frac{(1-t^n)^2}{1-qt^n}\right)V(t)+\frac{(q-1)}{2}\left(\prod_{n\geq 1}\frac{(1-t^n)^2}{1-qt^n}\right)((\mathrm{U}(t)-1)^2$$
$$-(\bar{\mathrm{U}}(t)-1)^2-(t)^2))$$

$$(4)\qquad (1+\sum_{n\geq 1}(d_n^+ - d_n^-)t^n)-(1+\sum_{n\geq 1}(\bar{d}_n^+ - \bar{d}_n^-)t^n)=0.$$

By a smiple computation we can check that the right hand side of the equation (1) coincides with the right hand side of the equation (3). (Use [4; p. 23, Corollary 2.10.]). Hence

$$(5)\qquad \Big(1+\sum_{n\geq 1}(c_n^+ + c_n^-)t^{2n}\Big)-\Big(1+\sum_{n\geq 1}(\bar{c}_n^+ + \bar{c}_n^-)t^{2n}\Big)$$
$$=\Big(1+\sum_{n\geq 1}(d_n^+ + d_n^-)t^{2n}\Big)-\Big(1+\sum_{n\geq 1}(\bar{d}_n^+ \bar{d}_n^-)t^{2n}\Big).$$

On the other hand, by [8, §6],

$$(6)\qquad \begin{aligned} 1+\sum_{n\geq 1}(c_n^+ + c_n^-)t^{2n} &= 1+\sum_{n\geq 1}(d_n^+ + d_n^-)t^{2n} \\ 1+\sum_{n\geq 1}(c_n^+ - c_n^-)t^{2n} &= 1+\sum_{n\geq 1}(d_n^+ - d_n^-)t^{2n} \end{aligned}$$

therefore

$$\begin{aligned} 1+\sum_{n\geq 1}(\bar{c}_n^+ + \bar{c}_n^-)t^{2n} &= 1+\sum_{n\geq 1}(\bar{d}_n^+ + \bar{d}_n^-)t^{2n} \\ 1+\sum_{n\geq 1}(\bar{c}_n^+ - \bar{c}_n^-)t^{2n} &= 1+\sum_{n\geq 1}(\bar{d}_n^+ - \bar{d}_n^-)t^{2n} \end{aligned}$$

by (2), (4) and (5). Therefore $\bar{c}_n^\varepsilon = \bar{d}_n^\varepsilon$ for $\varepsilon = +$ or $-$, proving the statement for $G = \mathrm{CO}_{2n}^{\varepsilon,0}$. The remaining cases are treated in a similar way.

## References

[1] Asai, T., Unipotent class functions of split special orthogonal groups $\mathrm{SO}_{2n}^+$ over finite fields, Comm. Algebra, **12** (1984), 517–615.
[2] ——, The unipotent class functions on the symplectic groups and odd orthogonal groups over finite fields, Comm. Algebra, **12** (1984), 617–645.
[3] ——, The unipotent class functions of exceptional groups over finite fields, to appear in Comm. Algebra.
[4] Andrews, G. E., The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Addison-Wesley, 1976.
[5] ——, Partitions, $q$-series and Lusztig-Macdonald-Wall conjectures, Invent. Math., **47** (1977), 91–101.
[6] Kawanaka, N., On the irreducible characters of finite unitary groups, J. Math. Soc. Japan, **29** (1977), 425–450.
[7] ——, Liftings of irreducible characters of finite classical groups, I. J. Fac. Sci. Univ. Tokyo, **28** (1982), 851–861.
[8] Lusztig, G., Irreducible representations of finite classical groups, Invent. Math., **43** (1977), 125–175.
[9] ——, Characters of reductive groups over a finite field, Ann. of Math. Studies, to appear.
[10] Springer, T. A. and Steinberg, R., Conjugacy classes, in Lecture Notes in Math., **131** Springer-Verlag, 1970.
[11] Satake, I., Classification Theory of Semi-Simple Algebraic Groups, Marcel Dekker, Inc., New York, 1971.
[12] Shinoda, K., The characters of Weil representations associated to finite fields, J. Algebra, **66** (1980), 251–280.

[13] Wall, G. E., On the conjugacy classes in the unitary, sympletic and orthogonal groups, J. Austral. Math. Soc., **3** (1963), 1–62.

*Department of Mathematics*
*Nara University of Education*
*Nara 630, Japan*