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A Local Formula for Springer's Representation

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To the memory of Takehiko Miyata

§ 1. Introduction

In [5], we have obtained a certain formula for the action of a simple reflection on the fundamental cycles in Springer's representation of a Weyl group. This formula has confirmed Joseph's conjecture given in [6] and has been used by Joseph [7] in order to give a unified proof for the irreducibility of the associated variety of a primitive ideal (see also [1], [8]). Furthermore Kashiwara and Tanisaki [8] have used a variant of our formula in order to establish a lucid connection between the characteristic cycles of certain holonomic systems on a flag manifold and Springer's representation (see also [13]). Their work has stimulated us to formulate a local formula better than the one given in [5]. The present formula given in Theorem in § 4 seems to be more transparent and usable. In this short note, this simpler version of a local formula will be reported, although the essential part of the proof is the same as the earlier one [5].

§ 2. Springer's representation

Let G be a connected complex semisimple algebraic group, B a Borel subgroup, fixed once and for all, and $X=G/B$ the flag manifold of G. Fix a maximal torus T in B and let $W=N_G(T)/T$ be the corresponding Weyl group. Let N be the nilpotent variety, i.e., the closed subvariety of the Lie algebra Lie G of G consisting of all nilpotent elements. We then have Springer's resolution

$$p\colon T^*X \longrightarrow N.$$

Here T^*X is the cotangent bundle of X which can be identified with the incidence subvariety

$$\{(gB, x) \in X \times N;\ x \in \operatorname{Lie} gBg^{-1}\}$$

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and p is the composite of the maps

$$T^*X \hookrightarrow X \times N \xrightarrow{pr_2} N.$$

We now assume that a locally closed subvariety V is given in N. Putting $Z = p^{-1}(V)$ in T^*X, we consider the rational Borel-Moore homology group $H_*(Z, \mathbf{Q})$ of Z. Throughout this report, the homology group $H_*(\ , \mathbf{Q})$ always means the rational Borel-Moore homology group, which is naturally isomorphic to the dual of $H_c^*(\ , \mathbf{Q})$, the rational compact support cohomology group. Then $H_*(Z, \mathbf{Q})$ is endowed with an action of the Weyl group W as follows. Lusztig [10] endowed the direct image $Rp_*\mathbf{Q}_{T^*X}$ of the constant sheaf $\mathbf{Q}_{T^*X}$ with the W-action as an object in the derived category of $\mathbf{Q}$-sheaves on N, using the method of the middle intersection cohomology of Deligne-Goresky-MacPherson [3] (see also [2], [4]). Thus the hypercohomology with compact support $\mathbf{H}_c^*(V, Rp_*\mathbf{Q}_{T^*X})$ is endowed with the W-action. But then we have the isomorphism

$$H_c^*(Z, \mathbf{Q}) \simeq \mathbf{H}_c^*(V, Rp_*\mathbf{Q}_{T^*X}).$$

Thus we have the dual action of W on $H_*(Z, \mathbf{Q})$. This W-module is considered as a generalization of Springer's original W-module [12] (see [4]).

§ 3. The simple reflections

Let $d = \dim Z$ and denote by $I(Z)$ the set of d-dimensional irreducible components of Z. An irreducible component $C \in I(Z)$ determines the fundamental cycle $[C]$ in the top dimensional homology $H_{2d}(Z, \mathbf{Q})$ and $\{[C];\ C \in I(Z)\}$ forms a basis of $H_{2d}(Z, \mathbf{Q})$.

The Borel subgroup B fixed in § 2 determines the canonical set S of generators in W so that for $s \in S$, $P_s = B \cup BsB$ is a semisimple rank one parabolic subgroup. Our purpose is to describe the action of $s \in S$ on the fundamental cycle $[C]$ in $H_{2d}(Z, \mathbf{Q})$. For this, we introduce the following situation. If we put $X_s = G/P_s$, then we have a $\mathbf{P}^1$-bundle $X \to X_s$. In the product map $X \times N \to X_s \times N$, let

$$Y_s = \{(gP_s, x) \in X_s \times N;\ x \in \operatorname{Lie} gP_s g^{-1}\}$$

be the image of T^*X in $X_s \times N$ and write

$$\phi_s\colon T^*X \longrightarrow Y_s.$$

We thus have a factorization of p,

$$T^*X \xrightarrow{\phi_s} Y_s \longrightarrow N.$$

Note that a fiber of ϕ_s is isomorphic to $\mathbf{P}^1$ or $\mathbf{P}^0$. We call $C \in I(Z)$ *s-hoirzontal* (resp. *s-vertical*) when $\dim \phi_s(C)=d$ (resp. $\dim \phi_s(C)=d-1$). We note that if C is s-horizontal (resp. s-vertical), then $C \to \phi_s(C)$ is birationally isomorphic (resp. a $\mathbf{P}^1$-bundle).

We now make the following identification

$$T^*X_s \simeq \{(gP_s, x) \in X_s \times N;\ x \in \operatorname{Lie} Ru(gP_s g^{-1})\} \subset Y_s$$

where T^*X_s is the cotangent bundle of X_s and $Ru(gP_s g^{-1})$ is the unipotent radical of $gP_s g^{-1}$. We then have the following commutative diagram of the canonical maps.

$$\begin{array}{ccc} T^*X & \xleftarrow{\rho_s} & X \times_{X_s} T^*X_s \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \varpi_s} \\ Y_s & \hookleftarrow & T^*X_s \end{array}$$

Note that ϖ_s is a $\mathbf{P}^1$-bundle maximally embedded in ϕ_s, i.e., for

$$T^*X_s \subsetneqq U \subset Y_s,$$

ϕ_s is no more a $\mathbf{P}^1$-bundle over U. Thus C is s-vertical if and only if C is contained in the image of ρ_s. Hence if we put $E_s = \rho_s^{-1}(Z)$, then $E_s \to \varpi_s(E_s)$ is a $\mathbf{P}^1$-bundle and $H_{2d}(E_s, \mathbf{Q})$ is spanned by the $[C]$'s for s-vertical components $C \in I(Z)$. We also have a short exact sequence

$$0 \longrightarrow H_{2d}(E_s, \mathbf{Q}) \longrightarrow H_{2d}(Z, \mathbf{Q}) \longrightarrow H_{2d}(\phi_s(Z), \mathbf{Q}) \longrightarrow 0.$$

By considering the s-action on $R\phi_{s,*}\mathbf{Q}_Z$ on $\phi_s(Z)$, it can be seen that the above exact sequence is s-equivariant and s acts trivially on $H_{2d}(\phi_s(Z), \mathbf{Q})$. We thus have the following lemma.

Lemma 1. *Consider the s-action on $H_{2d}(Z, \mathbf{Q})$ defined in* § 2. *Then the image of $s-\mathrm{id}$ is contained in $H_{2d}(E_s, \mathbf{Q})$. Particularly, $s[C]=-[C]$ for C s-vertical.*

This lemma allows us to define the map

$$\mathrm{Var} = s - \mathrm{id}\colon H_{2d}(Z, \mathbf{Q}) \longrightarrow H_{2d}(E_s, \mathbf{Q})$$

which formally looks like the variation map in the theory of vanishing cycles. However, we do not know whether or not this is the "real" Var. Perhaps, the question is related to Slodowy's results [11]. As stated above, Var$=-2$ id on the subspace $H_{2d}(E_s, \mathbf{Q})$. Hence the remaining problem is to determine Var $[C]$ for an s-horizontal $C \in I(Z)$.

§ 4. Statement of the theorem

We shall now describe the s-action on s-horizontal cycles. For a variety U, we denote by $C_i(U)$ te group of i-dimensional cycles on U (formal linear combinations of irreducible closed subvarieties in U). Thus, for example, we have $H_{2d}(Z, \mathbf{Q}) \simeq C_d(Z) \otimes \mathbf{Q}$. In § 3, we have considered

$$T^*X \xleftarrow[\rho_s]{} X \times_{X_s} T^*X_s \xrightarrow[\varpi_s]{} T^*X_s$$

where ρ_s is embedding of codimension 1 and ϖ_s is a $\mathbf{P}^1$-bundle. Let $C_d(Z)_{s\text{-hor}}$ be the subgroup of $C_d(Z)$ spanned by the s-horizontal fundamental cycles in $C_d(Z)$. Since an s-horizontal C properly intersects $X \times_{X_s} T^*X_s$, we can define the homomorphism

$$\rho_s^*\colon C_d(Z)_{s\text{-hor}} \longrightarrow C_{d-1}(X \times_{X_s} T^*X_s)$$

which factorizes as

$$C_d(Z)_{s\text{-hor}} \longrightarrow C_{d-1}(E_s) \subset C_{d-1}(X \times_{X_s} T^*X_s).$$

Actually, if C is s-horizontal, then by definition

$$\rho_s^*[C] = \sum_D m(C \times_{T^*X} X \times_{X_s} T^*X_s \colon D)[D]$$

where D runs through irreducible $(d-1)$-dimensional subvarieties of $X \times_{X_s} T^*X_s$ and the coefficient $m(\ \colon D)$ is the multiplicity of D in the scheme $C \times_{T^*X} X \times_{X_s} T^*X_s$.

By the projection ϖ_s, we can also define the two homomorphisms

$$\varpi_{s,*}\colon C_{d-1}(X \times_{X_s} T^*X_s) \longrightarrow C_{d-1}(T^*X_s)$$

and

$$\varpi_s^*\colon C_{d-1}(T^*X_s) \longrightarrow C_d(X \times_{X_s} T^*X_s).$$

Here

$$\varpi_{s,*}[D] = \deg(\varpi_s \,|\, D)[\varpi_s(D)]$$

for an irreducible D in $X \times_{X_s} T^*X_s$ (deg( ) is the degree of a map), and

$$\varpi_s^*[E] = [\varpi_s^{-1}(E)]$$

for an irreducible E in T^*X_s. Since $\varpi_s\colon E_s\to\varpi_s(E_s)\subset T^*X_s$ is a $\mathbf{P}^1$-bundle,

$$\varpi_s^*\varpi_{s,*}(C_{d-1}(E_s))\subset C_d(E_s).$$

We thus have a homomorphism

$$\varpi_s^*\varpi_{s,*}\rho_s^*\colon C_d(Z)_{s\text{-hor}}\longrightarrow C_d(E_s).$$

Note that $C_d(E_s)\otimes\mathbf{Q}\simeq H_{2d}(E_s,\mathbf{Q})$.

Theorem. *Let $s\in S$ be a simple reflection and $[C]\in C_d(Z)_{s\text{-hor}}$ an s-horizontal cycle. Then*

$$s[C]=[C]+\varpi_s^*\varpi_{s,*}\rho_s^*[C],$$

in other words, $\mathrm{Var}\,|\,C_d(Z)_{s\text{-hor}}=\varpi_s^*\varpi_{s,*}\rho_s^*$.

Remark. Theorem is true also for the relative situation. We start from a map $f\colon V\to N$ and put $Z=V\times_N T^*X\to T^*X$. Then $H_c^*(Z,\mathbf{Q})\simeq\mathbf{H}_c^*(V,f^*Rp_*\mathbf{Q}_{T^*X})$ has the W-action and hence $H_{2d}(Z,\mathbf{Q})$. All data can be pulled back to this relative situation and the same formula holds. Our proof is valid to this relative case.

Comment. It is desirable and plausible to be able to extend the definition of ρ_s^* justifying $\mathrm{Var}=\varpi_s^*\varpi_{s,*}\rho_s^*$ on the whole space $H_{2d}(Z,\mathbf{Q})$. For this, one likely has to find a nice functorial ambient space containing Z and to control the intersection theory.

§ 5. Applications

We have started from the choice of a subvariety V in the nilpotent variety N, or more generally, a map $V\to N$. We shall deduce some of the earlier results [5] when we make particular choices of V. For this purpose and for the later use, we write the diagram given in § 3 in terms of associated fiber bundles.

We have already fixed a simple reflection s in S and hence the parabolic subgroup $P_s=B\cup BsB$. Put

$$\mathfrak{n}=\mathrm{Lie}\,Ru(B),\qquad \mathfrak{n}_s=\mathrm{Lie}\,Ru(P_s),$$

the Lie algebras of the unipotent radicals of B and P_s. Denote by N_s the nilpotent variety of the Lie algebra of a Levi part of P_s. Then $N_s\times\mathfrak{n}_s$ is the nilpotent variety of Lie P_s. In general, for a closed subgroup H in G and an H-variety A, we denote by $G\times^H A$ the associated fiber bundle over G/H with standard fiber A. We then have the natural isomorphisms

$$T^*X \simeq G\times^B \mathfrak{n}, \qquad T^*X_s \simeq G\times^{P_s}\mathfrak{n}_s.$$

Similarly we easily have

$$X\times_{X_s}T^*X_s \simeq G\times^B\mathfrak{n}_s, \qquad Y_s\simeq G\times^{P_s}(N_s\times\mathfrak{n}_s).$$

The embedding $\rho_s\colon X\times_{X_s}T^*X_s \hookrightarrow T^*X$ is realized as

$$\rho_s\colon G\times^B\mathfrak{n}_s \hookrightarrow G\times^B\mathfrak{n}$$

associated to the B-equivariant embedding $\mathfrak{n}_s\hookrightarrow\mathfrak{n}$. The embedding $T^*X_s\hookrightarrow Y_s$ is also realized as

$$G\times^{P_s}\mathfrak{n}_s\hookrightarrow G\times^{P_s}(N_s\times\mathfrak{n}_s)$$

associated to the P_s-equivariant embedding

$$\mathfrak{n}_s\xrightarrow{\sim}0\times\mathfrak{n}_s\hookrightarrow N_s\times\mathfrak{n}_s.$$

We thus have the natural diagram corresponding to that given in § 3.

$$\begin{tikzcd} G\times^B\mathfrak{n} \arrow[d, "\phi_s"'] & G\times^B\mathfrak{n}_s \arrow[l, hook', "\rho_s"'] \arrow[d, "\varpi_s"] \\ G\times^{P_s}(N_s\times\mathfrak{n}_s) & G\times^{P_s}\mathfrak{n}_s \arrow[l, hook'] \end{tikzcd}$$

As the associated bundles over $X_s=G/P_s$, the above diagram is associated to the following diagram of the fibers.

$$\begin{tikzcd} T^*\mathbf{P}^1\times\mathfrak{n}_s\simeq P_s\times^B\mathfrak{n} \arrow[d] & P_s\times^B\mathfrak{n}_s\simeq\mathbf{P}^1\times\mathfrak{n}_s \arrow[l, hook'] \arrow[d] \\ N_s\times\mathfrak{n}_s & \mathfrak{n}_s \arrow[l, hook'] \end{tikzcd}$$

We now make particular choices of V in N.

Case (i). Let $V=O$ be a nilpotent orbit in N. Then

$$Z=p^{-1}(O)=\{(g,x)\in G\times^B\mathfrak{n};\ \mathrm{Ad}(g)x\in O\}.$$

Each irreducible component U of $O\cap\mathfrak{n}$ is stable under the B-action and each irreducible component of Z is isomorphic to $G\times^BU\subset G\times^B\mathfrak{n}$ for some $U\in I(O\cap\mathfrak{n})$. The correspondence

$$I(O\cap\mathfrak{n})\ni U\longmapsto G\times^BU\in I(Z)$$

is bijective. In this correspondence, $G\times^BU$ is s-vertical if and only if

$U \subset \mathfrak{n}_s$. For an s-horizontal $G\times^B U$, the cycle $\rho_s^*[G\times^B U]$ can be calculated through its fiber-level intersection $U\times_{\mathfrak{n}}\mathfrak{n}_s$. The resulting formula for $\varpi_s^*\varpi_{s,*}\rho_s^*[G\times^B U]$ then coincides with Joseph's one in [6] described in the geometry of $O\cap\mathfrak{n}$ ([5; § 3]). In this case the W-module $H_{2d}(Z, \mathbf{Q})$ ($d=\dim X+\frac{1}{2}\dim(O\cap\mathfrak{n})$) is irreducible. This construction fits the conjecture by Borho-Brylinski ([1], [14]) that the characteristic variety of a primitive ideal will be irreducible and coincide with the closure of some irreducible component of Z.

Case (ii). Let $V=\{x\}$ in N be a point. Then

$$p^{-1}(x)\simeq X^x=\{gB\in X;\ \mathrm{Ad}(g^{-1})x\in\mathfrak{n}\}.$$

The W-module $H_*(X^x, \mathbf{Q})$ is the original form given by Springer in [12]. To analyse this case, we use the relative version of Theorem (Remark to Theorem). Let $O=O(x)$ be the G-orbit of x and

$$\pi\colon \tilde{O}=G/Z^\circ(x)\longrightarrow O=G/Z(x)$$

be a covering space over O where $Z(x)$ is the stabilizer of x and $Z^\circ(x)$ is its identity component. Then

$$p^{-1}(O)\simeq G\times^{Z(x)}X^x \quad \text{and} \quad Z=\tilde{O}\times_N T^*X\simeq G\times^{Z^\circ(x)}X^x.$$

Put $d=\dim Z$, $d(x)=\dim X^x$ ($\dim O=d-d(x)$). We then have natural isomorphisms

$$\begin{aligned} H_c^{2d}(Z, \mathbf{Q}) &\simeq \mathbf{H}_c^{2d}(\tilde{O}, \pi^*Rp_*\mathbf{Q}_{T^*X}) \\ &\simeq H_c^{2\dim O}(\tilde{O}, \pi^*R^{2d(x)}p^*\mathbf{Q}_{T^*X}) \\ &\simeq H^0(\tilde{O}, (\pi^*R^{2d(x)}p_*\mathbf{Q})^\vee)^\vee \quad \text{(Poincaré duality)} \\ &\simeq (R^{2d(x)}p_*\mathbf{Q})_x \\ &\simeq H^{2d(x)}(X^x, \mathbf{Q}). \end{aligned}$$

($A^\vee$ denotes the dual of A. $\pi^*R^{2d(x)}p_*\mathbf{Q}$ is a constant sheaf on $\tilde{O}$.) Thus $H_{2d}(Z, \mathbf{Q})\simeq H_{2d(x)}(X^x, \mathbf{Q})$ as W-modules. Furthermore, an irreducible component of Z is given by $G\times^{Z^\circ(x)}C$ for some irreducible component C of X^x and $G\times^{Z^\circ(x)}C=G\times^B U$ for some irreducible component U of $\tilde{O}\times_N\mathfrak{n}$. By this, we have the bijective correspondence

$$I(X^x)\simeq I(Z)\simeq I(\tilde{O}\times_N\mathfrak{n}).$$

Applying the relative version of Theorem, we can calculate the s-action on each fundamental cycle in terms of $\tilde{O}\times_N\mathfrak{n}$. We then have [5; Theorem 2].

Case (iii). Let (G, K) be a complexified symmetric pair and

$$\operatorname{Lie} G = \operatorname{Lie} K \oplus \mathfrak{m}$$

the corresponding Cartan decomposition. Let $V = \mathfrak{m} \cap N$ and $Z = p^{-1}(V)$. Then Z has a partition

$$Z = \coprod_O T_O^* X$$

where $O \subset X$ runs through all K-orbits in X and $T_O^* X$ denotes the conormal bundle of O in X. The finiteness of the K-orbits in X was proved by Matsuki. Thus $I(Z) = \{\overline{T_O^* X};\ O \in K\backslash X\}$. Kashiwara and Tanisaki have defined a natural W-homomorphism from the Grothendieck group of certain Harish-Chandra $(\operatorname{Lie} G, K)$-modules into $H_{2d}(Z, \mathbf{Q})$ ([8], [13]). There our formula for the s-action in this case naturally occurs.

§ 6. The first reduction

The remaining part of this report will be devoted to the proof of Theorem. We are given a map $V \to N$ and a simple reflection s in W. The problem is to determine the s-action $s[C]$ in $H_{2d}(Z, \mathbf{Q})$ for an s-horizontal component C in $I(Z)$.

Consider the pull-back

$$\begin{array}{ccccc} Z & \xrightarrow{\phi} & V_s & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ T^*X & \xrightarrow[\phi_s]{} & Y_s & \longrightarrow & N. \end{array} \qquad (V_s = V \times_N Y_s)$$

As is seen in § 5, ϕ_s is given by the natural map $G \times^B \mathfrak{n} \to G \times^{P_s} (N_s \times \mathfrak{n}_s)$ which is the bundle morphism over $X_s = G/P_s$ associated to the fiber map

$$P_s \times^B \mathfrak{n} \simeq T^*\mathbf{P}^1 \times \mathfrak{n}_s \longrightarrow N_s \times \mathfrak{n}_s.$$

Hence the s-action on $R\phi_{s,*}\mathbf{Q}_{T^*X}$ comes from the involution on

$$R\phi_*^{\mathrm{SL}_2}\mathbf{Q}_{T^*\mathbf{P}^1} \quad \text{for } \phi^{\mathrm{SL}_2}\colon T^*\mathbf{P}^1 \longrightarrow N_s$$

(the SL_2-case). By definition the s-action on $H_{2d}(Z, \mathbf{Q})$ coincides with that on the dual of $\mathbf{H}_c^{2d}(V_s, R\phi_{s,*}\mathbf{Q}_{T^*X})$. Hence for an s-horizontal C, we may consider the submodule

$$H_{2d}(\phi^{-1}\phi(C), \mathbf{Q}) \simeq \mathbf{H}_c^{2d}(\phi(C), R\phi_{s,*}\mathbf{Q}_{T^*X})^\vee \quad \text{in } H_{2d}(Z, \mathbf{Q}).$$

The proof of Theorem can thus been reduced to the following statement

($\phi(C)$ being replaced by V).

Statement. *Let V be an irreducible variety and f: $V \to Y_s$ satisfy $f(V) \not\subset T^*X_s$ = singular locus of Y_s. Let C be the unique irreducible component of $Z = V \times_{Y_s} T^*X$ such that $\phi(C) = V$ where ϕ: $Z \to V$. (Note that $T^*X \to Y_s$ is a resolution.) Consider the s-action on*

$$H_{2d}(Z, \mathbf{Q}) \simeq \mathbf{H}_c^{2d}(V, f^*R\phi_{s,*}\mathbf{Q}_{T^*X})^\vee.$$

Then

$$s[C] = [C] + \varpi^*\varpi_*\rho^*[C].$$

Here

$$\begin{tikzcd} Z \times_{T^*X} X \times_{X^s} T^*X_s \arrow[r, hook, "\rho"] \arrow[d, "\varpi"] & Z \arrow[d, "\phi"] \\ V \times_{Y_s} T^*X_s \arrow[r, hook] & V \end{tikzcd}$$

is the pull-back of ρ_s, ϖ_s in § 3.

The proof of Statement can be reduced to a simpler one. For a nonempty open set U in V, $H_{2d}(\phi^{-1}(U), \mathbf{Q})$ is a quotient of $H_{2d}(Z, \mathbf{Q})$ modulo the span of the fundamental cycles which are not contained in $\phi^{-1}(U)$ as s-modules. Hence by definition we can replace V by smaller open coverings. Taking a small open set A in X_s, we can then replace T^*X by $T^*\mathbf{P}^1 \times \mathfrak{n}_s \times A$, Y_s by $N_s \times \mathfrak{n}_s \times A$, T^*X_s by $\mathfrak{n}_s \times A$ and $X \times_{X_s} T^*X_s$ by $\mathbf{P}^1 \times \mathfrak{n}_s \times A$. But then, as was noticed already, our diagram comes from the SL_2-situation

$$\begin{tikzcd} T^*\mathbf{P}^1 \arrow[d, "\phi"] & \mathbf{P}^1 \arrow[l, hook', "\rho"'] \arrow[d, "\varpi"] \\ N_s & \{0\}. \arrow[l, hook'] \end{tikzcd} \quad (\times \mathfrak{n}_s \times A)$$

Note that $R\phi^*\mathbf{Q}_{T^*\mathbf{P}^1} \simeq \mathbf{Q}_N \oplus \mathbf{Q}_{\{0\}}[-2]$ where s acts trivially on $\mathbf{Q}_N$ and -1 on $\mathbf{Q}_{\{0\}}[-2]$ (see [2]). (We have put $N = N_s$ and denote by $\mathbf{Q}_{\{0\}}[-2]$ the shift in degree -2 of the sykscraper sheaf $\mathbf{Q}_{\{0\}}$ at 0.) Thus, in order to establish Statement, it suffices to prove the following lemma.

Lemma 2. *Let $N = \{(x, y, z) \in \mathbf{C}^3;\ x^2 + yz = 0\}$ be the quadratic cone and ϕ: $T^*\mathbf{P}^1 \to N$ be the blowing-up at $0 \in N$. Let V be an irreducible variety with a map f: $V \to N$ such that $f(V) \neq \{0\}$. Let*

$$\begin{tikzcd} Z \arrow[d, "\phi"'] & E = V_0 \times \mathbf{P}^1 \arrow[l, hook', "\rho"'] \arrow[d, "\varpi"] \\ V & V_0 = f^{-1}(0) \arrow[l, hook'] \end{tikzcd}$$

be the pull-back of the above diagram. Let C be the irreducible component of Z such that $\phi(C) = V$. Then the action of the involution s on $[C] \in H_{2d}(Z, \mathbf{Q})$ is described as

$$S[C] = [C] + \varpi^* \varpi_* \rho^* [C].$$

The remaining part will be devoted to the proof of this lemma.

§ 7. The second reduction; functoriality

We keep the situation and the notations as in Lemma 2. Let V' be an irreducible variety and $V' \to V$ a finite morphism. By pulling back the diagram in Lemma 2, we have the following diagram.

$$\begin{tikzcd}
Z' \arrow[ddd, "\phi'"'] \arrow[dr] & & & E' \arrow[lll, hook', "\rho'"'] \arrow[ddd, "\varpi'"] \arrow[dr] & & \\
& Z \arrow[ddd, "\phi"] \arrow[dr] & & & E \arrow[lll, hook', "\rho"'] \arrow[ddd, "\varpi"] \arrow[dr] & \\
& & T^*\mathbf{P}^1 \arrow[ddd] & & & \mathbf{P}^1 \arrow[lll, hook'] \arrow[ddd] \\
V' \arrow[dr] & & & V'_0 \arrow[lll, hook'] \arrow[dr] & & \\
& V \arrow[dr, "f"'] & & & V_0 \arrow[lll, hook'] \arrow[dr] & \\
& & N & & & \{0\} \arrow[lll, hook']
\end{tikzcd}$$

Note that we have surjective s-homomorphisms

$$\begin{tikzcd} H_{2d}(Z', \mathbf{Q}) \arrow[r, two heads] & H_{2d}(Z, \mathbf{Q}) \\ H_{2d}(E', \mathbf{Q}) \arrow[u, hook] \arrow[r, two heads] & H_{2d}(E, \mathbf{Q}). \arrow[u, hook] \end{tikzcd}$$

Let C' be the unique irreducible component of Z' such that $\phi'(C')=V'$. Thus C' is mapped surjectively onto C in Z. We then have the following functorial property.

Lemma 3. *For $C' \in I(Z')$ as above, we have*

$$\alpha\varpi'^{,*}\varpi'_{*}\rho'^{,*}[C']=\varpi^{*}\varpi_{*}\rho^{*}[C]$$

where α: $H_{2d}(E', \mathbf{Q}) \longrightarrow H_{2d}(E, \mathbf{Q})$.

Proof. Denote by $C_i(\ )$ the group of i-dimensional cycles as in § 4. It suffices to prove the commutativity of the following diagram.

$$\begin{array}{ccc}
C_d(C') & \longrightarrow & C_d(C) \\
{\scriptstyle \rho'^{,*}}\downarrow & ① & \downarrow{\scriptstyle \rho^{*}} \\
C_{d-1}(E') & \longrightarrow & C_{d-1}(E) \\
{\scriptstyle \varpi'_{*}}\downarrow & ② & \downarrow{\scriptstyle \varpi_{*}} \\
C_{d-1}(V'_0) & \longrightarrow & C_{d-1}(V_0) \\
{\scriptstyle \varpi'^{,*}}\downarrow & ③ & \downarrow{\scriptstyle \varpi^{*}} \\
C_d(E') & \longrightarrow & C_d(E)
\end{array}$$

(Note that $H_{2d}(Z, \mathbf{Q})_{s\text{-hor}} \simeq C_d(C)\otimes\mathbf{Q}$, $H_{2d}(E, \mathbf{Q}) \simeq C_d(E)\otimes\mathbf{Q}$ etc..) Since $E' \to V'_0$ is the pull-back of the $\mathbf{P}^1$-bundle $E \to V_0$, the commutativity in ② and ③ is clear. The commutativity in ① follows from Lemma 7 in [5].

q.e.d.

§ 8. The third reduction; the case of analytic curves

By Lemma 3, in order to prove Lemma 2, it suffices to prove it for some finite cover $V' \to V$ ($H_{2d}(Z', \mathbf{Q}) \to H_{2d}(Z, \mathbf{Q})$ is a surjective s-homomorphism). Hence, by taking the normalization and localizing the situation even more, if necessary, we may assume that V is smooth and $V_0 = f^{-1}(0)$ is a smooth subvariety of codimension 1 (if not, $H_{2d}(E, \mathbf{Q})=0$).

Since the problem is topological and local, we may assume from now on that V is a d-dimensional polydisk D^d and V_0 is a $(d-1)$-dimensional polydisk D^{d-1} such that $V = V_0 \times D$. Fix a point $x_0 \in V_0$ and consider the embedding $D \rightleftarrows x_0 \times D \subset V$. For f: $V \to N$, we define f_0: $V \to N$ by $f_0(x, t) = f(x_0, t)$ for $(x, t) \in V_0 \times D = V$. Then, for $Rp_{*}\mathbf{Q}_{T^*\mathbf{P}^1}$ (p: $T^*\mathbf{P}^1 \to N$), we have $f^*Rp_{*}\mathbf{Q}_{T^*\mathbf{P}^1} \simeq f_0^*Rp_{*}\mathbf{Q}_{T^*\mathbf{P}^1}$ (quasi-isomorphism). On the other hand, $Z \simeq V \cup \mathbf{P}^1 \times V_0$ ($V \cap (\mathbf{P}^1 \times V_0) = \infty \times V_0$) and $V \simeq C$. Define f': $D \to N$ by $f'(t) = f(x_0, t)$. Then

$$Z_D = D \times_N T^*\mathbf{P}^1 \simeq D \cup \mathbf{P}^1 \quad (D \cap \mathbf{P}^1 = \{\infty\})$$

and $H_2(Z_D, \mathbf{Q})$ has the s-action by f'. By these discussions, we have an s-isomorphism

$$H_2(Z_D, \mathbf{Q}) \xrightarrow{\sim} H_{2d}(Z, \mathbf{Q})$$

sending $[D]$ to $[C]$ and $[\mathbf{P}^1]$ to $[\mathbf{P}^1 \times V_0]$.

On the other hand, it can be seen that under this isomorphism the map $\varpi^* \varpi_* \rho^*$ is preserved. Thus for the proof of Lemma 2, we may finally assume that V is a one-dimensional disk D and $f: D \to N$ satisfies $f^{-1}(0)=0$. For this case, we use the identification of our s-action with Kazhdan-Lusztig's one [9]. Kazhdan-Lusztig's action can be directly computed and turns out to coincide with our formula ([5; Lemma 13 and Appendix]). Thus the proof of Lemma 2 is completed and so is that of Theorem in § 4.

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