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On the Dimension of Spaces of Automorphic Cohomology

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It has recently been shown (in response to a question of Wells and Wolf [16]) that the dimension of the space of L^2-Γ-automorphic cohomology of any flag domain D is finite [18]. Here Γ is a discrete subgroup, with co-finite volume, of a connected semisimple Lie group of automorphisms of D. This work concerns the computation of that dimension, at least in the rank 1 case where an explicit L^2-index formula is available [1]. We prove, in particular, the *existence* of non-zero, square-integrable automorphic cohomology classes. Such existence questions have previously been settled (via the Atiyah-Singer or holomorphic Lefschetz formulas, for example) often, but not exclusively, when D reduces to a bounded Hermitian domain and when Γ is co-compact. The space of automorphic cohomology then reduces to a space of automorphic *forms* such as that considered, for example, in [6].

§ 1. Introduction

Let $X=G^{\mathbf{C}}/P$ be a complex flag manifold where P is a parabolic subgroup of a complex connected semisimple Lie group $G^{\mathbf{C}}$. Let G be a non-compact connected real form of $G^{\mathbf{C}}$ such that $V=G\cap P$ is compact. Then $D=G/V$ is a *flag domain* [20], i.e. an open real orbit in X with compact isotropy. D therefore carries a G-invariant holomorphic structure induced from X. Also if $E_\pi \to D$ is a homogeneous vector bundle over D induced by an irreducible representation π of V then E_π carries a G-invariant holomorphic structure. However, in general, E_π may have no global holomorphic sections, so in particular there may be no E_π-valued automorphic forms on D corresponding to a given discrete subgroup Γ of G. There is however the more general notion (due to Griffiths [3], [13]) of E_π-valued *automorphic cohomology* on D. Namely, if E_π is *non-degenerate* (in the sense of (3.3) below), if s is the dimension of a maximal compact subvariety of D, and if $H^*(D, \mathcal{O}E_\pi)$ is the cohomology of D with coefficients

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in the sheaf $\mathcal{O}E_\pi$ of germs of local holomorphic sections of E_π then, by a result of Schmid [10], [16], $H^q(D, \mathcal{O}E_\pi)=0$ for $q\neq s$ and $H^s(D, \mathcal{O}E_\pi)$ is an infinite dimensional Fréchet G module. The subspace $H^s(D, \mathcal{O}E_\pi)^\Gamma$ of Γ-invariant cohomology classes is the Γ-automorphic cohomology of D.

In [18] we established finite-dimensionality of the subspace $H^s_2(D, \mathcal{O}E_\pi)^\Gamma$ of square-integrable classes in $H^s(D, \mathcal{O}E_\pi)^\Gamma$ (cf. (3.7), (3.8) below). It is yet an open problem, raised in [16], to prove whether or not the full space $H^s(D, \mathcal{O}E_\pi)^\Gamma$ is finite-dimensional. In the present paper we compute the dimension of $H^s_2(D, \mathcal{O}E_\pi)^\Gamma$ in the case when the real rank of G is 1; this covers the important example of $D=$the period matrix domain $SO_e(2n, 1)/U(n)$. Apart from the Hermitian case our dimension formula is rather quite simple; i.e. it involves no Γ-cuspidal terms. The main results presented here are Theorems 3.9, 4.7, and 4.12. These depend, firstly, on a vanishing theorem which we develop for the L^2-cohomologies of Hotta's elliptic complex [5], though they could be obtained via a shorter route. Since the vanishing theorem, Theorem 2.16 below, is of independent interest (it is the best possible) we have therefore so written Section 2 as to make it completely independent of the rest of the paper.

We take this opportunity to express our heart-felt thanks to the mathematics faculty of Sophia University for their many kindnesses and for providing us the pleasant and stimulating environment, and resources, to conduct this research.

§ 2. The Hotta complex

In this section we recall the elliptic complex (a generalization of the Dolbeault complex) constructed by Hotta [5] whose "bootstrap" is the Dirac operator. We prove a sharp vanishing theorem for the L^2-cohomologies of this complex. Applications to automorphic cohomology are given in Sections 3, 4.

Let K be a maximal compact subgroup of G which contains a Cartan subgroup H of G. We denote by g, k, h the complexifications of the Lie algebras g_0, k_0, h_0 of G, K, H respectively. Let $(\,,)$ denote the Killing form of g, let $g_0=k_0+p_0$ be a Cartan decomposition of g_0 where p_0 is the orthocomplement of k_0 in g_0 with respect to $(\,,)$, and let p denote the complexification of p_0. We shall write Δ for the set of non-zero roots of (g, h), and for $Q\subset\Delta$ we shall write $\langle Q\rangle$ for the sum $\sum_{\alpha\in Q}\alpha$. Let Δ_k, Δ_n denote the set of compact, non-compact roots, respectively. Thus if g_β is the root space of $\beta\in\Delta$, $\beta\in\Delta_k \Longleftrightarrow g_\beta\subset k$; $\Delta_n=\Delta-\Delta_k$. We assume that $G^{\mathbf{C}}$ is simply connected. The character group of H is then identified with the lattice

$$(2.1)\qquad \mathscr{L}=\{\lambda\in \mathrm{Hom}_{\mathbf{R}}(\sqrt{-1}\,h_0, \mathbf{R})\,|\,\lambda \text{ is integral}\}.$$

Here of course $\mathbf{R}$ is the field of real numbers and integrality means that $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbf{Z}$, the ring of integers, for each α in Δ. If $\Sigma^+ \subset \Delta$ is a system of positive roots let Σ_k^+, $\Sigma_n^+ = \Sigma^+ \cap \Delta_k$, $\Sigma^+ \cap \Delta_n$, respectively, and let

$$(2.2) \qquad \mathscr{L}(\Sigma_k^+) = \{\lambda \in \mathscr{L} \mid (\lambda, \alpha) \geq 0 \quad \text{for } \alpha \in \Sigma_k^+\}.$$

For $\lambda \in \mathscr{L}(\Sigma_k^+)$ let V_λ be the irreducible K module with Σ_k^+-highest weight λ. Let

$$(2.3) \qquad \mathscr{L}_0(\Sigma_k^+) = \left\{\lambda \in \mathrm{Hom}_{\mathbf{R}}(\sqrt{-1}\, h_0, \mathbf{R}) \,\middle|\, \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbf{Z}^+ \quad \text{for } \alpha \in \Sigma_k^+\right\}$$

where $\mathbf{Z}^+$ is the set of non-negative integers, and let V_λ be the irreducible k module with Σ_k^+-highest weight λ for $\lambda \in \mathscr{L}_0(\Sigma_k^+)$. The 1/2-spin modules for k will be denoted by $S^\pm$, with the convention that $\{\delta_n(\Sigma^+) - \langle Q\rangle \mid Q \subset \Sigma_n^+, (-1)^{|Q|} = \pm 1\}$ is the set of weights of $S^\pm$, where $2\delta_n(\Sigma^+) = \langle \Sigma_n^+\rangle$, and where $|Q|$ is the cardinality of a set Q. For $\lambda \in \mathscr{L}$ such that $\lambda + \delta_n(\Sigma^+) \in \mathscr{L}_0(\Sigma_k^+)$ the k-representations $S^\pm \otimes V_{\lambda+\delta_n(\Sigma^+)}$ integrate to representations of K. Thus we can form the induced homogeneous C^∞ vector bundles $E_\lambda^\pm$, $E_\lambda \to G/K$ with fibers $S^\pm \otimes V_{\lambda+\delta_n(\Sigma^+)}$, $S \otimes V_{\lambda+\delta_n(\Sigma^+)}$, where $S = S^+ \oplus S^-$ and assuming, whenever necessary (without loss of generality), that G/K is a spin manifold we can consider the twisted Dirac operators $D_\lambda^\pm$, D_λ on G/K: $D_\lambda : \Gamma^\infty E_\lambda \to \Gamma^\infty E_\lambda$, $D_\lambda^\pm = D_\lambda|_{\Gamma^\infty E_\lambda^\pm} : \Gamma^\infty E_\lambda^\pm \to \Gamma^\infty E_\lambda^\mp$, [9], where Γ^∞ denotes the space of C^∞ sections. Let $m = \frac{1}{2} \dim G/K$, $2\delta(\Sigma^+) = \langle \Sigma^+\rangle$, and let Ω be the Casimir operator of g.

Theorem 2.4 (Lemma 3.3 of [5]). *Let $\lambda \in \mathscr{L}$ such that $\lambda + \delta_n(\Sigma^+) \in \mathscr{L}_0(\Sigma_k^+)$ as above. Then there is a direct sum K module decomposition*

$$(2.5) \qquad S^\pm \otimes V_{\lambda+\delta_n(\Sigma^+)} = \sum_{(-1)^q = \pm 1} V^q_{\lambda+2\delta_n(\Sigma^+)}$$

and a sequence of first order G-invariant differential operators $D^q : \Gamma^\infty E_{\lambda,q} \to \Gamma^\infty E_{\lambda,q+1}$, where $E_{\lambda,q} \to G/K$ is the homogeneous vector bundle induced by $V^q_{\lambda+2\delta_n(\Sigma^+)}$, $0 \leq q \leq m-1$, $V^0_{\lambda+2\delta_n(\Sigma^+)} = V_{\lambda+2\delta_n(\Sigma^+)}$, such that

$$0 \to \Gamma^\infty E_{\lambda,0} \xrightarrow{D^0} \Gamma^\infty E_{\lambda,1} \xrightarrow{D^1} \cdots \xrightarrow{D^{m-1}} \Gamma^\infty E_{\lambda,m} \to 0$$

is an elliptic complex. If $(D^q)^* : \Gamma^\infty E_{\lambda,q+1} \to \Gamma^\infty E_{\lambda,q}$ is the formal adjoint of D^q (for suitable metrics on the $E_{\lambda,q}$ induced by K-invariant inner products on the $V^q_{\lambda+2\delta_n(\Sigma^+)}$) and $\square^q = (D^q)^* D^q + D^{q-1}(D^{q-1})^* : \Gamma^\infty E_{\lambda,q} \to \Gamma^\infty E_{\lambda,q}$ is the corresponding Laplacian then $D + D^* : \Gamma^\infty \sum_{(-1)^q=1} E_{\lambda,q} \to \Gamma^\infty \sum_{(-1)^q=-1} E_{\lambda,q}$ is the Dirac operator $D_\lambda^+ : \Gamma^\infty E_\lambda^+ \to \Gamma^\infty E_\lambda^-$ (under the identification (2.5)) *and*

$$\Box^q = -\Omega + (\lambda, \lambda + 2\delta(\Sigma^+))1 \tag{2.6}$$

on $\Gamma^\infty E_{\lambda, q}$.

Now let $\Gamma \subset G$ be a finitely generated discrete torsion-free subgroup. Choosing an invariant measure dx on $\Gamma \backslash G$ induced by Haar measure on G, and letting $(\Gamma^\infty E_{\lambda, q})^\Gamma$ denote the space of Γ-invariant sections in $\Gamma^\infty E_{\lambda, q}$ we have the usual inner product

$$\langle s_1, s_2 \rangle_\Gamma = \int_{\Gamma \backslash G} \langle s_1, s_2 \rangle dx \tag{2.7}$$

on the subspace of compactly supported Γ-invariant sections s_1, s_2 where $\langle\ ,\ \rangle$ is a K-invariant inner product on $V^q_{\lambda + 2\delta_n(\Sigma^+)}$. Let $L^2(E_{\lambda, q})^\Gamma$ be the Hilbert space completion of the latter subspace; i.e. $L^2(E_{\lambda, q})^\Gamma$ is the space of L^2-sections of $\Gamma \backslash E_{\lambda, q}$. The G-invariant operator $\Box^q$ (which is Γ-invariant in particular) descends to a differential operator $\Box^q_\Gamma$ on $\Gamma \backslash E_{\lambda, q}$. The q^{th}-L^2-cohomology of the complex $\{E_{\lambda, q}, D^q\}$ with respect to Γ is defined by

$$H^q_2(\Gamma \backslash E_\lambda) = \{s \in L^2(E_{\lambda, q})^\Gamma \mid \Box^q_\Gamma s = 0 \text{ in the sense of distributions}\}. \tag{2.8}$$

Thus $H^q_2(\Gamma \backslash E_\lambda)$ is the L^2-kernel of $\Box^q_\Gamma$. By our assumptions on Γ the Riemannian metric on $\Gamma \backslash G/K$ is complete and thus, as pointed out in [5], one has

Theorem 2.9. *The q^{th}-L^2-cohomology space $H^q_2(\Gamma \backslash E_\lambda)$ coincides with the space*

$$H^q_2(E_\lambda)^\Gamma \overset{\text{def.}}{=} \{s \in (\Gamma^\infty E_{\lambda, q})^\Gamma \mid D^q s = (D^{q-1})^* s = 0, \qquad \|s\|^2_\Gamma < \infty\}. \tag{2.10}$$

The elliptic operators $\Box^q_\Gamma$ are *locally invariant*; i.e. they admit a G-invariant lift to $E_{\lambda, q}$—via $\Box^q$. Hence if Γ is a lattice in G, i.e. $\Gamma \backslash G$ has a finite G-invariant volume (in particular Γ is then finitely generated so that Theorem 2.9 applies), we can apply a recent theorem of Moscovici (Theorem 2.1 of [8]) to conclude that $\Box^q_\Gamma$ has a finite dimensional L^2-kernel. More precisely we have the following

Theorem 2.11. *Let $\Gamma \subset G$ be a torsion-free lattice. Assume in addition that Γ is subject to the mild technical condition*[1] *of Langlands'* [7] *(also cf.* (2.21) *of* [18]) *so that under the right regular representation of G, $L^2(\Gamma \backslash G)$*

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[1] In the rank 1 case this condition is automatically satisfied, as Warner points out [12]. It is also satisfied if G has no compact simple factors—as was pointed out to the author by Prof. M. Osborne.

(and accordingly $L^2(E_{\lambda,q})^\Gamma$) decomposes into a discrete and continuous spectrum:

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$$
$$L^2(E_{\lambda,q})^\Gamma = L^2_d(E_{\lambda,q})^\Gamma \oplus L^2_c(E_{\lambda,q})^\Gamma. \tag{2.12}$$

Then the q^{th}-L^2-cohomology space $H^q_2(\Gamma\backslash E_\lambda)$ in (2.8) (*which coincides with $H^q_2(E_\lambda)^\Gamma$ in* (2.10) *by Theorem* 2.9) *is finite-dimensional and also*

$$H^q_2(\Gamma\backslash E_\lambda) = \text{the } L^2\text{-kernel of } \square^q_\Gamma \text{ on } L^2_d(E_{\lambda,q})^\Gamma. \tag{2.13}$$

Write $\hat{G}$ for the set of equivalence classes of irreducible unitary representations of G and write $\hat{G}_d$ for the subset of elements of $\hat{G}$ occurring in the discrete spectrum of $L^2(\Gamma\backslash G)$:

$$L^2_d(\Gamma\backslash G) = \sum_{\pi\in\hat{G}_d} m_\pi(\Gamma)\pi \qquad \text{(direct sum)} \tag{2.14}$$

where $m_\pi(\Gamma)$ is the (finite) multiplicity of π. Let H_π be the Hilbert space of $\pi \in \hat{G}$.

Corollary 2.15. *For Γ as in the statement of Theorem* 2.11, $q \geq 0$, *and λ as in Theorem* 2.4

$$\dim H^q_2(\Gamma\backslash E_\lambda) = \sum_{\pi\in\hat{G}_d} m_\pi(\Gamma) \dim \operatorname{Hom}_K(H_\pi, V^q_{\lambda+2\delta_n(\Sigma^+)})$$
$$\pi(\Omega) = (\lambda, \lambda+2\delta(\Sigma^+))1.$$

Proof. Using (2.13) and (2.14), $H^q_2(\Gamma\backslash E_\lambda) = \sum_{\pi\in\hat{G}_d} m_\pi(\Gamma) \ker \pi(\square^q_\Gamma)$, where $\pi(\square^q_\Gamma)$: $(H_\pi \otimes V^q_{\lambda+2\delta(\Sigma^+)})^K \to (H_\pi \otimes V^q_{\lambda+2\delta(\Sigma^+)})^K$ is given by $\pi(\square^q_\Gamma) = -\pi(\Omega) + (\lambda, \lambda+2\delta(\Sigma^+))1$; see (2.6). Of course $\pi(\Omega)$ is a scalar multiple of 1, say $\pi(\Omega) = c_\pi 1$. Hence $\operatorname{Ker} \pi(\square^q_\Gamma)$ is zero unless $c_\pi = (\lambda, \lambda+2\delta(\Sigma^+))1$, in which case it is the full space. That is $H^q_2(\Gamma\backslash E_\lambda) = \sum_{\pi\in\hat{G}_d} m_\pi(\Gamma)(H_\pi \otimes V^q_{\lambda+2\delta(\Sigma^+)})^K$, $\pi(\Omega) = (\lambda, \lambda+2\delta(\Sigma^+))1$. If π^* is the contragradient of π, then $m_\pi(\Gamma) = m_{\pi^*}(\Gamma)$ and $\dim (H_\pi \otimes V^q_{\lambda+2\delta(\Sigma^+)})^K = \dim \operatorname{Hom}_K (H_{\pi^*}, V^q_{\lambda+2\delta(\Sigma^+)})$ so that Corollary 2.15 follows.

The following vanishing theorem improves the vanishing theorem obtained in Section 6 of [5].

Theorem 2.16. *Let $\lambda \in \mathscr{L}$ such that $\lambda + \delta_n(\Sigma^+) \in \mathscr{L}_0(\Sigma^+_k)$. Suppose $(\lambda, \alpha) > 0$ for every α in Σ^+_n. Then $H^q_2(\Gamma\backslash E_\lambda) = 0$ for $q > 0$ (again for Γ satisfying the conditions of Theorem* 2.11). *Moreover* $\dim H^0_2(\Gamma\backslash E_\lambda)$ *equals the multiplicity $m_{\pi_{\lambda+\delta(\Sigma^+)}}(\Gamma)$ (in the discrete spectrum of $L^2(\Gamma\backslash G)$) of Harish-Chandra's discrete series representation $\pi_{\lambda+\delta(\Sigma^+)}$* [2] *corresponding to the* *regular element* $\lambda+\delta(\Sigma^+)$; *cf. remarks preceding* (2.18) *below.*

We base the proof of Theorem 2.16 on Corollary 2.15 and on the following result, which is a special case of a more general result proved in [17]. See Corollary 2.9 and Theorem 2.13 there.

Theorem 2.17. *Let* $\lambda \in \mathscr{L}$ *such that* $(\lambda+\delta(\Sigma^+), \alpha)>0$ *for every* α *in* Σ_k^+ *and* $(\lambda, \beta)>0$ *for every* β *in* Σ_n^+. *Let* $\pi \in \hat{G}$ *such that* $\pi(\Omega)=(\lambda, \lambda+2\delta(\Sigma^+))1$. *Then* $\mathrm{Hom}_K(H_\pi, S^-\otimes V_{\lambda+\delta_n(\Sigma^+)})=0$. *If* $\mathrm{Hom}_K(H_\pi, S^+\otimes V_{\lambda+\delta_n(\Sigma^+)})\neq 0$, π *is unitarily equivalent to* $\pi_{\lambda+\delta(\Sigma^+)}$ (*in which case* $\dim \mathrm{Hom}_K(H_\pi, S^+\otimes V_{\lambda+\delta_n(\Sigma^+)})=1$). *In particular* (*by Schmid's lowest K-type theorem*) $\pi|_K$ *contains no K-type of the form* $V_{\lambda+2\delta_n(\Sigma^+)-\langle Q\rangle}$, *where* $Q\subset\Sigma_n^+$ *is non-empty, and* $\pi|_K$ *contains* $V_{\lambda+2\delta_n(\Sigma^+)}$ *exactly once.*

Proof of Theorem 2.16. Suppose $H_2^q(\Gamma\backslash E_\lambda)\neq 0$. Then by Corollary 2.15 $\mathrm{Hom}_K(H_\pi, V^q_{\lambda+2\delta_n(\Sigma^+)})\neq 0$ for some $\pi\in\hat{G}_d$ satisfying $\pi(\Omega)=(\lambda, \lambda+2\delta(\Sigma^+))1$. Thus there is a K-type V_μ contained in $\pi|_K$ and in $V^q_{\lambda+2\delta_n(\Sigma^+)}$. By (2.5) $V_\mu\subset S^\pm\otimes V_{\lambda+\delta_n(\Sigma^+)}$. Since $\mathrm{Hom}_K(H_\pi, S^-\otimes V_{\lambda+\delta_n(\Sigma^+)})=0$ by Theorem 2.17 we actually have $V_\mu \overset{(\mathrm{i})}{\subset} S^+\otimes V_{\lambda+\delta_n(\Sigma^+)}$; i.e. $\mathrm{Hom}_K(H_\pi, S^+\otimes V_{\lambda+\delta_n(\Sigma^+)})\neq 0$. By Theorem 2.17, again, $\pi=\pi_{\lambda+\delta(\Sigma^+)}$ and μ cannot have the form $\mu=\lambda+2\delta_n(\Sigma^+)-\langle Q\rangle$ for $Q\subset\Sigma_n^+$, $Q\neq\phi$. But by (i), $\mu=a$ weight of $S^+ +\lambda+\delta_n(\Sigma^+)=\delta_n(\Sigma^+)-\langle Q\rangle+\lambda+\delta_n(\Sigma^+)=\lambda+2\delta_n(\Sigma^+)-\langle Q\rangle$, where $Q\subset\Sigma_n^+$, $(-1)^{|Q|}=1$. Moreover $Q\neq\phi$ for $q>0$ since then $\mu\neq\lambda+2\delta_n(\Sigma^+)$. This forces $H_2^q(\Gamma\backslash E_\lambda)=0$ for $q>0$. Our argument, in conjunction with Corollary 2.15, shows that

$$\dim H_2^0(\Gamma\backslash E_\lambda)=m_{\pi_{\lambda+\delta(\Sigma^+)}}(\Gamma)\dim \mathrm{Hom}_K(H_{\pi_{\lambda+\delta(\Sigma^+)}}, V^0_{\lambda+2\delta_n(\Sigma^+)})=m_{\pi_{\lambda+\delta(\Sigma^+)}}(\Gamma),$$

since $V^0_{\lambda+2\delta_n(\Sigma^+)}=V_{\lambda+2\delta_n(\Sigma^+)}$ is contained in $\pi_{\lambda+\delta(\Sigma^+)}|_K$ exactly once. This proves Theorem 2.16.

Remarks. The discrete series representation $\pi_{\lambda+\delta(\Sigma^+)}$ corresponds to the character θ_λ given on the compact Cartan subgroup H by the formula

$$(2.18)\quad \theta_\lambda(\exp x)=\frac{(-1)^m \operatorname{sgn}\prod_{\alpha\in\Sigma^+}(\lambda+\delta(\Sigma^+),\alpha)\sum_{\sigma\in W(K,H)}\det\sigma e^{\sigma(\lambda+\delta(\Sigma^+))(x)}}{\prod_{\alpha\in\Sigma^+}(e^{\alpha(x)/2}-e^{-\alpha(x)/2})}$$

for $x\in h_0$, where $W(K, H)$ is the Weyl group of (K, H) [2]. $\pi_{\lambda+\delta(\Sigma^+)}$ satisfies $\pi_{\lambda+\delta(\Sigma^+)}(\Omega)=(\lambda, \lambda+2\delta(\Sigma^+))1$.

The Dirac operators $D_\lambda^\pm$ which we considered earlier also descend to locally invariant elliptic differential operators ${}_\Gamma D_\lambda^\pm$ on $\Gamma\backslash G/K$ which have finite L^2-kernels, again by Moscovici's theorem [8]. The L^2-index of ${}_\Gamma D_\lambda^+$ is defined by

$$\text{ind}\,({}_\Gamma D_\lambda^+) = \dim L^2\text{-Ker}_\Gamma D_\lambda^+ - \dim L^2\text{-Ker}_\Gamma D_\lambda^-. \tag{2.19}$$

A consequence of Theorem 2.4 is

$$\sum_{q=0}^{m} (-1)^q \dim H_2^q(\Gamma\backslash E_\lambda) = \text{ind}\,({}_\Gamma D_\lambda^+) \tag{2.20}$$

since $\square^q$ and $D^q + (D^{q-1})^*$ have the same L^2-kernel. The vanishing Theorem 2.16, therefore gives

Corollary 2.21. *In Theorem* 2.16 *we also have* $\dim H_2^0(\Gamma\backslash E_\lambda) = \text{ind}({}_\Gamma D_\lambda^+)$.

Remark. In Theorem 2.7 of [19] we have proved that, in particular, for λ satisfying Theorem 2.16

$$m_{\pi_{\lambda+\delta(\Sigma^+)}}(\Gamma) \equiv \text{ind}\,({}_\Gamma D_\lambda^+). \tag{2.22}$$

Thus Corollary 2.21 also follows by (2.22).

The λ's which we shall consider in later applications will satisfy, in addition, the so-called $\sharp$ condition:

$$\begin{aligned} &(\lambda + \delta_n(\Sigma^+) + \delta(\Sigma^+) - \langle Q\rangle, \alpha) \geq 0 \\ &\text{for every } \alpha \text{ in } \Sigma_k^+ \text{ and } Q \subset \Sigma_n^+. \end{aligned} \tag{2.23}$$

Under the $\sharp$ condition the first differential operator $D^0\colon \Gamma^\infty E_{\lambda,0} \to \Gamma^\infty E_{\lambda,1}$ in Theorem 2.4 coincides with Schmid's differential operator $\mathscr{D} = \mathscr{D}(\Sigma^+)$, constructed using the positive system Σ^+; see [11]; also cf. Section 2 of [18]. If, moreover, Σ^+ satisfies an "admissibility" condition the above elliptic complex coincides with the (cohomologically constructed) complex of [4]. In summary, (with some slight changes in notation), Theorems 2.9, 2.16 and Corollary 2.21 yield the following

Theorem 2.24. *Let $\Gamma \subset G$ be a torsion-free lattice as in Theorem* 2.11. *Let $\lambda \in \mathscr{L}$ such that $\lambda + \delta_n(\Sigma^+) \in \mathscr{L}_0(\Sigma_k^+)$* (*see* (2.3), *and such that λ satisfies the $\sharp$ condition* (2.23). *Let $E_\lambda \to G/K$ be the homogeneous vector bundle over G/K induced by the irreducible K module $V_{\lambda+2\delta_n(\Sigma^+)}$ with Σ_k^+-highest weight $\lambda + 2\delta_n(\Sigma^+)$. Let $H_2^0(E_\lambda)^\Gamma = \{s \in \Gamma^\infty E_\lambda | s$ is Γ-invariant, $\mathscr{D}(\Sigma^+)s = 0$, and $\|s\|^2 < \infty\}$; cf.* (2.7). *Then $H_2^0(E_\lambda)^\Gamma$ coincides with the* (*finite-dimensional*) *L^2-kernel of $\square_\Gamma$ on the L^2-sections of $\Gamma\backslash E_\lambda$, where $\square_\Gamma$ is the descent of $\square = \mathscr{D}(\Sigma^+)^*\mathscr{D}(\Sigma^+)$ to $\Gamma\backslash E_\lambda$. Suppose moreover that $(\lambda, \alpha) > 0$ for every*

α in Σ_n^+. Then dim $H_2^0(E_\lambda)^\Gamma = m_{\pi_{\lambda+\delta(\Sigma^+)}}(\Gamma)$ (*cf.* (2.18)) $=$ *the L^2-index of the Dirac operator ${}_\Gamma D_\lambda^+$: $\Gamma^\infty(\Gamma\backslash E_\lambda^+) \to \Gamma^\infty(\Gamma\backslash E_\lambda^-)$, where the bundles $E_\lambda^\pm \to G/K$ are induced by the K modules $S^\pm \otimes V_{\lambda+\delta_n(\Sigma^+)}$.*

§ 3. A general dimension formula

Before restricting attention to the rank 1 case altogether we express the dimension of automorphic cohomology, more generally, as a discrete series multiplicity in the discrete spectrum of $L^2(\Gamma\backslash G)$ or, equivalently, as the L^2-index of a twisted Dirac operator.

We retain the notation of Sections 1, 2 and denote by $\underset{\sim}{p}$, v_0 the Lie algebras of P, V. Choose a system of positive roots $\Delta^+ \subset \Delta$ such that

(3.1) $$\underset{\sim}{p} \supset \text{ the Borel subalgebra } \underset{\sim}{b} = h + \sum_{\alpha \in \Delta^+} g_{-\alpha}.$$

We can arrange the inclusions $H \subset V \subset K$ and write

(3.2) $\underset{\sim}{p} = v \oplus \underset{\sim}{n}$ where v (the reductive part of $\underset{\sim}{p}$) is the complexification of v_0, $v = h + \sum_{\alpha \in \Delta_v} g_\alpha$, $\underset{\sim}{n} = \sum_{\alpha \in \Delta^+ - \Delta_v} g_{-\alpha} =$ the unipotent radical of $\underset{\sim}{p}$,

and Δ_v is the set of roots of (v, h); $\Delta_v \subset \Delta_k$. With Δ^+ fixed we shall always write $2\delta = \langle \Delta^+ \rangle$, $2\delta_k = \langle \Delta_k^+ \rangle$, $2\delta_n = \langle \Delta_n^+ \rangle$, $\Delta_v^+ = \Delta^+ \cap \Delta_v$. Next let $E_{\pi_\lambda} \to D$ be a homogeneous (necessarily holomorphic) vector bundle over D induced by an irreducible representation π_λ of V with Δ_v^+-highest weight λ. We always assume that E_{π_λ} is *non-degenerate*; i.e. λ satisfies

(3.3) $$(\lambda + \delta_k + \langle Q \rangle, \alpha) > 0 \qquad \text{for all } \alpha \in \Delta_v^+$$
$$\text{and}$$
$$(\lambda + \delta_k + \langle Q \rangle, \alpha) < 0 \qquad \text{for all } \alpha \in \Delta_k^+ - \Delta_v^+$$
$$\text{for arbitrary } Q \subset \Delta_n^+.$$

Let W_k, W_v be the Weyl groups of (k, h), (v, h) respectively; W_k coincides with $W(K, H)$ in the notation of (2.18). Let $w \subset W_k$ be the unique element such that

(3.4) $$(w(\lambda + \delta_k), \alpha) < 0 \qquad \text{for every } \alpha \in \Delta_k^+$$

and define $\kappa \in W_k$, $\nu \in h^*$ (the dual space) by

(3.5) $$\kappa \Delta_k^+ = -\Delta_k^+, \qquad \nu = w(\lambda + \delta_k) + \delta_k.$$

Then by Corollary 2.14 of [18] one has (3.6) and (3.7) below:

(3.6) $$w \in W_v,\ w\Delta_v^+ = -\Delta_v^+,\quad w\Delta_n^+ = \Delta_n^+,\quad w(\Delta_k^+ - \Delta_v^+) = \Delta_k^+ - \Delta_v^+,$$
$$\text{and } (\nu + \langle Q\rangle, \alpha) \leq 0 \quad \text{for } \alpha \in \Delta_k^+,\quad Q \subset \Delta_n^+.$$

Moreover if s is the dimension of the maximal compact complex subvariety $Y = K/V$ of D, and $E_\nu \to G/K$ is the homogeneous vector bundle over G/K induced by the irreducible K module with Δ_k^+-*lowest* weight ν (by the Borel-Weil theorem the latter module can be taken to be $H^S(Y, \mathcal{O}E_{\pi_\lambda})$), then the Γ-automorphic cohomology $H^S(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ of D is given, up to isomorphism [15], by

(3.7) $H^S(D, \mathcal{O}E_{\pi_\lambda})^\Gamma =$ the Γ-invariant C^∞ sections s of E_ν such that $\mathscr{D}s = 0$

where $\mathscr{D}$ is Schmid's differential operator constructed relative to the choice of positive system $\Sigma^+ \overset{\text{(ii)}}{=} -\kappa\Delta^+$, and $\Gamma \subset G$ is a torsion-free discrete subgroup. Again if we choose an invariant measure on $\Gamma\backslash G$ induced by Haar measure on G and Hermitian metrics along the fibers of E_ν induced by a K-invariant unitary structure on the inducing module $H^S(Y, \mathcal{O}E_{\pi_\lambda})$ then we have an inner product $\langle\, ,\rangle_\Gamma$, given as in (2.7), on the compactly supported Γ-invariant C^∞ sections of E_ν. We define the L^2-Γ-automorphic cohomology $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ by

(3.8) $$H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma = \{s \in H^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma \mid \|s\|^2 < \infty\};\ \text{see (3.7)}.$$

Define $\lambda_1 = \kappa\nu - 2\delta_n(\Sigma^+) = \kappa\nu + \kappa 2\delta_n$ (by (ii)) $\in \mathscr{L}$. Then $\lambda_1 + \delta_n(\Sigma^+) = \kappa(\nu + \delta_n) \overset{\text{(iii)}}{=} \kappa(\nu + 2\delta_n) - \kappa\delta_n \Rightarrow (\lambda_1 + \delta_n(\Sigma^+), \alpha) \geq 0$ for $\alpha \in \Delta_k^+$ by (3.6) (where we take $Q = \Delta_n^+$). That is, noting that $\Sigma_k^+ = \Delta_k^+$ of course, we can write the Δ_k^+-*highest* weight $\kappa\nu$ (cf. (3.5)) as $\kappa\nu = \lambda_1 + 2\delta_n(\Sigma^+)$, where $\lambda_1 \in \mathscr{L}$ such that $\lambda_1 + \delta_n(\Sigma^+) \in \mathscr{L}_0(\Sigma_k^+)$. Moreover for $\alpha \in \Delta_k^+$ and $Q_1 \subset \Sigma_n^+ = -\kappa\Delta_n^+$ (again by (ii)), using $\kappa\delta_k = -\delta_k$ and $Q_1 = -\kappa Q$, $Q \subset \Delta_n^+$, we see that $(\lambda_1 + \delta_n(\Sigma^+) + \delta(\Sigma^+) - \langle Q_1\rangle, \alpha) = (\kappa\nu + \delta_k + \kappa\langle Q\rangle, \alpha) = (\nu + \langle Q\rangle - \delta_k, \kappa\alpha) > 0$ by (3.6). In other words λ_1 also satisfies the $\sharp$ condition (2.23) and hence Theorem 2.24 is applicable. By (3.8) $H_2^s(D, \mathcal{O}E_{\pi_\lambda}) \equiv H_2^0(E_{\lambda_1})^\Gamma$, in the notation of Theorem 2.24. The condition $(\lambda_1, \alpha) \overset{\text{(iv)}}{>} 0$ for α in Σ_n^+ translates to the condition $(\kappa\nu + \kappa 2\delta_n, -\kappa\alpha) > 0$ for α in Δ_n^+; i.e. $(\nu + 2\delta_n, \Delta_n^+) < 0$; i.e. (by (3.5)) $(w(\lambda + \delta_k) + \delta_k + 2\delta_n, \Delta_n^+) < 0$. Also $\lambda_1 + \delta(\Sigma^+) = \kappa(\nu + \delta_n) + \delta_k = \kappa[w(\lambda + \delta_k) + \delta_k + \delta_n] + \delta_k = \kappa[w(\lambda + \delta_k) + \delta_n]$ (again since $\kappa\delta_k = -\delta_k$) $= \kappa w(\lambda + \delta)$, since $\delta_n = w\delta_n$ by (3.6). That is, $\lambda_1 + \delta(\Sigma^+)$ and $\lambda + \delta$ lie in the same W_k orbit, which implies that the corresponding discrete series representations $\pi_{\lambda_1 + \delta(\Sigma^+)}$ and $\pi_{\lambda+\delta}$ are unitarily equivalent. Therefore by Theorem 2.24 (for Δ^+ chosen as in (3.1)) we have

Theorem 3.9. *Let $E_{\pi_\lambda} \to D$ be a non-degenerate homogeneous vector* *bundle over D induced by an irreducible representation π_λ of V with Δ_v^+-highest weight λ (cf. (3.3)). Suppose $(w(\lambda+\delta_k)+\delta_k+2\delta_n, \alpha)<0$ for every α in Δ_n^+, where w is the Weyl group element given by (3.4) (also cf. (3.6)). Then if $\Gamma\subset G$ is a torsion-free lattice satisfying Langlands' condition, as in Theorem 2.11, (the latter condition is satisfied if for example Γ is arithmetic or if the real rank of G is 1) the L^2-Γ-automorphic cohomology $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ (see (3.8)). has dimension equal to the multiplicity $m_{\pi_{\lambda+\delta}}(\Gamma)$ of the discrete series representation $\pi_{\lambda+\delta}$ in the discrete spectrum of $L^2(\Gamma\backslash G)$ (cf. (2.12), (2.14)). The character θ_λ of $\pi_{\lambda+\delta}$ is given in (2.18); there replace Σ^+ by Δ^+. Moreover the multiplicity $m_{\pi_{\lambda+\delta}}(\Gamma)$ coincides with the L^2-index (cf. (2.19)) of the Dirac operator ${}_\Gamma D_\lambda^+\lambda$: $\Gamma^\infty(\Gamma\backslash E_\lambda^+)\to\Gamma^\infty(\Gamma\backslash E_\lambda^-)$, where the bundles $E_\lambda^\pm\to G/K$ are induced by the K-modules $S^\pm\otimes V_{\lambda_1+\delta_n(\Sigma^+)}$, with $\lambda_1=\kappa\nu-2\delta_n(\Sigma^+)=\kappa(\nu+\delta_n)$, for κ, ν given by (3.5)[2], $\Sigma^+=-\kappa\Delta^+=\Delta_k^+\cup-\kappa\Delta_n^+$, and where the weights of the $\frac{1}{2}$-spin modules $S^\pm$ are $\{\delta_n(\Sigma^+)-\langle Q_1\rangle|(-1)^{|Q_1|}=\pm1, Q_1\subset \Delta_n^+\}=\{-\kappa(\delta_n-Q)|(-1)^{|Q|}=\pm1, Q\subset\Delta_n^+\}$; $\lambda_1+\delta(\Sigma^+)=\kappa w(\lambda+\delta)$.*

§ 4. Dimension formula for rank 1 groups

Theorem 3.9 reduces the dimension computation to an L^2-index computation for the Dirac operator. For rank one groups the latter (nontrivial) computation has been carried out by Barbasch and Moscovici in [1], using the Selberg trace formula developed in [12]. We therefore assume henceforth that G is simple and the real rank of G is 1. The symmetric space G/K then has strictly negative sectional curvature and coincides with one of the four hyperbolic spaces $SO_e(2n, 1)/SO(2n)$, $SU(n, 1)/U(n)$, $Sp(n, 1)/(Sp(n)\times Sp(1))$, of $F_4/\mathrm{Spin}(9)$. If $\Gamma\subset G$ is a lattice we shall assume as in [1], the following condition (which implies in particular that Γ is torsion-free): the group generated by the eigenvalues of any $\gamma\in\Gamma$ contains no roots of unity. Γ is then called *neat*. The Iwasawa decomposition $G=KAN$ gives rise to a standard normalization of Haar measure on G: $\int_G f(x)dx=\int_K\int_A\int_N f(kan)e^{2\rho(\log a)}\,dkdadn$ for $f\in C_c(G)$ where the Lie algebra of A is a maximal abelian (1-dimensional) subspace of p_0, the Lie algebra of N is a sum of positive restricted root spaces, and 2ρ is the sum (with multiplicity) of those corresponding positive restricted roots. Then if $\lambda'\in\mathscr{L}$ is a regular element the formal degree $d_{\lambda'}$ of the corresponding discrete series representation $\pi_{\lambda'}$ takes the form

$$d_{\lambda'}=c(G)\Big|\prod_{\alpha\in\Delta^+}(\lambda', \alpha)\Big| \tag{4.1}$$

where

[2] We have already observed that $\lambda_1+\delta_n(\Sigma^+)\in\mathscr{L}_0(\Delta_k^+)$.

$$\text{(4.2)} \qquad \frac{1}{c(G)}=(2\pi)^m 2^{m-1/2} \prod_{\alpha\in\Delta_k^+}(\delta_k, \alpha).$$

If $G\neq \mathrm{SU}(1, 1)$ and $\mathrm{SU}(2n, 1)$ the formula for the L^2-index of the Dirac operator is surprisingly simple. Namely, using the notation of Theorem 3.9, one has

$$\text{(4.3)} \qquad \operatorname{ind}({}_\Gamma D_\lambda^+)=c(G)\prod_{\alpha\in\Sigma^+=-\kappa\Delta^+}(\lambda_1+\delta(\Sigma^+), \alpha) \operatorname{vol}(\Gamma\backslash G)$$

by the key result Theorem 7.1(a) of [1], where we are using the fact that $\lambda_1+\delta(\Sigma^+)$ is *regular*. In fact $(\lambda_1+\delta(\Sigma^+), \alpha)\overset{(\mathrm{v})}{>}0$ for every α in Σ^+, under the hypotheses of Theorem 3.9 (compare the inequality (iv) following (3.8)). In applying results of [1] we take $\mu=\lambda_1+\delta_n(\Sigma^+)$, $\psi_c=\Delta_k^+=\Sigma_k^+$, $\rho_c=\delta_k$, $\psi=\Sigma^+$, to match the notation there. Of course $\operatorname{vol}(\Gamma\backslash G)$ means the G-invariant volume of $\Gamma\backslash G$. Since $\lambda_1+\delta(\Sigma^+)=\kappa w(\lambda+\delta)$ (by Theorem 3.9) and since $\Sigma_n^+=-\kappa\Delta_n^+$ the inequality (v) along with $w\Delta_n^+=\Delta_n^+$ in (3.6) forces the inequality

$$\text{(4.4)} \qquad (\lambda+\delta, \alpha)<0 \text{ for every } \alpha \text{ n } \Delta_n^+.$$

Writing $\Delta^+=\Delta_n^+\cup\Delta_k^+=\Delta_n^+\cup\Delta_v^+\cup\Delta_k^+-\Delta_v^+$ and using (3.6) we obtain

$$\text{(4.5)} \qquad \prod_{\alpha\in\Sigma^+}(\lambda_1+\delta(\Sigma^+), \alpha)$$
$$=(-1)^{m+s}\prod_{\alpha\in\Delta_n^+}(\lambda+\delta, \alpha)\prod_{\alpha\in\Delta_v^+}(\lambda+\delta, \alpha)\prod_{\alpha\in\Delta_k^+-\Delta_v^+}(\lambda+\delta, \alpha)$$

since $m=|\Delta_n^+|$ and $s=|\Delta_k^+-\Delta_v^+|$. That is (3.3) and (4.4) imply

$$\text{(4.6)} \qquad \prod_{\alpha\in\Sigma^+}(\lambda_1+\delta(\Sigma^+), \alpha)=\Big|\prod_{\alpha\in\Delta^+}(\lambda+\delta, \alpha)\Big|.$$

This combined with (4.3) and Theorem 3.9 yields the following main result.

Theorem 4.7. *Let $E_{\pi_\lambda}\to D$ be a non-degenerate homogeneous vector bundle over a flag domain $D=G/V$ where π_λ is an irreducible representation (which induces E_{π_λ}) of V with Δ_v^+-highest weight λ and G is one of the rank* 1 *simple groups* $\mathrm{SO}_e(2n, 1)$ *($n\geq 2$),* $\mathrm{SU}(n, 1)$ *(n odd, $n\neq 1$),* $\mathrm{Sp}(n, 1)$ *(n arbitrary), or F_4. Suppose λ satisfies*

$$\text{(4.8)} \qquad (w(\lambda+\delta_k)+\delta_k+2\delta_n, \alpha)<0 \qquad \textit{for every } \alpha \textit{ in } \Delta_n^+$$

where w is the Weyl group element given by (3.4) *(also cf.* (3.6)*). Then if Γ is a neat lattice in G the dimension of the L^2-Γ-automorphic cohomology $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ in* (3.8) *is given by*

$$(4.9)\qquad \dim H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma = c(G)\left|\prod_{\alpha\in\Delta^+}(\lambda+\delta, \alpha)\right|\operatorname{vol}(\Gamma\backslash G).$$

$c(G)$ is specified in (4.2) (*for the above normalization of Haar measure on G*) *and Δ^+ is chosen according to* (3.1). *In particular $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma \neq 0$; i.e. there exists non-zero square integrable automorphic cohomology classes on D.*

Remark. The coefficient $c(G)|\prod_{\alpha\in\Delta^+}(\lambda+\delta, \alpha)|$ of the volume of $\Gamma\backslash G$ in formula (4.9) is the formal degree of the discrete series representation $\pi_{\lambda+\delta}$ of G corresponding to the regular element $\lambda+\delta$; see (4.1). In the Hermitian case when $G=\mathrm{SU}(n, 1)$, s is *zero* and $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ therefore is a space of square integrable automorphic forms on $D=\mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n)\times \mathrm{U}(1))$.

An important example of a flag domain, apart from the classical bounded Hermitian symmetric domains or the Cartan domains G/H, is the period matrix domain $D=\mathrm{SO}_e(2n, 1)/\mathrm{U}(n)$ (or more generally the domain $D_{n,r}=\mathrm{SO}_e(2n, r)/(\mathrm{U}(n)\times\mathrm{SO}(r))$). Here P is a maximal parabolic subgroup of G^{C}, K/V is the compact irreducible Hermitian symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$ and Δ^+ in (3.1) can be chosen so that $\Delta_n^+=\{\alpha_j\}_{j=1}^n$, $\Delta_k^+=\{\alpha_j\pm\alpha_i \,|\, j>i\}$, $\Delta_v^+=\{\alpha_j-\alpha_i \,|\, j>i\}$, with $(\alpha_i, \alpha_j)\overset{(\mathrm{vi})}{=}\dfrac{\delta_{ij}}{2(2n-1)}$; cf. Section 3 of [18]. For an irreducible representation π_λ of $V=\mathrm{U}(n)$ with Δ_v^+-highest weight λ one has $\lambda=\sum_{j=1}^n m_j\alpha_j$, with $m_1\le m_2\le\cdots\le m_n$ and $2m_i$, $m_j\pm m_k \in \mathbf{Z}$ for $1\le i\le n, j>k$.

Proposition 4.10. *The induced bundle $E_{\pi_\lambda}\to D$ is non-degenerate if and only if* (i) $m_1<m_2<\cdots<m_n$ *and* (ii) $m_n+n+m_{n-1}+n-1<0$.

Proof. If E_{π_λ} is non-degenerate apply (3.3) firstly with $Q=\{\alpha_i\}\subset\Delta_n^+$ and $\alpha_{i+1}-\alpha_i \in \Delta_v^+$. Noting that $\lambda+\delta_k=\sum_{l=1}^n(m_l+l-1)\alpha_l$ we get $0<(\lambda+\delta_k+\alpha_i, \alpha_{i+1}-\alpha_i)=(m_{i+1}-m_i)/2(2n-1)$ by (vi); i.e. $m_{i+1}>m_i$ for $1\le i\le n-1\Rightarrow$(i). Secondly, taking $Q=\{\alpha_{n-1}, \alpha_n\}\subset\Delta_n^+$ and $\alpha_n+\alpha_{n-1}\in\Delta_k^+-\Delta_v^+$ in (3.3) we obtain (ii) by a similar argument. Conversely, assume (i), (ii) and let $Q\subset\Delta_n^+$ be arbitrary. Then $(\langle Q\rangle, \alpha_j-\alpha_i)=\sum_{\alpha\in Q}(\alpha, \alpha_j)-\sum_{\alpha\in Q}(\alpha, \alpha_i)\ge -1/2(2n-1)$ and $(\langle Q\rangle, \alpha_j+\alpha_i)\le 2/2(2n-1)$ by (vi). Hence for $j>i$, i.e. $\alpha_j-\alpha_i \in \Delta_v^+$, $\alpha_j+\alpha_i\in\Delta_k^+-\Delta_v^+$, $(\lambda+\delta_k+\langle Q\rangle, \alpha_j-\alpha_i)=(\lambda+\delta_k, \alpha_j-\alpha_i)+(\langle Q\rangle, \alpha_j-\alpha_i)=(m_j-m_i+(j-i))/2(2n-1)+(\langle Q\rangle, \alpha_j-\alpha_i)$ (again by (vi)) $\ge(m_j-m_i+j-i-1)/2(2n-1)>0$ by (i). Similarly $(\lambda+\delta_k+\langle Q\rangle, \alpha_j+\alpha_i)=(m_j+m_i+j+i-2)/2(2n-1)+(\langle Q\rangle, \alpha_j-\alpha_i)\le(m_j+m_i+j+i)/2(2n-1)<0$ by (ii) since $m_j+m_i+j+i\le m_n+m_{n-1}+n+n-1$. Thus (3.3) follows. Q.E.D.

By (3.10) of [18], $w(\lambda+\delta_k)+\delta_k+2\delta_n=\sum_{l=1}^n(m_{n-l+1}+n)\alpha_l$ for w in

(3.6). From (vi) it follows that (4.8) holds$\Leftrightarrow m_{n-j+1}+n<0$ for $1\leq j\leq n \Leftrightarrow m_n+n<0$. The last inequality implies (ii) of Proposition 4.10. Also from the above data

$$\prod_{\alpha\in\Delta+}(\alpha_k, \alpha)=\left[\frac{1}{2(2n-1)}\right]^{n(n-1)}\prod_{1\leq i<j\leq n}(j+i-2)(j-i),$$

$$(4.11)\qquad \prod_{\alpha\in\Delta+}(\lambda+\delta, \alpha)=\left[\frac{1}{2(2n-1)}\right]^{n^2}\prod_{j=1}^{n}\left(m_j+j-\frac{1}{2}\right)$$

$$\cdot\prod_{1\leq i<j\leq n}(m_j+m_i+j+i-1)(m_j-m_i+j-i).$$

$c(G)$ is now determined in (4.2) since $m=n$. Therefore by Theorem 4.7 and Proposition 4.10 we can state

Theorem 4.12. *Suppose $m_1<m_2<\cdots<m_n$ and $m_n+n<0$ for $\lambda=\sum_{j=1}^{n} m_j\alpha_j$ as above, where D is the period matrix domain $\mathrm{SO}_e(2n, 1)/\mathrm{U}(n)$, $(n\geq 2)$. Then if $\Gamma\subset \mathrm{SO}_e(2n, 1)$ is a neat lattice, the dimension of the L^2-Γ-automorphic cohomology $H_2^s(D, \mathcal{O}E_{\pi_\lambda})^\Gamma$ in (3.8) is non-zero and is given by*

$$(4.13)\qquad \begin{aligned} &c(n)\prod_{j=1}^{n}\left|m_j+j-\frac{1}{2}\right| \\ &\cdot\prod_{1\leq i<j\leq n}\frac{|m_j+m_i+j+i-1|(m_j-m_i+j-i)}{(j+i-2)(j-i)}\operatorname{vol}(\Gamma\backslash G)\end{aligned}$$

where $1/c(n)=(2\pi)^n 2^{n-1/2}(2(2n-1))^n$, $s=n(n-1)/2$, and where Haar measure on $G=\mathrm{SO}_e(2n, 1)$ is normalized as earlier.

For $G=\mathrm{SU}(n, 1)$ with n even, $n\neq 2$, there is an additional term which contributes to the dimension formula in (4.9); there are two such terms when $n=2$ (and when $G=\mathrm{SU}(1, 1)$). The extra terms (or term) account for the presence of cusps, as one would expect. For example assume $G=\mathrm{SU}(2n, 1)$; this case is not covered by Theorem 4.7. Then by Proposition 4.6 of [1] there is a unipotent contribution to the Selberg trace formula of the form $\pm C_2(\Gamma)c_n \dim V_\mu$, where as above we take $\mu=\lambda_1+\delta_n(\Sigma^+)$, where c_n is a positive constant depending on G and $C_2(\Gamma)$ is a positive constant depending on the number of cusps of Γ, (c_n and $C_2(\Gamma)$ are explicitly known) and were the sign $\pm$ is determined as follows. Let $z_0=\sqrt{-1}$ diagonal $(I_{2n-1}, -2n)\in h_0$. Here h_0 consists of the diagonal matrices in $g=sl(2n+1, C)$ with pure imaginary entries and I_{2n-1} is the $(2n-1)^2$ identity matrix. We choose $\Delta^+=\{\alpha_{ij}|1\leq i<j\leq 2n+1\}$, using the usual notation; $\Delta_n^+=\{\alpha_{i\,2n+1}|1\leq i\leq 2n\}$. Choosing the G-invariant complex structure on

G/K which is compatible with Δ^+ we have $\kappa\Delta_n^+=\Delta_n^+$, since $\kappa\in W_k$, and hence $\Sigma_n^+=-\Delta_n^+$. The sign $\pm$ is given by $\pm 1=sgn\sum_{\alpha\in\Sigma_n^+}\alpha(z_0)=$ $\mathrm{sgn}\,(\sqrt{-1}(1+2n))^{2n}=(-1)^n$. We remark that the factor $(2n+1)^n$ which appears on page 38 of [1] should be corrected to read $(2n+1)^{2n}$. By Theorem 7.1(a) of [1] and Theorem 3.9 above, for Γ=a neat lattice in $G=\mathrm{SU}(2n, 1)$ and $n\neq 1$

$$\dim H_2^0(D, \mathcal{O}E_{\pi_\lambda})^\Gamma=c(G)\Big|\prod_{\alpha\in\Delta^+}(\lambda+\delta, \alpha)\Big|\operatorname{vol}(\Gamma\backslash G)+\text{the unipotent contribution} \tag{4.14}$$

where the unipotent contribution $=(-1)^n C_2(\Gamma)c_n \dim V_{\lambda_1+\delta_n(\Sigma^+)}$. Here $\dim V_{\lambda_1+\delta_n(\Sigma^+)}=\prod_{\alpha\in\Delta_k^+}(\lambda+\delta, \alpha)/\prod_{\alpha\in\Delta_k^+}(\delta_k, \alpha)$ as we note that with $V=K$ $D=\mathrm{SU}(2n, 1)/\mathrm{S}(\mathrm{U}(2n)\times\mathrm{U}(1))$, $\Delta_v^+=\Delta_k^+$, and $\lambda_1+\delta(\Sigma^+)=\lambda+\delta$ since now w in (3.6) coincides with κ in (3.5); i.e. $\kappa w=\kappa^2=1$. The conditions on the Δ_k^+-highest weight λ in (4.14) are the non-degenerate condition (3.3) as usual and the condition (4.8). These, in the present case, simplify as follows:

$$\begin{aligned}&(\lambda+\delta_k+\langle Q\rangle, \alpha)>0 \quad \text{for every } \alpha \text{ in } \Delta_k^+ \text{ and } Q\subset\Delta_n^+,\\ &\text{and } (\lambda+2\delta_n, \alpha)<0 \quad \text{for every } \alpha \text{ in } \Delta_n^+.\end{aligned} \tag{4.15}$$

In the two remaining cases $G=\mathrm{SU}(2, 1)$, $\mathrm{SU}(1, 1)$ a third term, in addition to the unipotent contribution, contributes to the right hand side of (4.14). This term (the *weighted* unipotent contribution to the Selberg trace formula) has the form (see Theorem 7.1 (a) of [1]) $\pm\frac{1}{2}(-1)^{1+m}c(\Gamma)$ $\sum_{\sigma\in W_k}(\det\sigma)\,\mathrm{sgn}\,(K(\lambda+\delta))$ where $K(\lambda+\delta)$ is an integer uniquely determined by the regular element $\lambda+\delta$ ($\lambda+\delta$ also uniquely determines the sign $\pm$) and $c(\Gamma)$ is the exact number of cusps of Γ. The dimension formula is now obtained (applying Theorem 3.9 again) for all the rank 1 groups.

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