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Foliations
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On the Conjugation of Local Diffeomorphisms Infinitely Tangent to the Identity

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§ 0. Introduction

Let G be the group of germs at $0 \in \boldsymbol{R}$ of smooth orientation preserving diffeomorphisms of $\boldsymbol{R}$. When we study transversely orientable codimension one foliation, the group G plays an important role. In fact, the isomorphism problems of certain foliations are deeply related to the conjugacy problems of elements of G.

Let G_∞ be the normal subgroup of G which consists of the elements infinitely tangent to the identity at 0.

Concerning the conjugacy problem, we have the following well-known result due to Sternberg [7] and Takens [8]:

If f and g are elements of $G-G_\infty$ with $f^{-1}\circ g \in G_\infty$, then f is conjugate to g by an element of G_∞.

Then, the problem which is left to us is:

For two elements f and g of G_∞, when is f conjugate to g in G_∞ (or in G)?

Now, consider the submonoid G_∞^c of G_∞ consisting of the germ of the identity of $\boldsymbol{R}$ and all the elements f of G_∞ such that $f(x)=x$ for $x\leq 0$ and $f(x)<x$ for $x>0$.

The main purpose of this paper is to give a sufficient condition under which two elements of G_∞^c are conjugate.

Our main result is the following.

Theorem 2.4. *Let f be an element of G_∞^c with $\alpha(f)\neq 1$. ($\alpha(f)$ is a non-negative number ($\in [0, 1]$) defined in Section 1.) Let g be an element of G_∞^c satisfying the following $(*)_s$ for $s>(2-\alpha(f))/(1-\alpha(f))^2$;*

$$(*)_s \quad |f(x)-g(x)|\leq C\{x-f(x)\}^s$$

for any $x(\in \boldsymbol{R})$ near 0. Here, the constant C depends on f, g and s. Then, there exists a diffeomorphism h of $\boldsymbol{R}$ such that

(i) $g=h^{-1}\circ f\circ h$ *(in a neighbourhood of 0),*

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(ii) *h is of class* C^∞ *on* $(0, +\infty)$ *such that* $h|_{(-\infty,0]}=\mathrm{I}|_{(-\infty,0]}$, *and*

(iii) $h(0)=0$, $D^1h(0)=1$ *and* $D^rh(0)=0$ *for* $1<r<(1-\alpha(f))^2\cdot s-(2-\alpha(f))$.

This paper is organized as follows.

We begin Section 1 with defining the number $\alpha(f)$ for an element $f \in G^c_\infty$. We see that the number α is invariant under conjugations by elements of G_∞(or G). We also show the existence of an element $f \in G^c_\infty$ with $\alpha(f)=\alpha$ for any $\alpha \in [0, 1]$. This implies that there are uncountably many conjugacy classes in G^c_∞ (Corollary 1.6). Since these f's are not conjugate even by elements of G, we see that there are uncountably many Reeb foliations which are not C^∞ isomorphic to each other (Theorem 1.7).

In Section 2, we study the properties of elements $g \in G^c_\infty$ sufficiently close to an element $f \in G^c_\infty$. We prove our main result Theorem 2.4, which says that an element $g \in G^c_\infty$ "sufficiently close" to $f \in G^c_\infty$ is C^r-conjugate to f.

In Section 3, as an application of Theorem 2.4, we give an alternative proof of the perfectness of G_∞ which is originally due to Sergeraert [6]. We show that, for any element $f \in G_\infty$, there exists $g \in G^c_\infty$ such that $g\circ f \in G^c_\infty$ (Proposition 3.2). We can in fact construct an element $g \in G^c_\infty$ with $\alpha(g)=0$ so that g and $g\circ f$ satisfy the condition $(*)_s$ of Theorem 2.4 for any S. This implies that f is written as a commutator.

In Section 4, using Proposition 3.2, we show that the natural inclusion

$$j\colon G^c_\infty \times \bar{G}^c_\infty \longrightarrow G_\infty$$

induces isomorphisms on their homology groups. Here,

$$\bar{G}^c_\infty=\{(-\mathrm{I})\circ f\circ(-\mathrm{I});\, f \in G^c_\infty\}.$$

We introduce some notations.

Let $f(a, b, c)$ and $g(a, b, c)$ be real valued functions on $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$. Following Sergeraert [5], an inequality

$$f(a, b, c) \underset{(b,c)}{\leq} g(a, b, c)$$

means that, for any b and c, there exists a constant $C_{b,c}$ such that

$$f(a, b, c) \leq C_{b,c}\cdot g(a, b, c)$$

for any a.

Let f be a function on an open subset of $\boldsymbol{R}$. We denote by $D^rf(x)$ the r-th derivative of f at x. By $|f|^A_r$, we mean $\sup_{\substack{0\leq s\leq r\\ x\in A}}|D^sf(x)|$, where A is a subset of $\boldsymbol{R}$. When $A=\boldsymbol{R}$, we simply write $|f|_r$.

We write I for the identity of $\boldsymbol{R}$, and $f \circ g$ for the composition of f and g.

§ 1. The invariant α

Let $\mathrm{Diff}_+^\infty(\boldsymbol{R}, 0)$ be the group of the orientation preserving diffeomorphisms $\tilde{f}$ of $\boldsymbol{R}$ with $\tilde{f}(0)=0$. We consider the subgroup D_∞ of $\mathrm{Diff}_+^\infty(\boldsymbol{R}, 0)$ consisting of $\tilde{f}$ which satisfies

(i) $\tilde{f}(0)=0$, $D^1\tilde{f}(0)=1$ and $D^r\tilde{f}(0)=0$ for any $r \geq 2$, and

(ii) $\tilde{f}(x)=x+b$ for sufficiently large $x \gg 0$, where b is a constant.

Let D_∞^c be the submonoid of D_∞ consisting of the identity and the elements $\tilde{f}$ satisfying $\tilde{f}(x)=x$ for $x \leq 0$ and $\tilde{f}(x)<x$ for $x>0$. Then, we have the following exact sequences of groups and monoids:

$$\begin{tikzcd} D_\infty \arrow[r, "\pi"] & G_\infty \arrow[r] & 1 \\ D_\infty^c \arrow[u, hook] \arrow[r, "\pi"] & G_\infty^c \arrow[u, hook] \arrow[r] & 1, \end{tikzcd}$$

where $\pi(\tilde{f})=$ the germ of $\tilde{f}$ at 0.

We define a number $\alpha(\tilde{f}) \in [0, 1]$ for an element $\tilde{f} \in D_\infty^c$.

Definition 1.1.

$$\alpha(\tilde{f})=\inf \{\alpha \in [0, 1];\ \Delta_0^{\tilde{f}}(x) \underset{(\tilde{f})}{\leqq} \{\Delta^{\tilde{f}}(x)\}^{1-\alpha} \text{ for } x \in \boldsymbol{R}\},$$

where $\Delta^{\tilde{f}}(x)=x-\tilde{f}(x)$ and $\Delta_0^{\tilde{f}}(x)=\sup_{0 \leqq y \leqq x} \Delta^{\tilde{f}}(y)$. We note that the number $\alpha(\tilde{f})$ depends only on the germ of $\tilde{f}$ at 0. Hence, for an element $f \in G_\infty^c$, we can define the number $\alpha(f)$ to be $\alpha(\tilde{f})$ for some $\tilde{f} \in D_\infty^c$ with $\pi(\tilde{f})=f$.

We also introduce a mapping

$$\alpha_*\colon D_\infty^c \longrightarrow [0, 1] \times \{\min, \inf\}.$$

The mapping α_* is defined by $\alpha_*(\tilde{f})=(\alpha(\tilde{f}), \min)$, if $\alpha_*(\tilde{f})$ attains the minimum value. Otherwise, we define $\alpha_*(\tilde{f})=(\alpha(\tilde{f}), \inf)$. We can also consider α_* as a mapping from G_∞^c to $[0, 1] \times \{\min, \inf\}$ in a natural way, and we have the following commutative diagram:

$$\begin{tikzcd} D_\infty^c \arrow[r, "\alpha_*"] \arrow[d, "\pi"'] & {[0, 1] \times \{\min, \inf\}} \\ G_\infty^c \arrow[ur, "\alpha_*"'] & \end{tikzcd}$$

The following theorem motivates the definition of α.

Theorem 1.2. (Sergeraert [6]) *For an element $f \in D^c_\infty$, there exists a unique C^1 vector field $\xi=\xi(x)d/dx$, with $\xi(x)=0$ for $x\leq 0$ and of class C^∞ on $(0, \infty)$, such that f is the time one map of ξ. Moreover, if $\alpha(f)<1/r$ $(r\geq 2)$, $\xi(x)$ is of class C^r at 0.*

First, we show that α takes any value of [0, 1].
Let φ be a C^∞ function of $\boldsymbol{R}$ such that

$$\begin{cases} 0\leq\varphi(x)\leq 1 & \text{for } x\in\boldsymbol{R}, \\ \varphi(x)=1 & \text{for } x\leqq\frac{1}{6}, \\ \varphi(x)=0 & \text{for } x\geqq\frac{5}{6}, \\ -2\leq D^1\varphi(x)\leq 0 & \text{for } x\in\boldsymbol{R}. \end{cases}$$

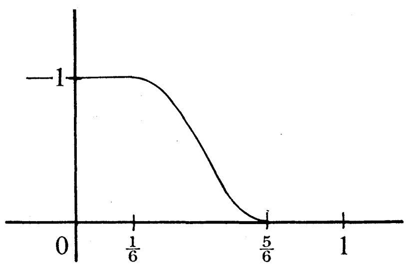

Fig. 1. The graph of φ.

Put

$$A_{\alpha,\min}=\alpha_*^{-1}(\alpha, \min)\subset D^c_\infty \quad \text{and} \quad A_{\alpha,\inf}=\alpha_*^{-1}(\alpha, \inf)\subset D^c_\infty.$$

Concerning the sets $A_{\alpha,\min}$ and $A_{\alpha,\inf}$, we have the following proposition.

Proposition 1.3. *The sets $A_{\alpha,\inf}$ ($\alpha\in[0, 1)$) and $A_{\alpha,\min}$ ($\alpha\in[0, 1]$) are not empty. (Note that $A_{1,\inf}$ is empty by definition.)*

Proof. First, we show that $A_{\alpha,\inf}$ $(0\leqq\alpha<1)$ is non-empty. We define a function h on (0, 1] as follows. We fix the following numbers: for $n=0, 1, 2, \cdots$,

$$\begin{cases} x_n=3^{-n} \\ y_n=2\cdot 3^{-n-1} \\ \beta_n=(1-\alpha)\cdot\left(\dfrac{n+1}{n+2}\right) \\ a_n=\frac{1}{6}\cdot\exp(-3^n) \\ b_n=(a_n)^{1/\beta^n}. \end{cases}$$

Note that $0<\beta_n<1$ and that $b_n<a_n$. On $[x_{n+1}, x_n]=[3^{-n-1}, 3^{-n}]$, we define h by

$$h(x)=\begin{cases} b_{n+1}+(a_n-b_{n+1})\cdot\varphi\left(\dfrac{y_n-x}{y_n-x_{n+1}}\right) & \text{for } x\in[x_{n+1}, y_n] \\ b_n+(a_n-b_n)\cdot\varphi\left(\dfrac{x-y_n}{x_n-y_n}\right) & \text{for } x\in[y_n, x_n]. \end{cases}$$

It is easy to see that $h(x)$ is a C^∞ function on (0, 1]. We can calculate the r-th derivative ($r\geqq 1$), and we have

$$\begin{aligned} \sup_{x\in[x_{n+1},x_n]}|D^r h(x)| &\leqq \sup\left\{\frac{a_n-b_n}{(x_n-y_n)^r}, \frac{a_n-b_{n+1}}{(y_n-x_{n+1})^r}\right\}\cdot\sup_x|D^r\varphi(x)| \\ &\leqq \tfrac{1}{6}\{\exp(-3^n)\}\cdot 3^{(n+1)r}\cdot\sup_x|D^r\varphi(x)|. \end{aligned}$$

Hence, $D^r h(x)\to 0$ as $x\to+0$ and h is of class C^∞ and flat at 0. Put

$$f(x)=\begin{cases} x & \text{for } x\leqq 0, \\ x-h(x) & \text{for } 0<x\leqq 1, \\ x-(\frac{1}{6}\cdot e^{-1})^{2/1-\alpha} & \text{for } 1\leqq x. \end{cases}$$

Since

$$\begin{aligned} \sup_{x\in[x_{n+1},x_n]}|D^1 h(x)| &\leqq \tfrac{1}{6}\{\exp(-3^n)\}\cdot 3^{n+1}\cdot\sup_x|D^1\varphi(x)| \\ &\leqq 3^n\cdot\exp(-3^n) \\ &\leqq e^{-1}, \end{aligned}$$

f is an element of D^c_∞.

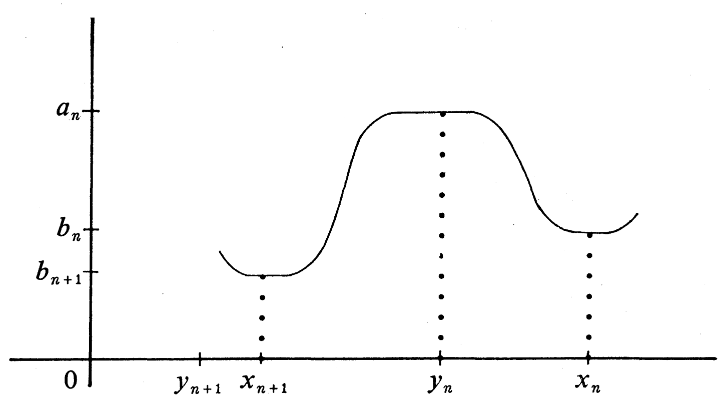

Fig. 2. The graph of h.

We can calculate $\alpha(f)$ as follows. For $x \in [y_{n+1}, y_n]$, we have

$$\max_{0 \leq y \leq x} \Delta^f(x) = \max \{h(x), h(y_{n+1})\},$$

where

$$h(y_{n+1}) = a_{n+1} = (b_{n+1})^{\beta_{n+1}} \leqq \{h(x)\}^{\beta_{n+1}}.$$

Hence, we have $\Delta_0^f(x) \leqq \{\Delta^f(x)\}^{\beta_{n+1}}$ on $[y_{n+1}, y_n]$. By the choice of β_n, we have $\alpha(f) \leqq \alpha$. Since $\Delta_0^f(x_n) = \{\Delta^f(x_n)\}^{\beta_n}$, $1-\beta_n > \alpha$ and $\lim_{n \to +\infty} (1-\beta_n) = \alpha$, f is an element of $A_{\alpha, \inf}$.

For $A_{\alpha,\min}$ $(\alpha \neq 1)$, we can construct a diffeomorphism belonging to $A_{\alpha,\min}$ similarly by taking $\beta = 1-\alpha$ in place of β_n.

When $\alpha = 1$, we replace β_n by $1/n$, that is, we replace b_n by $(a_n)^n$; then, the diffeomorphism f belongs to $A_{1,\min}$.

This completes the proof of Proposition 1.3.

Remark. The diffeomorphism given in [6], Theorem 4.1, is also an element of $A_{1,\min}$.

We give other properties of α_*.

Lemma 1.4. *Let f and g be elements of D_∞^c. If they satisfy the inequalities*

$$\Delta^f(x) \underset{(f,g)}{\leq} \Delta^g(x) \underset{(f,g)}{\leq} \Delta^f(x)$$

for any $x \in \boldsymbol{R}$, then $\alpha_*(f) = \alpha_*(g)$.

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} \Delta_0^g(x) &= \sup_{0 \leq y \leq x} \Delta^g(y) \\ &\underset{(f,g)}{\leq} \sup_{0 \leq y \leq x} \Delta^f(y) \\ &\underset{(f)}{\leq} \{\Delta^f(x)\}^{1-\alpha(f)-\varepsilon} \\ &\underset{(f,g)}{\leq} \{\Delta^g(x)\}^{1-\alpha(f)-\varepsilon}. \end{aligned}$$

This shows $\alpha(g) \leq \alpha(f)$. By the above calculation, it is obvious that $\alpha(f)$ attains the minimum value if and only if $\alpha(g)$ does so. This completes the proof.

Lemma 1.5. *For $f \in D_\infty^c$ and $h \in D_\infty$, we have $\alpha_*(f) = \alpha_*(h^{-1} \circ f \circ h)$.*

Proof. For any $\varepsilon > 0$, we have

$$\begin{aligned} \Delta_0^{h^{-1} \circ f \circ h}(x) &\underset{(h)}{\leq} \Delta_0^f(h(x)) \\ &\underset{(f)}{\leq} \{\Delta^f(h(x))\}^{1-\alpha(f)-\varepsilon} \\ &\underset{(f,h)}{\leq} \{\Delta^{h^{-1} \circ f \circ h}(x)\}^{1-\alpha(f)-\varepsilon}. \end{aligned}$$

This implies $\alpha(f)=\alpha(h^{-1}\circ f\circ h)$. Moreover, it is also obvious that $\alpha_*(f)=\alpha_*(h^{-1}\circ f\circ h)$ as in Lemma 1.4.

Combining Lemma 1.3 with Lemma 1.5, we obtain the following Corollary 1.6.

Corollary 1.6. *The set of conjugacy classes of G^c_∞ has the cardinal number of the continuum.*

Remark. The formula of Lemma 1.5 holds if h is the germ of a C^1 diffeomorphism which fixes the origin.

Corollary 1.6 and Remark imply the following theorem.

Theorem 1.7. *There are uncountably many C^∞ isomorphism classes of Reeb foliations.*

Here, by a Reeb foliation $\mathscr{F}_R$, we mean a transversely oriented smooth foliation of S^3 whose leaves are diffeomorphic to $\boldsymbol{R}^2$ except a compact leaf diffeomorphic to a torus T^2.

For the proof, it is sufficient to recall the following well-known fact concerning the holonomy of the torus T^2.

The holonomy of the compact leaf T^2 is a group homomorphism

$$\mathscr{H}^{\mathscr{F}_R}_{T^2}\colon \pi_1(T^2)\longrightarrow G_\infty.$$

Moreover, we can find the generators $a, b\in\pi_1(T^2)$ such that $\mathscr{H}^{\mathscr{F}_R}_{T^2}(a)\in G^c_\infty$ and $\mathscr{H}^{\mathscr{F}_R}_{T^2}(b)\in\bar{G}^c_\infty$. It is well-known that two Reeb foliations $\mathscr{F}_{R_1}$ and $\mathscr{F}_{R_2}$ are isomorphic if and only if $\mathscr{H}^{\mathscr{F}_{R_1}}_{T^2}(a)$ and $\mathscr{H}^{\mathscr{F}_{R_1}}_{T^2}(b)$ are simultaneously conjugate to $\mathscr{H}^{\mathscr{F}_{R_2}}_{T^2}(a)$ and $\mathscr{H}^{\mathscr{F}_{R_2}}_{T^2}(b)$ in G. On the other hand, for any element (f, g) of $G^c_\infty\times\bar{G}^c_\infty$, it is easy to construct a Reeb foliation $\mathscr{F}_R$ such that $\mathscr{H}^{\mathscr{F}_R}_{T^2}(a)=f$ and $\mathscr{H}^{\mathscr{F}_R}_{T^2}(b)=g$. These facts insure the theorem.

Lemma 1.8. *Let f and g be elements of D^c_∞ such that $f\circ g=g\circ f$. Then, $\alpha_*(f)=\alpha_*(g)$.*

Proof. By Kopell [4] and Sergeraert (Theorem 1.2), there are positive real numbers s, t and a C^1 vector field $\xi=\xi(x)d/dx$ with $\xi(x)<0$ for $x>0$ and $\xi(x)=0$ for $x\leq 0$ such that the diffeomorphisms f and g are the time s map and the time t map of ξ, respectively. We can assume $s<t$. Take a positive integer K such that $t<K\cdot s$. Then, we have

$$\Delta^f(x)\leq\Delta^g(x)\leq\Delta^{f^K}(x)\underset{(f,K)}{\leq}\Delta^f(x).$$

Hence, Lemma 1.8 follows from Lemma 1.4.

Remark. From Lemma 1.5 arises the following problem:

Does the invariant α_* determine the conjugacy class completely?

This is not the case for diffeomorphisms f with $\alpha(f)=1$. Sergeraert gives a diffeomorphism f with $\alpha(f)=1$ which does not have a root ([6], Theorem 4.1). Since $\alpha(f)=1$, we have $\alpha(f\circ f)=1$ by Lemma 1.8. It is obvious, however, that f is not conjugate to $f\circ f$.

Lemma 1.9. *For any two elements $f, g \in D^c_\infty$, we have the following inequality*:

$$\alpha(f\circ g)\leq \operatorname{Max}\{\alpha(f), \alpha(g)\}.$$

Proof. We can assume that $0\leq\alpha(g)\leq\alpha(f)\leq 1$. Put $\alpha=\alpha(f)$ and $\beta=\alpha(g)$. Then, for any $\varepsilon>0$, we have

$$\begin{aligned}\Delta_0^{f\circ g}(x) &\leq \Delta_0^f(x)+\Delta_0^g(x)\\ &\underset{(f,g)}{\leq} \{\Delta^f(x)\}^{1-\alpha-\varepsilon}+\{\Delta^g(x)\}^{1-\beta-\varepsilon}\\ &\leq \{\Delta^{f\circ g}(x)\}^{1-\alpha-\varepsilon}+\{\Delta^{f\circ g}(x)\}^{1-\beta-\varepsilon}\\ &\leq 2\{\Delta^{f\circ g}(x)\}^{1-\alpha-\varepsilon}.\end{aligned}$$

The third inequality holds because $\Delta^f(x)\leq\Delta^{f\circ g}(x)$ and $\Delta^g(x)\leq\Delta^{f\circ g}(x)$. This shows the desired inequality.

Remark. For any $\alpha \in [0, 1]$,

$$\{f \in D^c_\infty : \alpha(f)<\alpha\} \quad\text{and}\quad \{f\in D^c_\infty : \alpha(f)\leq\alpha\}$$

are submonoids of D^c_∞, invariant under the conjugation by elements of G. In particular, $\{f\in D^c_\infty : \alpha(f)=0\}$ is a submonoid. Note that, by Theorem 1.2, any $f\in D^c_\infty$ with $\alpha(f)=0$ is the time one map of a C^∞ vector field on $\boldsymbol{R}$.

§ 2. On the C^r-conjugation in G^c_∞

In this section, we prove our main result, Theorem 2.4, which gives a sufficient condition for C^r-conjugation in G^c_∞. The following proposition is due to Sergeraert [6] which is useful for us.

Proposition 2.1. *Let f be an element of D^c_∞. By Theorem* 1.2, *there is a unique C^1 vector field $\xi=\xi(x)d/dx$ of which f is the time one map.*

Then, for any $x>0$, we have the followings.

(i) $D^1 f^n(x)=\dfrac{\xi(x_n)}{\xi(x)}$, *where* $x_n=f^n(x)$.

(ii) $\lim_{x \to +0} \frac{\xi(x)}{\Delta^f(x)} = -1.$

(iii) $|D^r f^n(x)| \underset{(f,r)}{\leqq} D^1 f^n(x) \cdot \frac{x^{r-1}}{\{\xi(x)\}^{r-1}} = \frac{\xi(x_n)}{\{\xi(x)\}^r} \cdot x^{r-1},$

for $r \geqq 1$.

(iv) $|D^r f^{-n}(x)| \underset{(f,r)}{\leqq} \frac{1}{\{D^1 f^n(x_{-n})\}^r} \left\{ \frac{x_{-n}}{\xi(x_{-n})} \right\}^{r-1} = \frac{\xi(x_{-n})}{\{\xi(x)\}^r} \cdot (x_{-n})^{r-1},$

for $r \geqq 1$.

Remark. We use the above inequalities later in the following forms: For $f \in G^c_\infty$, choose $\tilde{f} \in D^c_\infty$ representing f appropriately. Then, we have, for any $x \in \boldsymbol{R}$ and any $r \geq 1$,

(iii)′ $|D^r \tilde{f}^n(x)| \underset{(\tilde{f},r)}{\leq} \frac{\Delta^{\tilde{f}}(x_n)}{\{\Delta^{\tilde{f}}(x)\}^r} x^{r-1}$

(iv)′ $|D^r \tilde{f}^{-n}(x)| \underset{(\tilde{f},r)}{\leq} \frac{\Delta^{\tilde{f}}(x_{-n})}{\{\Delta^{\tilde{f}}(x)\}^r} (x_{-n})^{r-1}.$

The following propositions will be used frequently later.

Proposition 2.2. *Let f and g be elements of D^c_∞. If there exists $\varepsilon > 0$ such that*

$$|f(x) - g(x)| \underset{(f,g,\varepsilon)}{\leq} \{\Delta^f(x)\}^{1+\varepsilon}$$

for any $x \in \boldsymbol{R}$, then we have:

(1) $\Delta^f(x) \underset{(f,g)}{\leq} \Delta^g(x) \underset{(f,g)}{\leq} \Delta^f(x).$

(2) *$g(x) \geq f \circ f(x)$ and $f(x) \geq g \circ g(x)$ for any x sufficiently close to* 0.

Proof. The inequalities (1) follows directly from the assumption. Put $y(x) = g(x) - f \circ f(x)$. Then, we have

$$\begin{aligned} y(x) &= f(x) - f \circ f(x) + g(x) - f(x) \\ &\geq f'(\theta) \cdot \Delta^f(x) - C\{\Delta^f(x)\}^{1+\varepsilon}, \end{aligned}$$

where $f(x) \leq \theta \leq x$ and C is a positive constant. This shows that $g(x) \geq f \circ f(x)$ for x sufficiently close to 0. On the other hand, from (1), we have

$$|f(x) - g(x)| \underset{(f,g,\varepsilon)}{\leq} \{\Delta^g(x)\}^{1+\varepsilon}.$$

By changing f and g in the above formula, we also have $f(x) \geq g \circ g(x)$ for x sufficiently close to 0. This completes the proof.

Proposition 2.3. *Let $f \in D_\infty^c$ with $\alpha(f) \neq 1$ and $g \in D_\infty^c$. Suppose that:*

$$|f(x)-g(x)| \underset{(f,g,s)}{\leq} \{\Delta^f(x)\}^s$$

for $s>(2-\alpha(f))/(1-\alpha(f))$. Then, $\{g^{-k}\circ f^k(1)\}_{0\leq k<+\infty}$ is bounded.

Proof. By the mean value theorem, we have

$$|g^{-k}\circ f^k(1)-g^{-k-1}\circ f^{k+1}(1)|=|D^1(g^{-k-1})(\theta)\cdot(g-f)\circ f^k(1)|,$$

where $f^{k+2}(1)\leq\theta\leq f^k(1)$ for sufficiently large k (Proposition 2.2). By Proposition 2.1, we have

$$D^1(g^{-k-1})(\theta)\underset{(g)}{\leq}\frac{\Delta^g(g^{-k-1}(\theta))}{\Delta^g(\theta)},$$

where

$$\begin{aligned}\Delta^g(g^{-k-1}(\theta)) &= g^{-k-1}(\theta)-g^{-k}(\theta)\\ &\leq g^{-k-1}(\theta)\\ &\leq g^{-k-1}\circ f^k(1)\\ &\underset{(g)}{\leq} g^{-k}\circ f^k(1)\end{aligned}$$

and, by Proposition 2.2 (1), $\Delta^g(\theta)\underset{(f,g)}{\geq}\Delta^f(\theta)$. Put $\alpha(f)=\alpha$. Since $\alpha<1$, for sufficiently small $\varepsilon>0$ with $1-\alpha-\varepsilon>0$, we have

$$\begin{aligned}\Delta^f(\theta) &\underset{(f)}{\geq} \{\Delta_0^f(\theta)\}^{1/(1-\alpha-\varepsilon)}\\ &\geq \{\Delta_0^f(f^{k+2}(1))\}^{1/(1-\alpha-\varepsilon)}\\ &\underset{(f,\alpha)}{\geq} \{\Delta_0^f(f^k(1))\}^{1/(1-\alpha-\varepsilon)}\\ &\geq \{\Delta^f(f^k(1))\}^{1/(1-\alpha-\varepsilon)}.\end{aligned}$$

Hence, $D^1(g^{-k-1})(\theta)\underset{(f,g)}{\leq} g^{-k}\circ f^k(1)\cdot\{\Delta^f(f^k(1))\}^{-1/(1-\alpha-\varepsilon)}$ $(\varepsilon>0)$. From this inequality and the assumption of the proposition, we have

$$|g^{-k}\circ f^k(1)-g^{-k-1}\circ f^{k+1}(1)|\leq A\{g^{-k}\circ f^k(1)\}\cdot\{\Delta^f(f^k(1))\}^{s-(1-\alpha-\varepsilon)^{-1}},$$

where $A>0$ is a constant determined by f and g. Therefore, we have

$$\begin{aligned}g^{-k-1}\circ f^{k+1}(1) &= \prod_{j=0}^{k}\frac{g^{-j-1}\circ f^{j+1}(1)}{g^{-j}\circ f^j(1)}\\ &\leq \prod_{j=0}^{\infty}(1+A\{\Delta^f(f^j(1))\}^{s-(1-\alpha-\varepsilon)^{-1}})\end{aligned}$$

$$\leqq \exp\left(A\left(\sum_{j=0}^{\infty}\{\Delta^f(f^j(1))\}^{s-(1-\alpha-\varepsilon)^{-1}}\right)\right)$$
$$\leqq \exp A,$$

where we used $s-(1-\alpha-\varepsilon)^{-1}>1$. q.e.d.

Now, we state the main theorem.

Theorem 2.4. *Let f be an element of G^c_∞ with $\alpha(f)\neq 1$. Let g be an element of G^c_∞ satisfying the following $(*)_s$ for $s>(2-\alpha(f))/(1-\alpha(f))^2$;*

$$(*)_s \quad |f(x)-g(x)| \underset{(f,g,s)}{\leqq} \{x-f(x)\}^s$$

for any $x(\in \boldsymbol{R})$ near 0.

Then, there exists a diffeomorphism h of $\boldsymbol{R}$ such that

(i) $g=h^{-1}\circ f\circ h$ (in a neighbourhood of 0),

(ii) h is of class C^∞ on $(0,+\infty)$ such that $h|_{(-\infty,0]}=\mathrm{I}|_{(-\infty,0]}$, and

(iii) $h(0)=0$, $D^1h(0)=1$ and $D^rh(0)=0$ for $1<r<(1-\alpha(f))^2\cdot s-(2-\alpha(f))$.

Hence, if g satisfies $(*)_s$ for any s, then, h is of class C^∞ at 0. (Note that, by Proposition 2.2, $\alpha_*(f)=\alpha_*(g)$.)

To prove Theorem 2.4, we take elements $\tilde{f}$ and $\tilde{g}$ of D^c_∞ such that $\pi(\tilde{f})=f$ and $\pi(\tilde{g})=g$ appropriately. It is sufficient to show the existence of a diffeomorphism h of $\boldsymbol{R}$ which satisfies (i), (ii) and (iii) with respect to $\tilde{f}$ and $\tilde{g}$.

The proof is divided into three steps.

In the first step, we construct an approximating sequence $\{g_k\}$ which converges to $\tilde{g}$.

In the second step, we construct a sequence $\{h_k\}$ such that $g_k\circ h_k=h_k\circ\tilde{f}$.

Finally, in the third step, we prove that the sequence $\{h_k\}$ converges to a diffeomorphism h, which satisfies (i), (ii) and (iii).

To simplify the notations, hereafter, we write f and g instead of $\tilde{f}$ and $\tilde{g}$, respectively, and we put $\alpha=\alpha(f)$.

Step 1. (The approximating sequence $\{g_k\}$.)

Let $\varphi(x)$ be a C^∞ function defined in Section 1. Define C^∞ functions φ_k $(k=1, 2, \cdots)$ on $\boldsymbol{R}$ by

$$\varphi_k(x)=\varphi\left(\frac{x-a_{k+1}}{a_k-a_{k+1}}\right),$$

where $a_k=f^k(1)$. We define C^∞ functions $\{g_k\}_{k=1,2,\ldots}$ on $\boldsymbol{R}$ by

$$g_k(x)=\varphi_k(x)\cdot f(x)+\{1-\varphi_k(x)\}\cdot g(x).$$

The following lemma shows that g_k is a diffeomorphism of $\boldsymbol{R}$ for sufficiently large k and that g_k converges to g.

Lemma 2.5.

(i) $g_k(x)=\begin{cases} f(x) & \text{for } x\leq a_{k+1} \\ g(x) & \text{for } x\geq a_k, \end{cases}$

(ii) *there are constants* $0<m<M$ *such that, for any sufficiently large* k, $m\leq D^1 g_k(x)\leq M$ *for any* $x\in\boldsymbol{R}$.

Proof. The assertion (i) is obvious by the definition.
For (ii), we have positive constants C_0 and C_1 such that

$$\begin{aligned} D^1 g_k(x) &\geq \min\{D^1 f(x), D^1 g(x)\}-\frac{C_0}{a_k-a_{k+1}}\cdot|f-g|_0^{[a_{k+1},a_k]} \\ &\geq \min\{D^1 f(x), D^1 g(x)\}-\frac{C_1}{a_k-a_{k+1}}\cdot\{|f-\mathrm{I}|_0^{[a_{k+1},a_k]}\}^s. \end{aligned}$$

Moreover,

$$|f-\mathrm{I}|_0^{[a_{k+1},a_k]}\leq \Delta_0^f(a_k)\underset{(f)}{\leq}\{\Delta^f(a_k)\}^{(1-\alpha-\varepsilon)},$$

($\varepsilon>0$). Since $(1-\alpha-\varepsilon)s-1>0$ for sufficiently small $\varepsilon>0$, we have $D^1 g_k(x)\geq m>0$ for sufficiently large k and any $x\in\boldsymbol{R}$. In a similar way, we can prove $D^1 g_k(x)\leq M$ for sufficiently large k. This completes the proof of the lemma.

By replacing f by g_{k_0} with sufficiently large k_0 if necessary, we may assume that the inequality of Lemma 2.5 (ii) holds for any $k\geq 1$.
The following lemma is an estimate on the norms of $g_{k+1}-g_k$.

Lemma 2.6. *For the diffeomorphisms* $\{g_k\}$, *we have*;

(i) $|g_{k+1}-g_k|_0 \underset{(f,g)}{\leq} (a_k-a_{k+1})^{s(1-\alpha-\varepsilon)}$

for any $\varepsilon>0$, *and*

(ii) $|g_{k+1}-g_k|_r \underset{(f,g,r)}{\leq} (a_k-a_{k+1})^{-r}$ *for* $r\geq 1$.

Proof. Note that $g_{k+1}-g_k=(\varphi_{k+1}-\varphi_k)\cdot(f-g)$.
Then,

$$\begin{aligned} |g_{k+1}-g_k|_0 &= |(\varphi_{k+1}-\varphi_k)\cdot(f-g)|_0 \\ &\leq |f-g|_0^{[a_{k+2},a_k]} \end{aligned}$$

$$
\begin{aligned}
&\underset{(f,g,s)}{\leqq} \{|f-\mathrm{I}|_0^{[a_{k+2},a_k]}\}^s \\
&\leqq \{\Delta_0^f(a_k)\}^s \\
&\underset{(f,g,s)}{\leqq} \{\Delta^f(a_k)\}^{s(1-\alpha-\varepsilon)} \quad (\varepsilon>0).
\end{aligned}
$$

On the other hand, for $r \geq 1$,

$$|D^r \varphi_k|_0 \underset{(\varphi,r)}{\leq} (a_k - a_{k+1})^{-r}.$$

Then, we have

$$
\begin{aligned}
|g_{k+1}-g_k|_r &= |(\varphi_{k+1}-\varphi_k)\cdot(f-g)|_r \\
&\underset{(f,g,r)}{\leq} \frac{1}{(a_k-a_{k+1})^r}. \qquad \text{q.e.d.}
\end{aligned}
$$

Step 2. (The sequence $\{h_k\}$.)

Put

$$h_k(x) = g_k^{-n} \circ f^n(x) \quad (k=1, 2, \cdots),$$

where n is an integer such that $f^n(x) \leq a_{k+1}$. Since $g_k = f$ on $[0, a_{k+1}]$ (Lemma 2.5 (i)), h_k is well-defined.

By definition, we see that

(i) $\operatorname{supp} h_k = cl\{x \in \boldsymbol{R} \mid h_k(x) \neq x\}$ is contained in $[a_{k+1}, +\infty)$,

(ii) $g_k \circ h_k = h_k \circ f$.

Now, we have the following lemma which is useful for calculating norms of $h_{k+1} - h_k$. By changing f by g_{k_0} with sufficiently large k_0 again if necessary, we may assume that the inequality of Proposition 2.2 (2) holds for $x \in [0, 1]$.

Lemma 2.7. *The set* $\{g_k^{-n} \circ f^n(x) \mid k \geq 1, n \geq 0$ *and* $x \in [0, 1]\}$ *is bounded.*

Proof. It is sufficient to prove that $\{g_k^{-n} \circ f^n(1) \mid k \geq 1, n \geq 0\}$ is bounded. We note that, for $0 \leq n \leq k$, $g_k^{-n} \circ f^n(1) = g^{-n} \circ f^n(1)$, which is bounded by Proposition 2.3.

For $k+1 \leq n$, we have, by Proposition 2.2,

$$
\begin{aligned}
g_k^{-n} \circ f^n(1) &= g_k^{-(k+1)} \circ f^{k+1}(1) \\
&\leq g^{-k+1} \circ g_k^{-2} \circ f^{k+1}(1) \\
&\leq g^{-k+1} \circ f^{k-3}(1) \\
&\leq g^{-2}(g^{-k+3} \circ f^{k-3}(1)),
\end{aligned}
$$

which is bounded by Proposition 2.3. This completes the proof.

Now, we have the following estimates on the sequence $\{h_k\}$.

Lemma 2.8. *For sufficiently small* $\varepsilon>0$, *we have*:

(i) $|h_k-h_{k+1}|_0^{[0,1]} \underset{(f,g,s)}{\leqq} (a_k-a_{k+1})^{(1-\alpha-\varepsilon)s-(1-\alpha-\varepsilon)^{-1}}$,

(ii) $|h_k-h_{k+1}|_r^{[0,1]} \underset{(f,g,r)}{\leqq} (a_k-a_{k+1})^{-r/(1-\alpha-\varepsilon)}$,

for $r\geqq 1$.

Proof. Note that $h_k(x)=h_{k+1}(x)=x$ for $x\leqq a_{k+2}$. For $a_{k+2}\leq x$ and the first n such that $a_{k+3}\leq f^n(x)\leq a_{k+2}$, we have

$$\begin{aligned}|h_k(x)-h_{k+1}(x)| &= |(g_k^{-n}-g_{k+1}^{-n})\circ f^n(x)|\\ &= |(g^{-n+6}\circ g_k^{-6}-g^{-n+6}\circ g_{k+1}^{-6})\circ f^n(x)|\\ &\leq D^1(g^{-n+6})(\theta_x)\cdot|(g_k^{-6}\circ g_{k+1}^{6}-\mathrm{I})\circ g_{k+1}^{-6}\circ f^n(x)|,\end{aligned}$$

where $\theta_x\in[\min\{g_k^{-6}\circ f^n(x), g_{k+1}^{-6}\circ f^n(x)\}, \max\{g_k^{-6}\circ f^n(x), g_{k+1}^{-6}\circ f^n(x)\}]$. By Proposition 2.1 and Lemma 2.7, we have

$$\begin{aligned}D^1(g^{-n+6})(\theta_x) &\underset{(g)}{\leq} \frac{\Delta^g(g^{-n+6}(\theta_x))}{\Delta^g(\theta_x)}\\ &\underset{(f,g)}{\leq} \{\Delta^f(a_k)\}^{-1/(1-\alpha-\varepsilon)} \quad (\varepsilon>0),\end{aligned}$$

where the second inequality follows by an argument similar to that in the proof of Proposition 2.3. On the other hand, we have

$$\begin{aligned}|g_{k+1}^{-6}\circ g_k^{6}-\mathrm{I}|_0 &\underset{(f,g)}{\leq} |g_{k+1}-g_k|_0\\ &= |(\varphi_{k+1}-\varphi_k)\cdot(f-g)|_0\\ &\underset{(f,g,s)}{\leq} \{\Delta^f(a_k)\}^{(1-\alpha-\varepsilon)s}, \quad (\varepsilon>0).\end{aligned}$$

These two inequalities imply the assertion (i) of the lemma. As to (ii), we have, for $r\geq 1$ and $x\in[0,1]$,

$$|D^r h_k(x)| \underset{(r)}{\leq} \sum_{\substack{\boldsymbol{r}=(r_1,\cdots,r_j)\\ 1\leq j\leq r}} |(D^j g_k^{-n})\circ f^n(x)|\cdot|D^{r_1}f^n(x)|\cdots|D^{r_j}f^n(x)|,$$

where the sum is taken over all partitions $\boldsymbol{r}=(r_1,\cdots,r_j)$ with $r_1+\cdots+r_j=r$ and $r_i\geq 1$ $(1\leq i\leq j)$. By Proposition 2.1 and Lemma 2.7, we have

$$\begin{aligned}|D^r h_k(x)| &\underset{(f,g,r)}{\leq} \sum_{1\leq j\leq r}\frac{\{g_k^{-n}\circ f^n(x)\}^j\cdot\{\Delta^f(f^n(x))\}^j}{\{\Delta^{g_k}(f^n(x))\}^j\cdot\{\Delta^f(x)\}^r}\\ &\underset{(f,g,r)}{\leq} \frac{1}{\{\Delta^f(x)\}^r},\end{aligned}$$

because $\Delta^{g_k}(f^n(x)) \underset{(f,g)}{\geq} \Delta^f(f^n(x))$ (Proposition 2.2 (1)).

Moreover, for $x \geq a_{k+2}$, we have, in a similar way to that in the proof of Proposition 2.3,

$$\Delta^f(x) \underset{(f)}{\geq} \{\Delta^f(a_k)\}^{1/(1-\alpha-\varepsilon)}.$$

This completes the proof of the lemma.

Lemma 2.9. *For $0<c<1$ and sufficiently small $\varepsilon>0$, we have*

$$|h_{k+1}-h_k|_r^{[c,1]} \underset{(f,g,r,c)}{\leq} 1,$$

for $r \geq 1$.

Proof. In Lemma 2.8, we have the inequality for $0 \leqq j \leqq r$,

$$|D^j(h_k-h_{k+1})(x)| \underset{(f,g,r)}{\leq} \frac{1}{\{\Delta^f(x)\}^r},$$

where the right hand side is bounded on $[c, 1]$.

Step 3. (The convergence of $\{h_k\}$.)

Now, we complete the proof of Theorem 2.4.

To estimate the C^r norm of h_k, we deform h_k to a function with compact support.

Choose a sufficiently small $\varepsilon_0>0$, and define a C^∞ function β by

$$\beta(x)=\varphi\left(\frac{x-(1-\varepsilon_0)}{\varepsilon_0}\right),$$

where φ is a C^∞ function given in Section 1.

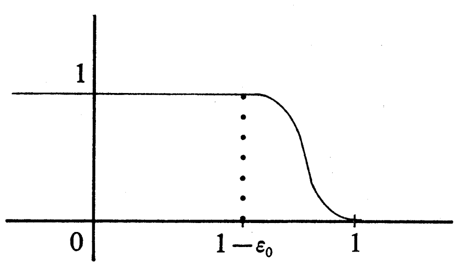

Fig. 3. The graph of β.

Using this $\beta(x)$, we define C^∞ functions $\tilde{h}_k$ by $\tilde{h}_k(x)=\beta(x)\cdot h_k(x)$.

Lemma 2.10.

(1) $\tilde{h}_k(x)=h_k(x)$ *for* $x \in [0, 1-\varepsilon_0]$.

(2) $|\tilde{h}_k-\tilde{h}_{k+1}|_0 \underset{(f,g,s)}{\leqq} (a_k-a_{k+1})^{(1-\alpha-\varepsilon)s-(1-\alpha-\varepsilon)^{-1}}$

and

$$|\tilde{h}_k-\tilde{h}_{k+1}|_r \underset{(f,g,r)}{\leqq} (a_k-a_{k+1})^{-r/(1-\alpha-\varepsilon)}$$

for $r\geqq 1$ *and sufficiently small* $\varepsilon>0$.

(3) *The sequence* $\{\tilde{h}_k\}$ *converges with respect to the* C^r *topology for* $0\leqq r<s(1-\alpha)^2-(2-\alpha)$.

Proof. The assertion (1) is obvious by the definition of $\tilde{h}_k$. As to (2), we show it easily by the formula $\tilde{h}_k-\tilde{h}_{k+1}=\beta\cdot(h_k-h_{k+1})$ and Lemma 2.8. For (3), since $r<s(1-\alpha)^2-(2-\alpha)$, we can choose a sufficiently small positive real ε and a sufficiently large integer $n\geq 0$ such that

$$\tilde{\beta}=\left\{(1-\alpha-\varepsilon)s-\frac{1}{1-\alpha-\varepsilon}\right\}\cdot\left(1-\frac{r}{n}\right)-\frac{r}{1-\alpha-\varepsilon}\geq 1.$$

Then, by the interpolation theorem (Hörmander [3]), we have

$$\begin{aligned}|\tilde{h}_k-\tilde{h}_{k+1}|_r &\underset{(r,n)}{\leq} \{|\tilde{h}_k-\tilde{h}_{k+1}|_0\}^{(n-r)/n}\cdot\{|\tilde{h}_k-\tilde{h}_{k+1}|_n\}^{r/n}\\ &\underset{(f,g,n,r)}{\leq} (a_k-a_{k+1})^{\tilde{\beta}}.\end{aligned}$$

This insures the convergence of the sequence $\{\tilde{h}_k\}$ with respect to the C^r-topology. This completes the proof of Lemma 2.10.

By Lemma 2.10, the sequence $\{h_k\}$ converges to some C^r-diffeomorphism h on [0, 1] for $0\leq r\leq s(1-\alpha)^2-(2-\alpha)$. Hence, $\{h_k\}$ converges on $[0, +\infty)$.

Since h_k is the identity on $[0, a_{k+1}]$, we have

$$\begin{aligned}|h-\mathrm{I}|_r^{[0,a_k+1]} &\leq \sum_{t\geq k}(a_t-a_{t+1})^{\tilde{\beta}}\\ &\leq a_k.\end{aligned}$$

Thus, we have (iii) of Theorem 2.4.

To show Theorem 2.4 (ii), for a small positive real c, we define a C^∞ function γ_c by

$$\gamma_c(x)=\left\{1-\varphi\left(\frac{x-c}{\varepsilon_1}\right)\right\}\cdot\varphi\left(\frac{x-(1-\varepsilon_1)}{\varepsilon_1}\right)$$

for sufficiently small $\varepsilon_1>0$.

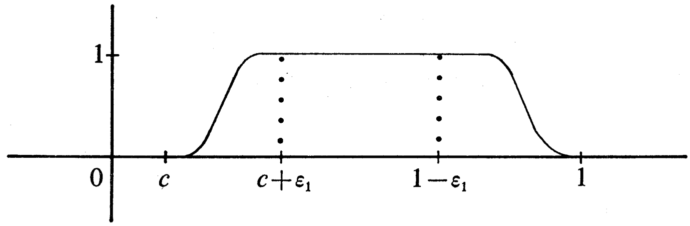

Fig. 4. The graph of γ_c.

We consider a C^∞ function $\tilde{\tilde{h}}_k$, which is defined by $\tilde{\tilde{h}}_k(x)=\gamma_c(x)\cdot h_k(x)$. Then, we have the following lemma corresponding to Lemma 2.10.

Lemma 2.11.

(1) $\tilde{\tilde{h}}_k(x)=h_k(x)$ *for* $x\in[c+\varepsilon_1, 1-\varepsilon_1]$.

(2) $|\tilde{\tilde{h}}_k-\tilde{\tilde{h}}_{k+1}|_0 \underset{(f,g,s)}{\leq} (a_k-a_{k+1})^{(1-\alpha-\varepsilon)s-(1-\alpha-\varepsilon)-1}$

and $|\tilde{\tilde{h}}_k-\tilde{\tilde{h}}_{k+1}|_r \underset{(f,g,r,c)}{\leq} 1$, *where* $r\geq 1$ *and sufficiently small* $\varepsilon>0$.

(3) *The sequence* $\{\tilde{\tilde{h}}_k\}$ *converges with respect to the* C^∞*-topology.*

We can show this in a way similar to that of the proof of Lemma 2.10 by using Lemma 2.9. Theorem 2.4 (ii) follows from Lemma 2.11 and we complete the proof of Theorem 2.4.

Remark. The argument used in the proof cannot be applied to the case where $\alpha(f)=1$. The author does not know whether or not Theorem 2.4 holds in this case.

§ 3. On a theorem of Sergeraert

In this section, we show that Theorem 2.4 can be applied to giving an alternative proof of the following theorem due to Sergeraert [6].

Theorem 3.1. *For any* $f\in G_\infty$, *there exist* $g\in G_\infty^c$ *and* $h\in G_\infty^c$ *such that* $f=g^{-1}\circ h^{-1}\circ g\circ h$.

In fact, the following proposition together with Theorem 2.4 implies Theorem 3.1.

Proposition 3.2. *For a finite number of diffeomorphisms* $f_1, f_2, \cdots, f_N$ *in* D_∞, *there exists* $g\in D_\infty^c$ *such that*

(0) $\alpha(g)=0$

(1) $g\circ f_i\in D_\infty^c$

(2) $|x-f_i(x)| \underset{(f_i,g,s)}{\leq} \{\Delta^g(x)\}^s$

for any integer $s\geq 0$, $x\in \boldsymbol{R}$ *and* $i=1, 2, \cdots, N$.

Proof. We prove this proposition in the case $N=1$. The case when $N\geq 2$ can be proved similarly.

We choose a sequence of positive numbers $\{a_n\}_{n=2,3,\dots}$ and a sequence of diffeomorphisms $\{g_n\}_{n=2,3,\dots}$ of $[0,+\infty)$ such that the following conditions hold:

(i) $0<a_{n+1}<a_n$ and $\lim_{n\to\infty} a_n=0$,

(ii) $g_n(0)=0$, $D^1g_n(0)=1$, $D^rg_n(0)=0$ $(2\leqq r\leqq n)$ and $D^{n+1}g_n(0)\neq 0$.

(iii) $g_n(x)\leqq g_{n+1}(x)<x$ and $g_n(x)<f^{-1}(x)$,

(iv) $g_n(x)=g_{n+1}(x)$ for $x\geqq a_n$,

(v) $|x-f(x)|\leqq\{\Delta^{g_n}(x)\}^n$ for $x\leqq a_n$,

and $|x-f(x)|\leqq\{\Delta^{g_n}(x)\}^{n-1}$ for $x\leqq a_{n-1}$,

(vi) $|g_{n+1}-g_n|_{n-1}\leqq 2^{-n}$.

Then, $g|_{[0,+\infty)}$ will be obtained as $\lim_{n\to\infty} g_n$.

We construct sequences $\{a_n\}$ and $\{g_n\}$ inductively on n. First, for $n=2$, put $a_2=1$ and let g_2 be the time one map of the vector field $\xi_2=\xi_2(x)\,d/dx$, where $\xi_2(x)$ is a C^∞ function such that $\xi_2(x)<0$ for $x>0$, $\xi_2(x)=cx^3$ on some neighbourhood of 0 and g_2 satisfies the second part of (iii) together with the first part of (v).

Assume that we have chosen g_n which is the time one map of a vector field $\xi_n=\xi_n(x)\,d/dx$. Let $\eta(x)$ be a C^∞ function on $[0,+\infty)$ such that

$$\eta(x)\begin{cases} =2x & \text{if } 0\leqq x\leqq\frac{1}{4} \\ \in[0,1) & \text{if } \frac{1}{4}\leqq x\leqq 1 \\ =1 & \text{if } 1\leqq x. \end{cases}$$

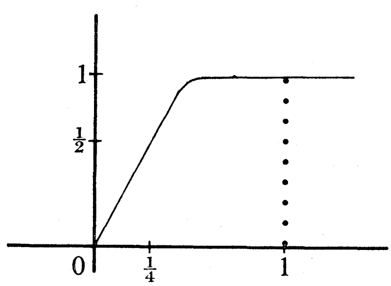

Fig. 5. The graph of $\eta(x)$.

Put $\xi_\varepsilon(x)=\eta(x/\varepsilon)\cdot\xi_n(x)$ for $\varepsilon>0$.

Then, the time one map g_ε of the vector field $\xi_\varepsilon=\xi_\varepsilon(x)\,d/dx$ satisfies (ii), (iii) and (vi) for sufficiently small $\varepsilon>0$. We note that $g_n(x)=g_\varepsilon(x)$ for $x\geqq g_n^{-1}(\varepsilon)$. This means that if we take ε smaller than $g_n(a_n)$, then $g_\varepsilon(x)$ satisfies the condition corresponding to (iv). Moreover, since

$D^{n+2}(\mathrm{I}-g_\varepsilon)(0)\neq 0$ and $x-g_n(x)\geq x-g_{\varepsilon'}(x)\geq x-g_\varepsilon(x)$ for $0<\varepsilon'<\varepsilon$, by taking $\varepsilon>0$ sufficiently small, we have

$$|x-f(x)|\leq\{\Delta^{g_\varepsilon}(x)\}^{n+1}\quad\text{for } x\leq\varepsilon$$

and

$$|x-f(x)|\leq\{\Delta^{g_\varepsilon}(x)\}^{n}\qquad\text{for } x\leq a_n.$$

Then, we put $a_{n+1}=\varepsilon$ and $g_{n+1}=g_\varepsilon$. Here, ε can be taken smaller than, for example, $\frac{1}{2}a_n$ so that $\{a_n\}$ converges to zero.

The desired diffeomorphism g is defined by

$$g(x)=\begin{cases} x & \text{for } x\leq 0\\ \lim\limits_{n\to+\infty} g_n(x) & \text{for } x\geq 0.\end{cases}$$

By (ii), (iii) and (vi), g belongs to D^c_∞. Since the function $x-g_n(x)$ is monotonously increasing, so is $x-g(x)$. Hence, we have $\alpha(g)=0$. It is obvious that $g\circ f\in D^c_\infty$ by (iii). Thus g satisfies the conditions (0) and (1). As to (2), it is enough to show that, for $s\geq 2$,

$$|x-f(x)|\leqq\{\Delta^{g_n}(x)\}^{s}\quad(*)$$

for any $x\leqq a_s$ and any $n\geqq s$. First, by (v), we have

$$|x-f(x)|\leqq\{\Delta^{g_s}(x)\}^{s}$$

and

$$|x-f(x)|\leqq\{\Delta^{g_{s+1}}(x)\}^{s}$$

for $x\leq a_s$. For g_{s+2}, (iv) and (v) insures that

$$g_{s+2}(x)=g_{s+1}(x)\quad\text{for } a_{s+1}\leq x\leq \text{[illegible]}$$

and

$$|x-f(x)|\leqq\{\Delta^{g_{s+2}}(x)\}^{s+1}\quad\text{for } x\leq a_{s+1}.$$

This shows that

$$|x-f(x)|\leq\{\Delta^{g_{s+2}}(x)\}^{s},\quad\text{for } x\leq a_s.$$

Iterating this procedure, we have $(*)$. This completes the proof of Proposition 3.2.

Remark. The strategy of the proof of Proposition 3.2 is due to Sergeraert [6].

§ 4. The monoid G_∞^c and the homology of G_∞^c

As we mentioned before, the group G_∞ and its submonoid G_∞^c are closely related to the theory of smooth foliations of codimension one. In this section, we show that the natural inclusion j: $G_\infty^c \times \bar{G}_\infty^c \to G_\infty$ induces isomorphisms of their Eilenberg-MacLane homology groups.

First, we recall the definition of the homology of a group and that of a monoid.

The Eilenberg-MacLane homology of a group G (simply, we say the homology of a group G) is the homology of a chain complex $\{C_q(G), \partial\}$, where $C_q(G)$ is the free Z-module generated by $G^q = G \times G \times \cdots \times G$ (q-times) for $q \geq 1$ and $C_0(G) = \mathrm{Z}$. The map ∂: $C_q(G) \to C_{q-1}(G)$ is defined by

$$\partial(g_1, \cdots, g_q) = (g_2, \cdots, g_q) + \sum_{i=1}^{q-1} (-1)^i (g_1, \cdots, g_i g_{i+1}, \cdots, g_q) + (-1)^q (g_1, \cdots, g_{q-1})$$

for $q \geqq 2$ and ∂: $C_1(G) \to C_0(G)$ is defined to be the zero map. The homology of a monoid M is defined in a similar way. Concerning the relation between the homology of groups and that of monoids, we have the following theorem.

Theorem 4.1. (H. Cartan and S. Eilenberg [1])
Let G be a group and M a submonoid of G such that each element of G has the form $x^{-1}y$ for some $x, y \in M$. Then, the homomorphisms

$$j_*: H_n(M) \longrightarrow H_n(G) \quad (n \geqq 0)$$

induced by the natural inclusion j: $M \to G$ are isomorphisms.

Applying this theorem to $G_\infty^c \times \bar{G}_\infty^c$ and G_∞, we have the following.

Proposition 4.2. *We have the isomorphisms*

$$j_*: H_n(G_\infty^c \times \bar{G}_\infty^c) \longrightarrow H_n(G_\infty) \quad (n \geqq 0),$$

which are induced by the natural inclusion

$$j: G_\infty^c \times \bar{G}_\infty^c \longrightarrow G_\infty.$$

The proof follows from Proposition 3.2.

Remark 1. If we consider the subgroup G'_∞ which consists of elements f of G_∞ such that $f(x) = x$ for any $x \leq 0$, then we have also the isomorphisms as above, induced by the natural inclusion j: $G_\infty^c \to G'_\infty$.

Remark 2. As we mentioned in Remark after Lemma 1.8, the monoid G^c_∞ contains an interesting submonoid $A_0=\{f\in G^c_\infty;\ (f)=0\}$. By Proposition 3.2 and Theorem 2.4, we can see that the inclusion

$$j\colon A_0 \hookrightarrow G^c_\infty$$

induces isomorphisms in homology groups.

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