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Determinacy of Analytic Foliation Germs

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In this paper we consider the determinacy problem for codim 1 complex analytic foliation germs. For function germs (smooth or analytic), the problem has been thoroughly worked out mostly by J. Mather and the results are widely known ([5], [6] see also [8], [14]). Let f be a smooth or analytic function germ at the origin 0 in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$ and let $J(f)$ denote the ideal generated by the partial derivatives of f in the ring of function germs at 0. We also denote by $\mathfrak{m}$ the maximal ideal of germs that are 0 at 0. Then if (a) $\mathfrak{m}^k \subset \mathfrak{m}J(f) + \mathfrak{m}^{k+1}$ for some natural number k, (b) f is (right) k-determined, i.e., for any germ g with the same k-jet as f, there is a germ ϕ of local diffeomorphism or of local biholomorphic map at 0 with $\phi(0)=0$ such that g is equal to the pull-back ϕ^*f of f by ϕ (g is right equivalent to f). Also, (b) implies that (c) $\mathfrak{m}^{k+1} \subset \mathfrak{m}J(f)$. The condition (c) can be referred to as "infinitesimal (right) k-determinacy", since $\mathfrak{m}J(f)$ and $\mathfrak{m}^{k+1}$ are interpreted as, respectively, the tangent spaces at f to the sets of germs right equivalent to f and of germs with the same k-jet as f. In general (c) does not imply (b). Hewever, (c) implies that (d) f is "locally k-determined", i.e., if a germ g has the same k-jet as f and is "close" to f, g is right equivalent to f. There are statements corresponding to the above in the right-left case. Note also that the problem is closely related to the unfolding theory.

The main result of this paper is (4.1) Theorem, which asserts a statement analogous to the implication (c)$\Rightarrow$(d) (local determinacy) for codim 1 foliation germs. As a special case, we also consider multiform functions. In this case we can generalize not only the local determinacy ((5.6) Theorem) but also, as already in the work of Cerveau and Mattei [1], the global determinacy (a)$\Rightarrow$(b) (Theorems (5.11) and (5.18)). The difficulty in the foliation case in general is caused by the fact that the associated algebraic objects have only vector space structures and may not be invariant under multiplication by function germs, which prevents us from using such an algebraic tool as Nakayama's lemma. Thus we obtain the local determinacy by actually solving some differential equations.

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In Section 1, we describe, for a given germ ω of holomorphic 1-form at 0 in $\boldsymbol{C}^n$, the tangent space at ω to the G_n-orbit of ω, i.e., the set of germs of holomorphic 1-forms that generate "analytically equivalent" modules as the one generated by ω ((1.5) Lemma). In Section 2, we recall the unfolding theory for codim 1 foliation germs and study its relation with infinitesimal transformations of such germs. Beside usual morphisms for unfoldings of foliation germs, which generalize (strict) right morphisms in the unfolding theory of function germs, we introduce RL-morphisms ((2.1) Definition), which turn out to generalize right-left morphisms in the function case. We also define some algebraic objects associated with a codim 1 foliation germ $F=(\omega)$. These are used to describe the classes of first order unfoldings of F under various types of equivalences as well as other infinitesimal conditions for F. Infinitesimal transformations of G_n on F define RL-trivial first order unfoldings of F ((2.6) Lemma, (2.12) Remark). We study transversality of unfoldings of foliation germs and its relation with (infinitesimal) versality in Section 3. Some results analogous to those in the function case are obtained ((3.5) Proposition, (3.10) Theorem). In Section 4, we prove that if a codim 1 foliation germ F is infinitesimally k-determined, then it is locally k-determined ((4.1) Theorem) as mentioned above. The problem is to solve the differential equations (4.5) and (4.6) for ϕ and u under some conditions. We can linearlize the equations by going from biholomorphic maps to vector fields as usual. Thus we solve (4.8) and (4.9) for ξ and g under the condition (4.10). We do this by the power series method. The infinitesimal k-determinacy guarantees the existence of formal solutions. As in [9], we compare the series with series obtained by modifying the one in Kodaira-Spencer [3]. We use the Malgrange privileged neighborhoods theorem [4] in our estimates. In Section 5, we mainly treat multiform functions. (5.6) Theorem gives a local determinacy result. If we combine it with Nakayama's lemma, we obtain the global determinacy (Theorems (5.11) and (5.18)).

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§ 1. The action of G_n on $\boldsymbol{\Omega}_n$

We denote by $\mathcal{O}_n$ the ring of germs of holomorphic functions at the origin 0 in $\boldsymbol{C}^n=\{(x_1, \cdots, x_n)\}$. The maximal ideal and the multiplicative group of units in $\mathcal{O}_n$ are denoted by $\mathfrak{m}$ and U_n, respectively. Also we denote by L_n the group of germs at 0 of local biholomorphic maps ϕ of $\boldsymbol{C}^n$ into itself with $\phi(0)=0$. The group L_n acts on U_n from the right by pull-back. We form the semi-direct product $U_n \rtimes L_n$, which is denoted by G_n. Thus as a set, G_n is the product $U_n \times L_n$ and the group multiplication in G_n is given by

$$(u', \phi')\cdot(u, \phi)=(u\phi^*u', \phi'\phi)$$

for u, u' in U_n and ϕ, ϕ' in L_n. If we denote by Ω_n the $\mathcal{O}_n$-module of germs at 0 of holomorphic 1-forms on C^n, the group G_n acts on Ω_n from the right by $(u, \phi)\omega=u\phi^*\omega$ for (u, ϕ) in G_n and ω in Ω_n.

For a germ f in $\mathcal{O}_n$, we denote by $j^k f$ the k-jet of f, i.e., the k-th Taylor polynomial of f at 0. We set $J_n^k=\{j^k f \mid f \in \mathcal{O}_n\}$. We also let $J^k(n, n)$ be the set of k-jets of germs of local holomorphic maps of C^n into itself leaving 0 fixed. Note that each of the sets J_n^k and $J^k(n, n)$ has a natural structure of complex Euclidean space. If $\omega=\sum_{i=1}^n f_i dx_i$, $f_i \in \mathcal{O}_n$, is a germ in Ω_n, we define the k-jet $j^k\omega$ of ω by

$$j^k\omega=\sum_{i=1}^n j^{k-1}f_i\cdot dx_i.$$

If we set $J^k\Omega_n=\{j^k\omega \mid \omega \in \Omega_n\}$, we have a map

$$\pi_k\colon \Omega_n \longrightarrow J^k\Omega_n \tag{1.1}$$

sending ω to $j^k\omega$. The set $J^k\Omega_n$ has also a natural structure of complex Euclidean space. Furthermore, we introduce the set $J^k\Omega(n)$ of k-jets of germs of local holomorphic 1-forms on C^n. The set $J^k\Omega(n)$ is naturally identified with $J^k\Omega_n \times C^n$. We let

$$\pi\colon J^k\Omega(n) \longrightarrow J^k\Omega_n \tag{1.2}$$

be the canonical projection. If we set $U_n^k=\{j^k u \mid u \in U_n\}$ and $L_n^k=\{j^k\phi \mid \phi \in L_n\}$, each of them has a natural structure of complex Lie group. Using the natural action of L_n^k on U_n^k, we form the semi-direct product

$$G_n^k=U_n^k \rtimes L_n^k.$$

Thus as a set G_n^k is the product $U_n^k \times L_n^k$, which is embedded as an open set in $J_n^k \times J^k(n, n)$. The group multiplication in G_n^k is given by

$$(j^k u', j^k\phi')\cdot(j^k u, j^k\phi)=(j^k(u\phi^*u'), j^k(\phi'\phi)).$$

With these, G_n^k is a complex Lie group, which acts on $J^k\Omega_n$ from the right by

$$(j^k u, j^k\phi)\cdot j^k\omega=j^k(u\phi^*\omega).$$

Let Θ_n denote the $\mathcal{O}_n$-module of germs at 0 of local holomorphic vector fields on $\boldsymbol{C}^n$. For X in Θ_n and ω in Ω_n, we have the Lie derivative $L_X\omega$ of ω with respect to X. If

$$X=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial x_i},\quad \xi_i\in\mathcal{O}_n\quad\text{and}\quad \omega=\sum_{i=1}^{n}f_i dx_i,\quad f_i\in\mathcal{O}_n,$$

then

$$(1.3)\qquad L_X\omega=\sum_{i=1}^{n}f_i d\xi_i+\sum_{i,j=1}^{n}\xi_i\frac{\partial f_j}{\partial x_i}dx_j.$$

For an ideal I in $\mathcal{O}_n$, we set

$$L_I(\omega)=\{L_X\omega\,|\,X\in I\cdot\Theta_n\},$$

which is a $\boldsymbol{C}$-vector space.

(1.4) **Remark.** If $\omega=df$, then $L_I(\omega)=d(I\cdot J(f))$, where $J(f)$ is the Jacobian ideal of f; the ideal generated by $\partial f/\partial x_1,\cdots,\partial f/\partial x_n$.

Take a germ ω in Ω_n and set $z=j^k\omega$. The holomorphic tangent space $T_z(J^k\Omega_n)$ of $J^k\Omega_n$ at z is identified with $J^k\Omega_n$. Now we compute the holomorphic tangent space $T_z(G_n^k z)$ at z of the orbit $G_n^k z$.

(1.5) **Lemma.** *We have*

$$T_z(G_n^k z)=\pi_k(L_{\mathfrak{m}}(\omega)+\mathcal{O}_n\omega),$$

where $\mathcal{O}_n\omega=\{g\omega\,|\,g\in\mathcal{O}_n\}$.

Proof. This is done as in the function case ([8] Ch. 7, Main Lemma III, [14] Lemma 2.8). Let $\mu_z\colon G_n^k\to J^k\Omega_n$ be the holomorphic map defined by $\mu_z(\gamma)=\gamma z$ for γ in G_n^k. Then we have $T_z(G_n^k z)=d\mu_z(T_e(G_n^k))$, where e denotes the identity in G_n^k. We think of G_n^k as a subset in $J^k\times J^k(n,n)$, which is a complex Euclidean space. Hence $T_e(G_n^k)$ is also a subspace in $J^k\times J^k(n,n)$. Thus for a vector v in $T_e(G_n^k)$, we may write

$$v=(j^k g,j^k\xi),$$

where g is in $\mathcal{O}_n$ and ξ is a germ of local holomorphic map of $\boldsymbol{C}^n$ into itself with $\xi(0)=0$. For t in a neighborhood U of 0 in $\boldsymbol{C}^n$, we set

$$u_t(x)=1+tg(x)\quad\text{and}\quad \phi_t(x)=x+t\xi(x).$$

Then we have a holomorphic map

$$\alpha_v\colon U\longrightarrow G_n^k$$

defined by $\alpha_v(t)=(j^k u_t,j^k\phi_t)$. Its differential

$$d\alpha_v\colon T_0(V) \longrightarrow T_e(G_n^k)$$

sends the tangent vector d/dt to v. We compute

$$\begin{aligned}
d\mu_z(v) &= d(\mu_z \circ \alpha_v)\left(\frac{d}{dt}\right) \\
&= \frac{d}{dt}(j^k(u_t\phi_t^*\omega))\Big|_{t=0} \\
&= \pi_k\left(\frac{d}{dt}(u_t\phi_t^*\omega)\Big|_{t=0}\right) \\
&= \pi_k\left(u_0\cdot\frac{d}{dt}(\phi_t^*\omega)\Big|_{t=0} + \frac{du_t}{dt}\Big|_{t=0}\cdot\phi_0^*\omega\right) \\
&= \pi_k(L_X\omega + g\omega),
\end{aligned}$$

where X is a germ in $\mathfrak{m}\Theta_n$ given by $X=\sum_{i=1}^n \xi_i(\partial/\partial x_i)$. Conversely, it is clear that every element of the form $\pi_k(L_X\omega+g\omega)$ is in the image of $d\mu_z$.

Q.E.D.

(1.6) **Remark.** In view of the above lemma, we may think of the space $L_{\mathfrak{m}}(\omega)+\mathcal{O}_n\omega$ as the "tangent space" at ω of the orbit $G_n\omega$ in Ω_n.

§ 2. Infinitesimal transformations and unfoldings of codim 1 foliations

For generalities on complex analytic foliations with singularities and their unfoldings, we refer to [9] and [10]. Let $F=(\omega)$ be a codim 1 foliation germ at 0 in C^n, i.e., F is a rank 1 free sub-$\mathcal{O}_n$-module of Ω_n with a generator ω satisfying the integrability condition $d\omega\wedge\omega=0$. The germ at 0 of the analytic set $\{x\,|\,\omega(x)=0\}$ is denoted by $S(\omega)$ (or by $S(F)$) and called the singular set of F. We always assume that codim $S(F)\geq 2$ ([10] (5.1) Lemma, [11] (1.1) Lemma). An unfolding of $F=(\omega)$ is a codim 1 foliation germ $\mathcal{F}=(\tilde{\omega})$ at 0 in $C^n\times C^m=\{(x,t)\}$ having a generator $\tilde{\omega}$ with $\iota^*\tilde{\omega}=\omega$, where ι denotes the embedding of C^n into $C^n\times C^m$ given by $\iota(x)=(x,0)$. We call C^m the parameter space of $\mathcal{F}$.

(2.1) **Definition.** Let $\mathcal{F}=(\tilde{\omega})$ and $\mathcal{F}'=(\tilde{\omega}')$ be two unfoldings of F with parameter spaces C^m and $C^l=\{(s_1,\cdots,s_l)\}$, respectively.

(I) A morphism from $\mathcal{F}'$ to $\mathcal{F}$ is a triple (Φ,ψ,u) such that

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc}
(C^n\times C^l,0) & \xrightarrow{\Phi} & (C^n\times C^m,0) \\
\downarrow & & \downarrow \\
(C^l,0) & \xrightarrow{\psi} & (C^m,0)
\end{array}$$

commutative, where the vertical maps are the projections. u is a unit in $\mathcal{O}_{n+l}$.

(b) $\Phi(x, 0)=(x, 0)$ and $u(x, 0)=1$.

(c) $u\tilde{\omega}'=\Phi^*\tilde{\omega}$.

(II) An RL-morphism from $\mathscr{F}'$ to $\mathscr{F}$ is a quadruple (Φ, ψ, u, α), where Φ, ψ and u are germs satisfying (a) and (b) in (I) and $\alpha=(\alpha_1, \cdots, \alpha_l)$ is a germ in $\mathcal{O}_{n+l}^l$. Instead of (c), we require

(c)$'$ $u\tilde{\omega}'=\Phi^*\tilde{\omega}+\sum_{k=1}^l \alpha_k ds_k$.

A morphism or an RL-morphism is said to be strong, if we further have

$$\Phi(0, s)=(0, \psi(s)).$$

(2.2) **Remark.** The notions of a morphism and an RL-morphism are generalizations of a (strict) right morphism and an right-left morphism, respectively, in the unfolding theory of function germs ([6], [14] Definitions 3.2 and 3.3, [11] (3.1) Definition and (3.11) Remark, [13]).

A first order unfolding of $F=(\omega)$ is a rank 1 free sub-$\mathcal{O}_{n+1}$-module $\mathscr{F}^{(1)}=(\tilde{\omega})$ of Ω_{n+1} with a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$, where ι denotes the inclusion of $C^n=\{x\}$ in $C^n\times C=\{(x, t)\}$ given by $\iota(x)=(x, 0)$, and that $d\tilde{\omega}\wedge\tilde{\omega}\equiv 0 \mod t^2, tdt$ (integrable to the first order). If we write

$$\tilde{\omega}\equiv\omega+\omega^{(1)}t+h^{(1)}dt \qquad \mod t^2,\ tdt$$

with $\omega^{(1)}$ in Ω_n and $h^{(1)}$ in $\mathcal{O}_n$, it is not difficult to show that the first order integrability is equivalent to

$$h^{(1)}d\omega+(\omega^{(1)}-dh^{(1)})\wedge\omega=0. \tag{2.3}$$

If $\tilde{\omega}'$ is another generator of $\mathscr{F}^{(1)}$ with $\iota^*\tilde{\omega}'=\omega$, write

$$\tilde{\omega}'\equiv\omega+\omega^{(1)\prime}t+h^{(1)\prime}dt \qquad \mod t^2,\ tdt.$$

There is a unit u in $\mathcal{O}_{n+1}$ of the form

$$u\equiv 1+u^{(1)}t \qquad \mod t^2$$

with $u^{(1)}$ in $\mathcal{O}_n$, such that $\tilde{\omega}'=u\tilde{\omega}$. Then

$$h^{(1)\prime}=h^{(1)} \quad \text{and} \quad \omega^{(1)\prime}=\omega^{(1)}+u^{(1)}\omega.$$

Thus if we set

$$I(\omega)=\{h\in\mathcal{O}_n \mid hd\omega=\eta\wedge\omega \text{ for some } \eta \text{ in } \Omega_n\},$$

each first order unfolding of $F=(\omega)$ determines an element in $I(\omega)$ and vice versa. We also set

$$J(\omega)=\{h \in \mathcal{O}_n \mid h=\langle X, \omega\rangle \text{ for some } X \text{ in } \Theta_n\},$$
$$K(\omega)=\{\alpha \in \mathcal{O}_n \mid \alpha d\omega=d\alpha\wedge\omega\} \quad \text{and}$$
$$\Omega(\omega)=\{\theta \in \Omega_n \mid \theta\wedge\omega=dh\wedge\omega-hd\omega \text{ for some } h \text{ in } \mathcal{O}_n\},$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing of vector fields and 1-forms. If $\omega=\sum_{i=1}^n f_i dx_i$, $f_i \in \mathcal{O}_n$, then $J(\omega)$ is the ideal in $\mathcal{O}_n$ generated by $f_1, \cdots, f_n$. The set $K(\omega)$ is in the ideal $I(\omega)$ and is the $\boldsymbol{C}$-vector space of integrating factors of ω ([1] p. 34). The set $\Omega(\omega)$ also forms a $\boldsymbol{C}$-vector space.

(2.4) **Remark.** If ω' is another generator of F, then $\omega'=u\omega$ for some unit u in $\mathcal{O}_n$. We have $I(\omega)=I(\omega')$ and $J(\omega)=J(\omega')$. The correspondence $g\mapsto ug$ gives an isomorphism of $K(\omega)$ onto $K(\omega')$ and the correspondence $\theta\mapsto u\theta$ gives isomorphisms of $\Omega(\omega)$ onto $\Omega(\omega')$ and of $L_I(\omega)+\mathcal{O}_n\omega$ onto $L_I(\omega')+\mathcal{O}_n\omega'$ for any ideal I in $\mathcal{O}_n$.

(2.5) **Remark.** If $\omega=df$, $f \in \mathcal{O}_n$, then $I(\omega)=\mathcal{O}_n$, $J(\omega)=J(f)$. We also have $K(\omega)=\{\alpha \in \mathcal{O}_n \mid d\alpha\wedge df=0\}$. If we assume $f(0)=0$, then by the factorization theorem in [7] p. 472, we may write

$$K(df)=f^*\mathcal{O}_1,$$

since the condition codim $S(df)\geq 2$ implies that f is power free.

If $F=(\omega)$ is a codim 1 foliation germ, the set

$$\mathcal{G}(F)=\{X \in \Theta_n \mid L_X\omega=g\omega \text{ for some } g \text{ in } \mathcal{O}_n\}$$

is independent of the chosen generator ω of F and forms a Lie algebra with respect to the Poisson bracket of vector fields. We call it the Lie algebra of infinitesimal automorphisms of F. Also we consider the annihilator

$$F^a=\{X \in \Theta_n \mid \langle X, \omega\rangle=0\}$$

of F.

(2.6) **Lemma.** *We have*

$$\langle X, \omega\rangle d\omega+(L_X\omega-d\langle X, \omega\rangle)\wedge\omega=0$$

for any germ X in Θ_n.

Proof. If we denote by ι_X the inner product by X, we have

$$L_X\omega=d\langle X, \omega\rangle+\iota_X d\omega.$$

On the other hand, if we apply ι_X to $d\omega\wedge\omega=0$, we get

$$(\iota_X d\omega)\wedge\omega+\langle X, \omega\rangle d\omega=0.$$

Hence we obtain the identity. Q.E.D.

(2.8) **Corollary.** $J(\omega)\subset I(\omega)$.

(2.9) **Corollary.** *For any germ X in Θ_n, $L_X\omega$ is in $\Omega(\omega)$, in particular*

$$L_I(\omega)+\mathcal{O}_n\omega\subset\Omega(\omega)$$

for any ideal I in $\mathcal{O}_n$.

From the condition codim $S(F)\geq 2$, if $\eta\wedge\omega=0$ for η in Ω_n, then $\eta=g\omega$ for some g in $\mathcal{O}_n$. Hence

(2.10) **Corollary.** *F^a is a Lie subalgebra of $\mathcal{G}(F)$ and there is an exact sequence of $\boldsymbol{C}$-vector spaces*

$$0\longrightarrow F^a\longrightarrow\mathcal{G}(F)\xrightarrow{\rho}J(\omega)\cap K(\omega)\longrightarrow 0,$$

where the map ρ sends X in $\mathcal{G}(F)$ to $\langle X, \omega\rangle$.

The ideal $I(\omega)$ and the vector space $\Omega(\omega)$ are related by the following

(2.11) **Proposition.** *For any ideal I in $\mathcal{O}_n$, there is an isomorphism of $\boldsymbol{C}$-vector spaces*

$$I(\omega)/I\cdot J(\omega)+K(\omega)\simeq\Omega(\omega)/L_I(\omega)+\mathcal{O}_n\omega.$$

Proof. If θ is in $\Omega(\omega)$, there is a germ h in $\mathcal{O}_n$ such that $\theta\wedge\omega=dh\wedge\omega-hd\omega$. Such an h is determined uniquely modulo $K(\omega)$. Thus the correspondence $\theta\mapsto[h]$ defines a surjective $\boldsymbol{C}$-linear map

$$\Omega(\omega)\longrightarrow I(\omega)/I\cdot J(\omega)+K(\omega).$$

Moreover, if h is in $I\cdot J(\omega)+K(\omega)$, we may write

$$h=\langle X, \omega\rangle+\alpha \quad \text{for } X \text{ in } I\cdot\Theta_n \text{ and } \alpha \text{ in } K(\omega).$$

Then by (2.6) Lemma, we have

$$\theta\wedge\omega=(L_X\omega)\wedge\omega.$$

Hence $\theta=L_X\omega+g\omega$ for some g in $\mathcal{O}_n$. Q.E.D.

(2.12) **Remark.** The quotients $I(\omega)/J(\omega)$ and $I(\omega)/J(\omega)+K(\omega)$ are interpreted, respectively, as the sets of isomorphism classes and RL-isomorphism classes of first order unfoldings of $F=(\omega)$ ([10] § 6, [9] 1, [13]). If $\mathscr{F}=(\tilde{\omega})$ is an unfolding of F with parameter space $C^m=\{(t_1, \cdots, t_m)\}$, we expand $\tilde{\omega}$ as a power series in t;

$$\tilde{\omega}=\omega+\sum_{j=1}^{m}\omega^{(1j)}t_j+\sum_{j=1}^{m}h_j dt_j+\text{higher order terms} \tag{2.13}$$

with $\omega^{(1j)}$ in Ω_n and h_j in $\mathcal{O}_n$. We say that $\mathscr{F}$ is infinitesimally versal if the classes $[h_1], \cdots, [h_m]$ of $h_1, \cdots, h_m$ in $I(\omega)/J(\omega)$ span the vector space and that $\mathscr{F}$ is infinitesimally RL-versal if the classes $[h_1], \cdots, [h_m]$ span the vector space $I(\omega)/J(\omega)+K(\omega)$ or equivalently the classes $[\omega^{(11)}], \cdots, [\omega^{(1m)}]$ span the vector space $\Omega(\omega)/L_{\mathcal{O}_n}(\omega)+\mathcal{O}_n\omega$. It is proved that if $\mathscr{F}$ is infinitesimally versal or infinitesimally RL-versal, then $\mathscr{F}$ is versal or RL-versal, respectively ([9], [13]).

(2.14) **Remark.** By (1.6) Remark and (2.9) Corollary, it is reasonable to say that a germ in Ω_n which is "close" to ω may not possibly be in the G_n-orbit of ω unless it is connected to ω by some unfolding of ω.

§ 3. Transversal unfoldings

Let $F=(\omega)$ be a codim 1 foliation germ at 0 in C^n and let $\mathscr{F}=(\tilde{\omega})$ be an unfolding of F with parameter space C^m. We take a representative of $\tilde{\omega}$ in some neighborhood of 0 in $C^n\times C^m$ and for each t near 0 in C^m and x near 0 in C^n, let $\omega_{t,x}$ denote the germ at x of the 1-form $\omega_t=\iota_t^*\tilde{\omega}$, where ι_t denotes the embedding $C^n\hookrightarrow C^n\times C^m$ given by $\iota_t(x)=(x, t)$. We define the map germ

$$j_1^k\tilde{\omega}\colon (C^n\times C^m, 0)\longrightarrow(J^k\Omega(n), j^k\omega)$$

by $j_1^k\tilde{\omega}(x, t)=j^k\omega_{t,x}$. Consider the differential of the composition of $j_1^k\tilde{\omega}$ and π in (1.2):

$$d(\pi\circ j_1^k\tilde{\omega})\colon T_0(C^n\times C^m)\longrightarrow T_z(J^k\Omega_n), \qquad z=j^k\omega.$$

(3.1) **Definition.** An unfolding $\mathscr{F}$ of $F=(\omega)$ is k-transversal if it has a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$ and that

$$d(\pi\circ j_1^k\tilde{\omega})(T_0(C^n\times C^m))+T_z(G_n^k z)=\pi_k\Omega(\omega), \qquad z=j^k\omega.$$

For $X=\partial/\partial x_i$, we denote $L_X\omega$ simply by $L_i\omega$. If $\mathscr{F}=(\tilde{\omega})$ is an unfolding of $F=(\omega)$ with parameter space C^m, we write $\tilde{\omega}$ as (2.13). A straightforward computation shows the following

(3.2) **Lemma.** *We have*

$$d(\pi \circ j_1^k \tilde{\omega})\left(\frac{\partial}{\partial x_i}\right) = j^k(L_i\omega), \qquad i=1, \cdots, n,$$

$$d(\pi \circ j_1^k \tilde{\omega})\left(\frac{\partial}{\partial t_j}\right) = j^k\omega^{(1j)}, \qquad j=1, \cdots, m.$$

(3.3) **Remark.** Let $\tilde{\omega}'$ be another generator of $\mathscr{F}$ with $\iota^*\tilde{\omega}'=\omega$. Using Lemmas (1.4) and (3.2), it is not difficult to show that if $\tilde{\omega}$ satisfies the condition in (3.1), so does $\tilde{\omega}'$.

For vectors $v_1, \cdots, v_r$ in a $\boldsymbol{C}$-vector space V, we denote by $[v_1, \cdots, v_r]_{\boldsymbol{C}}$ the subspace in V spanned by $v_1, \cdots, v_r$. Also for $F=(\omega)$, we set

(3.4) $\quad I^{(k+1)}(\omega)=\{h \in \mathcal{O}_n \mid hd\omega+(\theta-dh)\wedge\omega=0$ for some θ in $\pi_k^{-1}(0)\}$.

(3.5) **Proposition.** *For an unfolding $\mathscr{F}=(\tilde{\omega})$ of $F=(\omega)$, the following three conditions are equivalent*:

(a) *$\mathscr{F}$ is k-transversal.*

(b) $\Omega(\omega)=[\omega^{(11)}, \cdots, \omega^{(1m)}]_{\boldsymbol{C}}+L_{\mathcal{O}_n}(\omega)+\mathcal{O}_n\omega+\pi_k^{-1}(0)\cap\Omega(\omega)$.

(c) $I(\omega)=[h_1, \cdots, h_m]_{\boldsymbol{C}}+J(\omega)+K(\omega)+I^{(k+1)}(\omega)$.

Proof. By Lemmas (1.4) and (3.2), $\mathscr{F}$ is k-transversal if and only if

$$\pi_k\Omega(\omega)=\pi_k([L_1\omega, \cdots, L_n\omega]_{\boldsymbol{C}}+[\omega^{(11)}, \cdots, \omega^{(1m)}]_{\boldsymbol{C}}+L_{\mathfrak{m}}(\omega)+\mathcal{O}_n(\omega).$$

Noting that $[L_1\omega, \cdots, L_n\omega]_{\boldsymbol{C}}+L_{\mathfrak{m}}(\omega)=L_{\mathcal{O}_n}(\omega)$, we see that (a) and (b) are equivalent. The equivalence of (b) and (c) is straightforward. Q.E.D.

It is easily seen that

$$\mathfrak{m}^{k+1}\cdot I(\omega)\subset I^{(k+1)}(\omega).$$

Here we propose the following

(3.6) **Question.** $I^{(k+1)}(\omega)\subset J(\omega)+K(\omega)+\mathfrak{m}^{k+1}\cdot I(\omega)$?

Thus if (3.6) is true, the conditions in (3.5) Proposition are all equivalent to

$$I(\omega)=[h_1, \cdots, h_m]_{\boldsymbol{C}}+J(\omega)+K(\omega)+\mathfrak{m}^{k+1}\cdot I(\omega).$$

(3.7) **Remark.** If $\omega=df, f \in \mathfrak{m}$, then $I(\omega)=\mathcal{O}_n$, $J(\omega)=J(f)$, $K(\omega)=f^*\mathcal{O}_1$ and

$$I^{(k+1)}(\omega)=\{h \in \mathcal{O}_n \mid j^k(dh-gdf)=0 \text{ for some } g \in \mathcal{O}_n\}.$$

We denote the last space by $I^{(k+1)}(f)$. Let $\tilde{f}$ be an unfolding of f, i.e., a germ in $\mathcal{O}_{n+m}$ for some m such that $\tilde{f}(x, 0)=f(x)$. We set $f^{(1_j)}(x)=\partial\tilde{f}/\partial x_j(x, 0)$ for $j=1, \cdots, m$. Also, if we let $\tilde{\omega}=d\tilde{f}$, $\mathscr{F}=(\tilde{\omega})$ is an unfolding of $F=(df)$ and h_j in (2.13) is equal to $f^{(1_j)}$. Hence by (3.5), $\mathscr{F}$ is a k-transversal unfolding of F if and only if

$$\mathcal{O}_n=[f^{(1_1)}, \cdots, f^{(1_m)}]_C+J(f)+f^*\mathcal{O}_1+I^{(k+1)}(f).$$

Moreover, if (3.6) is true, then this is equivalent to

$$\mathcal{O}_n=[f^{(1_1)}, \cdots, f^{(1_m)}]_C+J(f)+f^*\mathcal{O}_1+\mathfrak{m}^{k+1},$$

which is exactly the condition that $\tilde{f}$ be a right-left k-transversal unfolding of f ([4] Lemma 3.13).

Now we consider the relation between k-transversality and (infinitesimal) versality.

(3.8) **Definition.** An unfolding $\mathscr{F}$ of $F=(\omega)$ with parameter space C^m is k-trivial if it has a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$ and that

$$j^k\omega_t=j^k\omega \quad \text{for all } t \text{ near } 0 \text{ in } C^m,$$

where $\omega_t=\iota_t^*\tilde{\omega}$, ι_t denotes the embedding $\iota_t(x)=(x, t)$ of C^n into $C^n\times C^m$ and $\iota=\iota_0$.

Let $\mathscr{F}$ be a k-trivial unfolding of F and let $\tilde{\omega}$ be a generator satisfying the conditions in (3.8). If we write $\tilde{\omega}$ as (2.13), we see that the $\omega^{(1_j)}$'s are in $\pi_k^{-1}(0)\cap\Omega(\omega)$. Conversely, any element in $\pi_k^{-1}(0)\cap\Omega(\omega)$ determines a (may not be unique) first order unfolding which is k-trivial. Hence we give the following (cf. (1.6) Remark).

(3.9) **Definition.** A codim 1 foliation germ $F=(\omega)$ is infinitesimally k-determined if

$$\pi_k^{-1}(0)\cap\Omega(\omega)\subset L_{\mathfrak{m}}(\omega)+\mathcal{O}_n\omega.$$

Note that the above condition does not depend on the choice of the generator ω of F. Also the condition is equivalent to

$$I^{(k+1)}(\omega)\subset\mathfrak{m}J(\omega)+K(\omega).$$

The following is a direct consequence of (3.5) Proposition (see also (2.12) Remark).

(3.10) **Theorem.** *Let $\mathscr{F}$ be an unfolding of a codim* 1 *foliation germ F.*

(a) *If $\mathscr{F}$ is infinitesimally RL-versal, then it is k-transversal for all k.*

(b) *Suppose that F is infinitesimally k-determined for some k. If $\mathscr{F}$ is k-transversal, then it is infinitesimally RL-versal, thus it is an RL-versal unfolding of F.*

§ 4. Determinacy of foliation germs

If $\mathscr{F}=(\tilde{\omega})$ is an unfolding (with parameter space $\boldsymbol{C}$) of a codim 1 foliation germ $F=(\omega)$ at 0 in $\boldsymbol{C}^n$, let ι_t be the embedding of $\boldsymbol{C}^n$ into $\boldsymbol{C}^n\times\boldsymbol{C}$ defined by $\iota_t(x)=(x, t)$ and set $\omega_t=\iota_t^*\tilde{\omega}$ as in the previous sections. In this section, we prove the following

(4.1) **Theorem.** *Let $F=(\omega)$ be a codim 1 foliation germ at 0 in $\boldsymbol{C}^n$. Suppose that*

(i) $\dim_{\boldsymbol{C}} K(\omega)/\mathfrak{m}J(\omega)\cap K(\omega)<+\infty$ *and that*

(ii) *F is infinitesimally k-determined,* i.e.,

$$\pi_k^{-1}(0)\cap\Omega(\omega)\subset L_{\mathfrak{m}}(\omega)+\mathcal{O}_n\omega$$

for some non-negative integer k. Then every k-trivial unfolding of F is strongly RL-isomorphic to the trivial unfolding F. More precisely, for any unfolding $\mathscr{F}=(\tilde{\omega})$ of F with parameter space $\boldsymbol{C}$ such that $j^k\omega_t=j^k\omega$ for all t near 0 in $\boldsymbol{C}$, there exists a triple (Φ, u, α) such that

(a) *Φ is a holomorphic map germ making the diagram*

$$\begin{array}{ccc} (\boldsymbol{C}^n\times\boldsymbol{C}, 0) & \xrightarrow{\Phi} & (\boldsymbol{C}^n\times\boldsymbol{C}, 0) \\ \downarrow & & \downarrow \\ (\boldsymbol{C}, 0) & \xrightarrow{\mathrm{id}} & (\boldsymbol{C}, 0) \end{array}$$

commutative, where the vertical maps are the projections. u is a unit in $\mathcal{O}_{n+1}$ and α is in $\mathcal{O}_{n+1}$.

(b) *$\Phi(x, 0)=(x, 0)$, $\Phi(0, t)=(0, t)$ and $u(x, 0)=1$.*

(c) *$u\Phi^*(\tilde{\omega}+\alpha dt)=\omega$,*

where we think of ω as a germ at 0 in $\boldsymbol{C}^n\times\boldsymbol{C}$.

(4.2) **Remark.** From the condition (a) above, we may write $\Phi(x, t)=(\phi(x, t), t)$ for some holomorphic map germ ϕ: $(\boldsymbol{C}^n\times\boldsymbol{C}, 0)\to(\boldsymbol{C}^n, 0)$ and the condition (b) is equivalent to

(b′) $\phi(x, 0)=x$, $\phi(0, t)=0$ and $u(x, 0)=1$.

Hence if we set, for each t near 0, $\phi_t=\iota_t^*\phi$ and $u_t=\iota_t^*u$, then ϕ_t is in L_n and u_t is in U_n. From the condition (c), we have $u_t\phi_t^*\omega_t=\omega$, which means that ω_t is in the G_n orbit of ω. Thus the foliation $F_t=(\omega_t)$ is equivalent to $F=(\omega)$.

(4.3) **Remark.** Consider the following conditions

(A) $\dim_C I(\omega)/J(\omega) < +\infty$,
(B) $\dim_C I(\omega)/\mathfrak{m}J(\omega) < +\infty$,
(C) $\dim_C I(\omega)/\mathfrak{m}J(\omega) + K(\omega) < +\infty$,
(D) $\dim_C K(\omega)/\mathfrak{m}J(\omega) \cap K(\omega) < +\infty$,
(E) $\mathfrak{m}^{k+1} \cdot I(\omega) \subset \mathfrak{m}J(\omega)$ for some k,
(F) $\mathfrak{m}^{k+1} \cdot I(\omega) \subset \mathfrak{m}J(\omega) + K(\omega)$ for some k,
(G) $I^{(k+1)}(\omega) \subset \mathfrak{m}J(\omega) + K(\omega)$ for some k,
(H) $\pi_k^{-1}(0) \cap \Omega(\omega) \subset L_{\mathfrak{m}}(\omega) + \mathcal{O}_n\omega$ for some k.

Then we have the implications

$$\begin{tikzcd}
& & (C) \arrow[dr, Rightarrow, dashed, shift left] & & & \\
(A) \arrow[r, Leftrightarrow] & (B) \arrow[ur, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Leftrightarrow] & & (F) \arrow[ul, Rightarrow, shift left] \arrow[r, Rightarrow, dashed, shift right] \arrow[dl, Rightarrow, dashed, shift right] & (G) \arrow[l, Rightarrow, shift right] \arrow[r, Leftrightarrow] & (H). \\
& (D) & (E) \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

In fact, (A)$\Leftrightarrow$(B) and (B)$\Rightarrow$(C) are obvious. Since $K(\omega)/\mathfrak{m}J(\omega) \cap K(\omega) \simeq \mathfrak{m}J(\omega) + K(\omega)/\mathfrak{m}J(\omega)$, (B)$\Rightarrow$(D). From (B), we have $\mathfrak{m}^{k+1} \cdot I(\omega) \subset \mathfrak{m}J(\omega) + \mathfrak{m}^{k+2} \cdot I(\omega)$ for some k. Then using Nakayama's lemma, we get (E). We have (E)$\Rightarrow$(B) and (F)$\Rightarrow$(C) since $I(\omega)$ is finitely generated. The implications (G)$\Rightarrow$(F) and (G)$\Leftrightarrow$(H) are obvious (see Section 3). If the answer to (3.6) is yes, then (F) implies (G). We also propose

Question. (C)$\Rightarrow$(F)$\Rightarrow$(E)?

(4.4) **Remark.** If we assume that $K(\omega) \subset \mathfrak{m}J(\omega)$ in (4.1) Theorem, the subsequent proof shows that we may let $\alpha = 0$. Thus in this case, $\mathscr{F}$ is strongly isomorphic to the trivial unfolding F.

The rest of this section is spent for the proof of the theorem.

Proof of (4.1) *Theorem.* Let $\mathscr{F} = (\tilde{\omega})$ be an arbitrarily given k-trivial unfolding of F with parameter space C and let $h(\in \mathcal{O}_{n+1})$ be the coefficient of dt in $\tilde{\omega}$. We set $h_t = \iota_t^* h$ for each t near 0. If we also set $\alpha_t = \iota_t^* \alpha$, then the condition (c) is equivalent to

$$u_t \phi_t^* \omega_t = \omega \quad \text{and} \tag{4.5}$$

$$\langle Z_t, \phi_t^* \omega_t \rangle + \phi_t^* h_t + \phi_t^* \alpha_t = 0, \tag{4.6}$$

where Z_t denotes the vector field $\sum_{i=1}^n \partial\phi_i/\partial t\,(x, t)\,\partial/\partial x_i$. If $\phi(x, 0) = x$ and $u(x, 0) = 1$, then (4.5) holds when $t = 0$. Hence we may replace (4.5) by the equation obtained by differentiating the both sides with respect to t;

$$\frac{\partial u_t}{\partial t}\cdot\phi_t^*\omega_t+u_t\cdot\frac{\partial}{\partial t}(\phi_t^*\omega_t)=0. \tag{4.5$'$}$$

(4.7) **Lemma.** *Suppose there exist a holomorphic map germ* $\xi\colon(C^n\times C, 0)\to(C^n, 0)$ *and a germ* g *in* $\mathcal{O}_{n+1}$ *such that*

$$g_t\cdot\omega_t+L_{X_t}\omega_t+\frac{\partial}{\partial t}\omega_t=0, \tag{4.8}$$

$$\langle X_t, \omega_t\rangle+h_t+\alpha_t=0 \quad \textit{and} \tag{4.9}$$

$$\xi(0, t)=0, \tag{4.10}$$

where

$$X_t=\sum_{i=1}^n \xi_i(x, t)\frac{\partial}{\partial x_i} \quad \textit{and} \quad g_t=\iota_t^* g,$$

then there exist ϕ, u *and* α *satisfying the conditions of the theorem.*

Proof. First we solve the differential equations

$$\frac{\partial\phi}{\partial t}(x, t)=\xi(\phi(x, t), t) \quad \text{and} \quad \frac{\partial v}{\partial t}(x, t)=g(\phi(x, t), t) \tag{4.11}$$

for ϕ and v under the initial condition $\phi(x, 0)=x$ and $v(x, 0)=0$ and set $u(x, t)=e^{v(x,t)}$. Then using

$$\phi_t^*(g_t\cdot\omega_t)=\frac{1}{u_t}\frac{\partial u_t}{\partial t}\cdot\phi_t^*\omega_t \quad \text{and}$$

$$\phi_t^*\left(L_{X_t}\omega_t+\frac{\partial}{\partial t}\omega_t\right)=\frac{\partial}{\partial t}(\phi_t^*\omega_t),$$

we get (4.5)$'$ from (4.8). Also, since

$$\phi_t^*\langle X_t, \omega_t\rangle=\langle Z_t, \phi_t^*\omega_t\rangle,$$

we get (4.6) from (4.9). If we set $x=0$ in the first equation in (4.11), we get

$$\frac{\partial\phi}{\partial t}(0, t)=\xi(\phi(0, t), t) \qquad \text{with } \phi(0, 0)=0.$$

The above equation has also 0 as a solution. Hence by the uniqueness of solution, we have $\phi(0, t)=0$. Q.E.D.

Thus it suffices to solve (4.8) and (4.9) for ξ (or $X_t=\sum_{i=1}^n \xi_i(x, t)\partial/\partial x_i$), g and α under the condition (4.10). First we find solutions as formal power series in t, then we show the existence of convergent solutions.

We express $\xi(x, t)$, $g(x, t)$ and $\alpha(x, t)$ as power series in t;

$$\xi(x, t)=\sum_{p\geq 0} \xi^{(p)}(x)t^p, \qquad g(x, t)=\sum_{p\geq 0} g^{(p)}(x)t^p \quad \text{and}$$
$$\alpha(x, t)=\sum_{p\geq 0} \alpha^{(p)}(x)t^p.$$

In general, for a series $\sigma(x, t)=\sum_{p\geq 0}\sigma^{(p)}(x)t^p$, we set

$$\sigma^{|p}(x, t)=\sum_{q=0}^{p} \sigma^{(q)}(x)t^q \quad \text{and} \quad [\sigma(x, t)]_p=\sigma^{(p)}(x)t^p.$$

If we set

$$X^{(p)}=\sum_{i=1}^{n} \xi_i^{(p)}\frac{\partial}{\partial x_i} \quad \text{and} \quad X^{|p}=\sum_{q=0}^{p} X^{(q)}t^q,$$

then (4.8) and (4.9) are equivalent to the congruences

$$(4.8)_p \qquad g^{|p}\cdot\omega_t+L_{X^{|p}}\omega_t+\frac{\partial}{\partial t}\omega_t\underset{p}{\equiv}0 \quad \text{and}$$

$$(4.9)_p \qquad \langle X^{|p}, \omega_t\rangle+\alpha^{|p}+h_t\underset{p}{\equiv}0,$$

for all $p\geq 0$, where $\underset{p}{\equiv}$ denotes the equality mod t^{p+1}. Also, (4.10) is equivalent to

$$(4.10)_p \qquad \xi_i^{(q)}\in \mathfrak{m}, \quad 1\leq i\leq n, \quad 0\leq q\leq p$$

for all $p\geq 0$.

Now from the integrability $d\tilde{\omega}\wedge\tilde{\omega}=0$ of $\tilde{\omega}$, we get

$$(4.12) \qquad h_t d_x\omega_t+\left(\frac{\partial}{\partial t}\omega_t-d_x h_t\right)\wedge\omega_t=0,$$

where d_x denotes the exterior derivative with respect to $x=(x_1, \cdots, x_n)$. If we write

$$\omega_t=\sum_{p\geq 0}\omega^{(p)}t^p, \qquad \omega^{(0)}=\omega,$$

then the condition that $j^k\omega_t=j^k\omega$ for all t near 0 implies that

$$(4.13) \qquad \omega^{(p)}\in\pi_k^{-1}(0) \qquad \text{for } p\geq 1.$$

The following is proved in [1] p. 47, Théorème 5.1 and [13].

(4.14) **Lemma.** *For any germ β in $K(\omega)$, there is a unique germ $\tilde{\beta}$ in $K(\tilde{\omega})$ that unfolds β.*

Thus $\tilde{\beta}$ is a germ in $\mathcal{O}_{n+1}$ such that $\tilde{\beta}(x, 0)=\beta(x)$ and that $\tilde{\beta}d\tilde{\omega}=d\tilde{\beta}\wedge\tilde{\omega}$. If we write $\tilde{\beta}(x, t)=\sum_{p\geq 0}\beta^{(p)}(x)t^p$, from the last condition, we have

$$\tilde{\beta}|^p d_x\omega_t \underset{p}{\equiv} d_x\tilde{\beta}|^p \wedge\omega_t \qquad \text{for all } p\geq 0,$$

which implies that

$$(4.15)\qquad (\beta^{(p+1)}d\omega-d\beta^{(p+1)}\wedge\omega)t^{p+1}=-[\tilde{\beta}|^p d_x\omega_t-d_x\tilde{\beta}|^p\wedge\omega_t]_{p+1} \qquad \text{for all } p\geq 0.$$

$(4.16)_p$ *Induction hypothesis.* There exist $\xi^{(q)}$ in $\mathcal{O}_n^n$, $g^{(q)}$ and $\alpha^{(q)}$ in $\mathcal{O}_n$ and $\tilde{\alpha}^{(q)}$ in $\mathcal{O}_{n+1}$ for $q=0, \cdots, p$ such that $(4.8)_p$, $(4.9)_p$ and $(4.10)_p$ hold and that if we write $\tilde{\alpha}^{(q)}=\sum_{r\geq 0}\alpha^{(q,r)}(x)t^r$, then

(a) $$\alpha^{(q,0)}\in K(\omega) \quad \text{and} \quad \tilde{\alpha}^{(q)}\in K(\tilde{\omega}),$$

(b) $$\alpha^{(q)}=\sum_{r=0}^{q}\alpha^{(r,q-r)}.$$

For the existence of formal solutions, it suffices to show that $(4.16)_0$ holds and that $(4.16)_p$ implies $(4.16)_{p+1}$. First we note that the following follows from (2.6) Lemma.

(4.17) **Lemma.** *Suppose that*

$$ed\omega+(\theta-de)\wedge\omega=0$$

for e in $\mathcal{O}_n$ and θ in Ω_n. If θ is in $L_{\mathfrak{m}}(\omega)+\mathcal{O}_n\omega$, then there exist g_0 in $\mathcal{O}_n$, X_0 in $\mathfrak{m}\Theta_n$ and α_0 in $K(\omega)$ such that

$$g_0\omega+L_{X_0}+\theta=0 \quad \textit{and} \quad \langle X_0, \omega\rangle+\alpha_0+e=0.$$

Thus e is in $\mathfrak{m}J(\omega)+K(\omega)$.

(4.18) **Lemma.** $(4.16)_0$ *holds.*

Proof. $(4.8)_0$ and $(4.9)_0$ read

$$g^{(0)}\omega+L_{X^{(0)}}\omega+\omega^{(1)}=0 \quad \text{and} \quad \langle X^{(0)}, \omega\rangle+\alpha^{(0)}+h^{(1)}=0,$$

where $h^{(1)}(x)=h(x, 0)$. Now from (4.12), we have

$$h^{(1)}d\omega+(\omega^{(1)}-dh^{(1)})\wedge\omega=0.$$

On the other hand, $\omega^{(1)}$ is in $\pi_k^{-1}(0)\cap\Omega(\omega)$ ((4.13)). Using the assumption (ii) of (4.1) Theorem, we see that (4.17) Lemma implies the existence of $g^{(0)}$, $X^{(0)}$ and $\alpha^{(0)}$ satisfying the two equations above. We set $\alpha^{(0,0)}=\alpha^{(0)}$ and let $\tilde{\alpha}^{(0)}=\sum_{r\geq 0}\alpha^{(0,r)}t^r$ be the unfolding of $\alpha^{(0,0)}$ with $\tilde{\alpha}^{(0)}\in K(\tilde{\omega})$ (cf. (4.14)).

(4.19) **Lemma.** $(4.16)_p$ *implies* $(4.16)_{p+1}$.

Proof. Suppose $(4.16)_p$ holds and set

$$\theta^{(p+1)}t^{p+1}=\left[g^{|p}\omega_t+L_{X|p}\omega_t+\frac{\partial}{\partial t}\omega_t\right]_{p+1}\quad\text{and}$$

$$\begin{aligned}e^{(p+1)}t^{p+1}&=[\langle X^{|p},\omega_t\rangle+\alpha^{|p}+h_t]_{p+1}\\&=[\langle X^{|p},\omega_t\rangle+h_t]_{p+1}.\end{aligned}$$

Then $(4.8)_{p+1}$ and $(4.9)_{p+1}$ are equivalent to

$$g^{(p+1)}\cdot\omega+L_{X^{(p+1)}}\omega+\theta^{(p+1)}=0\quad\text{and}\tag{4.8*}$$

$$\langle X^{(p+1)},\omega\rangle+\alpha^{(p+1)}+e^{(p+1)}=0.\tag{4.9*}$$

Thus we look for $g^{(p+1)}$, $X^{(p+1)}$ and $\alpha^{(p+1)}$ satisfying (4.8)* and (4.9)*. First we set

$$\tau^{(p+1)}=e^{(p+1)}d\omega+(\theta^{(p+1)}-de^{(p+1)})\wedge\omega$$

and compute this quantity. We claim that the following three identities hold:

$$\begin{aligned}&\left[h_t d\omega+\left(\frac{\partial}{\partial t}\omega_t-d_xh_t\right)\wedge\omega\right]_{p+1}\\&\quad=-\left[h_t^{|p}d\omega_t+\left(\frac{\partial}{\partial t}\omega_t-d_xh_t\right)^{|p}\wedge\omega_t\right]_{p+1},\end{aligned}\tag{4.20}$$

$$\begin{aligned}&[\langle X^{|p},\omega_t\rangle d\omega+(L_{X|p}\omega_t-d_x\langle X^{|p},\omega_t\rangle)\wedge\omega]_{p+1}\\&\quad=-[(\langle X^{|p},\omega_t\rangle)^{|p}d\omega_t+(L_{X|p}\omega_t-d_x\langle X^{|p},\omega_t\rangle)^{|p}\wedge\omega_t]_{p+1}\end{aligned}\tag{4.21}$$

and

$$[g^{|p}\omega_t\wedge\omega]_{p+1}=-[(g^{|p}\omega_t)^{|p}\wedge\omega_t]_{p+1}.\tag{4.22}$$

In fact, (4.20) follows from (4.12), (4.21) follows from (2.6) Lemma with

ω and X replaced by ω_t and $X^{|p}$, respectively, and (4.22) follows from $g^{|p}\omega_t\wedge\omega_t=0$. Now substituting (4.20), (4.21) and (4.22) in the expression of $\theta^{(p+1)}$ and $e^{(p+1)}$ and using $(4.8)_p$ and $(4.9)_p$, we obtain

$$\tau^{(p+1)}t^{p+1}=[\alpha^{|p}d_x\omega_t-d_x\alpha^{|p}\wedge\omega_t]_{p+1}.$$

We have

$$\begin{aligned}\alpha^{|p}&=\sum_{q=0}^{p}\alpha^{(q)}t^q=\sum_{q=0}^{p}\sum_{r=0}^{q}\alpha^{(r,q-r)}t^q\\&=\sum_{q=0}^{p}\left(\sum_{r=0}^{p-q}\alpha^{(q,r)}t^r\right)t^q=\sum_{q=0}^{p}(\tilde{\alpha}^{(q)})^{|p-q}t^q.\end{aligned}$$

Thus applying (4.15) to each $\tilde{\alpha}^{(q)}$ and noting that the term of order $p-q+1$ in $\tilde{\alpha}^{(q)}$ is $\alpha^{(q,p-q+1)}t^{p-q+1}$, we obtain

$$\tau^{(p+1)}=-\sum_{q=0}^{p}(\alpha^{(q,p-q+1)}d\omega-d\alpha^{(q,p-q+1)}\wedge\omega). \tag{4.23}$$

Thus if we set $e=e^{(p+1)}+\sum_{q=0}^{p}\alpha^{(q,p-q+1)}$, then

$$ed\omega+(\theta^{(p+1)}-de)\wedge\omega=0.$$

This shows that $\theta^{(p+1)}$ is in $\Omega(\omega)$. On the other hand, by (4.13), $\theta^{(p+1)}$ is also in $\pi_k^{-1}(0)$. Hence

$$\theta^{(p+1)}\in\pi_k^{-1}(0)\cap\Omega(\omega)\subset L_{\mathrm{m}}(\omega)+\mathcal{O}_n\omega.$$

Therefore, by (4.17) Lemma, there exist g_0, X_0 and α_0 such that

$$g_0\cdot\omega+L_{X_0}\omega+\theta^{(p+1)}=0\quad\text{and}\quad\langle X_0,\omega\rangle+\alpha_0+e=0.$$

We set $g^{(p+1)}=g_0$, $X^{(p+1)}=X_0$, $\alpha^{(p+1,0)}=\alpha_0$ and

$$\alpha^{(p+1)}=\alpha_0+\sum_{q=0}^{p}\alpha^{(q,p-q+1)}=\sum_{q=0}^{p+1}\alpha^{(q,p-q+1)}$$

and let $\tilde{\alpha}^{(p+1)}$ be the unfolding of $\alpha^{(p+1,0)}$ with $\tilde{\alpha}^{(p+1)}\in K(\tilde{\omega})$. Then $(4.16)_{p+1}$ holds. Q.E.D.

Now we prove the existence of convergent solutions. First we recall the Malgrange privileged neighborhoods theorem ([4]). For an n-tuple $\rho=(\rho_1,\cdots,\rho_n)$ of positive real numbers, we set

$$P(\rho)=\{x\in\boldsymbol{C}^n\,|\,|x_i|\leq\rho_i,\ 1\leq i\leq n\}.$$

For a germ f in $\mathcal{O}_n$, we write $f(x)=\sum_{|\alpha|\geq 0}a_\alpha x^\alpha$, where $\alpha=(\alpha_1,\cdots,\alpha_n)$ is an n-tuple of non-negative integers, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ as usual, and set $|f|_\rho = \sum_{|\alpha| \geq 0} a_\alpha \rho^\alpha$. If $f = (f_1, \cdots, f_r)$ is in $\mathcal{O}_n^r$, we set $|f|_\rho = \sum_{i=1}^r |f_i|_\rho$. The $\mathcal{O}_n$-modules Ω_n and Θ_n are both naturally identified with $\mathcal{O}_n^n$.

We fix a basis $[\beta_1], \cdots, [\beta_N]$ of the $\boldsymbol{C}$-vector space $K(\omega)/\mathfrak{m}J(\omega) \cap K(\omega)$. We choose open neighborhoods U and V of the origins in $\boldsymbol{C}^n$ and $\boldsymbol{C}$, respectively, so that the germs $\omega, \beta_1, \cdots, \beta_N$ have representatives on U and that the germ $\tilde{\omega}$ has a representative on $U \times V$. Consider the $\mathcal{O}_n$-homomorphisms

$$\lambda\colon \Theta_n \longrightarrow \mathcal{O}_n, \qquad \lambda(X) = \langle X, \omega \rangle,$$

$$\mu\colon \Theta_n^n \longrightarrow \mathcal{O}_n, \qquad \mu(X_1, \cdots, X_n) = \sum_{i=1}^n x_i \langle X_i, \omega \rangle \quad \text{and}$$

$$\nu\colon \mathcal{O}_n \longrightarrow \Omega_n, \qquad \nu(g) = g\omega.$$

By Malgrange [4] Théorème (1.1), there exists ρ such that $P(\rho) \subset U$ and that the homomorphisms λ, μ and ν have fissions simultaneously adapted to ρ, i.e., we have

(4.24) **Lemma.** *There exist $\rho = (\rho_1, \cdots, \rho_n)$, $\rho_i > 0$, and a positive constant K such that $P(\rho) \subset U$ and that*

(a) *every germ e in $J(\omega)$ ($= \operatorname{Im} \lambda$) can be written as*

$$e = \langle X_0, \omega \rangle \qquad \text{for } X_0 \text{ in } \Theta_n \text{ with}$$
$$|X_0|_{a\rho} \leq K|e|_{a\rho} \qquad \text{for } \tfrac{1}{2} \leq a \leq 1,$$

(b) *every germ e in $\mathfrak{m}J(\omega)$ ($= \operatorname{Im} \mu$) can be written as*

$$e = \sum_{i=1}^n x_i \langle X_i, \omega \rangle \qquad \text{for } (X_1, \cdots, X_n) \text{ in } \Theta_n^n \text{ with}$$
$$|X_i|_{a\rho} \leq K|e|_{a\rho} \qquad \text{for } \tfrac{1}{2} \leq a \leq 1,$$

(c) *every germ θ in $\mathcal{O}_n\omega$ ($= \operatorname{Im} \nu$) can be written as $\theta = g_0\omega$ for g_0 in $\mathcal{O}_n$ with*

$$|g_0|_{a\rho} \leq K|\theta|_{a\rho} \qquad \text{for } \tfrac{1}{2} \leq a \leq 1.$$

Note that g_0 in (4.24) (c) is uniquely determined by θ, since ν is injective.

We choose ρ with the properties in (4.24) Lemma and fix it once for all.

(4.25) **Lemma.** *There is a positive constant K_1 such that every germ e in $\mathfrak{m}J(\omega)$ can be written as $e = \langle X_0, \omega \rangle$ for X_0 in $\mathfrak{m}\Theta_n$ with*

$$|X|_{a\rho}\leq K_1|e|_{a\rho} \qquad \textit{for } \tfrac{1}{2}\leq a\leq 1.$$

Proof. Let $(X_1, \cdots, X_n)$ be as in (4.24) (b). If we set $e_i=\langle X_i, \omega\rangle$ for $i=1, \cdots, n$, then

$$|e_i|_{a\rho}\leq K_0|X_i|_{a\rho} \qquad \tfrac{1}{2}\leq a\leq 1,$$

where $K_0=|\omega|_\rho$. On the other hand, by (4.24) (a), there exist $X_1', \cdots, X_n'$ in Θ_n such that $e_i=\langle X_i', \omega\rangle$ and that

$$|X_i'|_{a\rho}\leq K|e_i|_{a\rho}, \qquad \tfrac{1}{2}\leq a\leq 1.$$

If we set $X_0=\sum_{i=1}^n x_i X_i'$, then $X_0\in \mathfrak{m}\Theta_n$, $\langle X_0, \omega\rangle=e$ and

$$|X_0|_{a\rho}\leq\sum_{i=1}^n|x_i|_{a\rho}|X_i'|_{a\rho}\leq K_0K^2\sum_{i=1}^n\rho_i\cdot|e|_{a\rho}.$$

Thus $K_1=K_0K^2\sum_{i=1}^n\rho_i$ satisfies the requirement of the lemma. Q.E.D.

Let $\sigma=\sum\sigma^{(p)}(x)t^p$ be a series with $\sigma^{(p)}$ in $\mathcal{O}_n^r$ and let $\sum a^{(p)}t^p$ be a series with $a^{(p)}$ positive real numbers. We say that $\sum a^{(p)}t^p$ dominates σ in $P(\rho)$ and write

$$\sum\sigma^{(p)}t^p\ll\sum a^{(p)}t^p \qquad \text{in } P(\rho)$$

if $|\sigma^{(p)}|_\rho\leq a^{(p)}$ for all $p\geq 0$. Consider the series ([3] p. 291, [2] p. 50, here we set $b=c^{1/3}$)

$$A(t)=\frac{1}{16c^{2/3}}\sum_{p\geq 1}\frac{c^p}{p^2}t^p,$$

where c is a positive constant to be determined later. We let $A'(t)$ be the series obtained by differentiating $A(t)$ with respect to t;

$$A'(t)=\frac{1}{16c^{2/3}}\sum_{p\geq 0}\frac{c^{p+1}}{p+1}t^p.$$

We have ([3] (19), [2] Lemma 3.6)

$$A(t)^2\ll\frac{1}{c^{2/3}}A(t). \tag{4.26}$$

From this we get

$$A'(t)A(t)\ll\frac{1}{2c^{2/3}}A'(t). \tag{4.27}$$

We set $A_1(t)=c^{1/3}A(t)$, $A_2(t)=c^{2/3}A(t)$ and $A_2'(t)=c^{2/3}A'(t)$ and prove that there exist ξ (or $X_t=\sum_{i=1}^n \xi_i(x, t)\partial/\partial x_i$), g and α such that if we choose c sufficiently large, then the following estimates hold for all non-negative integers p:

$(4.28)_p$ $$\xi^{|p}-\xi^{(0)}\ll A_2\left(\frac{t}{1-a}\right) \quad \text{or equivalently}$$

$$X^{|p}-X^{(0)}\ll A_2\left(\frac{t}{1-a}\right),$$

$(4.29)_p$ $$\alpha^{|p}-\alpha^{(0)}\ll A_2\left(\frac{t}{1-a}\right) \quad \text{and}$$

$(4.30)_p$ $$g^{|p}\ll\frac{1}{1-a}A_2'\left(\frac{t}{1-a}\right) \quad \text{in } P(a\rho) \text{ for } \tfrac{1}{2}\leq a<1.$$

First we need some estimates. If we let $\tilde{\beta}_i$ be the unfolding of β_i with $\tilde{\beta}_i \in K(\tilde{\omega})$ for each $i=1, \cdots, N$, then we have ([13] (A.2) Theorem and (4.7) Remark)

(4.31) **Lemma.** *If c is sufficiently large, then*

$$\tilde{\beta}_i-\beta_i\ll c^{1/6}A\left(\frac{t}{1-a}\right) \quad \text{in } P(a\rho), \quad \tfrac{1}{2}\leq a<1.$$

Using (4.25) Lemma, the following is proved similarly as [9] (3.18) Lemma.

(4.32) **Lemma.** *There exists a positive constant K_2 such that every element e in $\mathfrak{m}J(\omega)+K(\omega)$ can be written as*

$$e=\langle X_0, \omega\rangle+\sum_{i=1}^{N} c_i\beta_i$$

for some X_0 in $\mathfrak{m}\Theta_n$ and constants c_i with

$$|X_0|_{a\rho}\leq K_2|e|_{a\rho}, \quad |c_i|\leq K_2|e|_{a\rho} \quad \text{for } \tfrac{1}{2}\leq a\leq 1.$$

By [4] Lemma (2.4), there exists a constant c_1 such that for every germ f in $\mathcal{O}_n$,

$$\left|\frac{\partial f}{\partial x_i}\right|_{a\rho}\leq\frac{c_1}{b-a}|f|_{b\rho} \quad \text{for } \tfrac{1}{2}\leq a<b\leq 1.$$

From this we have

(4.33) **Lemma.** *There exists a positive constant K_3 such that for every X_0 in Θ_n,*

$$|L_{X_0}\omega|_{a\rho}\leq\frac{K_3}{b-a}|X_0|_{b\rho},\quad \tfrac{1}{2}\leq a<b\leq 1.$$

The following is proved similarly as in [9] p. 42.

(4.34) **Lemma.** *If, for a series* $\sum \sigma^{(p)}t^p$,

$$\sum \sigma^{(p)}t^p \ll \frac{1}{b-a}A\left(\frac{t}{1-b}\right), \qquad \textit{in } P(a\rho) \textit{ for } \tfrac{1}{2}\leq a<b<1,$$

then

$$\sum \sigma^{(p)}t^p \ll \frac{K_4}{c}\frac{1}{1-a}A'\left(\frac{t}{1-a}\right), \quad \textit{in } P(a\rho) \textit{ for } \tfrac{1}{2}\leq a<1,$$

where K_4 *is a positive constant.*

$(4.35)_p$ *Induction hypothesis.* There exist $\xi^{(q)}$ in $\mathcal{O}_n^n$ (or $X^{(q)}$ in Θ_n), $g^{(q)}$ in $\mathcal{O}_n$ and constants $c_i^{(q)}$, $i=1,\cdots,N$, for $q=0,\cdots,p$ such that if we write $\sum_{i=1}^N c_i^{(q)}\tilde{\beta}_i=\sum_{r\geq 0}\alpha^{(q,r)}t^r$ and set $\alpha^{(q)}=\sum_{r=0}^q \alpha^{(r,q-r)}$, then $(4.8)_p$, $(4.9)_p$ and $(4.10)_p$ hold and that we have the estimates $(4.28)_p$, $(4.29)_p$, $(4.30)_p$ and

$$(4.36)_p \qquad \sum_{q=1}^p c_i^{(q)}t^q \ll A_1\left(\frac{t}{1-a}\right), \quad \tfrac{1}{2}\leq a<1.$$

(4.37) **Lemma.** *If we take c sufficiently large, then* $(4.35)_0$ *holds.*

Proof. As in (4.18) Lemma, $h^{(1)}$ is in $\mathfrak{m}J(\omega)+K(\omega)$. Hence by (4.32) Lemma, we may write

$$\langle X_0, \omega\rangle+\sum_{i=1}^N c_i\beta_i+h^{(1)}=0$$

for some X_0 in $\mathfrak{m}\Theta_n$ and constants c_i with

$$|X_0|_{a\rho}\leq K_2|h^{(1)}|_{a\rho},\quad |c_i|\leq K_2|h^{(1)}|_{a\rho},\quad \tfrac{1}{2}\leq a\leq 1.$$

If we set $X^{(0)}=X_0$, $c_i^{(0)}=c_i$ and $\alpha^{(0)}=\alpha^{(0,0)}=\sum_{i=1}^N c_i^{(0)}\beta_i$, then $(4.9)_0$ and $(4.10)_0$ hold. Using the identity

$$h^{(1)}d\omega+(\omega^{(1)}-dh^{(1)})\wedge\omega=0$$

and (2.6) Lemma, we get

$$(L_{X^{(0)}}\omega+\omega^{(1)})\wedge\omega=0.$$

Since codim $S(\omega)\geq 2$, this shows that $L_{X^{(0)}}\omega+\omega^{(1)}$ is in $\mathcal{O}_n\omega$. Thus by (4.24) Lemma (c), there exists g_0 in $\mathcal{O}_n$ such that $L_{X^{(0)}}\omega+\omega^{(1)}=g_0\omega$ and that

$$|g_0|_{a\rho}\leq K|L_{X^{(0)}}\omega+\omega^{(1)}|_{a\rho},\quad \tfrac{1}{2}\leq a\leq 1.$$

If we set $g^{(0)}=-g_0$, then we have $(4.8)_0$. Using (4.33) Lemma and the estimate for $X^{(0)}$ above, we get

$$|g^{(0)}|_{a\rho}\leq KK^{(1)}\left(\frac{K_2K_3}{b-a}+1\right),\quad \tfrac{1}{2}\leq a<b\leq 1,$$

where $K^{(1)}=\max\{|h^{(1)}|_\rho,\ |\omega^{(1)}|_\rho\}$. Thus we have

$$|g^{(0)}|_{a\rho}\leq\frac{K'}{1-a},\quad \tfrac{1}{2}\leq a<1$$

for some constant K'. On the other hand, the constant term in $1/(1-a)A_2'(t/(1-a))$ is $c/16\,(1/(1-a))$. Therefore, if c is sufficiently large, then we have the estimate $(4.30)_0$. Q.E.D.

(4.38) **Lemma.** *If we take c sufficiently large, then* $(4.35)_p$ *implies* $(4.35)_{p+1}$ *for all* $p\geq 0$.

Proof. Let $\theta^{(p+1)}$ and $e^{(p+1)}$ be as in the proof of (4.19) Lemma and set $e=e^{(p+1)}+\sum_{q=0}^{p}\alpha^{(q,p-q+1)}$. Then since e is in $\mathfrak{m}J(\omega)+K(\omega)$, by (4.32) Lemma, we have

$$\langle X_0,\omega\rangle+\sum_{i=1}^{N}c_i\beta_i+e=0$$

for some X_0 in $\mathfrak{m}\Theta_n$ and constants c_i with

$$(4.39)\qquad |X_0|_{a\rho}\leq K_2|e|_{a\rho},\quad |c_i|\leq K_2|e|_{a\rho},\quad \tfrac{1}{2}\leq a\leq 1.$$

If we set

$$X^{(p+1)}=X_0,\quad c_i^{(p+1)}=c_i,\quad \alpha^{(p+1,0)}=\sum_{i=1}^{N}c_i^{(p+1)}\beta_i\quad\text{and}$$

$$\alpha^{(p+1)}=\alpha^{(p+1,0)}+\sum_{q=0}^{p}\alpha^{(q,p-q+1)}=\sum_{q=0}^{p+1}\alpha^{(q,p-q+1)},$$

then $(4.9)_{p+1}$ and $(4.10)_{p+1}$ hold. Using the identity

$$ed\omega+(\theta^{(p+1)}-de)\wedge\omega=0$$

and (2.6) Lemma, we get

$$(L_{X^{(p+1)}}\omega+\theta^{(p+1)})\wedge\omega=0.$$

Thus by (4.24) Lemma (c), there exists g_0 in $\mathcal{O}_n$ such that $L_{X^{(p+1)}}\omega+\theta^{(p+1)}=g_0\omega$ and that

$$|g_0|_{a\rho}\leq K|L_{X^{(p+1)}}\omega+\theta^{(p+1)}|_{a\rho},\quad \tfrac{1}{2}\leq a\leq 1. \tag{4.40}$$

If we set $g^{(p+1)}=-g_0$, then we have $(4.8)_{p+1}$. Now we show the estimates $(4.28)_{p+1}$, $(4.29)_{p+1}$, $(4.30)_{p+1}$ and $(4.36)_{p+1}$. Since $\tilde{\omega}$ is holomorphic in $U\times V$, if c is sufficiently large, we have

$$\omega_t-\omega\ll A\left(\frac{t}{1-a}\right)\quad\text{and}$$

$$h_t-h^{(1)}\ll A\left(\frac{t}{1-a}\right)\quad\text{in } P(a\rho),\quad \tfrac{1}{2}\leq a<1.$$

First we estimate $\theta^{(p+1)}$ and $e^{(p+1)}$. Using $(4.30)_p$ and (4.27), we have

$$\begin{aligned}[g^{|p}\omega_t]_{p+1}=[g^{|p}(\omega_t-\omega)]_{p+1}&\ll\frac{1}{1-a}A_2'\left(\frac{t}{1-a}\right)A\left(\frac{t}{1-a}\right)\\ &\ll\frac{1}{2c^{2/3}}\,\frac{1}{1-a}A_2'\left(\frac{t}{1-a}\right).\end{aligned}$$

Recalling the expression (1.3), using $(4.28)_p$ and (4.26) and noting that $|X^{(0)}|_{a\rho}\leq K_2K^{(1)}$ (cf. (4.37)) and that $1/(1-a)<1/(1-b)$, we have

$$\begin{aligned}[L_{X^{|p}}\omega_t]_{p+1}&=[L_{X^{|p}}(\omega_t-\omega)]_{p+1}\\ &\ll\frac{nc_1}{b-a}\left\{A\left(\frac{t}{1-a}\right)\left(K_2K^{(1)}+A_2\left(\frac{t}{1-b}\right)\right)\right.\\ &\qquad\left.+\left(K_2K^{(1)}+A_2\left(\frac{t}{1-a}\right)\right)A\left(\frac{t}{1-b}\right)\right\}\\ &\ll\frac{2nc_1(K_2K^{(1)}+1)}{b-a}A\left(\frac{t}{1-b}\right).\end{aligned}$$

Hence by (4.34) Lemma,

$$[L_{X^{|p}}\omega_t]_{p+1}\ll\frac{2nc_1K_4(K_2K^{(1)}+1)}{c^{5/3}}\,\frac{1}{1-a}A_2'\left(\frac{t}{1-a}\right).$$

Also we have

$$\left[\frac{\partial}{\partial t}\omega_t\right]_{p+1}\ll\frac{1}{1-a}A'\left(\frac{t}{1-a}\right)=\frac{1}{c^{2/3}}\,\frac{1}{1-a}A_2'\left(\frac{t}{1-a}\right).$$

In what follows we assume that $c\geq 1$. Then from the above, we obtain

$$\theta^{(p+1)}t^{p+1} \ll \frac{K_5}{c^{2/3}} \frac{1}{1-a} A_2'\left(\frac{t}{1-a}\right) \quad \text{in } P(a\rho),\ \tfrac{1}{2} \leq a < 1 \tag{4.41}$$

for some constant K_5. Using $(4.28)_p$ and (4.26), we get

$$\begin{aligned} e^{(p+1)}t^{p+1} &= [\langle X^{|p}, \omega_t\rangle + h_t]_{p+1} = [\langle X^{|p}, \omega_t - \omega\rangle + h_t]_{p+1} \\ &\ll \left(K_2K^{(1)} + A_2\left(\frac{t}{1-a}\right)\right)A\left(\frac{t}{1-a}\right) + A\left(\frac{t}{1-a}\right) \\ &\ll \frac{K_2K^{(1)}+2}{c^{2/3}} A_2\left(\frac{t}{1-a}\right) \quad \text{in } P(a\rho),\ \tfrac{1}{2} \leq a < 1. \end{aligned} \tag{4.42}$$

Next we estimate $\sum_{q=0}^{p} \alpha^{(q,p-q+1)}$. Using $(4.36)_p$ and (4.31) and noting that $|c_i^{(0)}| \leq K_2K^{(1)}$ (cf. (4.37)), we have

$$\begin{aligned} \left(\sum_{q=0}^{p} \alpha^{(q,p-q+1)}\right)t^{p+1} &= \sum_{q=0}^{p}\sum_{i=1}^{N} c_i^{(q)}[\tilde{\beta}_i]_{p-q+1}t^q \\ &\ll N\left(K_2K^{(1)} + A_1\left(\frac{t}{1-a}\right)\right)c^{1/6}A\left(\frac{t}{1-a}\right) \\ &\ll N\left(\frac{K_2K^{(1)}}{c^{1/2}} + \frac{1}{c^{5/6}}\right)\mathrm{A}_2\left(\frac{t}{1-a}\right). \end{aligned}$$

From this and (4.42), we have

$$et^{p+1} \ll \frac{K_6}{c^{1/3}} A_2\left(\frac{t}{1-a}\right) = \frac{K_6}{c^{1/6}} A_1\left(\frac{t}{1-a}\right) \quad \text{in } P(a\rho),\ \tfrac{1}{2} \leq a < 1$$

for some constant K_6. Thus, by (4.39), if c is sufficiently large, then we have $(4.28)_{p+1}$ and $(4.36)_{p+1}$. Also, since

$$\alpha^{(p+1)} = \sum_{q=0}^{p+1} \alpha^{(q,p-q+1)} = \sum_{q=0}^{p} \alpha^{(q,p-q+1)} + \sum_{i=1}^{N} c_i^{(p+1)}\beta_i,$$

we have

$$\alpha^{(p+1)}t^{p+1} \ll \frac{K_7}{c^{1/3}} A_2\left(\frac{t}{1-a}\right)$$

for some constant K_7. Hence if $c^{1/3} \geq K_7$, we have $(4.29)_{p+1}$. Finally, using (4.33) and $(4.28)_{p+1}$, we have

$$L_{X^{(p+1)}}\omega \cdot t^{p+1} \ll \frac{K_3}{b-a} A_2\left(\frac{t}{1-b}\right).$$

Hence by (4.34) Lemma,

$$L_{X^{(p+1)}}\omega \cdot t^{p+1} \ll \frac{K_3K_4}{c} \frac{1}{1-a} A_2'\left(\frac{t}{1-a}\right).$$

Thus, by (4.40) and (4.41), if $K(K_5/c^{2/3}+K_3K_4/c)<1$, then we have $(4.30)_{p+1}$. Q.E.D.

§ 5. Some special cases

Except for the determinacy results (Theorems (5.1) and (5.6) below) for germs close to the given one (the "local" determinacy), the material in this section are essentially in [1]. (5.1) Theorem is independent of the rest of this section, where we treat multiform functions. We refine ((5.15) Lemma, (5.16) Corollary and (5.17) Lemma) some of the arguments in [1] and use (5.6) Theorem to obtain "global" determinacy (Theorems (5.11) and (5.18)). We also specify the order of determinacy.

If $\mathscr{F}=(\tilde{\omega})$ is an unfolding (with parameter space $\boldsymbol{C}$) of a codim 1 foliation germ $F=(\omega)$ at 0 in $\boldsymbol{C}^n$, we set $\omega_t=\iota_t^*\tilde{\omega}$ and $h_t=\iota_t^*h$ as before, where $h=\langle\partial/\partial t, \tilde{\omega}\rangle$ ($\in \mathcal{O}_{n+1}$).

(5.1) **Theorem.** *Let $F=(\omega)$ be a codim 1 foliation germ at 0 in $\boldsymbol{C}^n$. Suppose that*

$$\mathfrak{m}^{k+1}\cap I(\omega)\subset \mathfrak{m}J(\omega)$$

for some non-negative integer k. Then any unfolding $\mathscr{F}=(\tilde{\omega})$ of F with parameter space $\boldsymbol{C}$ such that $j^k\omega_t=j^k\omega$ and $j^kh_t=0$, for all t near 0 in $\boldsymbol{C}$, is strongly isomorphic to the trivial unfolding F, i.e., there exists a pair (Φ, u) with the properties (a) *and* (b) *in* (4.1) *Theorem and*

$$u\Phi^*\tilde{\omega}=\omega.$$

Proof. We prove that we may let $\alpha=0$ in (4.1) Theorem under the given condition. Thus we show that $(4.16)_p$ hold for all $p\geq 0$ with $\alpha^{(q)}=0$ and $\tilde{\alpha}^{(q)}=0$ for $q=0, \cdots, p$. First, from the assumption $j^kh_t=0$, we have

$$h^{(1)}\in \mathfrak{m}^{k+1}\cap I(\omega)\subset \mathfrak{m}J(\omega).$$

Hence we may let $\alpha^{(0)}=0$ in the proof of (4.18) Lemma. If we assume that $(4.16)_p$ holds with $\alpha^{(q)}=0$ and $\tilde{\alpha}^{(q)}=0$ for $q=0, \cdots, p$, then we have

$$e^{(p+1)}d\omega+(\theta^{(p+1)}-de^{(p+1)})\wedge\omega=0$$

in the proof of (4.19) Lemma. Thus we see that $e^{(p+1)}$ is in $I(\omega)$. On the other hand, from the assumption $j^k\omega_t=j^k\omega$ and $j^kh_t=0$, $e^{(p+1)}$ is also in $\mathfrak{m}^{k+1}$. Hence

$$e^{(p+1)}\in \mathfrak{m}^{k+1}\cap I(\omega)\subset \mathfrak{m}J(\omega).$$

Therefore, $(4.16)_{p+1}$ holds with $\alpha^{(p+1)}=0$ and $\tilde{\alpha}^{(p+1)}=0$. The proof of the existence of convergent solutions is a special case of that part of the proof of (4.1) Theorem. Q.E.D.

(5.2) **Remark.** If $F=(\omega)$ has an isolated singularity at 0 in C^n, then $\mathfrak{m}^k \subset J(\omega)$ for some k. Thus the above theorem is a generalization of [1] p. 149, Lemma 2.1. Note that their proof can be easily modified to obtain $\Phi=(\phi, t)$ with $\phi(0, t)=0$, i.e., we may choose $u_{j,1}, \cdots, u_{j,n}$ in [1] p. 150 from $(x)=\mathfrak{m}\mathcal{O}_{n+p}$.

Now we consider a multiform function $f=f_1^{\lambda_1}\cdots f_l^{\lambda_l}$, where $f_i \in \mathcal{O}_n$ and $\lambda_i \in C$, $i=1, \cdots, l$ ([1], [12]). If we set

$$\omega=f_1\cdots f_l \sum_{i=1}^{l} \lambda_i \frac{df_i}{f_i},$$

then ω is an integrable germ in Ω_n and the unfolding theory for f is closely related to the unfolding theory for the foliation germ $F=(\omega)$ ([12]). By regrouping the f_i's, if necessary, we may always assume that

$$\lambda_i \neq \lambda_j \quad (\neq 0) \quad \text{if } i\neq j. \tag{5.3}$$

We also assume that codim $S(\omega)\geq 2$, which implies that

(5.4) each f_i is reduced, i.e., for any non-unit g in $\mathcal{O}_n$, f_i is not divisible by g^2, and that

(5.5) f_i and f_j are relatively prime.

We set $F_i=f_1\cdots \hat{f}_i \cdots f_l$ (omit f_i), $i=1, \cdots, l$, and let $I(f)$ denote the ideal of $\mathcal{O}_n$ generated by $F_1, \cdots, F_l$ (if $l=1$, we set $I(f)=\mathcal{O}_n$). Furthermore, we set $J(f)=J(\omega)$ (the Jacobian ideal of f) and $C(f)=S(\omega)$ (the critical set of f). In general, we have $I(f)\subset I(\omega)$ (see the proof of [12] (1.7) Proposition). If the conditions in [12] (1.11) Theorem are satisfied, then $I(f)=I(\omega)$ and the unfolding theory for f is equivalent to that for $F=(\omega)$ ([12] Section 2).

(5.6) **Theorem.** *Let $f=f_1^{\lambda_1}\cdots f_l^{\lambda_l}$ be a multiform function with* (5.3) *and* codim $C(f)\geq 2$. *Suppose that*

$$\mathfrak{m}^{k+1}\cdot I(f)\subset \mathfrak{m}J(f).$$

Then if, for each $i=1, \cdots, l$, $\tilde{f}_i$ is an unfolding of f_i (with parameter space C) such that $j^k f_{i,t}=j^k f_i$ for all t near 0 in C, there exists a map germ Φ with the properties (a) *and* (b) *of* (4.1) *Theorem and*

$$\Phi^*\tilde{f}=f,$$

where $f_{i,t}=\iota_t^*\tilde{f}_i$ *and* $\tilde{f}=\tilde{f}_1^{\lambda_1}\cdots\tilde{f}_l^{\lambda_l}$. *Thus if we write* $\Phi(x,t)=(\phi(x,t),t)$ *and let* $\phi_t=\iota_t^*\phi$ *and* $f_t=f_{1,t}^{\lambda_1}\cdots f_{l,t}^{\lambda_l}$ *for* t *near* 0, *then* ϕ_t *is in* L_n *and*

$$\phi_t^* f_t=f.$$

Proof. If we let

$$\omega=f_1\cdots f_l\sum_{i=1}^{l}\lambda_i\frac{df_i}{f_i}\quad\text{and}\quad\tilde{\omega}=\tilde{f}_1\cdots\tilde{f}_l\sum_{i=1}^{l}\lambda_i\frac{d\tilde{f}_i}{\tilde{f}_i},$$

then $\mathscr{F}=(\tilde{\omega})$ is an unfolding of $F=(\omega)$. We set $\omega_t=\iota_t^*\tilde{\omega}$ and $h_t=\iota_t^*h$, $h=\langle\partial/\partial t,\tilde{\omega}\rangle$ as before. First we show that the following hold for all $p\geq 0$.

$(5.7)_p$ *Induction hypothesis.* There exist $X^{(q)}$ in $\mathfrak{m}\Theta_n$ and $\delta_i^{(q)}$ in $\mathcal{O}_n$ for $q=0,\cdots,p$ and $i=1,\cdots,l$ such that

$$(5.8)_p\qquad \langle X^{|p},\omega_t\rangle+h_t\underset{p}{\equiv}0,$$

$$(5.9)_p\qquad X^{|p}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}\underset{p}{\equiv}\delta_i^{|p}f_{i,t},\quad i=1,\cdots,l,\quad\text{and}$$

$$(5.10)_p\qquad \sum_{i=1}^{l}\lambda_i\delta_i^{|p}=0,$$

where

$$X^{|p}=\sum_{q=0}^{p}X^{(q)}t^q\quad\text{and}\quad\delta_i^{|p}=\sum_{q=0}^{p}\delta_i^{(q)}t^q.$$

We claim $(5.7)_0$ holds. In fact, if we set $h^{(1)}=h_0$ and

$$f_i^{(1)}=\left.\frac{\partial f_{i,t}}{\partial t}\right|_{t=0},$$

we have

$$h^{(1)}=f_1\cdots f_l\sum_{i=1}^{l}\frac{\lambda_i}{f_i}f_i^{(1)}=\sum_{i=1}^{l}\lambda_i f_i^{(1)}F_i.$$

From the assumption that $j^k f_{i,t}=j^k f_i$ for all t near 0, we see that $h^{(1)}\in\mathfrak{m}^{k+1}I(f)=\mathfrak{m}J(f)=\mathfrak{m}J(\omega)$. Hence there exists $X^{(0)}$ in $\mathfrak{m}\Theta_n$ such that

$$\langle X^{(0)},\omega\rangle+h^{(1)}=0.$$

This equation can be written as

$$\sum_{i=1}^{l}\lambda_i F_i(X^{(0)}f_i+f_i^{(1)})=0.$$

Thus, by (5.5), there exist $\delta_i^{(0)}$, $i=1,\cdots,l$, in $\mathcal{O}_n$ such that

$$X^{(0)}f_i+f_i^{(1)}=\delta_i^{(0)}f_i \quad \text{and} \quad \sum_{i=1}^{l}\lambda_i\delta_i^{(0)}=0.$$

Next we claim that $(5.7)_p$ implies $(5.7)_{p+1}$. In fact, if we set

$$e^{(p+1)}t^{p+1}=[\langle X^{|p}, \omega_t\rangle+h_t]_{p+1} \quad \text{and}$$
$$F_{i,t}=f_{1,t}\cdots\hat{f}_{i,t}\cdots f_{l,t} \quad (\text{omit } f_{i,t}),$$

then we have

$$\begin{aligned} e^{(p+1)}t^{p+1}&=\sum_{i=1}^{l}\left[\lambda_iF_{i,t}\left(X^{|p}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}\right)\right]_{p+1} \\ &=\sum_{i=1}^{l}\lambda_iF_i\left[X^{|p}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}\right]_{p+1} \\ &\quad+\sum_{i=1}^{l}\left[\lambda_iF_{i,t}\left(X^{|p}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}\right)^{|p}\right]_{p+1}. \end{aligned}$$

If we use $(5.9)_p$, the second term is equal to

$$\sum_{i=1}^{l}[\lambda_iF_{i,t}(\delta_i^{|p}f_{i,t})^{|p}]_{p+1}=\left[f_{1,t}\cdots f_{l,t}\sum_{i=1}^{l}\lambda_i\delta_i^{|p}\right]_{p+1}-\sum_{i=1}^{l}\lambda_iF_i[\delta_i^{|p}f_{i,t}]_{p+1},$$

in which the first term vanishes by $(5.10)_p$. Therefore, we obtain

$$e^{(p+1)}t^{p+1}=\sum_{i=1}^{l}\lambda_iF_i\left[X^{|p}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}-\delta_i^{|p}f_{i,t}\right]_{p+1}.$$

Since $X^{(q)}$, $q=0, \cdots, p$, are in $\mathfrak{m}\Theta_n$ and $j^kf_{i,t}=j^kf_i$ for t near 0, we see that

$$e^{(p+1)}\in\mathfrak{m}^{k+1}\cdot I(f)\subset\mathfrak{m}J(f)=\mathfrak{m}J(\omega)$$

(note that $[\delta_i^{|p}f_{i,t}]_{p+1}=[\delta_i^{|p}(f_{i,t}-f_i)]_{p+1}$).

Hence there exists $X^{(p+1)}$ in $\mathfrak{m}\Theta_n$ such that

$$\langle X^{(p+1)}, \omega\rangle+e^{(p+1)}=0.$$

This equality can also be written as

$$\sum_{i=1}^{l}\lambda_iF_i\left[X^{|p+1}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}-\delta_i^{|p}f_{i,t}\right]_{p+1}=0.$$

Thus, by (5.5), there exist $\delta_i^{(p+1)}$, $i=1, \cdots, l$, in $\mathcal{O}_n$ such that

$$\left[X^{|p+1}f_{i,t}+\frac{\partial f_{i,t}}{\partial t}-\delta_i^{|p}f_{i,t}\right]_{p+1}=\delta_i^{(p+1)}f_it^{p+1} \quad \text{and} \quad \sum_{i=1}^{l}\lambda_i\delta_i^{(p+1)}=0.$$

Therefore, $(5.7)_{p+1}$ holds.

Note that when $l=1$, $(5.7)_p$ hold for all $p\geq 0$ with $\delta^{(q)}=0$, $q=0, \cdots, p$.

After this, we are able to prove the theorem by either one of the following arguments (I) and (II):

(I) We can easily see that $(4.16)_p$ hold for all $p\geq 0$ with $\alpha^{(q)}=0$ and $\tilde{\alpha}^{(q)}=0$, $q=0, \cdots, p$, which implies the existence of Φ and u as formal power series in t such that

$$u\Phi^*\tilde{\omega}=\omega.$$

The proof of the existence of convergent solutions is a special case of that part of the proof of (4.1) Theorem. Thus there is an isomorphism (Φ, u) from the trivial unfolding (ω) to $\mathscr{F}=(\tilde{\omega})$. Then by [12] (2.4) Lemma, Φ is an isomorphism from f to $\tilde{f}$;

$$\Phi^*\tilde{f}=f.$$

(II) We can show, as a very special case of the convergence part of the proof of (4.1) Theorem, that there exists

$$X_t=\sum_{i=1}^{n}\xi_i(x, t)\frac{\partial}{\partial x_i} \quad \text{with } \xi_i \in \mathfrak{m}\mathcal{O}_{n+1}$$

such that

$$\langle X_t, \omega_t\rangle+h_t=0,$$

or if we set $X=X_t+\partial/\partial t$, $\langle X, \tilde{\omega}\rangle=0$. If we solve the differential equation

$$\frac{\partial\phi}{\partial t}(x, t)=\xi(\phi(x, t), t)$$

for ϕ under the initial condition $\phi(x, 0)=x$, and set $\Phi(x, t)=(\phi(x, t), t)$, then we have $\Phi^*\tilde{f}=f$. Q.E.D.

We say that a multiform function $f=f_1^{\lambda_1}\cdots f_l^{\lambda_l}$ is k-determined (de détermination finie faible d'ordre k in [1]) if for any germ $(g_1, \cdots, g_l)$ in $\mathcal{O}_n^l$ with $j^k g_i=j^k f_i$, $i=1, \cdots, l$, there is a germ ϕ in L_n such that $\phi^*g=f$, where $g=g_1^{\lambda_1}\cdots g_l^{\lambda_l}$. Also, f is finitely determined if f is k-determined for some non-negative integer k.

(5.11) **Theorem.** *Let $f=f_1^{\lambda_1}\cdots f_l^{\lambda_l}$ be a multiform function with* (5.3) *and* codim $C(f)\geq 2$. *Suppose that*

$$\mathfrak{m}^k\subset\mathfrak{m}J(f).$$

Then f is k-determined.

Proof. Let $(g_1, \cdots, g_l)$ be a germ in $\mathcal{O}_n^l$ with $j^k g_i = j^k f_i$, $i=1, \cdots, l$, and set

$$(5.12) \qquad \tilde{f}_i(x, t) = f_i(x) + t(g_i(x) - f_i(x)).$$

Also, for each t in C, we set $f_{i,t}(x) = \tilde{f}_i(x, t)$ and $f_t = (f_{1,t})^{\lambda_1} \cdots (f_{l,t})^{\lambda_l}$. By (5.6) Theorem and by the compactness of the interval $[0, 1]$ in C, it suffices to show that

$$\mathfrak{m}^{k+1} I(f_t) \subset \mathfrak{m} J(f_t)$$

for all t in $[0, 1]$. Now from (5.12), we have

$$J(f) \subset J(f_t) + \mathfrak{m}^k.$$

Hence we have

$$\mathfrak{m}^k \subset \mathfrak{m} J(f) \subset \mathfrak{m} J(f_t) + \mathfrak{m}^{k+1}.$$

By Nakayama's lemma, we have $\mathfrak{m}^k \subset \mathfrak{m} J(f_t)$. Since $\mathfrak{m}^{k+1} I(f_t) \subset \mathfrak{m}^k$, the theorem is proved.

(5.13) **Remark.** We may replace the condition in the above theorem by

$$\mathfrak{m}^k \subset \mathfrak{m} J(f) + \mathfrak{m}^{k+1}.$$

Since the critical set $C(f)$ of f is the analytic set of the ideal $J(f)$, we have

(5.14) **Corollary.** *If* $C(f) = \{0\}$ *or* ϕ, *or equivalently, if* $\dim_C \mathcal{O}/J(f) < +\infty$, *then f is finitely determined.*

For a multiform function $f = f_1^{\lambda_1} \cdots f_l^{\lambda_l}$, we set $\bar{f} = (f_1, \cdots, f_l)$ $(\in \mathcal{O}_n^l)$ and let $\chi(\bar{f})$, $Q(\bar{f})$, $N(\bar{f})$, $S(\bar{f})$ and $R(\bar{f})$ be as in pp. 157–158 of [1]. Thus if we naturally identify $\chi(\bar{f})$ with $\mathcal{O}_n^l$, then $Q(\bar{f})$ is the sub-$\mathcal{O}_n$-module of $\mathcal{O}_n^l$ generated by $(f_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, f_l)$, $N(\bar{f})$ is the sub-$\mathcal{O}_n$-module of $\mathcal{O}_n^l$ generated by

$$\left(\frac{\partial f_1}{\partial x_1}, \cdots, \frac{\partial f_l}{\partial x_1}\right), \cdots, \left(\frac{\partial f_1}{\partial x_n}, \cdots, \frac{\partial f_l}{\partial x_n}\right),$$

$S(\bar{f}) = N(\bar{f}) + Q(\bar{f})$ and $R(\bar{f}) = \chi(\bar{f})/S(\bar{f})$.

The following gives a more direct proof of a result obtained by combining Proposition 1.2 and Lemma 2.1 of [1] p. 154 and p. 158.

(5.15) **Lemma.** *There is a surjective $\mathcal{O}_n$-homomorphism*

$$I(f)/J(f) \longrightarrow R(\bar{f}).$$

Proof. Recall that $I(f)$ is generated by $F_1, \cdots, F_l$, where $F_i = f_1 \cdots \hat{f}_i \cdots f_l$. If $\sum_{i=1}^l a_i F_i = \sum_{i=1}^l a'_i F_i$, a_i, $a'_i \in \mathcal{O}_n$, then by (5.5), for each $i=1, \cdots, l$, $a'_i - a_i$ is divisible by f_i in $\mathcal{O}_n$. Thus the map

$$\rho: I(f) \longrightarrow \chi(\bar{f})/Q(\bar{f})$$

given by

$$\rho\left(\sum_{i=1}^l a_i F_i\right) = \left[\left(\frac{a_1}{\lambda_1}, \cdots, \frac{a_l}{\lambda_l}\right)\right]$$

is a well defined surjective $\mathcal{O}_n$-homomorphism. It is easily seen that $\rho(J(f)) \subset N(\bar{f})/N(\bar{f}) \cap Q(\bar{f})$. Thus ρ induces a surjective homomorphism $I(f)/J(f) \to \chi(\bar{f})/N(\bar{f}) + Q(\bar{f}) = R(\bar{f})$. Q.E.D.

(5.16) **Corollary.** *If* $\mathfrak{m}^k I(f) \subset J(f)$, *then* $\mathfrak{m}^k \chi(\bar{f}) \subset S(\bar{f})$.

The following is proved by modifying the arguments in [1] p. 158, Proposition 2.2 (cf. also [5]).

(5.17) **Lemma.** *For non-negative integers k and r, set $k_1 = k + r$ if $r > 0$ and $k_1 = k + 1$ if $r = 0$. Let $\bar{f} = (f_1, \cdots, f_l)$ be a germ in $\mathcal{O}_n^l$ such that*

$$\mathfrak{m}^k \chi(\bar{f}) \subset S(\bar{f}).$$

Then for any germ $\bar{g} = (g_1, \cdots, g_l)$ in $\mathcal{O}_n^l$ with $j^{k_1} g_i = j^{k_1} f_i$, $i = 1, \cdots, l$, there is a germ ϕ in L_n and units $u_1, \cdots, u_l$ in $\mathcal{O}_n$ with $j^r \phi = j^r x$ and $j^r u_i = 1$ such that

$$u_i \phi^* g_i = f_i, \quad i = 1, \cdots, l.$$

(5.18) **Theorem.** *Let $f = f_1^{\lambda_1} \cdots f_l^{\lambda_l}$ be a multiform function with* (5.3) *and* codim $C(f) \geq 2$. *Suppose that*

$$\mathfrak{m}^k I(f) \subset \mathfrak{m} J(f)$$

for some positive integer k. Then f is $2k$-determined.

Proof. By (5.16) Corollary, we have $\mathfrak{m}^k \chi(\bar{f}) \subset S(\bar{f})$. Given a germ $\bar{g} = (g_1, \cdots, g_l)$ in $\mathcal{O}_n^l$ with $j^{2k} g_i = j^{2k} f_i$. Then by (5.17) Lemma, there is a germ ϕ in L_n and units $u_1, \cdots, u_l$ in $\mathcal{O}_n$ with $j^k \phi = x$ and $j^k u_i = 1$ such that $u_i \phi^* g_i = f_i$. Since $j^k u_i^{-1} = 1$, we may assume from the beginning that $g_i = v_i f_i$ for some v_i in $\mathcal{O}_n$ with $j^k v_i = 1$, $i = 1, \cdots, l$. We set

(5.19) $$\tilde{f}_i(x, t) = f_i(x) + t(g_i(x) - f_i(x)) = (1 + t(v_i - 1)) f_i(x).$$

Also, for each t in $\boldsymbol{C}$, we set $f_{i,t}(x)=\tilde{f}_i(x, t)$ and $f_t=f_{1,t}^{\lambda_1}\cdots f_{l,t}^{\lambda_l}$. By (5.6) Theorem and by the compactness of $[0, 1]$ in $\boldsymbol{C}$, it suffices to show that

$$\mathfrak{m}^{k+1}I(f_t)\subset \mathfrak{m}J(f_t)$$

for all t in $[0, 1]$. Now by (5.19),

$$I(f)=I(f_t) \quad \text{and} \quad J(f)\subset J(f_t)+\mathfrak{m}^k I(f_t).$$

Hence we have

$$\mathfrak{m}^k I(f_t)=\mathfrak{m}^k I(f)\subset \mathfrak{m}J(f)\subset \mathfrak{m}J(f_t)+\mathfrak{m}^{k+1}I(f_t).$$

Thus by Nakayama's lemma, we have $\mathfrak{m}^k I(f_t)\subset \mathfrak{m}J(f_t)$. Since $\mathfrak{m}^{k+1}I(f_t) \subset \mathfrak{m}^k I(f_t)$, the theorem is proved.

(5.20) **Remark.** By a similar argument, we may prove that if

$$\mathfrak{m}^k I(f)\subset J(f)$$

for some non-negative integer k, then f is $(2k+1)$-determined.

(5.21) **Remark.** We may replace the condition in the above theorem by

$$\mathfrak{m}^k I(f)\subset \mathfrak{m}J(f)+\mathfrak{m}^{k+1}I(f).$$

Since the support of the $\mathcal{O}_n$-module $I(f)/J(f)$ is the strict critical set $C'(f)$ of f ([1] p. 154, Proposition 1.2), we have

(5.22) **Corollary** ([1] p. 161, Théorème 3.1). *If* $C'(f)=\{0\}$ *or* ϕ, *or equivalently, if* $\dim_{\boldsymbol{C}} I(f)/J(f)<+\infty$, *then* f *is finitely determined.*

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