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Average Euler Characteristic of Leaves of Codimension-One Foliations

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§ 1. Introduction

The purpose of this paper is to generalize the results of Phillips and Sullivan [3]. All the objects are of class C^∞ and all the manifolds are without boundary, unless stated otherwise.

Two Riemannian manifolds (F_1, g_1) and (F_2, g_2) are said to be *quasi-isometric* or to have the same *quasi-isometry type* if there exist a diffeomorphism $f\colon F_1 \to F_2$ and positive constants A, B such that

$$A \cdot \|v\|_{g_1} \leqq \|f_* v\|_{g_2} \leqq B \cdot \|v\|_{g_1}$$

for all $v \in TF_1$. Now let M be a closed manifold, $\mathscr{F}$ a foliation of M and F a leaf of $\mathscr{F}$. Take a Riemannian metric g of M and consider the induced metric $g|F$ of F. Then the quasi-isometry type of $(F, g|F)$ does not depend on the choice of g. We consider the following problem.

Problem A. When can a non-compact manifold with a given quasi-isometry type be realized as a leaf of a foliation of a closed manifold?

As a quasi-isometric invariant, we have the growth type of a Riemannian manifold. In [3], Phillips and Sullivan introduced a new invariant as follows. A non-compact Riemannian manifold (F, g) of dimension 2 is said to have *average Euler characteristic zero* if there exists a sequence $F_1 \subset F_2 \subset \cdots \subset F$ of compact connected submanifolds with boundary such that

(1) $\{F_i\}_{i \in N}$ is *comparable* to $\{D_r(x)\}_{r \in R^+}$ for some $x \in F$, that is, there are a constant Q and a sequence of radii $r_1, r_2, \cdots \to \infty$ satisfying $D_{r_i}(x) \subset F_i \subset D_{Qr_i}(x)$ for all i, where $D_r(x)$ denotes the set of $y \in F$ at distance $\leqq r$ from x,

(2) $\lim_{i \to \infty} \chi(F_i)/\mathrm{vol}\, F_i = 0$.

An answer to Problem A by Phillips and Sullivan is the following.

Theorem ([3]). *Let M be a closed manifold, $\mathscr{F}$ a 2-dimensional orientable foliation of M and F a non-compact leaf of $\mathscr{F}$. If $H_2(M; \boldsymbol{R}) = 0$ and*

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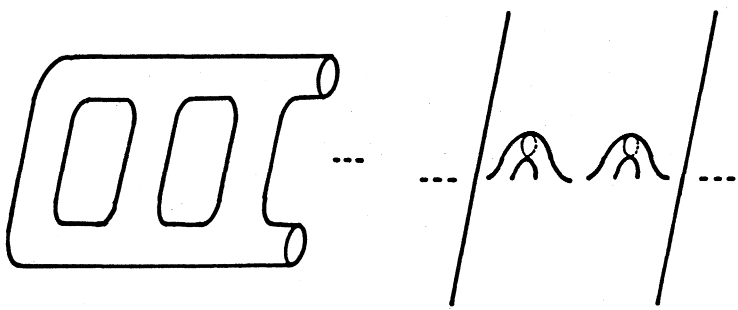
F has non-exponential growth, then F has average Euler characteristic zero.

We quote two examples from [3] in Figure 1.1. Both the Riemannian manifolds are diffeomorphic to $T^2 \# T^2 \# \cdots$. On one hand, Jacob's ladder has polynomial growth of degree 1 and does not have average Euler characteristic zero. On the other hand, Infinite Loch Ness Monster has polynomial growth of degree 2 and has average Euler characteristic zero. Furthermore Cantwell and Conlon [1] realized Infinite Loch Ness monster as a leaf of a codimension-one foliation of S^3.



(a) Jacob's ladder (b) Infinite Loch Ness monster

Fig. 1.1.

For a Riemannian manifold of dimension >2, the above definition of average Euler characteristic becomes meaningless. The reason is as follows. Let (F, g) be an arbitrary non-compact Riemannian manifold of dimension $n>2$ and $\{F_i\}_{i\in \boldsymbol{N}}$ a sequence, as above, comparable to $\{D_r(x)\}_{r\in \boldsymbol{R}^+}$ for some point $x \in F$. Let $\chi_i = \chi(F_i)$. When χ_i is positive, take disjoint submanifolds-with-boundary $K(1), \cdots, K(\chi_i)$ of $F-F_i$ diffeomorphic to $T^2\times D^{n-2}$ in a small neighborhood of ∂F and let F_i' be the submanifold with boundary obtained from F_i by connecting $K(j)$'s by small tubes in that neighborhood. It follows that $\chi(F_i')=0$. When χ_i is negative, make the similar construction as above by taking submanifolds-with-boundary diffeomorphic to $S^2\times D^{n-2}$. When $\chi_i=0$, let $F_i'=F_i$. Then the sequence $\{F_i'\}_{i\in \boldsymbol{N}}$ is clearly comparable to $\{D_r(x)\}_{r\in \boldsymbol{R}^+}$. Therefore (F, g) has average Euler characteristic zero.

In Section 2, we give a modified definition of the average Euler characteristic and prove that an odd dimensional non-compact Riemannian manifold, admitting a uniform triangulation and having non-exponential growth, has average Euler characteristic zero.

In Section 3, we compute the average Euler characteristic of almost periodic Riemannian manifolds and show that our definition has meaning. For example, Jacob's ladder multiplied by S^{2n} does not have average

Euler characteristic zero, but Infinite Loch Ness monster multiplied by S^{2n} does.

In Section 4, we prove the following.

Main Theorem. *Let M be a closed orientable manifold, $\mathscr{F}$ a transversely-orientable codimension-one foliation of M and F a non-compact leaf of $\mathscr{F}$. Suppose that*

(1) *$H_1(M; \boldsymbol{R})=0$ if* $\dim F$ *is even,*

(2) *F has non-exponential growth.*

Then F has average Euler characteristic zero.

§ 2. Average Euler characteristic

We give preliminary definitions.

Definition 2.1. Let (F, g) be a Riemannian manifold. A triangulation T of F is *uniform* if there exist positive constants v, V, d and N such that for each simplex σ of T except vertices,

$$v \leqq \operatorname{vol} \sigma \leqq V, \qquad \operatorname{diam} \sigma \leqq d,$$

and for each vertex α of T,

$$\sharp \{\sigma \mid \sigma \text{ is a simplex of } T \text{ containing } \alpha\} \leqq N.$$

Furthermore T is *hereditarily uniform* if there exists a sequence $T^{(0)}=T$, $T^{(1)}, \cdots$ of uniform triangulations of F such that

(1) $T^{(i+1)}$ is a derived subdivision of $T^{(i)}$ for all i,

(2) $\lim_{i\to\infty} d^{(i)}=0$, where we denote by $v^{(i)}$, $V^{(i)}$, $d^{(i)}$ and $N^{(i)}$ the constants corresponding to $T^{(i)}$.

Definition 2.2. Let T be a triangulation of a manifold F. A *T-submanifold* is a subcomplex of F with respect to T which is also a p.l. manifold (possibly with boundary).

Our modified definition of average Euler characteristic is the following.

Definition 2.3. A non-compact Riemannian manifold (F, g) of dimension >2 has *average Euler characteristic zero* if (F, g) has a hereditarily uniform triangulation T and there exists a sequence $F_1 \subset F_2 \subset \cdots \subset F$ of compact connected T-submanifolds with boundary such that

(1) $\{F_i\}_{i\in \boldsymbol{N}}$ is comparable to $\{D_r(x)\}_{r\in \boldsymbol{R}^+}$ for some $x \in F$ (this is the same as the 2-dimensional case in Section 1),

(2) $\lim_{i\to\infty} \operatorname{vol} \partial F_i / \operatorname{vol} F_i=0$,

(3) $\lim_{i\to\infty} \chi(F_i)/\operatorname{vol} F_i=0$.

We can easily show that the average Euler characteristic is a quasi-isometric invariant. We give a precise formulation as follows and we omit the proof.

Proposition 2.4. *Let (F_1, g_1) and (F_2, g_2) be quasi-isometric non-compact Riemannian manifolds. Then (F_1, g_1) has average Euler characteristic zero if and only if (F_2, g_2) does.*

As an analogy to the fact that every closed manifold of odd dimension has Euler characteristic zero, we have the following result for the average Euler characteristic.

Theorem 1. *Let (F, g) be a non-compact orientable Riemannian manifold of odd dimension. If (F, g) has a hereditarily uniform triangulation and has non-exponential growth, then (F, g) has average Euler characteristic zero.*

Proof. Let n be the dimension of F. Take a hereditarily uniform triangulation T of F and let v, V, d and N be the positive constants corresponding to T. We may suppose that T is a derived subdivision of some triangulation T_0 of F. Let T^* be the dual cell decomposition of T_0. Note that each cell of T^* is a subcomplex of T. Take a point $x_0 \in F$ and for $r>0$ denote by G_r the union of n-cells of T^* intersecting $D_r(x_0)$. Then G_r is a compact connected T-submanifold with boundary. Clearly $G_r \subset D_{r+2d}(x_0)$. Furthermore denote by $\partial^* G_r$ the union of simplices of T intersecting ∂G_r. It follows that $\partial^* G_r \subset D_{r+3d}(x_0) - D_{r-d}(x_0)$.

Lemma 2.5. $$\mathrm{vol}\, \partial^* G_r \geqq \frac{v}{(n+1)V}\, \mathrm{vol}\, \partial G_r.$$

Proof. Since the number of $(n-1)$-simplices of T contained in ∂G_r is not smaller than $\mathrm{vol}\, \partial G_r / V$ and each n-simplex contains $n+1$ $(n-1)$-simplices, the number of n-simplices of T contained in $\partial^* G_r$ is not smaller than $\mathrm{vol}\, \partial G_r/(n+1)V$. This implies the above inequality.

Now consider the sequence $\{G_{3kd}\}_{k \in N}$. The condition that (F, g) has non-exponential growth is used to prove the following.

Lemma 2.6. $\alpha = \liminf_{k \to \infty} \mathrm{vol}\, \partial G_{3kd} / \mathrm{vol}\, G_{3kd} = 0.$

Proof. Suppose that $\alpha > 0$. Then there exist $\kappa \in N$ and $P>0$ such that if $k \geqq \kappa$ then $\mathrm{vol}\, \partial G_{3kd}/\mathrm{vol}\, G_{3kd} \geqq P$. Let $v_k = \mathrm{vol}\, G_{3kd}$. It follows that

$$v_{k+1}-v_{k-1}\geqq \mathrm{vol}\,(D_{3kd+3d}(x_0)-D_{3kd-3d}(x_0))\geqq \mathrm{vol}\,\partial^* G_{3kd}$$
$$\geqq \frac{v}{(n+1)V}\,\mathrm{vol}\,\partial G_{3kd}\geqq \frac{vP}{(n+1)V}\,v_k.$$

Let $Q=vP/(n+1)V$. Then $v_{k+1}\geqq Qv_k+v_{k-1}$. By using these inequalities twice, we have

$$v_{k+2}\geqq Qv_{k+1}+v_k\geqq (Q^2v_k+Qv_{k-1})+v_k\geqq (1+Q^2)v_k.$$

Therefore for all $k\in N$ satisfying $k\geqq\kappa$,

$$v_k\geqq (1+Q^2)^{[k-\kappa]/2}v_\kappa.$$

Since $G_{3kd}\subset D_{3kd+2d}(x_0)$, it follows that

$$\mathrm{vol}\,D_{(3k+2)d}(x_0)\geqq (1+Q^2)^{[k-\kappa]/2}v_\kappa.$$

This implies that (F, g) has exponential growth, which is a contradiction. This completes the proof of Lemma 2.6.

According to Lemma 2.6, there exists a subsequence $\{F_i\}_{i\in N}$ of $\{G_{3kd}\}_{k\in N}$ such that $\lim_{i\to\infty}\mathrm{vol}\,\partial F_i/\mathrm{vol}\,F_i=0$. Determine $r_i>0$ by $F_i=G_{r_i}$. Then we have

$$D_{r_i}(x_0)\subset F_i\subset D_{r_i+2d}(x_0)\subset D_{2r_i}(x_0).$$

Therefore $\{F_i\}_{i\in N}$ is comparable to $\{D_r(x_0)\}_{r\in R^+}$. Thus the conditions (1) and (2) of Definition 2.3 are satisfied.

Consider the double W_i of F_i. Since W_i is an odd dimensional closed p.l. manifold, we have

$$\chi(W_i)=2\chi(F_i)-\chi(\partial F_i)=0.$$

Lemma 2.7. $|\chi(\partial F_i)|<(2^n/v)\cdot\mathrm{vol}\,\partial F_i$.

Proof. The number of $(n-1)$-simplices of T contained in ∂F_i is not greater than $\mathrm{vol}\,\partial F_i/v$. Since each $(n-1)$-simplices contains (2^n-1) simplices, the total number of simplices of T contained in ∂F_i is smaller than $(2^n/v)\,\mathrm{vol}\,\partial F_i$. This implies the above inequality.

Now we have

$$\limsup_{i\to\infty}|\chi(F_i)|/\mathrm{vol}\,F_i=\limsup_{i\to\infty}(1/2)|\chi(\partial F_i)|/\mathrm{vol}\,F_i$$
$$\leqq \limsup_{i\to\infty}(2^n/2v)\cdot\mathrm{vol}\,\partial F_i/\mathrm{vol}\,F_i=0.$$

This implies that $\lim_{i\to\infty}\chi(F_i)/\mathrm{vol}\,F_i=0$. Therefore (F, g) has average Euler characteristic zero.

§ 3. Almost periodic Riemannian manifolds

We introduce a class of Riemannian manifolds whose average Euler characteristics can be computed.

Definition 3.1. (a) A triplet $(\{P_j\}_{j=1}^{\nu}, \{C_j\}_{j=1}^{\mu}, \{\psi_C\}_{C\in\mathscr{C}})$ is called a *boundary-designated system* if

(1) P_j's are compact manifolds of the same dimension, say n,

(2) C_j's are closed connected manifolds of dimension $n-1$, and if $i\neq j$ then C_i and C_j are not diffeomorphic,

(3) $\mathscr{C}$ is the set of connected components of $\partial P_1, \cdots, \partial P_\nu$ and ψ_C is a diffeomorphism from C to some C_j.

(b) A non-compact Riemannian manifold (F, g) is *almost periodic* if there exist (i) a boundary-designated system $(\{P_j\}_{j=1}^{\nu}, \{C_j\}_{j=1}^{\mu}, \{\psi_C\}_{C\in\mathscr{C}})$, (ii) a Riemannian metric g_j of P_j for each j, (iii) positive constants A, B, and (iv) a covering $\{K_\lambda\}_{\lambda\in\Lambda}$ of F such that

(1) K_λ is a compact submanifold of F, $K_\lambda\cap K_\theta$ is a closed submanifold, and Int K_λ's are pairwise disjoint,

(2) for each $\lambda\in\Lambda$, there exists a diffeomorphism $\phi_\lambda\colon K_\lambda\to P_{j(\lambda)}$ satisfying

$$A\cdot\|v\|_g\leqq\|\phi_{\lambda*}v\|_{g_{j(\lambda)}}\leqq B\cdot\|v\|_g,\quad v\in TK_\lambda,$$

(3) for each connected component C of $K_\lambda\cap K_\theta$,

$$\psi_{\phi_\lambda(C)}\circ\phi_\lambda|C=\psi_{\phi_\theta(C)}\circ\phi_\theta|C.$$

(c) Let (F, g) satisfy the condition in (b). We call each (P_j, g_j) a *period*. A period (P_j, g_j) is *essential* if $\#\{\lambda\in\Lambda|j(\lambda)=j\}=\infty$. For $x\in F$ and $r>0$, let $f_j(r;x)=\#\{\lambda\in\Lambda|K_\lambda\subset D_r(x), j(\lambda)=j\}$. A period (P_j, g_j) is *frequent* if $\limsup_{r\to\infty} f_j(r;x)/\mathrm{vol}\,D_r(x)>0$ for some $x\in F$.

Note that for two quasi-isometric non-compact Riemannian manifolds, one of them is almost periodic if and only if the other is.

Proposition 3.2. *If a non-compact Riemannian manifold (F, g) is almost periodic, then (F, g) has a hereditarily uniform triangulation.*

Proof. We use notations in Definition 3.1. Firstly, for each C_j take a triangulation $T(C_j)$ of C_j (see Munkres [2] for example). Secondly for connected components C of each P_j, take the induced triangulations $\psi_C^*T(C_j)$ (where C_j is the range of ψ_C) and extend them to a triangulation $T(P_j)$ of P_j. Then the condition (3) in Definition 3.1 (b) guarantees the consistency of the induced triangulations $\phi_\lambda^*T(P_{j(\lambda)})$ of K_λ. Thus a triangulation T of F is obtained from $\phi_\lambda^*T(P_{j(\lambda)})$'s. Since T is determined by a finite number of triangulations $T(P_1), \cdots, T(P_\nu)$, it is easy to see that T is hereditarily uniform. This completes the proof of Proposition 3.2.

We can compute the average Euler characteristic of almost periodic Riemannian manifolds, as follows.

Theorem 2. *Let (F, g) be a non-compact connected orientable Riemannian manifold of even dimension, say $2n$. Suppose that (F, g) has non-exponential growth and is almost periodic. Take (P_j, g_j)'s and so forth as in Definition* 3.1.

(a) *If $\chi(P_j)=0$ for each frequent period (P_j, g_j), then (F, g) has average Euler characteristic zero.*

(b) *If $\chi(P_j)>0$ for each essential period (P_j, g_j), then (F, g) does not have average Euler characteristic zero.*

Proof. By Proposition 3.2, (F, g) has a hereditarily uniform triangulation T such that K_λ is a T-submanifold with boundary for all $\lambda \in \Lambda$. Let v, V, d and N be positive constants corresponding to T. Since (F, g) is almost periodic, there exist positive constants v_p, V_p, d_p and N_p such that for each $\lambda \in \Lambda$,

(1) $v_p \leqq \operatorname{vol} K_\lambda \leqq V_p$, $\operatorname{diam} K_\lambda \leqq d_p$,

(2) for each connected component C of ∂K_λ,

$$v_p \leqq \operatorname{vol} C \leqq V_p,$$

(3) $\#\{C \mid C \text{ is a connected component of } \partial K_\lambda\} \leqq N_p$. Now we have the rough estimates as in the following lemmas.

Lemma 3.3. *There exists a positive constant δ such that $D_\delta(x) \leqq 2V_p$ for all $x \in F$.*

Proof. For each P_j, take a collar R_j of ∂P_j. Let

$$\delta_j = \min\{d(x, y) \mid x \in \partial P_j,\ y \in \partial R_j - \partial P_j\} \quad \text{and} \quad \delta_0 = \min\{\delta_1, \cdots, \delta_\nu\}.$$

Furthermore let $\delta = \delta_0/2B$. Then we see that for each $x \in F$ the set $D_\delta(x)$ is contained in at most two K_λ's. Therefore $D_\delta(x) \leqq 2V_p$, which completes the proof of Lemma 3.3.

Lemma 3.4. *There exist positive constants ξ and η such that for each compact codimension-one T-submanifold S and for $r>0$,*

$$\operatorname{vol} N_r(S) \leqq \exp(\xi r + \eta) \cdot \operatorname{vol} S,$$

where $N_r(S)$ denotes the set of $x \in F$ at distance $\leqq r$ from S.

Proof. Since T is hereditarily uniform, there exists a sequence $T^{(0)}=T, T^{(1)}, \cdots$ of uniform triangulations of F as in Definition 2.1. As before we denote by $v^{(i)}$, $V^{(i)}$, $d^{(i)}$ and $N^{(i)}$ the constants corresponding to $T^{(i)}$. Since $\lim_{i\to\infty} d^{(i)}=0$, there exists $\iota \in N$ such that $d^{(\iota)}\leqq\delta/15$. Let $\bar{T}=T^{(\iota)}$ and denote by $\bar{T}^*$ the dual cell decomposition of $T^{(\iota-1)}$. Let $\bar{v}=v^{(\iota)}$, $\bar{V}=V^{(\iota)}$, $\bar{d}=d^{(\iota)}$ and $\bar{N}=N^{(\iota)}$. Denote by $\bar{\mathscr{V}}(S)$ the set of veritices of $\bar{T}$ contained in S. It follows that $N_{12\bar{d}}(S)\subset \cup\{D_{13\bar{d}}(\alpha)\,|\,\alpha \in \bar{\mathscr{V}}(S)\}$. Let $N^*(S)=\cup\{\sigma^*\,|\,\sigma^*$ is a $2n$-cell of $\bar{T}^*$ intersecting $N_{10\bar{d}}(S)\}$. Then $N^*(S)$ is a $\bar{T}$-submanifold with boundary, and we have $\partial N^*(S)\subset N_{12\bar{d}}(S)-N_{10\bar{d}}(S)$. Since

$$\begin{aligned}\operatorname{vol} N^*(S)&\leqq\operatorname{vol}\ \cup\{D_{13\bar{d}}(\alpha)\,|\,\alpha \in \bar{\mathscr{V}}(S)\}\\&\leqq 2V_p\cdot\#\bar{\mathscr{V}}(S)\leqq 2V_p\cdot(2n/\bar{v})\cdot\operatorname{vol} S,\end{aligned}$$

it follows that

$$\begin{aligned}\operatorname{vol}\partial N^*(S)&\leqq\bar{V}\cdot\#\{\sigma\,|\,\sigma \text{ is a } (2n-1)\text{-simplex}\subset\partial N^*(S)\}\\&\leqq\bar{V}\cdot\#\bar{\mathscr{V}}(N^*(S))\cdot\bar{N}\leqq\bar{N}\bar{V}\cdot(2n+1)\cdot\operatorname{vol} N^*(S)/\bar{v}\\&\leqq 4n(2n+1)(\bar{N}\bar{V}V_p/\bar{v}^2)\operatorname{vol} S.\end{aligned}$$

Let $K_1=\max\{2, 4nV_p/\bar{v}\}$ and $K_2=\max\{2, 4n(2n+1)\bar{N}\bar{V}V_p/\bar{v}^2\}$.

By the similar arguments as above for $S'=\partial N^*(S)$ in this time, we have

$$\operatorname{vol} N^*(S')\leqq K_1\operatorname{vol} S',\qquad \operatorname{vol}\partial N^*(S')\leqq K_2\operatorname{vol} S'.$$

By repeating this process $([r/10\bar{d}]+1)$-times, we have a $\bar{T}$-submanifold with boundary

$$N^*(S)\cup N^*(\partial N^*(S))\cup\cdots\cup N^*(\partial N^*(\cdots(N^*(\partial N^*(S)))\cdots))$$

containing $N_r(S)$. Let $r'=[r/10\bar{d}]+1$. Then we have

$$\begin{aligned}\operatorname{vol} N_r(S)&\leqq K_1(K_2+K_2^2+\cdots+K_2^{r'})\operatorname{vol} S\\&<(K_1K_2^2/(K_2-1))K_2^{r/10\bar{d}}\operatorname{vol} S.\end{aligned}$$

Let $\eta=\log(K_1K_2^2/(K_2-1))$ and $\xi=(\log K_2)/10\bar{d}$. Then it follows that $\operatorname{vol} N_r(S)\leqq\exp(\xi r+\eta)\operatorname{vol} S$, which completes the proof of Lemma 3.4.

(a) Let $x_0 \in F$. For each $r>0$, let $G_r=\cup\{K_\lambda\,|\,K_\lambda\cap D_r(x_0)\neq\emptyset\}$. Then G_r is a compact connected T-submanifold with boundary. As in the proof of Theorem 1, we have a subsequence $\{F_i\}_{i\in N}$ of $\{G_{3kd_p}\}_{k\in N}$ such that

(1) $\{F_i\}_{i\in N}$ is comparable to $\{D_r(x_0)\}_{r\in R^+}$,

(2) $\lim_{i\to\infty}\operatorname{vol}\partial F_i/\operatorname{vol} F_i=0$.

Determine r_i by $F_i = G_{r_i}$ and let $\partial_p D_{r_i}(x_0) = \cup \{K_\lambda \mid K_\lambda \cap \partial D_{r_i}(x_0) \neq \emptyset\}$. Clearly $\partial_p D_{r_i}(x_0) \subset N_{2d_p}(\partial F_i)$. Let $f_{ji} = \#\{K_\lambda \subset F_i \mid j(\lambda) = j\}$. Then we have

$$\begin{aligned}|f_j(r_i) - f_{ji}| &\leqq \#\{K_\lambda \mid K_\lambda \subset \partial_p D_{r_i}(x_0)\}\\ &\leqq (1/v_p) \operatorname{vol} N_{2d_p}(\partial F_i)\\ &\leqq (1/v_p) \exp(2d_p\xi + \eta) \operatorname{vol} \partial F_i.\end{aligned}$$

Lemma 3.5. $\lim_{i\to\infty} \operatorname{vol} D_{r_i}(x_0)/\operatorname{vol} F_i = 1$.

Proof. Since $F_i - D_{r_i}(x_0) \subset \partial_p D_{r_i}(x_0)$, it follows that

$$|\operatorname{vol} F_i - \operatorname{vol} D_{r_i}(x_0)| \leqq \operatorname{vol} N_{2d_p}(\partial F_i) \leqq \exp(2d_p\xi + \eta) \operatorname{vol} \partial F_i.$$

Therefore we have

$$\begin{aligned}&\limsup_{i\to\infty} |1 - (\operatorname{vol} D_{r_i}(x_0)/\operatorname{vol} F_i)|\\ &\qquad \leqq \limsup_{i\to\infty} \exp(2d_p\xi + \eta) \operatorname{vol} \partial F_i/\operatorname{vol} F_i = 0,\end{aligned}$$

which completes the proof of Lemma 3.5.

Now we have

$$\begin{aligned}\limsup_{i\to\infty} |\chi(F_i)|/\operatorname{vol} F_i &= \limsup_{i\to\infty} \left|\sum_{j=1}^{\nu} f_{ji}\chi(P_j)\right| \Big/ \operatorname{vol} F_i\\ &\leqq \limsup_{i\to\infty} \left(\sum_{j=1}^{\nu} (f_j(r_i) + (\exp(2d_p\xi+\eta)/v_p) \operatorname{vol} \partial F_i)|\chi(P_j)|\right) \Big/ \operatorname{vol} F_i\\ &\leqq \sum_{j=1}^{\nu} (\limsup_{i\to\infty} f_j(r_i)/\operatorname{vol}(D_{r_i}(x_0)) |\chi(P_j)| = 0,\end{aligned}$$

by the assumption of (a). Therefore it follows that

$$\lim_{i\to\infty} \chi(F_i)/\operatorname{vol} F_i = 0,$$

which completes the proof of (a).

(b) Suppose that (F, g) has average Euler characteristic zero, from which we will bring out a contradiction. By the assumption, there exist a hereditarily uniform triangulation T' of (F, g) and a sequence $F_1' \subset F_2' \subset \cdots \subset F$ of compact connected T'-submanifolds with boundary such that

(1) $\{F_i'\}_{i \in N}$ is comparable to $\{D_r(x_0')\}_{r \in R^+}$ for some $x_0' \in F$,

(2) $\lim_{i\to\infty} \operatorname{vol} \partial F_i'/\operatorname{vol} F_i' = 0$,

(3) $\lim_{i\to\infty} \chi(F_i')/\operatorname{vol} F_i' = 0$.

Let v', V', d' and N' be constants corresponding to T'. By taking a subdivision of T' if necessary, we may suppose that $2d' < \delta$. Denote by

F_i the union of cells of $\bar{T}^*$ intersecting F'_i. The proof of Lemma 3.4 implies that

$$\text{vol } N_r(\partial F_i) \leqq \exp(\xi r + \eta) \text{ vol } \partial F_i$$

for all $r>0$. Let $G_i = \cup \{K_\lambda \mid K_\lambda \cap F'_i \neq \emptyset\}$ and $H_i = \cup \{\sigma \mid \sigma$ is simplex of T' intersecting $G_i\}$. It follows that $F_i - F'_i \subset N_{2\bar{d}}(\partial F'_i)$, $G_i - F_i \subset N_{d_p}(\partial F_i)$ and $H_i - G_i \subset N_{d'}(\partial G_i)$.

Lemma 3.6. *There exists a constant K_3 such that*

$$\text{vol } \partial F_i \leqq K_3 \text{ vol } \partial F'_i$$

for all i.

Proof. Denote by $\partial^* F_i$ the union of simplices of $\bar{T}$ intersecting ∂F_i and by $\mathscr{V}'(\partial F'_i)$ the set of vertices of T' contained in $\partial F'_i$. It follows that

$$\partial^* F_i \subset N_{\bar{d}}(\partial F_i) \subset N_{3\bar{d}}(\partial F'_i) \subset \cup \{D_\delta(\alpha) \mid \alpha \in \mathscr{V}'(\partial F'_i)\}.$$

Since $\text{vol } \partial^* F_i \leqq 2V_p \cdot \# \mathscr{V}'(\partial F'_i)$ and $\# \mathscr{V}'(\partial F'_i) \leqq 2n \text{ vol } \partial F'_i / v'$, we have $\text{vol } \partial^* F_i \leqq (4nV_p/v') \text{ vol } \partial F'_i$. Since the number of $(2n-1)$-simplices of $\bar{T}$ contained in $\partial^* F_i$ is not greater than $(2n+1) \text{ vol } \partial^* F_i / \bar{v}$, it follows that

$$\begin{aligned} \text{vol } \partial F_i &\leqq \bar{V} \# \{\sigma \mid \sigma \text{ is a } (2n-1)\text{-simplex of } \bar{T} \text{ contained in } \partial F_i\} \\ &\leqq (2n+1)(\bar{V}/\bar{v}) \text{ vol } \partial^* F_i \\ &\leqq 4n(2n+1)(V_p \bar{V}/v'\bar{v}) \text{ vol } \partial F'_i. \end{aligned}$$

This completes the proof of Lemma 3.6.

Lemma 3.7. *There exists a constant K_4 such that*

$$|\chi(G_i) - \chi(F'_i)| < K_4 \text{ vol } \partial F'_i.$$

for all i.

Proof. By the way of choosing δ, the subset $H_i - G_i$ is contained in a collar W_i of ∂G_i. Let $\alpha: G_i - F'_i \to H_i - F'_i$ and $\beta: H_i - F'_i \to (G_i \cup W_i) - F'_i$ be the inclusion maps. Since $(\beta \circ \alpha)_*: H_l(G_i - F'_i; \boldsymbol{R}) \to H_l((G_i \cup W_i) - F'_i; \boldsymbol{R})$ is an isomorphism, $\alpha_*: H_l(G_i - F'_i; \boldsymbol{R}) \to H_l(H_i - F'_i; \boldsymbol{R})$ is injective. Therefore we have

$$\begin{aligned} |\chi(G_i) - \chi(F'_i)| &= |\chi(G_i - F'_i)| \\ &\leqq \sum_{l=0}^{2n} \dim H_l(G_i - F'_i; \boldsymbol{R}) \leqq \sum_{l=0}^{2n} \dim H_l(H_i - F'_i; \boldsymbol{R}) \\ &\leqq \# \{\sigma \mid \sigma \text{ is a simplex of } T' \text{ contained in } \mathrm{Cl}(H_i - F'_i)\}. \end{aligned}$$

Since $\mathrm{Cl}\,(H_i - F_i') \subset N_{d_p+d'}(\partial F_i)$, the number of simplices of T' contained in $\mathrm{Cl}\,(H_i - F_i')$ is not greater than $2^{2n+1} \cdot \mathrm{vol}\, N_{d_p+d'}(\partial F_i)/v'$. Since $\sum_{l=0}^{2n} \dim H_l(H_i - F_i'; \boldsymbol{R})$ is not greater than the number of simplices of T' contained in $\mathrm{Cl}\,(H_i - F_i')$, it follows that

$$\begin{aligned} |\chi(G_i) - \chi(F_i')| &\leqq 2^{2n+1} \,\mathrm{vol}\, N_{d_p+d'}(\partial F_i)/v' \\ &\leqq 2^{2n+1} \exp\,(\xi(d_p + d') + \eta)\, \mathrm{vol}\, \partial F_i/v' \\ &\leqq 2^{2n+1} \exp\,(\xi(d_p + d') + \eta) K_3\, \mathrm{vol}\, \partial F_i'/v'. \end{aligned}$$

This completes the proof of Lemma 3.7.

By Lemma 3.7, we have

$$\lim_{i\to\infty} \chi(G_i)/\mathrm{vol}\, F_i' = \lim_{i\to\infty} \chi(F_i')/\mathrm{vol}\, F_i' = 0.$$

Now we find a contradiction, as follows. Let $\chi = \min\,\{\chi(P_j) \mid (P_j, g_j)$ is an essential period$\}$. By the assumption of (b), it follows that $\chi > 0$. Let $\Lambda(i) = \{\lambda \mid K_\lambda \subset G_i\}$ and $K_5 = \sum\,\{|\chi(K_\lambda)|;\ K_\lambda$ corresponds to an inessential period$\}$. Clearly $K_5 < \infty$. Since $\#\Lambda(i) \geqq \mathrm{vol}\, F_i'/V_p$, we have

$$\chi(G_i) = \sum_{\lambda \in \Lambda(i)} \chi(K_\lambda) \geqq \chi \cdot \#\Lambda(i) - K_5 \geqq \chi \cdot (\mathrm{vol}\, F_i'/V_p) - K_5$$

and

$$\liminf_{i\to\infty} \chi(G_i)/\mathrm{vol}\, F_i' \geqq \chi/V_p > 0.$$

This contradiction completes the proof of (b).

Remark 3.8. By the similar arguments as in the proof of (b) of Theorem 2, we see that

(b$'$) If $\chi(P_j) < 0$ for each essential period (P_j, g_j), then (F, g) does not have average Euler characteristic zero.

Example 3.9. Jacob's ladder multiplied by S^{2n} is almost periodic and satisfy the condition of (b$'$) in Remark 3.8. Therefore it does not have average Euler characteristic zero. On the other hand, we can directly show that Infinite Loch Ness monster multiplied by S^{2n} has average Euler characteristic zero.

Remark 3.10. By a result of Tsuchiya [5], we can show that a leaf, of finite depth, of a codimension-one foliation of a closed manifold M with $H_1(M; \boldsymbol{R}) = 0$ satisfies the condition of (a) in Theorem 2. Therefore such a leaf has average Euler characteristic zero. This result and the above examples guaranteed the justice of our definition of average Euler characteristic and encouraged us. This result is included in Main Theorem since a leaf of finite depth has polynomial growth.

§ 4. The proof of Main Theorem

Let M be a closed orientable manifold of dimension n, $\mathscr{F}$ a transversely orientable codimension-one foliation of M and F a non-compact leaf of $\mathscr{F}$, satisfying the conditions (1) and (2) of Main Theorem. Take a Riemannian metric g of M. We take a vector field Z tangent to $\mathscr{F}$ generically with respect to $\mathscr{F}$ and denote by Zero (Z) the set $\{x \in M \mid Z(x)=0\}$. Then we may suppose the following conditions.

(1) Zero (Z) consists of a finite number of connected components $C_1, \cdots, C_\kappa$.

(2) For each C_k, there exists an imbedding $\psi_k: S^1 \to M$ with $C_k = \psi_k(S^1)$.

(3) Each C_k is transverse to F and tangent to $\mathscr{F}$ at at most finite points. Denote by $\Sigma_k(Z)$ the set of such points and let $\Sigma(Z)=\Sigma_1(Z) \cup \cdots \cup \Sigma_\kappa(Z)$.

(4) If C_k is tangent to $\mathscr{F}$ at z (that is, $z \in \Sigma_k(Z)$), then there exist a compact neighborhood $U(z)$ of z in M and a diffeomorphism $h: U(z) \to J^{n-1} \times J$ such that

$$\mathscr{F} \mid U(z) = h_z^*(\{J^{n-1} \times \{t\} \mid t \in J\}),$$
$$h_z(C_k \cap U(z)) = \{(t, 0, \cdots, 0, 4t^2) \mid t \in J\},$$

where $J=[-1, 1]$. Furthermore $U(z_1) \cap U(z_2)=\emptyset$ if $z_1 \neq z_2$. (See Figure 4.1.)

Denote by $\mathscr{L}_k$ the set of the closures of connected components of $C_k - \Sigma_k(Z)$. We number the elements of $\mathscr{L}_k$ in such a way that

$$\mathscr{L}_k = \{L_1^k, \cdots, L_{\nu(k)}^k\},$$
$$L_i^k \cap L_j^k \neq \emptyset \quad \text{if } j=i+1 \bmod \nu(k).$$

Denote by F_x the leaf of $\mathscr{F}$ passing through x, by $Z \mid F_x$ the vector field of F_x induced from Z, and by $I(x, Z \mid F_x)$ the index of $Z \mid F_x$ at x. Easily we have the following lemma and we omit the proof.

Lemma 4.1. (1) *If $x, y \in \operatorname{Int} L_i^k$, then $I(y, Z \mid F_y)=I(x, Z \mid F_x)$.*

(2) *If $x \in \operatorname{Int} L_i^k$ and $y \in \operatorname{Int} L_j^k$ where $j=i+1 \bmod \nu(k)$, then $I(y, Z \mid F_y) = -I(x, Z \mid F_x)$.*

Take a non-singular vector field X transverse to $\mathscr{F}$ such that each connected component of Zero $(Z) - \cup \{U(z) \mid z \in \Sigma(Z)\}$ is contained in an orbit of X and X is tangent to $h_z^{-1}(\partial J^{n-1} \times J)$ for all $z \in \Sigma(Z)$.

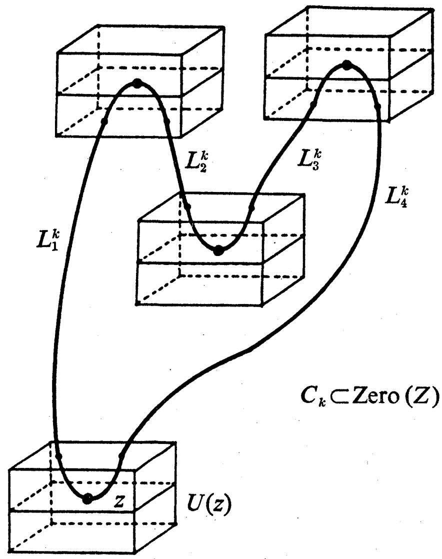


Fig. 4.1.

Definition 4.2. An $(\mathscr{F}, X)$-*box* is a compact subset B of M for which there exists a p.d. imbedding $\psi\colon D^{n-1}\times J\to M$ with $\psi(D^{n-1}\times J)=B$ such that

(1) for each $t\in J$, the image $\psi(D^{n-1}\times\{t\})$, called a *plaque* of B, is contained in a leaf of $\mathscr{F}$,

(2) for each $x\in D^{n-1}$, the image $\psi(\{x\}\times J)$, called a *pin* of B, is contained in an orbit of X.

For each $(\mathscr{F}, X)$-box B, let $\partial_{\mathscr{F}}B=\psi(D^{n-1}\times\{-1, 1\})$, $F(B)=\psi(D^{n-1}\times\{-1\})$, $C(B)=\psi(D^{n-1}\times\{1\})$ and $\partial_X(B)=\psi(\partial D^{n-1}\times J)$, where ψ is as in Definition 4.2. Clearly $U(z)$ is an $(\mathscr{F}, X)$-box for all $z\in\Sigma(Z)$. The proof of the following lemma is a tedious routine work and we omit it.

Lemma 4.3. *There exists a finite* $(\mathscr{F}, X)$*-box covering* $\mathscr{B}=\{B_\lambda\}_{\lambda\in\Lambda}$ *of* M *such that*

(1) $U(z)\in\mathscr{B}$ *for all* $z\in\Sigma(Z)$,

(2) Int B_λ's *are disjoint,*

(3) $\partial B_\lambda\subset\bigcup\{\partial B_\mu\,|\,\mu\in\Lambda, \mu\neq\lambda\}$, $\partial_{\mathscr{F}}B_\lambda\cap F=\emptyset$, *and* $\partial_X B_\lambda\cap \mathrm{Zero}\,(Z)=\emptyset$ *for all* $\lambda\in\Lambda$. (*See Figure* 4.2.)

Denote by $\mathrm{pr}_X^\lambda\colon B_\lambda\to F(B_\lambda)$ the projection along the pins of B_λ. By an induction on the dimension of strata of a certain stratifications of $\bigcup_{\lambda\in\Lambda}\partial F(B_\lambda)$ determined by $\mathscr{B}$ and by perturbing $\partial_X(B_\lambda)$'s if necessary, we can take a triangulation $T(\partial F(B_\lambda))$ of each $\partial F(B_\lambda)$ such that

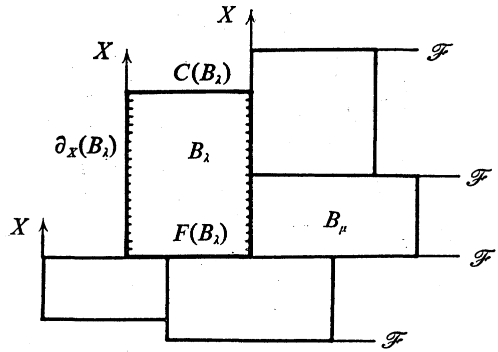


Fig. 4.2.

(1) for all $\lambda \in \Lambda$ and all $\Omega \subset \Lambda$, the intersection

$$\bigcap_{\mu \in \Omega} \mathrm{pr}_X^\lambda((\partial_X(B_\lambda) - C(B_\lambda)) \cap \partial_X(B_\mu))$$

is a subcomplex with respect to $T(\partial F(B_\lambda))$,

(2) Z is transverse to each simplex of $T(\partial F(B_\lambda))$. (See Figure 4.3.)

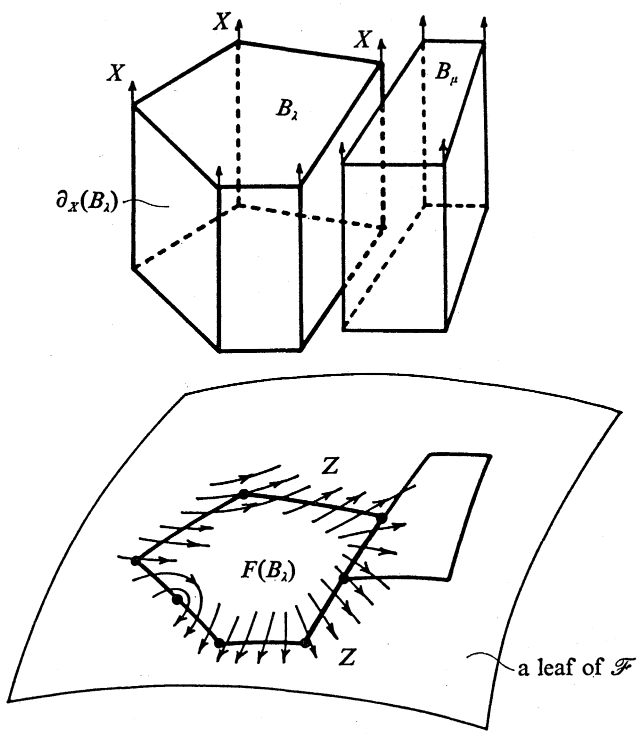


Fig. 4.3.

By the way of taking X, there exists a triangulation $T(F(B_\lambda))$ of $F(B_\lambda)$ such that

(1) $T(\partial F(B_\lambda))$ is the restriction of $T(F(B_\lambda))$ to $\partial F(B_\lambda)$,

(2) $\mathrm{pr}_X^\lambda(\mathrm{Zero}\,(Z)\cap B_\lambda)\subset \mathrm{Int}\,\sigma$ for some $(n-1)$-simplex σ of $T(F(B_\lambda))$.

For each plaque $P\subset B_\lambda - C(B_\lambda)$, let $T(P)$ be the induced triangulation $(\mathrm{pr}_X^\lambda | P)^* T(F(B_\lambda))$ of P. By decomposing B_λ's more finely and perturbing ∂B_λ's if necessary, we may suppose that Z is transverse to each simplex of $T(P)$. These triangulations determine a triangulation T of F. The finiteness of $\mathscr{B}$ implies that T is hereditarily uniform with respect to $g|F$. When $\dim F = n-1$ is odd, the proof of Main Theorem is completed by applying Theorem 1. Hereafter we suppose that $\dim F$ is even.

We may suppose that T is a derived subdivision of some triangulation T' of F and let F^* be the dual cell decomposition of T'. By the similar arguments as in the proof of Theorem 1, we have the following lemma and omit the proof.

Lemma 4.4. *There exists a sequence $F_1\subset F_2\subset\cdots\subset F$ of compact connected T-submanifolds with boundary such that*

(1) *$\{F_i\}_{i\in \boldsymbol{N}}$ is comparable to $\{D_r(x)\}_{r\in \boldsymbol{R}^+}$ for some $x\in F$,*

(2) $\lim_{i\to\infty} \mathrm{vol}\, \partial F_i/\mathrm{vol}\, F_i = 0$.

According to Plante [4], we have an *asymptotic homology class* $A_F \in H_{n-1}(M; \boldsymbol{R})$ of F constructed from $\{F_i\}_{i\in \boldsymbol{N}}$. Precisely this means that

(1) there exists a subsequence $\{F'_j\}_{j\in \boldsymbol{N}}$ of $\{F_i\}_{i\in \boldsymbol{N}}$ such that for any closed $(n-1)$-form η of M the sequence $\left\{\int_{F'_j}\eta \Big/ \mathrm{vol}\, F'_j\right\}_{j\in \boldsymbol{N}}$ is convergent,

(2) A_F is well-defined as an element in the dual of $H^{n-1}(M; \boldsymbol{R})$ by

$$A_F([\eta]) = \lim_{j\to\infty}\int_{F'_j}\eta \Big/ \mathrm{vol}\, F'_j \qquad \text{for } [\eta]\in H^{n-1}(M; \boldsymbol{R}).$$

Now we are going to construct an $(n-1)$-form e of M representing the Euler class of $T\mathscr{F}$. In the first place, we take a function $\alpha\colon D^{n-1}\to [0, \infty[$ such that

(1) $\alpha = 0$ in a neighborhood of ∂D^{n-1},

(2) $\alpha(x) = \alpha(y)$ if $x, y\in D^{n-1}$ and $\|x\| = \|y\|$,

(3) $\displaystyle\int_{D^{n-1}} \alpha\, dx_1\wedge\cdots\wedge dx_{n-1} = 1$,

where

$$D^{n-1} = \{x = (x_1, \cdots, x_{n-1})\in \boldsymbol{R}^{n-1} \mid \|x\| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}\leqq 1\}.$$

Let $\pi_1\colon D^{n-1}\times S^1\to D^{n-1}$ and $\pi_2\colon D^{n-1}\times \boldsymbol{R}\to D^{n-1}$ be the projections. We will use the induced $(n-1)$-forms $\xi_j = \pi_j^*(\alpha\, dx_1\wedge\cdots\wedge dx_{n-1})$, $j = 1, 2$.

Clearly $d\xi_1 = d\xi_2 = 0$.

Secondly consider the connected components $C_1, \cdots, C_\kappa$ of Zero(Z). For C_k transverse to $\mathscr{F}$, take a small tubular neighborhood W_1^k of C_k in M and an orientation preserving diffeomorphism $f_{k1}\colon W_1^k \to D^{n-1} \times S$ such that

(1) $\mathscr{F} \mid W_1^k = f_{k1}^*(\{D^{n-1} \times \{t\} \mid t \in S^1\})$,

(2) $f_{k1}(C_k) = \{0\} \times S^1$.

For C_k with $\Sigma_k(Z) \neq \emptyset$, we consider $\mathscr{L}_k$. For each $L_i^k \in \mathscr{L}_k$, take an arc K_i^k in M such that

(1) $K_i^k - \cup \{U(z) \mid z \in \Sigma(Z)\} = L_i^k - \cup \{U(z) \mid z \in \Sigma(Z)\}$,

(2) for $z \in \Sigma_k(Z)$ with $L_i^k \cap U(z) \neq \emptyset$,

$$K_i^k \cap U(z) = h_z^{-1}(\{(0, \cdots, 0)\} \times [-1/3, 0]) \cup \gamma_i^k,$$

where γ_i^k is a curve in $h_z^{-1}(J^{n-1} \times [0, 1])$ transverse to $\mathscr{F}$ connecting the points $L_i^k \cap \partial U(z)$ and z.

Furthermore take a small $\varepsilon > 0$ and a small tubular neighborhood W_i^k of K_i^k such that for $z \in \Sigma_k(Z)$ with $K_i^k \cap U(z) \neq \emptyset$,

$$W_i^k \cap h_z^{-1}(J^{n-1} \times [-1, 0]) = h_z^{-1}(D_\varepsilon^{n-1} \times [-1/2, 0])$$

where $D_\varepsilon^{n-1} = \{x \in \boldsymbol{R}^{n-1} \mid \|x\| \leqq \varepsilon\}$, and an orientation preserving imbedding $f_{ki}\colon W_i^k \to D^{n-1} \times \boldsymbol{R}$ such that

(1) $\mathscr{F} \mid W_i^k = f_{ki}^*(\{D^{n-1} \times \{t\} \mid t \in \boldsymbol{R}\})$,

(2) $f_{ki}(K_i^k) \subset \{0\} \times \boldsymbol{R}$,

(3) $f_{ki}(\partial W_i^k - \cup \{U(z) \mid z \in \Sigma(Z)\}) \subset \partial D^{n-1} \times \boldsymbol{R}$,

(4) for each $z \in \Sigma_k(Z)$ with $K_i^k \cap U(z) \neq \emptyset$,

$$\pi_2 \circ f_{ki}(h_z^{-1}(x_1, \cdots, x_{n-1}, t)) = (x_1/\varepsilon, \cdots, x_{n-1}/\varepsilon),$$

for $(x_1, \cdots, x_{n-1}) \in D^{n-1}$ and $t \in [-1/2, 0]$.

We may suppose that if $W_i^k \cap B_\lambda \neq \emptyset$, then for each plaque P of B_λ the intersection $W_i^k \cap P$ is contained in Int σ for some $(n-1)$-simplex σ of $T(P)$. See Figure 4.4.

By (1) of Lemma 4.1, we can attach an integer I_i^k to each $L_i^k \in \mathscr{L}_1 \cup \cdots \cup \mathscr{L}_k$ in such a way that $I_i^k = I(x, Z \mid F_x)$ for all $x \in \text{Int}\, L_i^k$. For each L_i^k, define an $(n-1)$-form η_{ki} of M by

$$\eta_{ki} \mid W_i^k = \begin{cases} f_{ki}^* \xi_1 & \text{if } \Sigma_k(Z) \neq \emptyset, \\ f_{ki}^* \xi_2 & \text{otherwise,} \end{cases}$$

$$\eta_{ki} \mid M - W_i^k = 0.$$

Note that if $\Sigma_k(Z) \neq \emptyset$, then η_{ki} is not continuous at $h_z^{-1}(D^{n-1} \times \{-1/2\})$ for all $z \in \Sigma_k(Z)$ with $U(z) \cap W_i^k \neq \emptyset$. At last, let $e = \sum_{k,i} I_i^k \eta_{ki}$. By (2)

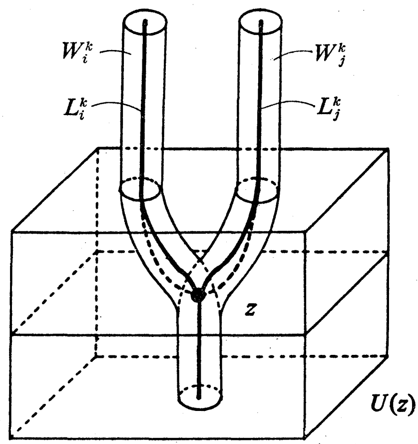


Fig. 4.4.

of Lemma 4.1, the discontinuities of η_{ki}'s are canceled and e is a smooth $(n-1)$-form of M. We have

$$de = \sum_{k,i} I_i^k d\eta_{ki} = \sum_{k,i} I_i^k f_{ki}^* d\xi_j = 0,$$

where $j=1$ or 2.

Lemma 4.5. *The closed form e represents the Euler class of $T\mathscr{F}$.*

Proof. Let $C \in H_{n-1}(M; Z)$. Using a singular chain c representing C, we can construct a simplicial complex K and a p.d. map $\phi: K \to M$ such that $\phi_*([K]) = C$, where $[K]$ is the homology class of K represented by the sum of all the $(n-1)$-simplices of K. We may suppose that for each B_λ the intersection $\phi(K) \cap \operatorname{Int} B_\lambda$ is written as $\bigcup_i \operatorname{Int} P_i^\lambda$ for a finite number of plaques $P_1^\lambda, \cdots, P_{\nu(\lambda)}^\lambda$ of B_λ and $\phi \mid \phi^{-1}(\operatorname{Int} P_j^\lambda): \phi^{-1}(\operatorname{Int} P_j^\lambda) \to \operatorname{Int} P_j^\lambda$ is a diffeomorphism for $j = 1, \cdots, \nu(\lambda)$. Furthermore we may suppose that $\phi(K) \cap (\cup \{U(z) \mid z \in \Sigma(Z)\}) = \emptyset$. Consider the induced vector bundle $\phi^* T\mathscr{F} \subset K \times T\mathscr{F}$ and the induced section $\phi^* Z: K \to \phi^* T\mathscr{F}$ defined by $\phi^* Z(x) = (x, Z(\phi(x)))$ for all $x \in K$. Clearly Zero $(\phi^* Z) = \phi^{-1}$ (Zero (Z)) is a finite set. Denote by $I(x, \phi^* Z)$ the index of $\phi^* Z$ at $x \in$ Zero $(\phi^* Z)$. For each plaque P of B_λ intersecting Zero $(Z) - \cup \{U(z) \mid z \in \Sigma(Z)\}$, we have

$$\begin{aligned} \int_P e &= \int_P I_i^k f_{ki}^* \xi_j = I_i^k \cdot \int_{D^{n-1}} \alpha \, dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= I_i^k = I(x, Z \mid F_x), \end{aligned}$$

for adequate k, i, j and $x \in P \subset L_i^k$. Therefore it follows that

$$\langle [K], \phi^*[e] \rangle = \int_{\phi(k)} e = \sum_{x \in \mathrm{Zero}(\phi^* Z)} I(x, \phi^* Z).$$

This means that $\phi^*[e]$ is the Euler class of $\phi^* T\mathscr{F}$. Therefore $[e]$ is the Euler class of $T\mathscr{F}$, which completes the proof of Lemma 4.5.

By the condition (1) of Main Theorem, it follows that $H_{n-1}(M; \boldsymbol{R}) = 0$. Therefore the asymptotic homology class A_F is zero. Since $A_F(e) = 0$, it follows that

$$0 = \lim_{j \to \infty} \int_{F'_j} e \Big/ \mathrm{vol}\, F'_j = \lim_{j \to \infty} \sum_{x \in F'_j} I(x, Z \mid F) / \mathrm{vol}\, F'_j,$$

where we use the convention that $I(x, Z \mid F) = 0$ for $x \in F$ with $Z(x) \neq 0$. We can estimate the difference between $\chi(F_i)$ and $\sum_{x \in F_i} I(x, Z \mid F)$ as follows.

Lemma 4.6. *There exists a positive constant Q such that*

$$|\chi(F_i) - \sum_{x \in F_i} I(x, Z \mid F)| < Q \,\mathrm{vol}\, \partial F_i.$$

Suppose that Lemma 4.6 is proved. Then we have

$$\begin{aligned}
0 &\leqq \limsup_{j \to \infty} |\chi(F'_j)| / \mathrm{vol}\, F'_j \\
&\leqq \limsup_{j \to \infty} \Big(\Big| \sum_{x \in F'_j} I(x, Z \mid F) \Big| + Q \,\mathrm{vol}\, \partial F'_j \Big) / \mathrm{vol}\, F'_j \\
&= \limsup_{j \to \infty} \Big| \sum_{x \in F'_j} I(x, Z \mid F) \Big| / \mathrm{vol}\, F'_j + Q \limsup_{j \to \infty} \mathrm{vol}\, \partial F'_j / \mathrm{vol}\, F'_j \\
&= 0.
\end{aligned}$$

Therefore $\lim_{j \to \infty} \chi(F'_j) / \mathrm{vol}\, F'_j = 0$. Since $\{F'_j\}_{j \in N}$ satisfies the conditions corresponding to (1) and (2) of Definition 2.3 also, $(F, g \mid F)$ has average Euler characteristic zero. The rest of this section is devoted to the proof of Lemma 4.6.

Since each $z \in \mathrm{Zero}\,(Z \mid F)$ is contained in Int σ for some $(n-1)$-simplex σ of T, there exists a positive constant δ^* such that

$$N_{\delta^*}(\partial F_i) \cap \mathrm{Zero}\,(Z \mid F) = \emptyset.$$

We are going to construct a smooth submanifold G_i in $N_{\delta^*}(\partial F_i) - \mathrm{Int}\, F_i$.

Denote by $\mathscr{O}$ (or $\mathscr{I}$) the set of $(n-2)$-simplices $\sigma \subset \partial F_i$ such that Z is outward (or inward, respectively) at Int σ from F_i. We call a simplex $\tau \subset \partial F_i$ *positive* (or *negative*) if all the $(n-1)$-simplices $\sigma \subset \partial F_i$ containing τ belong to $\mathscr{O}$ (or $\mathscr{I}$), and call τ *neutral* if τ is neither positive nor negative.

Let τ be a vertex in ∂F_i. Then there exists an imbedding u_τ of a small compact neighborhood $U(\tau)$ of τ in F into $\boldsymbol{R}^{n-1}$ with coordinates $y_1, \cdots, y_{n-1}$ such that

(1) $u_\tau(\tau)=(0, \cdots, 0)$, $u_\tau(U(\tau))=D_3^{n-1}$,

(2) $u_{\tau*}(Z(x))=\partial/\partial y_{n-1}$ for all $x \in U(\tau)$,

(3) for some bicollar $V(\tau)$ of $K(\tau)=u_\tau(\partial F_i \cap U(\tau)) \cap \partial D_3^{n-1}$ in ∂D_3^{n-1}, the image $u_\tau(\partial F_i \cap U(\tau))$ is contained in the cone

$$CV(\tau)=\{(tx_1, \cdots, tx_{n-1}) \,|\, 0 \leqq t \leqq 1,\ (x_1, \cdots, x_{n-1}) \in V(\tau)\}.$$

See Figure 4.5.

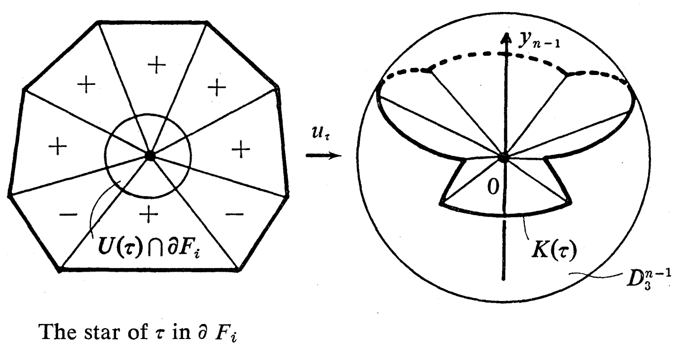


Fig. 4.5.

Since each simplex of ∂F_i is transverse to Z, the y_{n-1} axis does not intersect $CV(\tau)-\{(0, \cdots, 0)\}$. Let

$$E(a_1, \cdots, a_{n-1}; w)=\left\{(x_1, \cdots, x_{n-1}) \in \boldsymbol{R}^{n-1} \,\middle|\, \left(\sum_{j=1}^{n-2}(x_j-a_j)^2\right)\Big/ w^2 + (x_{n-1}-a_{n-1})^2/(3/2)^2=1\right\}.$$

Then there exists $w>0$ such that $E(0, \cdots, 0, 1; w) \cap CV(\tau) \subset D_1^{n-1}$ and $E(0, \cdots, 0, -1; w) \cap CV(\tau) \subset D_1^{n-1}$. Let $D(\tau)=u^{-1}(E(0, \cdots, 0, a; w))$ where $a=1$ if τ is negative, and otherwise $a=-1$. Then Z is inward (or outward) at $\partial(F_i \cup D(\tau)) \cap D(\tau)$ if τ is negative (or otherwise). The change from ∂F_i to $\partial(F_i \cup D(\tau))$ near τ is illustrated by Figure 4.6, where $+$ (or $-$) means outward (or inward).

Let τ be a 1-simplex in ∂F_i. Then there exists an imbedding u_τ of a small tubular neighborhood $U(\tau)$ of τ into $\boldsymbol{R}^{n-1}$ such that

(1) $u_\tau(t)=\{(t, 0, \cdots, 0) \,|\, t \in J\}$, $u_\tau(U(\tau))=J \times D_3^{n-2}$,

(2) $u_{\tau*}(Z(x))=\partial/\partial y_{n-1}$ for all $x \in U(\tau)$,

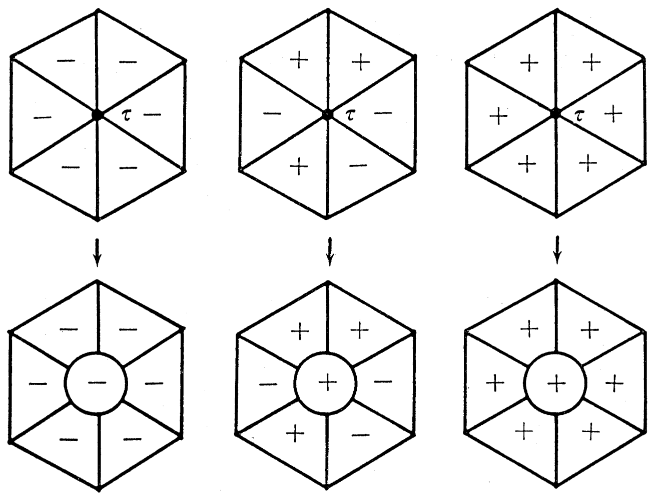


Fig. 4.6.

(3) for some bicollar $V(\tau)$ of $K(\tau)=u_\tau(\partial F_i \cap U(\tau)) \cap J \times \partial D_3^{n-2}$ in $J \times \partial D_3^{n-2}$, the subset $u_\tau(\partial F_i \cap U(\tau)) \cap \{t\} \times \boldsymbol{R}^{n-2}$ is contained in the cone $\{(t, sx_2, \cdots, sx_{n-1}) \mid 0 \leqq s \leqq 1, (t, x_2, \cdots, x_{n-1}) \in V(\tau) \cap \{t\} \times \boldsymbol{R}^{n-2}\}$. As in the case of vertex, we can take a thin tubular neighborhood $D(\tau)$ of τ in F such that Z is inward (or outward) at $\partial(F_i \cup D(\tau)) \cap D(\tau)$ except near the vertices $\in \tau$ if τ is negative (or otherwise).

By the similar way, we take a thin tubular neighborhood $D(\tau)$ for each simplex τ of dimension $<n-1$ and let $F_i^* = F_i \cup (\bigcup_\tau D(\tau))$. (We omit the details.) By a downward induction on the dimension of τ, we can round the corner of ∂F_i^* in an arbitrarily small neighborhood of the corner and obtain a smooth submanifold G_i. Let H_i be the compact submanifold of F such that $\partial H_i = G_i$ and $H_i \supset F_i$. Clearly F_i is a deformation retract of F_i^* and of H_i.

Denote by $I(\partial F_i^*)$ (or $I(G_i)$) the closure of the set of $x \in \partial F_i^*$ (or G_i) such that $Z(x)$ is inward. Each connected component I_j^* of $I(\partial F_i^*)$ is homeomorphic to $\bar{I}_j = \cup \{\sigma \in \mathscr{S} \mid I_j^* \cap \operatorname{Int} \sigma \neq \emptyset\}$, and corresponds to a connected component I_j of $I(G_i)$, which may be supposed to be a smooth submanifold-with-boundary of G_i. Glue $G_i \times J$ to H_i by identifying $G_i \times \{-1\}$ and $G_i = \partial H_i$ canonically. Take a function $\beta: J \to [0, 1]$ such that

(1) $\beta=0$ on $[-1, -1/3]$, $\beta=1$ on $[1/3, 1]$,

(2) $0<\beta<1$ on $]-1/3, 1/3[$,

and a function $\gamma: J \to [0, 1]$ such that

(1) $\gamma=0$ on $[-1, -2/3] \cup [2/3, 1]$,

(2) $\gamma=1$ on $[-1/3, 1/3]$,

(3) $0<\gamma<1$ on $]-2/3, -1/3[\cup]1/3, 2/3[$.

Take a vector field Y on G_i such that

(1) $Y=0$ outside of a neighborhood of $I(G_i)$,

(2) Y is outward at $\partial I(G_i)$ from $I(G_i)$,

(3) Y has $|\chi(I(G_i))|$ singular points of index ± 1 in Int $I(G_i)$.

Now let Z_i be the vector field of $H_i \cup G_i \times J$ defined by

(1) $Z_i | H_i = Z | H_i$,

(2) for each $(x, t) \in G_i \times J$,

$$Z_i(x, t)=\beta(t)\, \partial/\partial t+\beta(-t)Z(x)+\gamma(t)Y(x).$$

Then $Z_i | G_i \times J$ has just $|\chi(I(G_i))|$ singular points of index ± 1 and Z_i is outward at $G_i \times \{1\}$. It follows that $|\chi(F_i)-\sum_{x \in F_i} I(x, Z | F)|=|\chi(I(G_i))|$. Since $I(G_i)$ is homeomorphic to $I(\partial F_i^*)$, we have

$$\begin{aligned} |\chi(I(G_i))| &= |\chi(I(\partial F_i^*))| \\ &\leqq \sum_j \# \{\sigma \mid \sigma \text{ is a simplex } \subset \bar{I}_j\} \\ &\leqq N \# \{\sigma \mid \sigma \text{ is a simplex } \subset \partial F_i\} \\ &\leqq N(2^n/v) \operatorname{vol} \partial F_i, \end{aligned}$$

where N and v are the positive constants corresponding to the uniform triangulation T. This completes the proof of Lemma 4.6 and of Main Theorem.

Remark 4.7. By the above proof, we can replace the condition (1) of Main Theorem by the following weak condition.

(1)$'$ $\chi(T\mathscr{F})=0$ if dim F is even.

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