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A Sufficient Condition for the C^2-Reeb Stability of Noncompact Leaves of Codimension One Foliations

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Introduction

A leaf L of a codimension one foliation is called *stable* (in Reeb's sense) if it has a saturated neighborhood which is foliation-preservingly diffeomorphic to $L\times D^1$ with leaves $L\times\{t\}$, $t\in D^1$. The following problem naturally arises: Find topological conditions on a leaf L which imply the stability of L. In the case when L is a compact leaf of a C^1 foliation, a final result has been obtained by Thurston (See § 1). In this paper we deal with the case of noncompact leaves of C^2 foliations. Our main result generalizes a theorem of Cantwell-Conlon on surfaces [C-C 1] to higher dimensions.

The organization of this paper is as follows. In Section 1, the notion of C^r-stability property is introduced and some known facts are reviewed. In Section 2, results of this paper are stated precisely. Some of them are proved in Section 3. In Section 4, fundamental properties of plaque cycles are presented. In Section 5, Hector's theorem on the convergence of holonomy is stated. (The proof is given in the Appendix.) His theorem plays an important role in the proof of our main theorem, which appears in the last section.

Some results of this paper have already been announced in [I 1].

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§ 1. Definitions and known facts

Throughout this paper, foliations are assumed to be transversely oriented, of codimension one and with C^∞ leaves, unless otherwise specified.

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1.1 Definition. Let $\mathscr{F}$ be a foliation. A proper leaf L of $\mathscr{F}$ is *stable* if there exist a saturated neighborhood N of L and a foliation-preserving diffeomorphism

$$\psi\colon (L\times D^1, \{L\times\{t\}\}_{t\in D^1})\longrightarrow (N, \mathscr{F}\,|\,N)$$

such that $\psi\,|\,L\times\{0\}=\mathrm{id}_L$.

The following definitions are due to Cantwell-Conlon [C-C 1].

1.2 Definition. Let L be a C^∞ manifold without boundary. L has the *C^r-stability property* if, whenever L is diffeomorphic to a proper leaf of a C^r foliation of a closed C^∞ manifold, that leaf is stable.

1.3 Definition. Let L be a C^∞ manifold without boundary. L has the *weak C^r-stability property* if, whenever L is diffeomorphic to a proper leaf with trivial holonomy group of a C^r foliation of a closed C^∞ manifold, that leaf is stable.

The ultimate aim of the stability problem of proper leaves is to present a necessary and sufficient condition under which L has the C^r-stability property (or the weak C^r-stability property) in terms of the topology of L. In the case when L is a closed manifold (i.e. a compact leaf when realized as a leaf in some foliation), this problem has been almost completely solved by Thurston [Th]. He proved that for such L to have the C^r-stability property ($0\leqq r\leqq\infty$), it is necessary, and even sufficient if $r\geqq 1$, that $H^1(L;\boldsymbol{R})=0$. As for the C^0 case, the complete answer has not been obtained yet. (There is a partial answer of Plante [Pl, § 8], which states that if $H^1(L;\boldsymbol{R})=0$ and $\pi_1(L)$ has nonexponential growth, then L has the C^0-stability property.)

In contrast with the closed case, few results have been known in the case when L is an open manifold. In [I 2], the author showed that $\boldsymbol{R}^2$ has the C^0-stability property. Pixton [Pi] proved that $S^1\times\boldsymbol{R}$ does not have the C^1-stability property. Recently Cantwell-Conlon [C-C 1] proved that orientable surfaces without genus whose endsets are countable have the C^2-stability property. By the results of Pixton and Cantwell-Conlon, one sees that there is a difference between C^1- and C^2-stability properties. This is a phenomenon which never occurs in the case of closed manifolds.

As for the weak stability property, the following facts are known: The classical Reeb stability theorem says that every closed manifold has the weak C^0-stability property. It is shown in [I 2] that orientable surfaces with finitely generated fundamental groups have the weak C^0-stability property, and in [D 2] that if L satisfies the following condition, then L has the weak C^0-stability property; for every compact subset K of L, there is a compact codimension zero submanifold K' containing K such that $i_\sharp$:

$\pi_0(\partial K') \to \pi_0(L - \operatorname{Int} K')$ is bijective and $i_\#: \pi_1(\partial K') \to \pi_1(L - \operatorname{Int} K')$ is surjective on each component of $\partial K'$, where $i: \partial K' \to L - \operatorname{Int} K'$ is the inclusion map. In [C-C 1], it is proved that orientable surfaces with finite genus whose endsets are countable have the weak C^2-stability property.

§ 2. Statement of results

Let $\hat{H}^1(L; \boldsymbol{R})$ be the image of the canonical homomorphism $H_c^1(L; \boldsymbol{R}) \to H^1(L; \boldsymbol{R})$, where H_c^1 denotes the first cohomology group with compact supports. (Note that $\hat{H}^1(L; \boldsymbol{R})$ coincides with $H^1(L; \boldsymbol{R})$ when L is compact.) Then we have the following necessary condition for the stability property.

2.1 Proposition. *Suppose that L is a manifold which can be realized as a proper leaf of some C^r foliation $(0 \leqq r \leqq \infty)$. If L has the C^r-stability property, then $\hat{H}^1(L; \boldsymbol{R}) = 0$.*

The vanishing of $\hat{H}^1(L; \boldsymbol{R})$ is, however, not a sufficient condition as the following says.

2.2 Proposition. *$T^2 \times \boldsymbol{R}$ does not have the C^r-stability property* $(0 \leqq r \leqq \infty)$.

In order to obtain a sufficient condition, we impose a certain restriction on the behavior of ends of manifolds. First we need to define the notion of ends. Note that the definition given here is different from the usual one (c.f. [I 3]).

Denote by E the set of pairs $(M, \{U_i\}_{i=0}^\infty)$ where M is a manifold and $U_0 \supset U_1 \supset U_2 \supset \cdots$ is a decreasing sequence of nonempty, connected open subsets of M satisfying the following conditions:

1) $\operatorname{Int} \bar{U}_i = U_i$, and $\bar{U}_i - U_i$ is a compact codimension one submanifold of M,

2) $\bar{U}_{i+1} \subset U_i$,

3) $\bigcap_{i=0}^\infty \bar{U}_i = \phi$.

Introduce an equivalence relation among elements of E in the following way. Two elements $(M, \{U_i\})$ and $(M', \{U_i'\})$ of E are equivalent if there exist a connected open subset W (resp. W') of M (resp. M') and a diffeomorphism $f: W \to W'$ such that

1) U_i (resp. U_i') is contained in W (resp. W') for large i,

2) every $f(U_i)$ contains some U_j' and every U_i' contains some $f(U_j)$.

Let $\mathscr{E}$ be the set of equivalence classes of elements of E. Elements of $\mathscr{E}$ are called *ends*. If $(M, \{U_i\}_{i=0}^\infty)$ represents an end e, then e is called an *end of M* and U_i's are called *neighborhoods of e*.

An end e is *periodic* if e is represented by $(M, \{g^i(U)\})$, where U is a subset of M and g is a diffeomorphism of U *into* U. In this case, we denote e by $[U, g]$. We call $\bar{U}-g(U)$ a *period* of e and U a *periodic neighborhood* of e.

Now we define a class $\mathscr{P}$ of periodic ends as follows. $\mathscr{P}$ is described as a disjoint union of subsets $\mathscr{P}_k$ $(k=0, 1, 2, \cdots)$. We define $\mathscr{P}_k$ by the induction on k. An end e belongs to $\mathscr{P}_0$ if e is a periodic end, say $[N(e), g(e)]$, of compact period satisfying the following condition:

(*) $F(\pi_1(N(e)))$ is isomorphic to $\{1\}$ or $\boldsymbol{Z}$, and for each $k \geqq 0$, the homomorphism $F(\pi_1(g(e)^k(N(e)))) \to F(\pi_1(N(e)))$ induced from the inclusion map is bijective,

where $F(G)$ means the quotient group of a group G by the smallest normal subgroup containing all torsion elements of G. Suppose we have defined $\mathscr{P}_i$ for $0 \leqq i \leqq k-1$. An end e belongs to $\mathscr{P}_k$ if e is constructed in the following way: Let $e_0=[N(e_0), g(e_0)] \in \mathscr{P}_0$ such that $N(e_0)$ satisfies (*). Let B_j $(1 \leqq j \leqq s)$ be pairwise disjoint, codimension zero, compact connected submanifolds of Int $P(e_0)$, where $P(e_0)$ denotes the period of $N(e_0)$. Let $e_j \in \mathscr{P}_{k_j}$ $(0 \leqq k_j \leqq k-1$, and at least one of the k_j's is equal to $k-1)$ and $N(e_j)$ the periodic neighborhood of e_j constructed in the inductive definition such that $\partial N(e_j)$ is diffeomorphic to ∂B_j by a diffeomorphism $f_j\colon \partial N(e_j) \to \partial B_j$. Define $N(e)$ by

$$N(e)=(N(e_0)-\sum_{\substack{1 \leqq j \leqq s \\ k \in \boldsymbol{N}}} \operatorname{Int} g(e_0)^k(B_j)) \bigcup_{\partial} (\bigcup_{\substack{1 \leqq j \leqq s \\ k \in \boldsymbol{N}}} N_{j,k}),$$

where $N_{j,k}$ $(k \in \boldsymbol{N})$ are copies of $N(e_j)$, and $\bigcup_\partial$ means that $\partial N_{j,k}$ is identified with $\partial g(e_0)^k(B_j)$ by $g(e_0)^k \circ f_j$. Define $g(e)$ to be an extension of

$$g(e_0) \,|\, (N(e_0)-\bigcup_{\substack{1 \leqq j \leqq s \\ k \in \boldsymbol{N}}} \operatorname{Int} g(e_0)^k(B_j))$$

to $N(e)$. Then $e=[N(e), g(e)]$.

The main results of this paper are stated in the following.

2.3 Theorem. *If L is a C^∞ manifold such that all ends of L belong to $\mathscr{P}$, then L has the weak C^r-stability property* $(2 \leqq r \leqq \infty)$.

2.4 Theorem. *If L is a C^∞ manifold such that* $\hat{H}^1(L; \boldsymbol{R})=0$ *and that all ends of L belong to $\mathscr{P}$, then L has the C^r-stability property* $(2 \leqq r \leqq \infty)$.

2.5 Remark. If the condition (*) is dropped in the definition of $\mathscr{P}$, or if $r \leqq 1$, the conclusions of these theorems do not hold (c.f. (2.2) and [Pi]).

2.6 Remark. If L is an open 2-manifold such that all ends of L are planar and that the endset of L is of finite type, then all ends of L belong to $\mathscr{P}$. Therefore (2.3) and (2.4) are regarded as a generalization of the results of Cantwell-Conlon [C-C 1].

§ 3. Proof of propositions

Proof of (2.1). Let $\mathscr{F}$ be a C^r foliation ($0 \leqq r \leqq \infty$) of a closed C^∞ manifold M which has a proper leaf L. We show that if $\hat{H}^1(L; \boldsymbol{R}) \neq 0$, then we can modify $\mathscr{F}$ to obtain a new C^r foliation $\mathscr{F}'$ of M which has a nonstable proper leaf diffeomorphic to L. If L is not stable in $\mathscr{F}$, we are done. Suppose L is stable in $\mathscr{F}$. Then there exist a saturated tubular neighborhood N of L and a foliation-preserving C^r diffeomorphism

$$\psi: (L \times D^1, \{L \times \{t\}\}_{t \in D^1}) \longrightarrow (N, \mathscr{F} \mid N)$$

such that $\psi \mid L \times \{0\} = \mathrm{id}$. By the hypothesis that $\hat{H}^1(L; \boldsymbol{R}) \neq 0$, there exists a non-exact, closed C^∞ 1-form α such that the support of α is contained in a compact subset K of L. Let $f: D^1 \to \boldsymbol{R}$ be a C^∞ bump function whose support is contained in Int D^1. Let $\mathscr{G}$ be the foliation on $L \times D^1$ defined by $dt + f(t)\alpha$. Now define a foliation $\mathscr{F}'$ of M by $\mathscr{F}' = \psi\mathscr{G}$ on N and $\mathscr{F}' = \mathscr{F}$ on $M - N$. Then $\mathscr{F}'$ is the desired foliation. ($\mathscr{F}'$ is C^r, because $\mathscr{F} = \mathscr{F}'$ outside $\psi(K \times D^1)$.) This completes the proof.

3.1 Remark. Leaves of $\mathscr{F}'$ are only C^r. But according to the recent result of Hart [Ha], for $r \geqq 1$ we can make the leaves C^∞ by a C^r conjugation of $\mathscr{F}'$.

Proof of (2.2). We explicitly construct a C^∞ foliation of a closed manifold which has a non-stable proper leaf diffeomorphic to $T^2 \times \boldsymbol{R}$. Let A be a matrix in $SL(2, \boldsymbol{Z})$ such that $|\text{trace } A| > 2$. Let λ be the eigenvalue of A with $|\lambda| > 1$ and $\begin{pmatrix} a \\ b \end{pmatrix}$ an eigenvector of ${}^tA$ for λ. Let ξ be a C^∞ vector field on the interval $[0, 1]$ such that supp $\xi = [1/2, 1]$, and let ϕ_t ($t \in \boldsymbol{R}$) be the flow generated by ξ. Define maps $f: [0, 1] \to [0, 1/2]$ and $g_1, g_2: [0, 1] \to [0, 1]$ by

$$f(x) = \frac{x}{x+1},$$

$$g_1(x) = \prod_{n=0}^{\infty} f^n \circ \varphi_{a\lambda^{-n}} \circ f^{-n}(x),$$

and

$$g_2(x) = \prod_{n=0}^{\infty} f^n \circ \varphi_{b\lambda^{-n}} \circ f^{-n}(x).$$

3.2 Lemma. *g_1 and g_2 are C^∞ diffeomorphisms of* $[0, 1]$.

Proof. Since g_1 is the time 1 map of the vector field

$$\eta_1=\sum_{n=0}^{\infty} f_*^n(a\lambda^{-n}\xi),$$

in order to show the smoothness of g_1, it suffices to show that of η_1. It is clear that η_1 is C^∞ except at 0. By [Tsub. Lemma 7.5], for arbitrary r, $0<r<\infty$, we have

$$|f_*^n(a\lambda^{-n}\xi)|_r \leqq C_r|a|\lambda^{-n}n^{2(r-1)}|\xi|_r,$$

where C_r is a constant depending only on r. Since the right hand of this inequality converges to 0 as n goes to infinity, it follows that η_1 is C^∞ at 0. The smoothness of g_2 can be shown similarly. Lemma is proved.

Noting that $g_1 \circ g_2=g_2 \circ g_1$, we define a C^∞ foliation $\mathscr{F}_0$ of $T^2\times[0, 1]\times[0, 1]$ transverse to the first $[0, 1]$-factor by the total holonomy homomorphism ([I 3] p. 115)

$$\Phi\colon \pi_1(T^2\times[0, 1])\longrightarrow \mathrm{Diff}\,[0, 1]; \quad \Phi(\alpha)=g_1, \quad \Phi(\beta)=g_2,$$

where α and β are the standard generators of $\pi_1(T^2\times[0, 1])$. Let S be the manifold with corner obtained from $T^2\times[0, 1]\times[0, 1]$ by identifying $(x, s, 1)$ with $(Ax, f(s), 0)$ for $x \in T^2=\mathbf{R}^2/\mathbf{Z}^2$, $s \in [0, 1]$. We see that $\mathscr{F}_0$ induces a well-defined foliation of S. In fact, to see this, it suffices to show that $f\circ g_1\circ f^{-1}=g_1^p\circ g_2^r$ and $f\circ g_2\circ f^{-1}=g_1^q\circ g_2^s$ on $[0, 1/2]$, where $A=\begin{pmatrix} p & q \\ r & s\end{pmatrix}$. These equalities are easily verified by direct computations. We denote the resulting foliation by $\mathscr{F}_S$. $(S, \mathscr{F}_S)$ is an *irregular staircase* in the sense of Nishimori [N]. Recall that if $p\colon T^2\times[0, 1]\times[0, 1]\to S$ is the natural projection, $W(S)=p(T^2\times[1/2, 1]\times\{1\})$ is called the *wall* of S and $C(S)=p(T^2\times\{1\}\times[0, 1])$ the *ceiling* of S. Now we glue together $(T^2\times[1/2, 1]\times[0, 1], (\mathscr{F}_S\,|\,W(S))\times[0, 1])$, $(S_1, \mathscr{F}_{S_1})$ and $(S_2, \mathscr{F}_{S_2})$ as illustrated in Figure 1, where $(S_1, \mathscr{F}_{S_1})$ and $(S_2, \mathscr{F}_{S_2})$ are copies of $(S, \mathscr{F}_S)$.

More precisely, $T^2\times[1/2, 1]\times\{0\}$ is identified with $W(S_1)$ by $(x, s, 0)\sim p(x, s, 1)$, $T^2\times[1/2, 1]\times\{1\}$ is identified with $W(S_2)$ by $(x, s, 1)\sim p(x, s, 1)$, $T^2\times\{1/2\}\times[0, 1/3]$ is identified with $C(S_1)$ by $(Ax, 1/2, 3t)\sim p(x, 1, t)$, $T^2\times\{1/2\}\times[2/3, 1]$ is identified with $C(S_2)$ by $(Ax, 1/2, 3t-2)\sim p(x, 1, t)$, and $T^2\times\{1/2\}\times[1/3, 2/3]$ is identified with $T^2\times\{1\}\times[0, 1]$ by $(Ax, 1/2, 3t-1)\sim(x, 1, t)$. The resulting manifold is diffeomorphic to $E_A\times[0, 1]$, where E_A is the total space of the T^2-bundle over S^1 with monodromy A. The induced foliation $\mathscr{F}$ of $E_A\times[0, 1]$ has a non-stable proper leaf

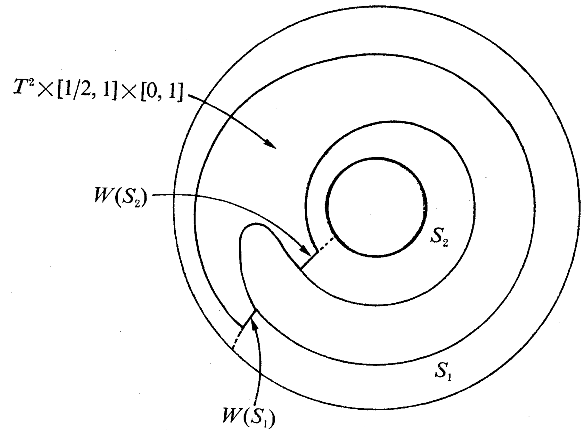

Fig. 1.

diffeomorphic to $T^2\times \boldsymbol{R}$. (The leaf which contains $C(S_1)$ and $C(S_2)$ is the required leaf.)

Let $\mathscr{F}'$ be the foliation of $E_A\times[0, 1]$ constructed in the same way as above, using diffeomorphisms f^{-1}, id and id instead of f, g_1 and g_2. Then we paste $(E_A\times[0, 1], \mathscr{F})$ and $(E_A\times[0, 1], \mathscr{F}')$ along the boundaries by the identity map to obtain a C^∞ foliation of $E_A\times S^1$ which has a non-stable proper leaf diffeomorphic to $T^2\times \boldsymbol{R}$. This completes the proof.

§ 4. Plaque chains and plaque cycles

Let M be an n-dimensional closed C^∞ manifold and $\mathscr{F}$ a C^r foliation of M. We fix a one-dimensional foliation $\mathscr{T}$ which is everywhere transverse to $\mathscr{F}$. A *foliation chart* (W, φ) for $\mathscr{F}$ is a pair of a compact submanifold W of M and a foliation-preserving C^r diffeomorphism

$$\varphi\colon (D^{n-1}\times D^1, \{D^{n-1}\times\{t\}\}_{t\in D^1}, \quad \{\{x\}\times D^1\}_{x\in D^{n-1}})\longrightarrow(W, \mathscr{F}\mid W, \mathscr{T}\mid W).$$

We call $\varphi(D^{n-1}\times\{t\})$, $t\in D^1$, a *plaque*.

The following lemma is fundamental and well known.

4.1 Lemma. *There exists a finite family* $\mathscr{W}=\{(W_i, \varphi_i)\}_{i=1}^m$ *of foliation charts for* $\mathscr{F}$ *satisfying the following conditions.*

i) $\{\mathrm{Int}\, W_i\}_{i=1}^m$ *is an open covering of* M.

ii) *If* $W_i\cup W_j\cup W_k$ *is connected, there exists a foliation chart* (W, φ) *such that* $W_i\cup W_j\cup W_k\subset \mathrm{Int}\, W$.

iii) *For each leaf* L *of* $\mathscr{F}$, *every nonempty finite intersection of plaques of* $\mathscr{W}$ *on* L *is homeomorphic to* D^{n-1}.

Sketch of Proof. Conditions i) and ii) are discussed in [R]. Here we only consider Condition iii). Fix a Riemannian metric on M. By a fundamental theorem in differential geometry, for every point x of M, there exists a neighborhood P_x of x in L_x (the leaf containing x) which is strictly geodesically convex with respect to the induced metric on L_x. Choose a "sufficiently thin" foliation chart W_x containing P_x as a plaque. Then all plaques of W_x are strictly geodesically convex, because strict geodesical convexity is an open condition. Take a suitable finite subcover $\mathscr{W}$ of $\{W_x\}_{x \in M}$. Then $\mathscr{W}$ clearly satisfies iii).

We choose a cover $\mathscr{W}$ as in this lemma. In the rest of this paper, a plaque always means a plaque of $\mathscr{W}$.

A *plaque chain* of length k which connects P_0 to P_k is a finite sequence $\{P_0, P_1, \cdots, P_k\}$ of plaques such that $P_{i-1} \cap P_i \neq \phi$ for $1 \leqq i \leqq k$. A *plaque cycle* of length k based at P_0 is a plaque chain $\{P_0, P_1, \cdots, P_k\}$ such that $P_k = P_0$. A plaque chain (resp. a plaque cycle) $\{P_0, P_1, \cdots, P_k\}$ is *simple* if $P_i \neq P_j$ for each i, j, $0 \leqq i < j \leqq k$ (resp. $0 \leqq i < j < k$). Among plaque chains and plaque cycles, the following operations are defined: For $\alpha = \{P_0, P_1, \cdots, P_k\}$ and $\beta = \{Q_0, Q_1, \cdots, Q_l\}$ with $P_k = Q_0$, the composition $\beta * \alpha = \{P_0, P_1, \cdots, P_k (= Q_0), Q_1, \cdots, Q_l\}$ and the inverse $\alpha^{-1} = \{P_k, P_{k-1}, \cdots, P_0\}$.

A *basic cycle* is a plaque cycle of the form $\beta^{-1} * \alpha * \beta$, where α is a simple plaque cycle and β is a simple plaque chain.

Two plaque cycles α_0 and α_k based at P_0 are *homotopic* if there is a sequence $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ of plaque cycles based at P_0 such that α_i is obtained from α_{i-1} by inserting a new plaque or by removing one of the plaques of α_{i-1}, for $1 \leqq i \leqq k$ (c.f. [Hi] p.145). Let L be a leaf of $\mathscr{F}$. Denote by $\mathscr{L}$ the set of all plaques of L. Let P_0 be a plaque of L and x_0 a point of P_0. Then by Condition iii) on $\mathscr{W}$, the group $\pi_1(\mathscr{L}, P_0)$ of homotopy classes of plaque cycles based at P_0 is naturally isomorphic to the ordinary fundamental group $\pi_1(L, x_0)$.

A plaque chain $\beta = \{Q_0, Q_1, \cdots, Q_l\}$ is a *subchain* of another plaque chain $\alpha = \{P_0, P_1, \cdots, P_k\}$ if there is an order preserving injection $\lambda: \{0, 1, \cdots, l\} \to \{0, 1, \cdots, k\}$ such that $Q_i = P_{\lambda(i)}$, $1 \leqq i \leqq l$. Similarly, a plaque cycle $\beta = \{Q_0, Q_1, \cdots, Q_l\}$ is a *subcycle* of another plaque cycle $\alpha = \{P_0, P_1, \cdots, P_k\}$ if there is a cyclic order preserving injection $\lambda: \{0, 1, \cdots, l\} \to \{0, 1, \cdots, k\}$ such that $Q_i = P_{\lambda(i)}$, $1 \leqq i \leqq l$.

We have the following easy lemmas.

4.2 Lemma (Simple chain lemma). *Every plaque chain which connects a plaque P to another plaque P' has a simple subchain which connects P to P'.*

Proof. Obvious.

4.3 Lemma (Simple cycle lemma). *Let H be a normal subgroup of $\pi_1(\mathscr{L})$. Then every plaque cycle whose homotopy class does not belong to H has a simple subcycle whose homotopy class does not belong to H.*

Remark. Since H is normal, we need not pay attention to the base plaque of plaque cycles in this lemma.

Proof. Let $\alpha=\{P_0, P_1, \cdots, P_k\}$ be a plaque cycle such that its homotopy class $[\alpha]$ does not belong to H. We prove the lemma by the induction on the length of α. If α is simple, we are done. So let us assume $P_i=P_j$ for some i, j, $0\leqq i<j<k$. If we put $\beta_1=\{P_0, P_1, \cdots, P_i\}$, $\beta_2=\{P_i, P_{i+1}, \cdots, P_j(=P_i)\}$ and $\beta_3=\{P_j(=P_i), P_{j+1}, \cdots, P_k\}$, then $[\alpha]=[\beta_3*\beta_2*\beta_1]=[\beta_3*\beta_2*\beta_3^{-1}*\beta_3*\beta_1]$. Since $[\alpha]\notin H$, it follows that either $[\beta_3*\beta_1]\notin H$ or $[\beta_2]\notin H$. Thus we have found a subcycle whose length is strictly less than the length of α and whose homotopy class does not belong to H. Lemma is proved.

Let P be a plaque and $\alpha=\{P_0, P_1, \cdots, P_k\}$ a plaque chain. We define the distance between P and α by the following.

$$d(P, \alpha)=\min\left\{\text{length}\,(\beta)\,\middle|\,\begin{array}{l}\beta \text{ is a plaque chain which connects}\\ P \text{ to one of the plaque of } \alpha.\end{array}\right\}.$$

§ 5. Hector's holonomy convergence theorem

In this section, we explain a result of Hector [H], which plays a crucial role in the proof of our main theorems.

Let M, $\mathscr{F}$ and $\mathscr{T}$ be as in Section 4. We assume that $\mathscr{F}$ is C^r $(2\leqq r\leqq\infty)$. Let L be a proper leaf of $\mathscr{F}$. Let U be the connected component of $M-L$ which contains the positive side of L (with respect to the transverse orientation of $\mathscr{F}$). Denote by $\hat{U}$ the Dippolito completion of U (i.e. Fix an arbitrary Riemannian metric on M. Then $\hat{U}$ is the completion of U with respect to the Riemannian metric of U induced from that of M) and by $\hat{i}\colon \hat{U}\to M$ the immersion induced from the inclusion map $i\colon U\to M$. Let $\hat{\mathscr{F}}=i^*\mathscr{F}$ and $\hat{\mathscr{T}}=i^*\mathscr{T}$.

According to [D 1], $\hat{U}$ is described as a union of $\hat{\mathscr{T}}$-saturated submanifolds K and A such that K is compact and each connected component A_i $(i=1, 2, \cdots, s)$ of A is a foliated interval bundle with noncompact base space. K is called a *nucleus* of $\hat{U}$ and each A_i is called an *arm* of $\hat{U}$.

Let $\hat{L}$ be the leaf in $\partial\hat{U}$ such that $\hat{i}(\hat{L})=L$ and that $\hat{L}$ has its positive side in $\hat{U}$. Let A_i be an arm of $\hat{U}$ which intersects $\hat{L}$. Then the total holonomy homomorphism

$$\Phi\colon \pi_1(\hat{\mathscr{L}}\cap A_i, P_0)\longrightarrow \mathrm{Diff}_+^r[0, 1]$$

determines the foliation $\mathscr{F}|A_i$ (c.f. [I 3] p. 115), where $\hat{\mathscr{P}}\cap A_i$ is the set of plaques of $\hat{L}$ contained in A_i and P_0 is a plaque in $\hat{\mathscr{P}}\cap A_i$.

5.1 Theorem (Hector's holonomy convergence theorem). *Let $\{\beta_n^{-1}*\alpha_n*\beta_n\}_{n\in N}$ be a sequence of basic cycles in $\hat{\mathscr{P}}\cap A_i$ based at P_0 such that $\lim_{n\to\infty} d(P_0, \alpha_n)=\infty$. Then $\Phi(\beta_n^{-1}*\alpha_n*\beta_n)$ converges to the identity uniformly.*

Proof is given in the appendix.

5.2 Remark. Hector's original result is actually stronger than (5.1). He proved the *C^1-convergence* of the total holonomy of basic cycles for *both sides* of proper or *semiproper exceptional* leaves. In this paper, however, we only need the result stated in the above.

§ 6. Proof of theorems

First we extend the notion of stability for subsets of proper leaves as follows.

6.1 Definition. Let M, $\mathscr{F}$ and $\mathscr{T}$ be as in Section 4. Let V be a submanifold (possibly with boundary) of a proper leaf L of $\mathscr{F}$. V is *stable* if there exist a submanifold (possibly with corner) N of M containing V and a foliation preserving diffeomorphism

$$\varphi\colon (V\times D^1, \{V\times\{t\}\}_{t\in D^1}, \quad \{\{x\}\times D^1\}_{x\in V})\longrightarrow(N, \mathscr{F}|N, \mathscr{T}|N)$$

such that $\varphi|V\times\{0\}=\mathrm{id}_V$.

Next we define stability of ends.

6.2 Definition. Let M and $\mathscr{F}$ be as in (6.1). Let L be a proper leaf of $\mathscr{F}$ and e an end of L. e is *stable* if there exists a neighborhood V of e in L such that V is stable.

6.3 Theorem. *Let $\mathscr{F}$ be a C^r foliation ($2\leqq r\leqq\infty$) of a closed C^∞ manifold M and L a proper leaf of $\mathscr{F}$. Let e be an end of L which belongs to $\mathscr{P}$. Then e is stable.*

Proof. We show that there is a neighborhood of e in L which is stable on the positive side. (The proof of the stability on the negative side is identical.)

Let $\hat{U}$ and $\hat{L}$ be as in Section 5. We suppose $e\in\mathscr{P}_k$ and prove the theorem by induction on k. Choose a periodic neighborhood V of e sufficiently small that V is contained in an arm of $\hat{U}$. Let P_0 be a plaque contained in V and x_0 a point of P_0. Denote by $\mathscr{V}$ the set of plaques contained in V.

We first consider the case when $k=0$. In this case, $\pi_1(V, x_0)/T$ is isomorphic to $\{1\}$ to $\boldsymbol{Z}$, where T is the smallest normal subgroup that contains all torsion elements. Note that the stability of V is equivalent to the triviality of the total holonomy homomorphism

$$\Phi\colon \pi_1(\mathscr{V}, P_0) \longrightarrow \mathrm{Diff}^2_+[0, 1].$$

There are two cases which possibly occur.

Case I. There is a sequence $\{\alpha_n\}$ of plaque cycles in $\mathscr{V}$ with $\lim_{n\to\infty} d(P_0, \alpha_n)=\infty$ such that $[\alpha_n]$ is nontrivial in $\pi_1(\mathscr{V})/T$ for each n.

Case II. For any sequence $\{\alpha_n\}$ of plaque cycles in $\mathscr{V}$ with $\lim_{n\to\infty} d(P_0, \alpha_n)=\infty$, there is a number n_0 such that if $n>n_0$ then $[\alpha_n]$ is trivial in $\pi_1(\mathscr{V})/T$.

In Case I, $\pi_1(\mathscr{V})/T$ is necessarily isomorphic to $\boldsymbol{Z}$. Since $\mathrm{Diff}^2_+[0, 1]$ contains no torsion elements, Φ factors through $\pi_1(\mathscr{V})/T$. Let γ be an element of $\pi_1(\mathscr{V}, P_0)$ which represents a generator of $\pi_1(\mathscr{V}, P_0)/T$ and f the image of γ under Φ. Applying to each α_n the Simple Cycle Lemma, we obtain for each n a simple subcycle $\hat{\alpha}_n$ of α_n such that

1) $\lim_{n\to\infty} d(P_0, \hat{\alpha}_n)=\infty$, and
2) $[\hat{\alpha}_n]\neq 0$ in $\pi_1(\mathscr{V})/T$.

Take an arbitrary plaque chain β_n which connects P_0 to one of the plaques of $\hat{\alpha}_n$. Making use of the Simple Chain Lemma, we have a simple plaque chain $\hat{\beta}_n$ which connects P_0 to $\hat{\alpha}_n$. Then $\{\hat{\beta}_n^{-1}*\hat{\alpha}_n*\hat{\beta}_n\}_{n\in N}$ is a sequence of basic cycles based at P_0 and contained in an arm of $\hat{U}$ with $\lim_{n\to\infty} d(P_0, \hat{\alpha}_n)=\infty$. By Hector's holonomy convergence theorem, we have that $\Phi(\hat{\beta}_n^{-1}*\hat{\alpha}_n*\hat{\beta}_n)$ converges to the identity uniformly. Since $\pi_1(\mathscr{V})/T$ is generated by γ, there is a nonzero integer k_n such that $[\hat{\beta}_n^{-1}*\hat{\alpha}_n*\hat{\beta}_n]=\gamma^{k_n}$. Hence $f^{k_n}\to \mathrm{id}$ uniformly as $n\to\infty$. Then

$$0\leqq \max_{0\leqq x\leqq 1} |f(x)-x| \leqq \max_{0\leqq x\leqq 1} |f^{k_n}(x)-x| \to 0,$$

which shows that f is the identity map. Thus Φ is trivial.

In Case II, by the condition $(*)$ in the definition of $\mathscr{P}$, we see that $\pi_1(\mathscr{P})/T$ vanishes. Hence Φ is obviously trivial.

This completes the proof of the case $k=0$.

Now suppose we have proved the theorem for ends belonging to $\mathscr{P}_i$, $0\leqq i\leqq k-1$. Let e be an end of L belonging to $\mathscr{P}_k$. We define a subgroup H of $\pi_1(\mathscr{V}, P_0)$ to be the smallest normal subgroup containing the following subset.

$$\left\{ [\beta^{-1}*\alpha*\beta] \;\middle|\; \begin{array}{l} 0\leqq i\leqq k-1. \\ W \text{ is a subset of } V \text{ which is a periodic neighborhood} \\ \text{of an end of } V \text{ belonging to } \mathscr{P}_i. \\ \alpha \text{ is a plaque cycle contained in } W. \\ \beta \text{ is a plaque chain contained in } V \text{ which connects} \\ P_0 \text{ to } \alpha. \end{array} \right\}$$

Let $p: \pi_1(\mathscr{V}, P_0) \to \pi_1(\mathscr{V}, P_0)/H$ be the canonical projection and T the smallest normal subgroup of $\pi_1(\mathscr{V}, P_0)/H$ containing all torsion elements. The following two cases may occur.

Case I. There is a sequence $\{\alpha_n\}$ of plaque cycles in $\mathscr{V}$ with $\lim_{n\to\infty} d(P_0, \alpha_n)=\infty$ such that $[\alpha_n]\neq 0$ in $\pi_1(\mathscr{V})/p^{-1}(T)$ for each n.

Case II. For any sequence $\{\alpha_n\}$ of plaque cycles in $\mathscr{V}$ with $\lim_{n\to\infty} d(P_0, \alpha_n)=\infty$, there is a number n_0 such that if $n>n_0$, then $[\alpha_n]=0$ in $\pi_1(\mathscr{V})/p^{-1}(T)$.

By the definition of the class $\mathscr{P}$ and the induction assumption, it is easily seen that $\pi_1(\mathscr{V})/p^{-1}(T)$ is isomorphic to $\{1\}$ or $\boldsymbol{Z}$ and that Φ factors through $\pi_1(\mathscr{V})/p^{-1}(T)$. Therefore the method used in the case when $k=0$ applies also to this case and shows that Φ is trivial. This completes the proof of (6.3).

Proof of (2.3). Suppose L is a proper leaf without holonomy of a C^2 foliation of a closed manifold such that all ends of L belong to $\mathscr{P}$. Then by (6.3), periodic neighborhoods of all ends of L are stable. That is, there is a compact codimension zero submanifold K of L such that $L-K$ is stable. Since by the assumption the holonomy group of L is trivial, it follows from the relative version of the classical Reeb Stability Theorem that K is stable. Hence L is stable. (2.3) is proved.

We give here a relative version of Thurston's result [Th], which is used in the proof of (2.4).

6.4 Theorem. *Let K be a compact codimension zero submanifold of a leaf of a C^r foliation ($1\leqq r\leqq\infty$) such that*

1) *each component of ∂K has trivial holonomy,*

2) *the restriction homomorphism $i^*: H^1(K; \boldsymbol{R}) \to H^1(\partial K; \boldsymbol{R})$ is injective.*

Then K is stable.

Proof. If K is not stable, by the Reeb Stability Theorem, the holonomy group $\mathscr{H}(K)$ of K is nontrivial. Then using Thurston's argument, we get a nontrivial homomorphism $h: \pi_1(K) \to \boldsymbol{R}$ which factors through $\mathscr{H}(K)$. By the assumption that $\mathscr{H}(\partial K)$ is trivial, we see that $h\circ i_\#$ is trivial, where $i_\#\colon \pi_1(\partial K)\to\pi_1(K)$ is the homomorphism induced from the inclusion map. In fact, this is because $h\circ i_\#$ factors through $\mathscr{H}(\partial K)$. Then by the assumption of the injectivity of i^*, we obtain that h is trivial. This contradicts the nontriviality of h. The proof is complete.

Proof of (2.4). Suppose L is a proper leaf of a C^2 foliation of a closed manifold such that $\hat{H}^1(L; \boldsymbol{R})=0$ and that all ends of L belong to $\mathscr{P}$. As in the proof of (2.3), it is shown that there exists a compact submanifold K of L such that $L-K$ is stable. In particular, ∂K has trivial holonomy. Now consider the following commutative diagram:

$$\begin{tikzcd} H^1(L, L-K) \arrow[r] \arrow[ddr, "j^*"'] & H^1_c(L) \arrow[d] \\ & \hat{H}^1(L)=0, \arrow[d, phantom, "\cap"] \\ & H^1(L) \end{tikzcd}$$

where $j\colon L\to(L, L-K)$ is the inclusion map. Then we see that j^* is a zero map. This implies that $i^*\colon H^1(K)\to H^1(\partial K)$ is injective. Thus Thurston's theorem applies and we can conclude that K is stable. Hence L itself is stable. This proves (2.4).

Appendix

As Hector's paper has not been published for a long time since I was first informed of his result, we give his proof of (5.1) here for self-containedness of this paper.

Let $R_i=\varphi_i(\{0\}\times D^1)$, where φ_i $(i=1, \cdots, m)$ are the foliation charts chosen in Section 4, and let R be the disjoint union of R_i's. We regard R as a subset of the reals $\boldsymbol{R}$ by an arbitrary orientation preserving embedding. Let γ_{ij}: (a subinterval of R_j)$\to$(a subinterval of R_i) be the *holonomy transitions*. Put $\Gamma_0=\{\gamma_{ij}\}_{i,j=1,\cdots,m}$. The pseudogroup Γ of local diffeomorphisms of R generated by Γ_0 is called the *holonomy pseudogroup* for $\mathscr{F}$.

Every plaque chain α induces an element of Γ, which we also write as $\Phi(\alpha)$ by abuse of notations. Since R is compact, one can find positive numbers A, B such that for every i, j $(1\leqq i, j\leqq m)$, and $t\in \operatorname{Dom}\gamma_{ij}$, $D\gamma_{ij}(t)\geqq A$ and $|D^2\gamma_{ij}(t)|\leqq B$.

The following is well known.

A.1 Lemma (Denjoy's inequality). *Let g be an element of Γ such that its domain J is connected.* If $g=g_k\circ\cdots\circ g_1$ $(g_i\in\Gamma_0, i=1, \cdots, k)$

and z, $w \in J$, *then*

$$\left|\log \frac{Dg(z)}{Dg(w)}\right| \leqq \frac{B}{A} \sum_{i=0}^{k-1} |z_i - w_i|,$$

where $z_i = g_i \circ \cdots \circ g_1(z)$, $z_0 = z$, $w_i = g_i \circ \cdots \circ g_1(w)$, $w_0 = w$.

A.2 Corollary. *Let* g, J, z *and* w *be as in* (A.1). *Then*

$$\left|\log \frac{Dg(z)}{Dg(w)}\right| \leqq \frac{B}{A} \sum_{i=0}^{k-1} \text{length } (J_i),$$

where $J_i = g_i \circ \cdots \circ g_1(J)$, $J_0 = J$.

A.3 Corollary. *Let* g, J *and* z *be as in* (A.1). *If* $g(J) = J$, *then*

$$|\log Dg(z)| \leqq \frac{B}{A} \sum_{i=0}^{k-1} \text{length } (J_i).$$

For the proof of these facts, see [C-C 2, p. 188].

Proof of (5.1). *Step* 1. Let $\theta = 3(B/A)$ length (R), $f_n = \Phi(\alpha_n)$, $g_n = \Phi(\beta_n)$, $h_n = g_n^{-1} \circ f_n \circ g_n$, $I = \operatorname{Dom} g_n$, $J_n = g_n(I) = \operatorname{Dom} f_n$, $f_n = f_{n,k_n} \circ \cdots \circ f_{n,1}$ $(f_{n,i} \in \Gamma_0)$, $g_n = g_{n,l_n} \circ \cdots \circ g_{n,1}$ $(g_{n,i} \in \Gamma_0)$, $I_{n,i} = g_{n,i} \circ \cdots \circ g_{n,1}(I)$, $J_{n,i} = f_{n,i} \circ \cdots \circ f_{n,1}(J_n)$. By (A.3),

$$|\log D(f_n^j)(z)| = \left|\sum_{l=1}^{j} \log Df_n(f_n^{l-1}(z))\right| \leqq j \frac{B}{A} \sum_{l=0}^{k_n-1} \text{length } (J_{n,i})$$

for $z \in J_n$ and $j \in N$. Since α_n is simple, $J_{n,i}$'s $(i = 0, 1, \cdots, k_n - 1)$ are pairwise disjoint in R. Hence by the assumption that $\lim_{n\to\infty} d(P_0, \alpha_n) = \infty$, one sees that $\lim_{n\to\infty} \sum_{i=0}^{k_n-1}$ length $(J_{n,i}) = 0$. Now define q_n by

$$q_n = \max\left\{j \in N \,\middle|\, j \frac{B}{A} \sum_{i=0}^{k_n-1} \text{length } (J_{n,i}) \leqq \frac{\theta}{2}\right\}.$$

Then $\lim_{n\to\infty} q_n = \infty$ and

(1) $$|\log Df_n^j(z)| \leqq \frac{\theta}{2} \qquad \text{for } z \in J_n,\ 1 \leqq j \leqq q_n.$$

Since β_n is simple, $I_{n,i}$'s $(i = 0, 1, \cdots, l_n - 1)$ are pairwise disjoint in R. Therefore

(2) $$\sum_{i=0}^{l_n-1} \text{length } (I_{n,i}) \leqq \text{length } (R).$$

Using (1) and (2), we have for $1 \leqq j \leqq q_n$, $x \in I$,

$$
\begin{aligned}
|\log Dh_n^j(x)| &\leqq \left|\log \frac{Dg_n(x)}{Dg_n(h_n^j(x))}\right| + |\log Df_n^j(g_n(x))| \\
&\leqq \frac{B}{A} \sum_{i=0}^{l_n-1} \text{length}\,(I_{n,i}) + |\log Df_n^j(g_n(x))| && \text{(A.2)} \\
&\leqq \frac{\theta}{3} + \frac{\theta}{2}. && \text{(1) and (2)}
\end{aligned}
$$

Hence

$$(3) \qquad |\log Dh_n^j(x)| \leqq \theta \qquad \text{for } 1 \leqq j \leqq q_n.$$

Step 2. We show the following inequality:

$$(4) \qquad |h_n(x) - x| \leqq \frac{e^\theta \,\text{length}\,(I)}{q_n} \qquad \text{for each } x \in I.$$

Note that (4) exactly means that h_n converges to the identity uniformly on I. Let $x \in I$. Then

$$(5) \qquad \sum_{j=1}^{q_n} |h_n^j(x) - h_n^{j-1}(x)| = |h_n^{q_n}(x) - x| \leqq \text{length}\,(I).$$

On the other hand, by the mean value theorem, there exists a point ξ_j in I such that

$$|h_n^j(x) - h_n^{j-1}(x)| = Dh_n^{j-1}(\xi_j)\,|h_n(x) - x|.$$

Then by (3), we have

$$(6) \qquad |h_n^j(x) - h_n^{j-1}(x)| \geqq e^{-\theta} |h_n(x) - x| \qquad \text{for each } 1 \leqq j \leqq q_n.$$

The inequalities (5) and (6) combine to produce (4). This completes the proof of (5.1).

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