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Asymptotic Cycles on Two Dimensional Manifolds

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Introduction

In [15], Schwartzman introduced the notion of asymptotic cycles, which represent how the trajectory of the flow rounds on the phase space in the sense of homology. In this paper, we study the asymptotic cycles for flows on 2-manifolds and show that they characterize the qualitative behaviour of trajectories. For this, we consider the asymptotic cycle of an individual trajectory, while Schwartzman defined it as a functional on the space of invariant probability measures to the one dimensional homology group.

Let us recall the definition: Let $\psi=\{\psi_t\}$ be a continuous flow on a compact manifold M and take a point p of M. Fix a Riemannian metric on M. Then for a positive real number T, there is a 1-cycle

$$\begin{aligned}\gamma_{T,p}=&(\text{the trajectory from } p \text{ to } \psi_T(p))\\ &+(\text{a minimal geodesic from } \psi_T(p) \text{ to } p).\end{aligned}$$

Definition. The *asymptotic cycle* of p is defined by

$$A(p)=\lim_{T\to\infty}\frac{1}{T}[\gamma_{T,p}]\in H_1(M;\boldsymbol{R})$$

if the limit exists.

It is easy to see that $A(p)$ is invariant under the flow, i.e. $A(p)=A(\psi_t(p))$ for every $t\in\boldsymbol{R}$, and that $A(p)$ does not depend on the choice of the Riemannian metric on M.

To compute the asymptotic cycle, we first establish a classification theorem for positive semi-trajectories (see section 1 for definitions).

Theorem 1. *Let $\psi=\{\psi_t\}$ be a continuous flow on a closed, orientable, two dimensional manifold and suppose that the number of singular points of ψ is finite. Then each positive semi-trajectory of this flow satisfies one and only one of the following.*

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(1) $L_+(p)$ *is one singular point.*
(2) $L_+(p)$ *approaches to one singular point.*
(3) $L_+(p)$ *winds round a circuit.*
(4) $L_+(p)$ *is a closed orbit.*
(5) $L_+(p)$ *winds round a closed orbit.*
(6) $L_+(p)$ *is locally dense.*
(7) $L_+(p)$ *is exceptional.*
(8) $L_+(p)$ *approaches to one exceptional domain.*

Note that the statement (8) above is justified by Proposition 1.5 in section 1.

To state further results, we divide one dimensional homology classes into three types.

Definition. An element $\alpha \in H_1(M; \boldsymbol{R})$ is *rational* if α is not 0 and is in the cone of $H_1(M; \boldsymbol{Z})$ in $H_1(M; \boldsymbol{R})$, i.e. there exist a positive number a and a non-zero integral homology class α' such that $\alpha = a\alpha'$, and α is *irrational* if α is neither rational nor 0.

Then the asymptotic cycles of the trajectories are computed as follows:

Theorem 2. *Let ψ be a continuous flow on a closed, orientable, two dimensional manifold and suppose that ψ has only a finite number of singular points. Then we have*

(i) *if p is of type* (1), (2) *or* (3), *then* $A(p)=0$,

(ii) *if p is of type* (4) *or* (5), *then* $A(p)=1/\tau[\gamma]$, *where γ is the closed orbit and τ is the minimal period, and*

(iii) *if p is of type* (6), (7) *or* (8) *and $A(p)$ exists, then $A(p)$ is either* 0 *or irrational.*

Corollary. *A flow as in Theorem* 2 *has a closed orbit which is not homologous to zero if and only if it has a rational asymptotic cycle.*

For flows on T^2, the asymptotic cycle always converges because of the unique ergodicity of irrational rotations of the circle.

Theorem 3. *Let ψ be a continuous flow on T^2 with only a finite number of singular points. Then for every point p of T^2, the asymptotic cycle $A(p)$ exists. Moreover, if p and q have the same ω-limit set, then $A(p)=A(q)$.*

There are many investigations of flows on two dimensional manifolds. Among these, Maĭer [10] is basic for this research. In this paper we study flows and their asymptotic cycles by using P-transformations, and this method is similar to those of Katok [7] and Arnoux [2], in which interval exchange transformations are used. (See Cornfeld et al. [5] for interval exchange transformations.) They deal with total dynamics under the assumption that the flow has an invariant measure positive on each non-empty open subset, or equivalently, the nonwandering set of the flow coincides with the total manifold. We deal with individual trajectories and this point of view is rather close to that of Plante [13] and Sullivan [16]. Finally we refer to Aranson-Grines [1] for a homotopy version of asymptotic cycles.

Contents are as follows: Section 1 is for definitions and basic results. In section 2, we study positive semi-trajectories near exceptional domains and give a proof of Theorem 1. In section 3, we introduce the notion of P-transformations and study their invariant measures. Using these, we give a proof of Theorem 2 in section 4. In section 5, we discuss the convergence of asymptotic cycles and prove Theorem 3. In the final section we define projective asymptotic cycles, which are invariant under time change of the flow.

A part of this paper is the author's master thesis at the University of Tokyo, 1978.

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§ 1. Preliminaries

Throughout this paper, manifolds are closed, orientable, and of dimension two, and flows are continuous. We also assume, for simplicity, that the flow has only a finite number of singular points.

We begin with definitions. Let $\psi=\{\psi_t\}$ be a flow on a manifold M. For a point p of M, we let $L_+(p)$ denote the (*positive*) *semi-trajectory* of p, i.e. $L_+(p)=\{\psi_t(p)\}_{t\geqq 0}$. A point p is called *positively* (resp. *negatively*) *Poisson stable* if the ω-limit set $\omega(p)$ (resp. the α-limit set $\alpha(p)$) of p contains p itself. The semi-trajectory $L_+(p)$ of p is called *positively Poisson stable* if so is p. Singular points and closed orbits are typical examples of such ones, and we divide the others into two types: a positively Poisson stable semi-trajectory is *locally dense* if its closure contains a non-empty open set and is *exceptional* if it is none of the three types above. A compact subset D of M is called an *exceptional domain* if there exists an exceptional semi-trajectory $L_+(p)$ such that $\overline{L_+(p)}=D$. (For the existence of exceptional semi-trajectories, see Denjoy [6], Cherry [4] and Schwartz [14]). A compact subset Γ of M is called a *circuit* if Γ consists of a finite number of singular points and (possibly countably many) trajectories, the α-limit set and the ω-limit set of each of which consist of singular points in Γ.

Let X be a closed interval or a circle. Then an embedding $\iota: X \to M$ is called *transverse* to the flow if the map $\bar{\iota}: X \times [-\varepsilon, \varepsilon] \to M$ defined by $\bar{\iota}(x, t) = \psi_t(\iota(x))$ is a topological embedding for a sufficiently small ε. In the former case, the embedding ι (or its image) is called a *local section.* The following theorem due to Bebutov and Whitney (see Whitney [19]) is fundamental for the study of continuous flows.

Theorem 1.1 (Bebutov, Whitney). *If p is a regular point, then there exists a local section through p.*

Corollary 1.2. *If $L_+(p)$ is exceptional, then there exists a local section $\iota: I \to M$ such that $\iota^{-1}(\overline{L_+(p)})$ is a Cantor subset of I.*

Finally we recall results of Maĭer [10], which will be needed later.

Lemma 1.3. *Let M be a two dimensional manifold of genus g and fix two distinct points p and q of M. Then for any $2g+2$ disjoint curves $C_1, \cdots, C_{2g+2}$ from p to q, there exist $i \neq j$ such that $C_i \cup C_j$ bounds a disk in M.*

Fix a continuous flow on a two dimensional manifold.

Proposition 1.4. *Let p_1, p_2 and p_3 satisfy $p_2 \in \omega(p_1)$ and $p_3 \in \omega(p_2)$ (resp. $p_3 \in \alpha(p_2)$). Then either p_2 is positively (resp. negatively) Poisson stable or p_3 is a singular point.*

Proposition 1.5. *Suppose that both of p and q are positively Poisson stable, $\overline{L_+(p)} \supset L_+(q)$, and q is not a singular point. Then $\overline{L_+(q)} \supset L_+(p)$.*

The proof of Lemma 1.3 is immediate. Though Propositions 1.4 and 1.5 were prove in Maĭer [10] without the assumption that the number of singularities is finite, we give proof in section 2 for self containedness.

§ 2. Classification of semi-trajectories

In this section we prove Theorem 1. For this, we need to study semi-trajectories near exceptional domains. First we prove the following

Proposition 2.1. *Suppose that the ω-limit set $\omega(p)$ of a point p contains an exceptional semi-trajectory $L_+(q)$. Then $\omega(p) = \overline{L_+(q)}$.*

Proof. By Theorem 1.1, there exists a transverse simple closed curve C such that $C \cap L_+(q) \neq \emptyset$. By Corollary 1.2, $C \cap \overline{L_+(q)}$ is a Cantor subset of C and we put $C - \overline{L_+(q)} = \bigcup_{i \geq 1} I_i$, where I_i's are mutually disjoint open intervals in C. Let φ be the Poincaré map with respect to C. We write $I_i \to I_j$ if there exists a point $x \in I_i$ such that $\varphi(x) \in I_j$. It may happen for an I_i that there does not exist an I_j with $I_i \to I_j$. We say that an interval I_n is *of type* (S) if there exists a point x_0 in I_n such that $\omega(x_0)$ consists of a single singular point and $(L_+(x_0) - \{x_0\}) \cap C = \emptyset$.

Assertion 2.2. *Suppose that for an interval I_n, there are two distinct intervals I_m and $I_{m'}$ such that $I_n \to I_m$ and $I_n \to I_{m'}$. Then I_n is of type* (S).

Proof. By the assumption, there exists a closed subinterval $[x_0, x_\infty] \subset I_n$ such that the curves $[x, \varphi(x)]$ on the orbit form a continuous family for $x \in [x_0, x_\infty)$ and $L_+(x_\infty) \cap C = \emptyset$. We prove that $\omega(x_\infty)$ consists of a single singular point. Suppose the contrary, i.e. some point z in $\omega(x_\infty)$ is regular. By Theorem 1.1, there is a transverse simple closed curve C' which intersects $L_+(x_\infty)$ and is disjoint from the arc $[x_0, \varphi(x_0)]$ as in Figure 1.

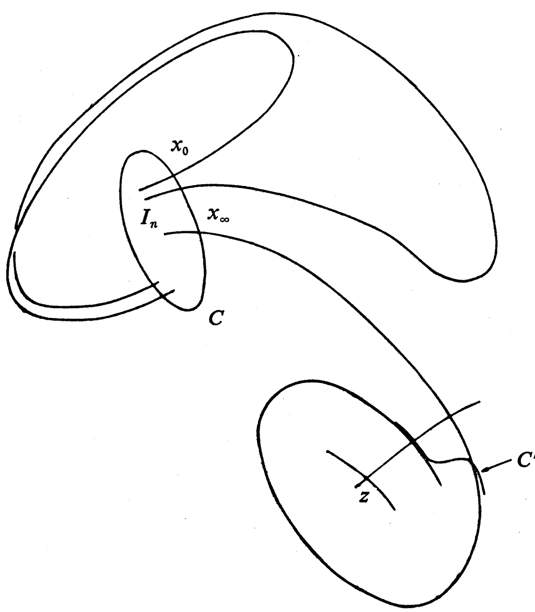

Fig. 1.

Let $\circ$ denote the intersection number in the resulting manifold obtained from M by cutting along C. Then $[x, \varphi(x)] \circ C' > 0$ for a point $x \in (x_0, x_\infty)$ sufficiently near x_∞. This contradicts $[x_0, \varphi(x_0)] \circ C' = 0$ and proves the assertion. □

Note that our assumption and Lemma 1.3 imply that the number of intervals of type (S) is finite and that the number $\#\{I_m;\, I_n \to I_m\}$ is finite for a fixed I_n. Therefore we have the following

Assertion 2.3. *The number of the intervals I_n such that there exist two intervals I_{m_1} and I_{m_k} of type* (S) *with $I_{m_1} \to I_{m_2} \to \cdots \to I_n \to \cdots \to I_{m_k}$ is finite.*

Now we proceed to prove the proposition. We can assume that $p \in C$. If $p \in \overline{L_+(q)}$, then $\omega(p) \subset \overline{L_+(q)}$. Thus we assume that $p \notin \overline{L_+(q)}$. By $\omega(p) \subset \overline{L_+(q)}$, the cardinal number of the set $L_+(p) \cap C$ is infinite and thus $\varphi^n(p)$ is defined for all n. Let I_{i_n} be the interval which contains $\varphi^n(p)$. Then

$$I_{i_1} \to I_{i_2} \to \cdots \to I_{i_k} \to \cdots.$$

Therefore by Assertion 2.3, we have

Assertion 2.4. *If the number N is sufficiently large, then I_{i_n} is not of type* (S) *for every $n \geqq N$.*

For a number N as in this assertion, the restricted flow $\psi_t|_{L_+(I_{i_N})}$ is the suspension of the map $\varphi|_{\bigcup_{k \geqq N} I_{i_k}}$. Here we use the notation $L_+(A) = \bigcup_{x \in A} L_+(x)$. Thus

$$\omega(p) = \omega(\varphi^N(p)) \subset L_+(\bigcap_{n \geqq 0} \overline{(\bigcup_{k>0} I_{i_{N+n+k}})}) = \overline{L_+(q)}.$$

This implies $\omega(p) = \overline{L_+(q)}$ and the proof is completed. □

Proof of Proposition 1.4. We only deal with the case that $p_2 \in \omega(p_1)$ and $p_3 \in \omega(p_2)$. Suppose that p_3 is regular. First we show

Assertion 2.5. *There exists a transverse simple closed curve which $L_+(p_2)$ intersects infinitely many times.*

Proof. Suppose that there is none. We take a transverse arc X through p_3 and construct disjoint curves C_i $i=1, 2, \cdots$ consisting of parts of $L_+(p_2)$ and parts of X as in Figure 2. Then each C_i is not homologous to zero because $L_+(p_1)$ intersects it infinitely many times. Thus there is a number N such theat for every $i \geqq N$, the curves C_i and C_{i+1} bounds an annulus A_i in M. Let $t_N < t_{N+1} < \cdots$ be the sequence such that $\psi_{t_i}(p_1) \in C_i$. Then $\psi_t(p_1)$ lies in A_i for $t_i \leqq t \leqq t_{i+1}$. Since $t_{i+1} - t_i$'s are bounded from below by a positive constant, $\psi_t(p_1)$ does not return to C_N for $t > t_N$. This contradicts $\omega(p_1) \supset L_+(p_2)$. □

Take a transverse curve C as in this assertion and let I_i $i=1, 2, \cdots$ be the connected components of $C - \overline{L_+(p_2)}$. Since in the proof of Pro-

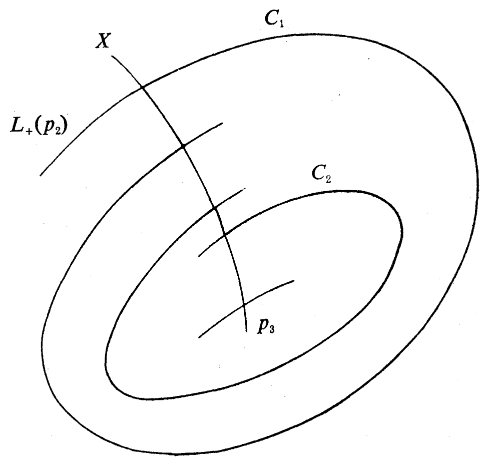

Fig. 2.

position 2.1 we do not use the assumption that $C \cap \overline{L_+(q)}$ is a Cantor set, it applies to this case. Thus for all but finite I_i's, any point in I_i does not return to I_i again. Hence if $L_+(p_2)$ is not positively Poisson stable, the ω-limit set of any point does not contain $L_+(p_2)$. This contradiction proves the proposition. □

Proof of Proposition 1.5. If $L_+(q)$ is a closed orbit, the result follows easily. If $L_+(q)$ is locally dense, then $\overline{L_+(q)}$ contains a non-empty open set. Since $\overline{L_+(p)} \supset \overline{L_+(q)}$, there is a t such that $\psi_t(p)$ is in this open set. Therefore $\overline{L_+(q)} \supset L_+(p)$ and thus $\overline{L_+(p)} = \overline{L_+(q)}$. If $L_+(q)$ is exceptional, Proposition 2.1 applies. □

Proof of Theorem 1. Let p be a point of M. If p is positively Poisson stable, then p is one of types (1), (4), (6) and (7) by definition. Suppose that p is not. Since $\omega(p)$ is a closed invariant set, it contains a positively Poisson stable semi-trajectory. We divide the proof into four cases according to types of such semi-trajectories in $\omega(p)$.

Case 1. *All the positively Poisson stable semi-trajectories in* $\omega(p)$ *are singular points.* Suppose that all the points of $\omega(p)$ are singular. Since the number of singular points is finite and $\omega(p)$ is connected, it follows that $\omega(p)$ consists of only one point and thus p is of type (2). Suppose the contrary. Then for every regular point q of $\omega(p)$, the ω-limit set and the α-limit set of q are singular points by Proposition 1.4. Thus $\omega(p)$ consists of trajectories, the α-limit set and the ω-limit set of each of which consists of singular points in $\omega(p)$. The cardinality of these trajectories is at most countable by Theorem 1.1 and by the second countability axiom for manifolds. Hence the point p is of type (3).

Case 2. *$\omega(p)$ contains a closed orbit.* It is clear that p is of type (5).

Case 3. *$\omega(p)$ contains a locally dense semi-trajectory.* Since $\omega(p)$ contains a non-empty open set, there is a number t such that $\psi_t(p) \in \omega(p)$. This contradicts the assumption that p is not positively Poisson stable.

Case 4. *$\omega(p)$ contains an exceptional semi-trajectory.* In this case p is of type (8), i.e. $\omega(p)$ is equal to one exceptional domain, by Proposition 2.1. Thus we have checked all the cases and proved the theorem. □

§ 3. P-transformations

In this section, we introduce the notion of P-transformations and study their invariant measures. P-transformations are naturally derived from locally dense semi-trajectories and for such P-transformations, we show the existence of invariant measures which are non-atomic and the supports of which are whole of S^1. Therefore each P-transformation thus derived is a so called minimal interval exchange transformation (see Cornfeld et al. [5]), while the latter is a priori assumed to leave Lebesgue measure invariant. From the view point of Katok [7] and Arnoux [2], there arise interval exchange transformations since the flow is given an invariant measure. We note that P-transformations are also useful for the study of exceptional semi-trajectories (section 4).

Let $L_+(p)$ be a locally dense positive semi-trajectory. Then using Theorem 1.1, we can construct a simple closed curve C transverse to the flow such that $\overline{L_+(p)}\supset C$. We consider the Poincaré map φ for this C. Since the map φ is defined in a dense subset $C\cap L_+(p)$, it happens that $(L_+(x)-\{x\})\cap C=\emptyset$ for a point x only if x is of type (2), i.e. $\omega(x)$ consists of a single singular point. By our assumption that the number of singular points is finite, Lemma 1.3 implies that the number of points x with $(L_+(x)-\{x\})\cap C=\emptyset$ is finite. Since the Poincaré map is continuous in the domain of definition, we get a P-transformation, which will be defined below, from $L_+(p)$ and C above.

Definition. A "map" $\varphi\colon S^1\to S^1$ is called a *P-transformation* if there are subsets $\{p_1, \cdots, p_k\}$ and $\{q_1, \cdots, q_k\}$ of S^1 of the same number k such that φ is an orientation preserving homeomorphism from $S^1-\{p_1, \cdots, p_k\}$ to $S^1-\{q_1, \cdots, q_k\}$. For a P-transformation φ, we put φ_R and $\varphi_L\colon S^1\to S^1$ to be the right and the left continuous extentions of φ respectively. These are one-to-one and onto maps. We put

$$S(\varphi)=\bigcup_{n\in Z}\varphi_R^n(\{p_1, \cdots, p_k\})=\bigcup_{n\in Z}\varphi_L^n(\{p_1, \cdots, p_k\})$$

and call a point of $S(\varphi)$ *singular* (*with respect to* φ) and a point of $S^1-S(\varphi)$ *regular* (*with respect to* φ).

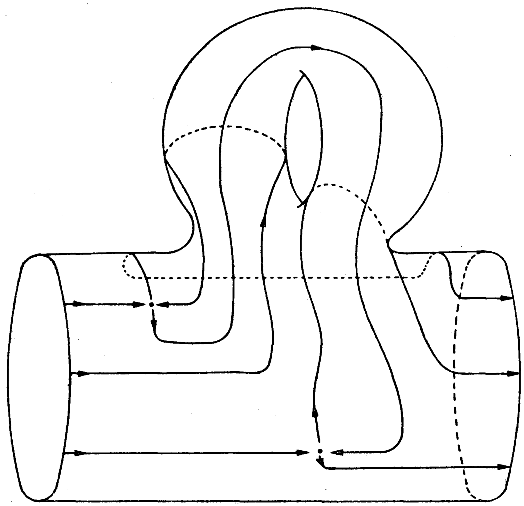

Fig. 3.

Remark 3.1. While P-transformations are naturally derived from flows on manifolds, conversely for a given P-transformation φ, we can construct a flow on a manifold and a simple closed curve C transverse to this flow such that the Poincaré map with respect to C coincides with this φ (Figure 3, see also Arnoux [2]).

The following is essentially the same as the theorem of section 1 of Keane [8] (see also Cornfeld et al. [5]).

Lemma 3.2. *Let φ: $S^1 \to S^1$ be a P-transformation and suppose that there exists a regular point x_0 with $\overline{\bigcup_{n\geqq 0} \varphi^n(x_0)} = S^1$. Then a subset of S^1 which is closed and invariant under φ_R (or φ_L) is either the whole space S^1 or the empty subset.*

Proof. Let A be a minimal set of φ_R-invariant non-empty closed subsets. It suffices to show that A coincides with S^1. Suppose the contrary. Then A is either finite or perfect. In the former case φ_R has a periodic point and thus does not have a dense orbit. In the latter case, by the proof of Proposition 2.1, φ_R does not have a dense orbit. Therefore in both cases, we have a contradiction and thus the lemma is proved. □

Lemma 3.3. *Let φ: $S^1 \to S^1$ be a P-transformation. Then either φ_R or φ_L has a non-trivial invariant measure.*

Proof. For a point $z \in S(\varphi)$, we put

$$l(z)=\exp(-\min\{|n|;\text{ there is a } j \text{ with } \varphi_R^n(p_j)=z\}).$$

Then $l=\sum_{z\in S(\varphi)} l(z)$ is finite. We identify $S^1\cong[0,1]/\sim$ and put $\hat{S}^1=[0,1+l]/\sim$. Then there exists a map $\iota\colon S^1\to\hat{S}^1$ defined by

$$\iota(x)=x+\sum_{0<z\leqq x,\ z\in S(\varphi)} l(z).$$

We put $B=\overline{\text{Image}(\iota)}\subset\hat{S}^1$ and define $p\colon B\to S^1$ by

$$p(u)=m(B\cap[0,u]),$$

where m is the standard measure of $\hat{S}^1$ with $m(\hat{S}^1)=1+l$. Then we have $p\circ\iota=\mathrm{id}_{S^1}$. We define a decomposition $B=B_R\cup B_L\cup B_0$ by

$$\begin{aligned} B_R&=\iota(S(\varphi))\\ B_L&=B-\text{Image}(\iota)\\ B_0&=\iota(S^1-S(\varphi))=B-(B_R\cup B_L). \end{aligned}$$

Then the map p is one-to-one on B_0 and two-to-one on $B_R\cup B_L$. We define a map $\tilde{\varphi}\colon B\to B$ which covers φ with respect to this p as follows:

$$(\tilde{\varphi}|_{B_0\cup B_R})(u)=\iota\circ\varphi_R\circ p(u),$$

and

$$(\tilde{\varphi}|_{B_L})(u)=\iota\circ\varphi_R\circ p(u)-l(\varphi_L\circ p(u)).$$

Then $\tilde{\varphi}$ is a homeomorphism, B_0, B_R and B_L are $\tilde{\varphi}$-invariant and the following diagram commutes. (The suffix R can be replaced by L.)

$$\begin{array}{ccc} B_0\cup B_R & \xrightarrow{\tilde{\varphi}} & B_0\cup B_R\\ p\downarrow & & p\downarrow\\ S^1 & \xrightarrow{\varphi_R} & S^1 \end{array}$$

Since B is a compact subset of $\hat{S}^1$, there exists a non-trivial $\tilde{\varphi}$-invariant measure μ on B by Bogoliouboff-Kryloff [3]. We split μ into

$$\mu=(\mu|_{B_0})+(\mu|_{B_R})+(\mu|_{B_L})$$

and then we have either $(\mu|_{B_0})+(\mu|_{B_R})\neq 0$ or $(\mu|_{B_0})+(\mu|_{B_L})\neq 0$. In the former case, we get a φ_R-invariant measure on S^1 by the above commuting diagram and in the latter case, a φ_L-invariant measure similarly. □

Remark 3.4. For a point u of B, any cluster point of the sequence $1/n\sum_{k=0}^{n-1}\tilde{\varphi}_*^k\delta_u$ gives a $\tilde{\varphi}$-invariant measure. Here δ denotes Dirac's point measure.

Corollary 3.5. *Let $\varphi\colon S^1\to S^1$ be a P-transformation and suppose that there exists a regular point x_0 such that $\overline{\bigcup_{n\geq 0}\varphi^n(x_0)}=S^1$. Then any φ_R-invariant (or φ_L-invariant) measure μ satisfies* $\operatorname{supp}(\mu)=S^1$ *and $\mu(S(\varphi))=0$. That is, φ is a μ-preserving map. Moreover, for each regular point x, any cluster point of the sequence $1/n\sum_{k=0}^{n-1}\tilde{\varphi}_*^k\delta_x$ gives a measure invariant under any of $\tilde{\varphi}$, $\tilde{\varphi}_R$ and $\tilde{\varphi}_L$.*

Proof. The former claim follows from that $S(\varphi)$ is at most countable, and the latter from the definition and Remark 3.4. □

Thus P-transformations as in this corollary are so called minimal interval exchange transformations.

§ 4. Asymptotic cycles

In this section, we compute asymptotic cycles of semi-trajectories of type (6), (7) and (8) and prove Theorem 2. First we deal with locally dense semi-trajectories and give an expression of the asymptotic cycle by a measure invariant under the P-transformation.

Let p be a point on a locally dense semi-trajectory. Take a transverse simple closed curve C through p and let $\varphi\colon C\to C$ be the P-transformation defined by the Poincaré map. Let $\{p_1,\cdots,p_k\}\subset C$ be the set of points where φ is not defined. For a point x of $C-\{p_1,\cdots,p_k\}$, we define $\tau(x)$ to be the first return time, i.e.

$$\tau(x)=\inf\{t>0;\ \psi_t(x)\in C\}.$$

Then the n-th return time of p is given by

$$T(n)=\sum_{k=0}^{n-1}\tau(\varphi^k(p)).$$

Fix a point p_∞ in C and let $\eta(x)$ denote the homology class in $H_1(M;\boldsymbol{R})$ represented by the curve

(from x to $\varphi(x)$ along the trajectory)
+(from $\varphi(x)$ to x along C avoiding p_∞).

Let α be a cluster point of $\{1/T[\gamma_{T,p}]\}$.

Lemma 4.1. *There exists a measure μ on C invariant under φ such that*

$$\alpha=\frac{\int_C \eta d\mu}{\int_C \tau d\mu}\in H_1(M;\boldsymbol{R}).$$

In particular α *is not zero if and only if* $\int_C \tau d\mu < \infty$.

Proof. Let $\alpha = \lim_{i\to\infty} 1/T_i[\gamma_{T_i,p}]$ and consider the sequence of integers $\{n_i\}$ such that $T(n_i) \leqq T_i < T(n_i+1)$. By Corollary 3.5, taking a subsequence if necessary, we get a φ-invariant measure μ on C defined by

$$\mu = \lim_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} \varphi_*^k \delta_p.$$

Since the sequence $1/n_i \sum_{k=0}^{n_i} \varphi_*^k \delta_p$ also converges to μ, we have

$$\lim_{i\to\infty} \frac{T_i}{n_i} = \int_C \tau d\mu.$$

By Lemma 1.3, there are at most finite number of homology classes in the relative homology group $H_1(M, C; \boldsymbol{Z})$ which are represented by the segments of trajectories. Therefore

$$\lim_{i\to\infty} \frac{1}{n_i} [\gamma_{T_i,p}] = \lim_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} \eta(\varphi^k(p)) = \int_C \eta d\mu$$

and we get the required expression of α. Since the intersection number $\eta(x) \circ [C] = 1$ for every $x \in C$, we know by the expression that $\alpha = 0$ if and only if $\alpha \circ [C] = \left(\int_C \tau d\mu\right)^{-1} = 0$. This completes the proof. □

Using the measure μ in Lemma 4.1, we introduce a coordinate on C by

$$C \longrightarrow \boldsymbol{R}/\boldsymbol{Z}$$
$$x \longmapsto \int_*^x d\mu.$$

Since φ is μ-preserving, there are numbers a_i $i = 1, \cdots, k$ such that

$$\varphi(x) = x + a_i \qquad \text{for} \quad p_i < x < p_{i+1}. \tag{4.2}$$

Take a point $y_i \in (p_i, p_{i+1})$ and let β_i be the homology class represented by the curve

(from y_i to $\varphi(y_i)$ along the trajectory)

+(from $\varphi(y_i)$ to y_i along C).

Then we have

Lemma 4.3. *Suppose that $\alpha \neq 0$ and thus $\alpha \circ [C] \neq 0$. Then for each i, the number a_i is given by*

$$a_i = \frac{\alpha \circ \beta_i}{\alpha \circ [C]},$$

where $\circ$ stands for the intersection number.

Now suppose that α is rational. Then by this lemma, all of a_i's are rational numbers and this contradicts that φ has a dense orbit. Therefore we have proved the following

Proposition 4.4. *Let p be a point on a locally dense semi-trajectory. Then any cluster point of $\{1/T\,[\gamma_{T,p}]\}$ is either 0 or irrational.*

Next, we deal with exceptional semi-trajectories and those which approach to exceptional domains.

Proposition 4.5. *Let $L_+(p)$ be an exceptional semi-trajectory. Then any cluster point of $\{1/T[\gamma_{T,p}]\}$ is either 0 or irrational.*

Proof. Take a transverse simple closed curve C through p. By Corollary 1.2, the set $C \cap \overline{L_+(p)}$ is a Cantor subset of C. Thus there is a non-atomic measure ν on C such that $\nu(C)=1$ and $\operatorname{supp}(\nu) = C \cap \overline{L_+(p)}$. We define a map $h\colon C \cap \overline{L_+(p)} \to S^1 \cong \boldsymbol{R}/\boldsymbol{Z}$ by $h(x) = \int_*^x d\nu$. Since h is at most two-to-one and satisfies $\overline{h(C \cap L_+(p))} = S^1$, the Poincaré map $\varphi\colon (C \cap L_+(p)) \to (C \cap L_+(p))$ extends via $h \circ \varphi \circ h^{-1}$ to a P-transformation $\varphi'\colon S^1 \to S^1$ with a dense orbit. Then any cluster point of $\{1/T[\gamma_{T,p}]\}$ is expressed as in Lemma 4.1 by a measure invariant under this φ', and conversely φ' is given as (4.2) by this measure. Thus this proposition is proved by Lemma 4.3 as Proposition 4.4 is. □

Corollary 4.6. *Let p be a point such that $L_+(p)$ approaches to an exceptional domain. Then any cluster point of $\{1/T[\gamma_{T,p}]\}$ is either 0 or irrational.*

Proof. Let $L_+(q)$ be an exceptional semi-trajectory with $\omega(p) = \overline{L_+(q)}$ and C be a transverse simple closed curve such that $C \cap \overline{L_+(q)} \neq \emptyset$. Let φ be the Poincaré map. We can assume that $p \in C$. Then any cluster point of $\{1/T[\gamma_{T,p}]\}$ is expressed as in Lemma 4.1 by a measure on C, which is given by some sequence $\{1/n_i \sum_{k=0}^{n_i-1} \varphi_*^k \delta_p\}$. Since $\omega(p) = \overline{L_+(q)}$, the support of such a measure is contained in $C \cap \overline{L_+(q)}$ and thus the proof of Proposition 4.5 applies. □

Finally we prove Theorem 2.

Proof of Theorem 2. The proof of cases (i) and (ii) are immediate and that of case (iii) is a consequence of Propositions 4.4, 4.5 and Corollary 4.6. □

§ 5. Convergence of asymptotic cycles

First we prove Theorem 3, which says that for flows on T^2, the asymptotic cycle of any point converges. Our proof is based on the unique ergodicity of irrational rotations of the circle. We remark that Nishimori [12] also developed a similar argument, while his objects are like projective asymptotic cycles in section 6 rather than original ones (see also Tamura [17]).

Proof of Theorem 3. Let ψ be a flow on T^2. First we show the convergence of asymptotic cycles. It suffices to deal with semi-trajectories of type (6), (7), and (8). Let $L_+(p)$ be a locally dense semi-trajectory.

Assertion 5.1. *Let* C *be a transverse simple closed curve such that* $C \subset \overline{L_+(p)}$. *Then the P-transformation* φ *defined by the Poincaré map with respect to* C *has a continuous extension.*

Proof. It suffices to show that $\varphi_R(x) = \varphi_L(x)$ for each $x \in C$. Suppose the contrary. Then no points in the interval $(\varphi_L(x), \varphi_R(x))$ are in the image of φ (see Figure 4). This contradicts $\overline{L_+(p)} \supset C$. □

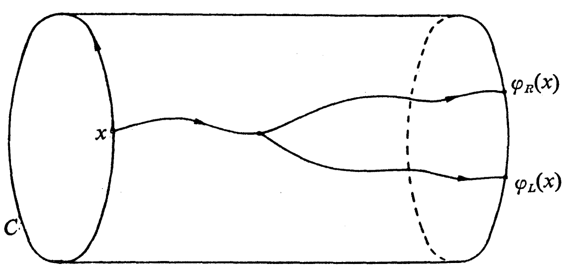

Fig. 4.

Let $\tilde{\varphi}$: $C \to C$ be the continuous extension of φ. Since $\tilde{\varphi}$ has a dense orbit, $\tilde{\varphi}$ is topologically conjugate to an irrational rotation of S^1. It is known that irrational rotations are uniquely ergodic (see Cornfeld et al. [5]) and thus so is $\tilde{\varphi}$. Any cluster point of the sequence $\{1/T[\gamma_{T,p}]\}$ is expressed as in Lemma 4.1 by this unique invariant measure, it follows that the asymptotic cycle of p converges.

Let $L_+(p)$ be an exceptional semi-trajectory. Let C be a transverse simple closed curve with $C \cap \overline{L_+(p)} \neq \emptyset$. Then as in the proof of Proposition 4.5, the Poincaré map φ with respect to this C induces a P-transformation φ' such that any cluster point of the sequence $\{1/T[\gamma_{T,p}]\}$is expressed by a φ'-invariant measure. An argument similar to Assertion 5.1 implies that φ' is uniquely ergodic and thus the asymptotic cycle of p converges.

The convergence of the asymptotic cycle in the case when $L_+(p)$ approaches to an exceptional domain reduces to the case above.

Finally we show that the asymptotic cycle of a semi-trajectory is determined by its ω-limit set. If the ω-limit set is one of a singular point, a circuit and a closed orbit, then the conclusion is obvious. Otherwise, the semi-trajectory is one of types (6), (7) and (8). Then its asymptotic cycle is represented by the unique invariant measure of a P-transformation that is determined by the ω-limit set. This completes the proof. □

As is seen by the above proof of Theorem 3, the convergence of the asymptotic cycle is closely related to the unique ergodicity of the P-transformation.

Proposition 5.2. *Let $L_+(p)$ be a locally dense semi-trajectory and $C \subset \overline{L_+(p)}$ be a transverse simple closed curve. Let $\varphi\colon C \to C$ be the Poincaré map and τ the first return time (see section 4). Then the following two conditions are equivalent.*

(a) *φ is uniquely ergodic with respect to a measure μ such that $\int_C \tau d\mu < \infty$.*

(b) *There exists an irrational homology class α such that for any point $q \in C$ with $\overline{L_+(q)} \supset C$, the asymptotic cycle of q exists and is equal to α.*

Proof. Since the proof that (a) implies (b) is the same as that of Theorem 3, we only show that (b) implies (a). It suffices to prove that any two ergodic probability measures coincide. Let μ and ν be such measures on C with respect to φ. Take an interval $I \subset C$. Since the orbit of a regular point is dense in C, there exist a point q and a positive integer n such that the interval $[q, \varphi^n(q)] \subset C$ approximates I. Let β be the homology class represented by the curve

$$\begin{aligned}&(\text{from } q \text{ to } \varphi^n(q) \text{ along the trajectory})\\ &\qquad\qquad + (\text{from } \varphi^n(q) \text{ to } q \text{ along } C).\end{aligned}$$

Since μ is an ergodic measure, for almost every point x with respect to μ, the sequence $1/n\sum_{k=0}^{n-1} \varphi_*^k \delta_x$ converges to μ. By the assumption that α is the asymptotic cycle of x, the homology class α corresponds to the measure

μ as in Lemma 4.1. Therefore, we see as in Lemma 4.3 that

$$\mu([q, \varphi^n(q)]) = \frac{\alpha \circ \beta}{\alpha \circ [C]}$$

where $\alpha \circ [C] \neq 0$ is guaranteed by the proof of Lemma 4.1. Since we can take $[q, \varphi^n(q)]$ arbitrarily close to I, the measure $\mu(I)$ is determined by the homology class α. Applying this argument to ν, we have $\mu(I) = \nu(I)$ for any closed interval I and thus $\mu = \nu$. This completes the proof. □

It is natural to expect that the asymptotic cycle converges for flows on any two dimensional manifold but this does not hold. In [9], Keynes and Newton constructed a minimal but not uniquely ergodic interval exchange transformation. Suspending their example as in Remark 3.1, we have a flow on a two dimensional manifold of genus 3 for which there exist two points which have the same ω-limit set (the whole manifold) but have distinct asymptotic cycles. T. Kamae informed the author that there also exists an interval exchange transformation such that the limit distribution of some point does not converge, and thus by suspending this, we have a flow on a two dimensional manifold for which the asymptotic cycle of some point does not converge. It is rather easy to construct such flows in higher dimensions.

In contrast with this, Masur [11] and Veech [18] proved that "almost all" interval exchange transformations with dense orbits are uniquely ergodic. Hence for "almost all" flows on two dimensional manifolds, the asymptotic cycle of any point converges.

As opposed to these general results, there are geometric criterions for the convergence of asymptotic cycles. Concluding this section, we give such a criterion: For a flow on a manifold M, a compact subset D of M is called a *recurrent domain* if either D is an exceptional domain or there exists a locally dense semi-trajectory $L_+(p)$ such that $D = \overline{L_+(p)}$.

Theorem 5.3. *Let ψ be a flow on a manifold M of genus g. Suppose that there exist g distinct recurrent domains. Then the asymptotic cycle converges for any point and it only depends on the ω-limit set.*

Proof. For each recurrent domain D, there exist two transverse simple closed curves C_1 and C_2 for ψ such that C_1 and C_2 intersect transversely at one point and are disjoint from other recurrent domains. Hence cutting M along these curves, we reduce this theorem to Theorem 3. □

There are similar conditions for flows under which the asymptotic cycle of any point converges, but we omit them since they are only variations of the above theorem.

§ 6. Projective asymptotic cycles

As is seen in sections 3, 4 and 5, the asymptotic cycle characterizes the behaviour of positive semi-trajectories in some sense. However, it may vary under time change of the flow (though only by a multiple by a scalar). It can happen that locally dense or exceptional semi-trajectories have 0 as the asymptotic cycle. To avoid this, we introduce the notion of projective asymptotic cycles and rewrite Theorems 2 and 3 in terms of them.

We being with definitions: We put

$$PH_1(M; \boldsymbol{R}) = \{0\} \cup (H_1(M; \boldsymbol{R}) - \{0\})/\sim$$

where $\alpha \sim \beta$ if there is a positive number a such that $\alpha = a\beta$, and we let π: $H_1(M; \boldsymbol{R}) \to PH_1(M; \boldsymbol{R})$ be the quotient map. The sets $\{0\}$ and $(H_1(M; \boldsymbol{R}) - \{0\})/\sim$ have natural topologies and $PH_1(M; \boldsymbol{R})$ is their disjoint union as a topological space. An element $\bar{\alpha}$ of $PH_1(M; \boldsymbol{R})$ is called *rational* and *irrational* if elements of $\pi^{-1}(\bar{\alpha})$ are rational and irrational, respectively.

Let $\psi = \{\psi_t\}$ be a flow on a compact manifold M and p be a point of M. Then as in the introduction, there exists a 1-cycle

$$\begin{aligned}\gamma_{T,p} = &(\text{the trajectory from } p \text{ to } \psi_T(p)) \\ &+ (\text{a minimal geodesic from } \psi_T(p) \text{ to } p)\end{aligned}$$

for a real number T. To be precise, we first define a *generalized projective asymptotic cycle* $GPA(p)$ of p, which is a closed subset of $PH_1(M; \boldsymbol{R})$, by the following. Let $\| \quad \|$ be a norm on $H_1(M; \boldsymbol{R})$.

(1) $GPA(p)$ contains 0 if there exists a sequence $\{T_i\}$ such that $T_i \to \infty$ and $\|[\gamma_{T_i,p}]\|$ is bounded.

(2) $GPA(p)$ contains $\bar{\alpha}$ with $\bar{\alpha} \in (H_1(M; \boldsymbol{R}) - \{0\})/\sim$ if there exists a sequence $\{T_i\}$ such that $T_i \to \infty$, $\|[\gamma_{T_i,p}]\| \to \infty$ and $\pi([\gamma_{T_i,p}]) \to \bar{\alpha}$.

When $GPA(p)$ consists of a single element, we say that the projective asymptotic cycle of p exists and this element, which will be denoted by $PA(p)$, is called the *projective asymptotic cycle* of p. By this notion, Theorems 2 and 3 are rewritten as follows.

Theorem 6.1. *Let ψ be a continuous flow on a closed, orientable, two dimensional manifold and suppose that ψ has only a finite number of singular points. Then we have*

(i) *if p is of type* (1) *or* (2), *then $PA(p) = 0$,*

(ii) *if p is of type* (3), *then $PA(p) = \pi([\Gamma])$, where Γ is the circuit to which $L_+(p)$ approaches,*

(iii) *if p is of type* (4) *or* (5), *then $PA(p) = \pi([\gamma])$, where γ is the closed orbit, and*

(iv) *if p is of type* (6), (7) *or* (8) *and $PA(p)$ exists, then $PA(p)$ is irrational.*

Note that if $L_+(p)$ winds round a circuit $\Gamma = \bigcup \gamma$, then all the tranjectories but finite in Γ satisfy $\alpha(\gamma) = \omega(\gamma)$ and $\gamma \cup \alpha(\gamma)$ is homologous to zero by Lemma 1.3. This justifies the statement (ii) above.

Theorem 6.2. *Let ψ be a continuous flow on T^2, which has only a finite number of singular points. Then for every point p of T^2, the projective asymptotic cycle $PA(p)$ exists. Moreover the set $PA(\psi) = \{PA(p); p \in T^2\}$ is one of the following*: $\{0\}$, $\{\bar{\alpha}\}$, $\{0, \bar{\alpha}\}$, $\{\bar{\alpha}, -\bar{\alpha}\}$, $\{0, \bar{\alpha}, -\bar{\alpha}\}$ *for a rational* $\bar{\alpha}$, *and* $\{\bar{\alpha}\}$, $\{0, \bar{\alpha}\}$ *for an irrational* $\bar{\alpha}$.

Proof of these are straightforward and we omit them.

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