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Foliations
pp. 325–339

Foliations of Seifert Fibered Spaces over S^2

Shigenori Matsumoto

Introduction

In his thesis [15], Thurston has shown that given an oriented S^1-bundle $p: V \to \Sigma$ over an oriented closed surface $\Sigma \neq T^2$, any transversely oriented, codimension one C^r foliation ($r \geq 2$) of V without compact leaves can be moved by a C^r-isotopy so as to be transverse to fibers of p. Recently this result has been extended by Eisenbud-Hirsch-Neumann [1] to Seifert fibrations over surfaces of nonzero genus.

In this paper we deal with the remaining case of Seifert fibered spaces over S^2, where we have a generalization to foliations without dead end components. For the definition of dead end components, see Section 1. This phenomenon reflects the vanishing of the first Betti number of S^2 and is characteristic to the present case. (One can construct counter-examples for S^1-bundles over surfaces of nonzero genus.) Our main result is the following.

Theorem 1. *Let $p: V \to S^2$ be an oriented Seifert fibration with at least four exceptional fibers and $\mathcal{F}$ a transversely oriented, codimension one C^r foliation ($r \geq 2$) of V. Then there is an isotopy $\{\phi_t\}$ of V such that $\phi_0 =$ id and $\phi_1^* \mathcal{F}$ is transverse to all the fibers of p, if and only if $\mathcal{F}$ has no dead end components.*

Corollary 2. *Let V and $\mathcal{F}$ be as in Theorem* 1. *If $\mathcal{F}$ does not have a compact leaf, then $\mathcal{F}$ is isotopic to the one which is transverse to all the fibers of p.*

Gazolas [3] has determined which Seifert fibrations admit transverse foliations. Together with Theorem 1, this determines which Seifert fibrations over S^2 admit foliations without dead end components.

The proof of the only if part of Theorem 1 is clear. The proof of the if part proceeds along the same line as in Levitt [4]. We consider a vertical incompressible torus; make it transverse to the foliation; and study the induced foliation on it. When it does not give us enough information, we consider two such tori intersecting along two closed

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curves.

Thus our argument, as it is, is not applicable to the case of three exceptional fibers, where there are no such tori. However when there are at most two exceptional fibers or when there are three and $\sum_{i=1}^{3} 1/\alpha_i > 1$, where $((\alpha_i, \beta_i), b)$ is the Seifert index (b is the Euler number), the total manifold V has a finite fundamental group (except $V = S^1 \times S^2$) and thus any foliation of V has a Reeb component. This shows that Theorem 1 holds (vacantly).

Also in the case of three exceptional fibers with the above sum equal to one, any transverse foliation has a compact leaf. This is shown as follows. The quotient group of $\pi_1(V)$ by the subgroup generated by the class of an ordinary fiber acts on $\boldsymbol{R}^2$ totally discontinuously as Euclidean motions. This implies that $\pi_1(V)$ has a polynomial growth (c.f. Milnor [6]). On the other hand, the Betti number of V is at most one. Thus the assertion is meant by Corollary 7.4 of Plante [10]. Further if $\sum_{i=1}^{3} \beta_i/\alpha_i + b \neq 0$, then V does not admit an incompressible surface. This implies the existence of a Reeb component. Thus Theorem 1 also holds in this case. However the author does not know whether it holds or not in the remaining case.

In Section 1, we give preparatory results necessary for the proof of Theorem 1. In Section 2, we show that a foliation without a dead end component is isotopic to the one which is transverse to a certain fiber. In Section 3, we show following Levitt [4] that it is further moved so as to be transverse to all the fibers. This is involved by the suggestion of the referee for the convenience of the reader. So far, however, our argument is not applicable to the special case where there are four exceptional fibers, all with Seifert index (2,1). This case is dealt with separately in Section 4.

All manifolds and maps in this paper are to be of class C^∞ unless made explicit to the contrary. Foliations we study are always transversely oriented, codimension one and at least of class C^2. A path or a curve is understood to be a continuous map from the unit interval. But we often confuse a path with its image. Both are denoted by the same letter.

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§ 1. Preparations

Here we prove an analogue of Théorème 2 of [4]. Our argument is a variant of that of [4] and the reader is recommended to consult [4] whenever necessary.

Novikov introduced in [8] the following equivalence relation among leaves of a transversely oriented foliation $\mathscr{F}$; two leaves L and L' are equivalent if and only if there exists a closed curve transverse to the foliation and passing through L and L'. The equivalence classes are either compact leaves or open saturated subsets whose boundaries are finite unions (maybe empty) of the former types. One calls the closures of latter types *Novikov components* of $\mathscr{F}$. They are either the total manifold or compact submanifolds. Among them there is defined a natural partial order. Minimal or maximal elements (other than the total manifold) are called *dead end components*. Notice that $\mathscr{F}$ admits a dead end component if and only if the total manifold is not a Novikov component, or equivalently if and only if there is an open saturated subset whose boundary leaves are transversely oriented simultaneously inwards or outwards.

Next following [4] we define well placed curves w.r.t. a foliation $\mathscr{G}$ of the 2-torus T^2.

Definition 1. A simple closed curve C of T^2 is called *well placed w.r.t.* $\mathscr{G}$, if one of the following three conditions is satisfied.

(a) C is transverse to $\mathscr{G}$.

(b) $\mathscr{G}$ is a trivial foliation by circles and C is a leaf of $\mathscr{G}$.

(c) $\mathscr{G}$ contains at least one Reeb component and C has the smallest possible number (nonzero) of points of tangency with $\mathscr{G}$ among its isotopy class.

The above three conditions are mutually exclusive and we call C according to the case well placed of type (a), (b) or (c).

Theorem 3. *Let V be a closed oriented 3-manifold, equipped with a foliation $\mathscr{F}$ without dead end components and let T_1 and T_2 be embedded tori in V. We suppose;*

(1.1) *T_1 and T_2 are incompressible, i.e. their inclusion maps into V induce injections on π_1.*

(1.2) *Neither T_1 nor T_2 is isotopic to a leaf of $\mathscr{F}$.*

(1.3) *T_1 and T_2 intersect transversely along two disjoint simple closed curves γ_1 and γ_2, which are homotopically nontrivial in V.*

(1.4) *Let σ and τ be paths in T_1 and T_2, respectively, connecting a point of γ_1 to a point of γ_2. Then the closed path $\tau\sigma^{-1}$ is homotopically nontrivial in V.*

Then one can find two embedded tori T_1' and T_2' isotopic to T_1 and T_2, respectively, such that

(1.5) *T_1' and T_2' are transverse to $\mathscr{F}$,*

(1.6) *T_1' and T_2' intersect along two simple closed curves which are well placed w.r.t. both $\mathscr{F}|_{T_1'}$ and $\mathscr{F}|_{T_2'}$.*

The proof of Theorem 3 reduces to the following lemma.

Lemma 4. *Assume besides the hypothesis of Theorem* 3 *that* T_1 *is transverse to* $\mathcal{F}$ *and that* γ_1 *and* γ_2 *are well placed w.r.t.* $\mathcal{F}|_{T_1}$. *Then* T_2 *is isotopic to a torus* T_2', *transverse to* $\mathcal{F}$, *such that* T_1 *and* T_2' *meet transversely along two disjoint simple closed curves, well placed w.r.t.* $\mathcal{F}|_{T_1}$ *(not necessarily w.r.t.* $\mathcal{F}|_{T_2'}$*).*

First we show why Lemma 4 implies Theorem 3. By a preliminary isotopy, we can make T_1 and T_2 of Theorem 3 to satisfy the assumption of Lemma 4. Apply Lemma 4. When $T_1 \cap T_2'$ is well placed of type (a) w.r.t. $\mathcal{F}|_{T_1}$, so is it w.r.t. $\mathcal{F}|_{T_2'}$ and nothing is left to prove. Suppose $T_1 \cap T_2'$ is well placed of type (b) w.r.t. $\mathcal{F}|_{T_1}$. If $\mathcal{F}|_{T_2'}$ is a trivial foliation, then $T_1 \cap T_2'$ is also well placed w.r.t. $\mathcal{F}|_{T_2'}$. Otherwise after a preliminary isotopy apply Lemma 4 again, this time so as to move T_1. Then $T_1 \cap T_2'$ becomes well placed of type (a) w.r.t. $\mathcal{F}|_{T_2'}$. Finally when $T_1 \cap T_2'$ is of type (c), we can use Lemma 4 successively (moving T_1 and T_2' alternately), until we get that $T_1 \cap T_2'$ is well placed w.r.t. both $\mathcal{F}|_{T_1}$ and $\mathcal{F}|_{T_2'}$. This is done by an induction on the number of points of tangency of the curve.

Proof of Lemma 4. When $T_1 \cap T_2$ is well placed of type (a) or (b) w.r.t. $\mathcal{F}|_{T_1}$, Lemma 4 can be obtained without much difficulty by just the same argument as in [4]. So we shall be concerned solely with the case where $T_1 \cap T_2$ is well placed of type (c) w.r.t. $\mathcal{F}|_{T_1}$. As is proved in [4], it suffices to show only the following.

(1.7) Let E be a disk (resp. annulus) in T_2 and D a disk (resp. annulus) in a leaf of $\mathcal{F}$ such that $D \cap E = \partial E = \partial D$. Suppose ∂E is transverse to T_1 and $D \cup E$ bounds an angular ball (resp. solid torus) B in V such that $B \cap T_2 = E$. Suppose also that D is isotopic to E keeping the boundary fixed in B. Then $B \cap T_1$ is a union of angular disks which intersect ∂E transversely at two points.

Proof of (1.7). Consider $T_1 \cap \partial B$. It is a union of simple closed curves. Note that none of them is contained in the interior of D. For otherwise there would exist a leaf curve of $\mathcal{F}|_{T_1}$ disjoint from $T_1 \cap T_2$, contradicting the hypothesis that $T_1 \cap T_2$ is well placed of type (c) w.r.t. $\mathcal{F}|_{T_1}$. Consider if there exists any, a component λ intersecting ∂E at some point x. Assume to fix the idea that $x \in \gamma_1$. Let α be a subarc of λ contained in E which connects x to some point y of ∂E and let β be a subarc of λ in D which connects x to some point z of ∂E. One has that $z \in \gamma_1$. If not, by choosing a curve δ in E connecting y to z such that $\alpha\delta\beta^{-1} \simeq 1$ in B, one would get a contradiction to the condition (1.4).

Note that β cannot intersect γ_2 because $B \cap T_2 = E$. Thus β must be a leaf curve in a Reeb component of $\mathcal{F}|_{T_1}$ which is homotopic to a subarc of γ_1 leaving the end points fixed. Notice also that α is a subarc of γ_1 having at least one point of tangency with $\mathcal{F}$ in its interior. This is due to the assumption that $\mathcal{F}$ be transversely oriented.

We call λ a curve of type (1) if $y=z$ and λ is null homotopic in T_1. Otherwise it is called a curve of type (2). A curve which is wholly contained in Int E is called a curve of type (3). Our aim is to show that all the components of $T_1 \cap \partial B$ are of type (1). For then all the components of $T_1 \cap B$ prove to be disks and the proof is completed.

Assume there were a curve of type (2) or (3), say λ_1. Suppose to fix the idea that $\lambda_1 \cap \gamma_1 \neq \phi$. We show;

(1.8) λ_1 is a composition of subarcs of γ_1 and leaf curves of $\mathcal{F}|_{T_1}$ isotopic to γ_1, keeping $\lambda_1 \cap \gamma_1$ fixed. All the other components of $T_1 \cap \partial B$ meeting γ_1 are of type (1).

Proof of (1.8). When λ_1 is of type (3), (1.8) holds trivially.

Suppose λ_1 is of type (2). Then as we have already shown, a leaf curve of λ_1 is homotopic to a subarc of γ_1 leaving the end points fixed and each component of $\lambda_1 \cap \gamma_1$ contains a point of tangency with $\mathcal{F}|_{T_1}$. This completes the proof of (1.8).

Now (1.8) implies that there are at most two curves of type (2) or (3), one meeting γ_1 and the other γ_2. They are homotopically nontrivial by the incompressibility of T_1. Consider the component C of $B \cap T_1$ containing λ_1. Because $\lambda_1 \not\simeq 1$, one of the other boundary components of C, say λ_2, must also be homotopically nontrivial. That is, λ_2 is of type (2) or (3). Thus $\lambda_2 \cap \gamma_2 = \phi$. But then by choosing a path σ in C connecting a point of γ_1 to a point of γ_2 and a path τ in E such that $\partial\sigma = \partial\tau$ and $\sigma\tau^{-1} \simeq 1$ in B, one would obtain a contradiction to the condition (1.4). This completes the proof of (1.7). q.e.d.

In the rest of this section, we study homotopical properties of a Seifert fibration V over S^2. Let $((\alpha_i, \beta_i), b)$ $(i=1, 2, \cdots, n)$ be the Seifert index of V (see [9]). Then $\pi_1(V)$ is given by

$$\pi_1(V) = \langle p_1, p_2, \cdots, p_n, h \mid p_i h = h p_i, p_i^{\alpha_i} h^{\beta_i} = 1 (\forall i), p_1 p_2 \cdots p_n = h^b \rangle,$$

where h stands for the class of an ordinary fiber and p_i a closed path around the i-th exceptional fiber. Let G be the quotient group of $\pi_1(V)$ by the subgroup generated by h. By certain abuse the class in G will be denoted by the same letter as an element of $\pi_1(V)$. The following lemma is concerned with G.

Lemma 5. *If the number n of the exceptional fibers is not less than* 4, *then*

(1.9) $(p_1p_2)^n p_2^{-1}(p_2p_3)^m \neq 1$ *for any integer n and m*,

(1.10) $(p_1p_2)^n(p_2p_3)^m \neq 1$ *except when* $n=m=0$.

Further if $n \geqq 5$ *or if* $n=4$ *and at least one of* α_i *is greater than* 2, *then*

(1.11) *for sufficiently large n and m*, $(p_1p_2)^n$ *and* $(p_2p_3)^m$ *freely generate a subgroup of G*.

Proof. First suppose that $n \geq 5$ or $n=4$ and at least one of α_i is greater than 2. Let

$$\tilde{G}=\langle u_1, \cdots, u_n \mid u_j^2=1, (u_ju_{j+1})^{\alpha_j}=1(\forall j)\rangle,$$

where we use the convention $u_{n+1}=u_1$. As is well known (see for example [5]) the homomorphism ψ: $G \to \tilde{G}$ defined by $\psi(p_j)=u_ju_{j+1}$ $(1 \leq j \leq n)$ is an injection onto a subgroup of index two. Take an n-gon Γ of the Poincaré disk such that the angle of $U_j \cap U_{j+1}$ is π/α_j $(1 \leq j \leq n)$, where U_j is the face of Γ (taken successively). Then $\tilde{G}$ acts on the Poincaré disk properly discontinuously so that its fundamental domain is Γ and that u_j acts as the symmetry about U_j. Let W_1 be the geodesic perpendicular to both U_1 and U_3. Then $\psi(p_1p_2)=u_1u_3$ is a hyperbolic motion with axis W_1, sending Γ to Γ' in Figure 1. Likewise $\psi(p_2p_3)=u_2u_4$ is a hyperbolic motion whose axis is the geodesic W_2 perpendicular to U_2 and U_4. u_2u_4 sends Γ to Γ''.

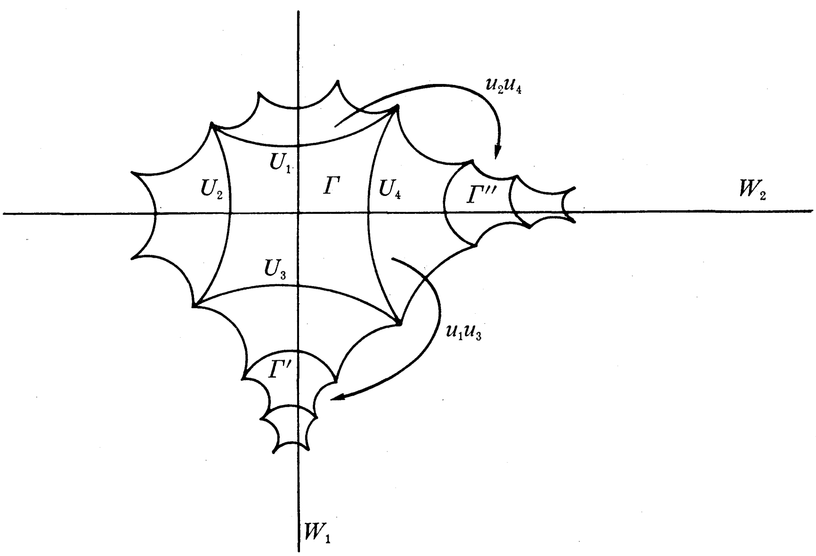

Fig. 1.

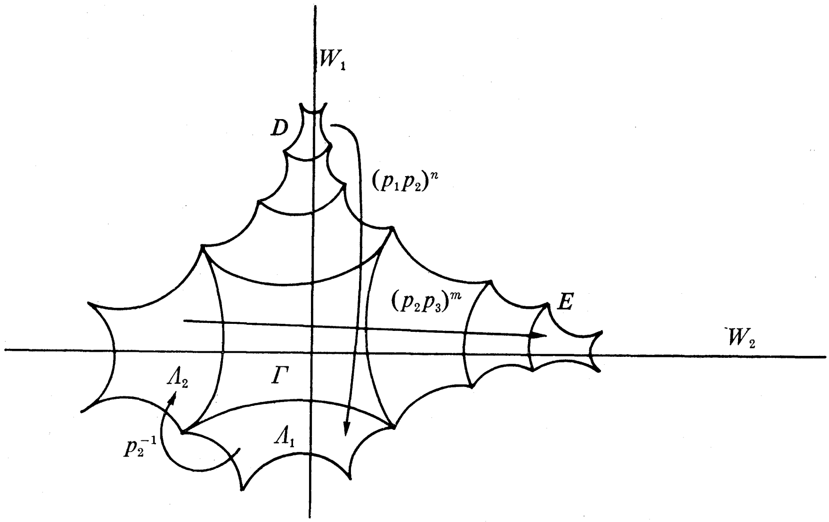

Fig. 2.

Now (1.10) is clear by Figure 1. (1.11) is a direct consequence of Klein's criterion ([2]). For the proof of (1.9), notice that $p_2^{-1}=u_3u_2$ is a rotation by $2\pi/\alpha_2$ around the vertex $U_2\cap U_3$ sending Λ_1 to Λ_2 in Figure 2. Hence $(p_1p_2)^n p_2^{-1}(p_2p_3)^m$ sends D to E in Figure 2. Thus it cannot be the identity.

Let us consider the remaining case, where $n=4$ and each $(\alpha_i, \beta_i)=(2, 1)$. Then G acts on the Euclidean plane instead of the Poincaré disk. But (1.9) and (1.10) are shown by the same argument as above. q.e.d.

§ 2. Proof of Theorem 1

In this section and the next we prove Theorem 1 (if part) except in a special case. First we summarize conditions of a foliation which lead to the existence of a Reeb component.

Theorem 6 ([8], [12]). *If a transversely oriented foliation $\mathcal{F}$ of a 3-manifold V satisfies one of the following conditions, then $\mathcal{F}$ has a Reeb component.*

(2.1) *Some leaf F of $\mathcal{F}$ is compressible, i.e., there exists a closed curve in F, null homotopic in V, but not in F.*

(2.2) *There exists a null transversal (a null homotopic closed curve transverse to the foliation).*

(2.3) *V is not prime.*

In this section we consider V and $\mathcal{F}$ of Theorem 1 and assume that $\mathcal{F}$ has no dead end components. We also assume that V has more than

4 exceptional fibers or has just 4 and at least one of α_j is greater than 2. Our purpose is to show the following.

(2.4) *$\mathcal{F}$ is moved by an isotopy so as to be transverse to a certain fiber of p.*

Let F be a compact leaf of $\mathcal{F}$. By (2.1), it is incompressible. Thus there is an (ambient) isotopy which makes F either vertical or horizontal (Waldhausen [16]), (A surface in V is called vertical if it is a union of fibers and called horizontal if it is transverse to the fibers.) In fact F is made horizontal. If not, F would be separating and thus $\mathcal{F}$ would have a dead end component.

Consider a vertical torus T in V and let V_1 and V_2 be the closures of the components of $V\backslash T$. Assume that both V_1 and V_2 contain at least two exceptional fibers, that is, T is incompressible. By Roussarie [13] the foliation we consider is isotoped so that it is transverse to T. Several cases are to be considered concerning the induced foliation $\mathcal{F}|_T$. But except the following two cases, one can easily obtain (2.4).

(2.5) $\mathcal{F}|_T$ is a trivial foliation by circles which are isotopic to ordinary fibers of V.

(2.6) $\mathcal{F}|_T$ has Reeb components whose boundary leaves are not isotopic to ordinary fibers of p.

Our goal is to show that both cases are in fact impossible. First let us consider the case (2.5). Consider a solid torus V_3 equipped with the trivial foliation $\mathcal{F}_0$ by meridean disks. Paste V_1 and V_3 so that $\mathcal{F}|_{V_1}$ and $\mathcal{F}_0$ match together to give a foliation $\tilde{\mathcal{F}}$ of the resultant manifold $\tilde{V}$. Clearly $\tilde{V}$ is not a prime manifold. Thus by (2.3), $\tilde{\mathcal{F}}$ must have a Reeb component. Note that the boundary torus is contained in V_1 and is isotopic to T. This leads to a contradiction because T is vertical.

In the remainder of this section we prove that the case (2.6) is also impossible. For this it does not suffice to consider one torus T. Instead we consider two tori T_1 and T_2 such that;

(2.7) T_1 and T_2 are vertical,

(2.8) they intersect along two ordinary fibers,

(2.9) three of the four components of $V\backslash(T_1 \cup T_2)$ contain exactly one exceptional fiber.

Further we may asume that $\pi_1(T_1)$ (resp. $\pi_1(T_2)$) is generated by h and p_1p_2 (resp. p_2p_3). Then by using Lemma 5, we get the following.

Lemma 7. *Let σ and τ be paths of T_1 and T_2 respectively such that $\partial\sigma=\partial\tau$ and $\sigma^{-1}\tau\simeq 1$ in V. Then σ is homotopic in T_1 to a path in $T_1 \cap T_2$ keeping the end points fixed. Especially when the two boundary points of σ lie in the different components of T_1 and T_2, we have $\sigma^{-1}\tau \not\simeq 1$ in V.*

Proof. If two boundary points of σ lie in the different components of $T_1 \cap T_2$, then $\sigma^{-1}\tau$ represents in the group G of Lemma 5 the class $(p_1p_2)^n p_2^{-1}(p_2p_3)^m$ and by (1.9) cannot be null homotopic in V. On the other hand if they lie in the same component, $\sigma^{-1}\tau$ represent $(p_1p_2)^n(p_2p_3)^m$. Thus $\sigma^{-1}\tau \simeq 1$ implies $n=0$. Recall that a component of $T_1 \cap T_2$ represents h. Also h is contained in the center of $\pi_1(V)$. This shows that σ followed by an appropriate path in $T_1 \cap T_2$ is null homotopic in V, hence in T_1 because T_1 is incompressible. q.e.d.

From the above lemma we get that the tori T_1 and T_2 satisfy the hypothesis of Theorem 3. Thus after an appropriate isotopy we can assume;

(2.10) T_1 and T_2 are transverse to $\mathscr{F}$,

(2.11) they meet along two disjoint simple closed curves γ_1 and γ_2, each isotopic to an ordinary fiber,

(2.12) γ_1 and γ_2 are well placed of type (c) w.r.t. T_1 and T_2.

Note that we are working in case (2.6) and thus we may assume (2.12). Also note that in the course of all the isotopies we have used the way of intersection of T_1 and T_2 does not change. So their homotopical properties remain valid.

Definition 2. Two points x and x' of the same component of $T_1 \cap T_2$ are called *symmetric w.r.t.* $\mathscr{F}|_{T_j}$ $(j=1, 2)$ if they are joined by a leaf curve of $\mathscr{F}|_{T_j}$ which is homotopic in T_j to a path in $T_1 \cap T_2$ keeping the end points fixed.

Symmetric points belong to the same leaf of a Reeb component of $\mathscr{F}|_{T_j}$.

Lemma 8. *Let α be a simple arc in a leaf L of $\mathscr{F}|_{T_1}$ and β_0 and β_1 closed leaves of $\mathscr{F}|_{T_2}$ such that $\alpha(0) \in \beta_0$, $\alpha(1) \in \beta_1$ and $\alpha(0)$ and $\alpha(1)$ are symmetric w.r.t. $\mathscr{F}|_{T_1}$. Then we have $L \cap \beta_0 = \alpha(0)$ and $L \cap \beta_1 = \alpha(1)$.*

Proof. *Step* 1. Consider the special case where Int $\alpha \cap (\beta_0 \cup \beta_1) = \phi$. Notice first that β_0 and β_1 are distinct. If they were identical, take a subarc σ of $\beta_0 = \beta_1$ joining $\alpha(0)$ to $\alpha(1)$. Then $\sigma^{-1}\alpha$ would be a leaf curve of $\mathscr{F}$ whose holonomy is orientation reversing.

Because all the leaves of $\mathscr{F}$ are incompressible and β_0 is not null homotopic in V, one can find an annulus A in a leaf F of $\mathscr{F}$ such that A is bounded by β_0 and β_1 and A contains α. L meets ∂A transversely in F. Because $T_1 \cap A$ is compact, $L \cap A$ consists of finite arcs. Suppose in way of contradiction that the conclusion of Lemma 8 is false. Then there exists a component δ of $L \cap A$ different from α. Obviously δ is homotopic to a curve of T_2, keeping the end points fixed, hence to a curve in $T_1 \cap T_2$ by Lemma 7. But in L, α also has this property. This contradicts the assumption that $T_1 \cap T_2$ are well placed.

Step 2. We shall reduce the general case to Step 1. That is, we shall prove under the hypothesis of Lemma 8 that Int $\alpha \cap (\beta_0 \cup \beta_1) = \phi$. The proof is an induction on the number of transverse intersection points of Int α and T_2. If it is zero, then the claim is trivially true. Suppose it is positive and assume in way of contradiction that there is a point $x \in$ Int $\alpha \cap \beta_0$. Let x' be the symmetric points of x w.r.t. $\mathscr{F}|_{T_1}$. Clearly $x' \in$ Int α, and the leaf β' of $\mathscr{F}|_{T_2}$ through x' is closed. If not, one would obtain a null transversal of $\mathscr{F}$. But this is prohibited by Theorem 6. Let α' be the subarc of α joining x and x'. Now by the induction hypothesis, we have $\alpha' \cap \beta_0 = x$ and $\alpha' \cap \beta' = x'$. Thus by Step 1, $\alpha \cap \beta_0 = x$. A contradiction. q.e.d.

Now let us complete the proof that case (2.6) is impossible. Take a point of tangency p of $T_1 \cap T_2$ with $\mathscr{F}$. Suppose p is contained in an open Reeb component C_i of $\mathscr{F}|_{T_i} (i=1, 2)$. Let I_i be the component of $C_i \cap T_1 \cap T_2$ containing p. For a point of $I_1 \cap I_2$, its symmetric points w.r.t. $\mathscr{F}|_{T_1}$ and $\mathscr{F}|_{T_2}$ are identical. If not, one would get a null transversal. Thus there are two possibilities.

(2.13) $I_2 \subsetneqq I_1$ or $I_1 \subsetneqq I_2$.

(2.14) $I_1 = I_2$.

We show in the below that in both of the above cases, we can find annuli A_i such that vol $(A_i) \to \infty$, while vol (∂A_i) is bounded.

Consider first case (2.13). Suppose $I_2 \subsetneqq I_1$. Then around p, one can find simple arcs α, β_0 and β_1 which satisfy the assumption of Lemma 8. (β_0 and β_1 are boundary leaves of C_2.) Let L be the leaf of $\mathscr{F}|_{T_1}$ containing α, and let x and y be the points of $(L \backslash \alpha) \cap T_2$ which are symmetric w.r.t. $\mathscr{F}|_{T_1}$.

Claim. A leaf of $\mathscr{F}|_{T_2}$ passing through x or y is closed.

Proof. First notice that the leaf through x and the leaf through y are simultaneously closed or no. Otherwise one would get a null transversal. Let α' be the subarc of L joining x and y and let σ be the subarc of $T_1 \cap T_2$ such that $\partial\sigma = \partial\alpha'$ and $\sigma \simeq \alpha'$ in T_1 keeping the end points fixed.

Suppose to start with that x and y are symmetric w.r.t. $\mathscr{F}|_{T_2}$ and the leaf curve δ of $\mathscr{F}|_{T_2}$ joining x and y are homotopic to σ in T_2. Then the closed curve composed of α' and δ is null homotopic in V, hence in a leaf of $\mathscr{F}$. But by Lemma 8, we have that β_0 intersects it transversely only at one point. A contradiction.

Next suppose that the leaf F and G through x and y respectively, of $\mathscr{F}|_{T_2}$ are noncompact but they do not satisfy the above assumption. In this case both F and G intersect σ infinitely many times. Let x_k (resp. y_k)

be the k-th intersection point of F (resp. G) with σ. Clearly x_k and y_k are symmetric w.r.t. $\mathscr{F}|_{T_1}$. Let δ_k be the leaf arc of $\mathscr{F}|_{T_1}$ joining x_k and y_k. There are infinitely many disjoint δ_k's. Thus some δ_k is disjoint from β_0. Now α', δ_k and leaf paths of $\mathscr{F}|_{T_2}$, one joining x and x_k and the other y and y_k constitute a null homotopic curve in a leaf of $\mathscr{F}$, which intersects β_0 at one point. A contradiction. q.e.d.

Now by what was shown, we can choose α_i, β_i and β_i' as in Lemma 8 so that the length of α_i is arbitrarily large. β_i and β_i' bound an annulus A_i which contains α_i, in a leaf of $\mathscr{F}$. $\mathrm{vol}(A_i)$ can be arbitrarily large, while $\mathrm{vol}(\partial A_i)$ is bounded.

Next let us consider case (2.14). Let z and z' be the end points of $I_1=I_2$ and let σ_i be the closed leaf of $\mathscr{F}|_{T_i}$ through z $(i=1,2)$. Let H be the subgroup of $\pi_1(\sigma_1\cup\sigma_2)$ generated freely by σ_1^n and σ_2^m for sufficiently large n and m. Then by Lemma 5, the inclusion map induces an injection of H into $\pi_1(V)$. Let $\Phi\colon H\to\mathrm{LocDiff}^r(I_1,z)$ be the one sided holonomy homomorphism. By Moussu [7], there exists a nontrivial element g of H such that $\Phi(g)$ has infinitely many fixed points $z_1, z_2, \cdots$, which tend monotonously to z. Let τ_i be the lift to a leaf of $\mathscr{F}|_{T_1\cup T_2}$ through z_i of a closed path in $\sigma_1\cup\sigma_2$ which represents g. Of course τ_i is a closed curve and is homotopically nontrivial in V. Let us take β_i, a homotopically nontrivial simple closed curve in τ_i. One can define in an obvious fashion the curve β_i' which is "*symmetric*" to β_i. It is also a simple closed curve.

By choosing appropriately two symmetric points $w_i\in\beta_i\cap T_1\cap T_2$ and $w_i'\in\beta_i'\cap T_1\cap T_2$ and a simple leaf curve α_i of either T_1 or T_2 joining w_i and w_i', one may assume that $\mathrm{Int}\,\alpha_i\cap(\beta_i\cup\beta_i')=\phi$. One has only to choose the shortest of all such curves. Now as in the previous case β_i and β_i' bound an annulus A_i in a leaf of $\mathscr{F}$ which contains α_i.

Thus we have obtained in both cases (2.13) and (2.14) annuli A_i which have the following properties.

(2.15)

i) $\mathrm{vol}(A_i)\to\infty$ and $\partial A_i=\beta_i\cup\beta_i'$ has a bounded volume.

ii) A_i contains α_i which is a leaf curve in a Reeb component of either $\mathscr{F}|_{T_1}$ or $\mathscr{F}|_{T_2}$.

iii) α_i is homotopic to an arc δ_i of $T_1\cap T_2$ leaving the end points fixed.

iv) β_i and β_i' bound an immersed annulus C_i in $T_1\cup T_2$. C_i contains δ_i.

By i) the sequence $\{A_i\}$ defines a transverse invariant measure μ. This is shown by an argument analogous to [11]. (Compare also [10]). In [11] it is also shown that the transverse invariant measure μ defines a class x of $H_2(V;\boldsymbol{R})$. We shall show $x=0$. Let x_i be the homology class represented by the immersed torus $A_i \cup C_i$. (The orientation is chosen so that it fits with the transverse orientation of A_i.) It is not difficult to show $x = \lim_{i\to\infty} x_i$. Now α_i and δ_i bound an (immersed) disk D_i in T_1 (or T_2). One can perform a surgery of $A_i \cup C_i$ along D_i. The resultant surface is an immersed copy of S^2, which represents the same homology class x_i. But the manifold V we are considering is aspherical. Thus $x_i = 0$. Hence we have $x = 0$.

By Theorem 6.6 of [11], the support of the invariant measure μ consists of compact leaves. These compact leaves are horizontal as we have shown before. It is easy to show that they are mutually isotopic. Thus the vanishing of the homology class x implies the existence of two compact leaves transversely oriented in the opposite directions. This implies the existence of a dead end component.

We have shown that (2.6) is impossible and thus (2.4) is now established.

§ 3. End of proof of Theorem 1

Here we give a proof essentially due to Thurston [15] of the following proposition, which completes the proof of Theorem 1 in the case treated in the previous section. Let $p\colon V \to S^2$ be an oriented Seifert fibration and let $\mathscr{F}$ be a foliation of V.

Proposition 9. *Suppose that $\mathscr{F}$ has no dead end components and it is transverse to a certain fiber α of p. Then there exists an isotopy $\{\varphi_t\}$ of diffeomorphisms of V such that $\varphi_0 = \mathrm{id}$ and $\varphi_1^* \mathscr{F}$ is transverse to all the fibers of p.*

Proof. Take a small tubular neighbourhood N of α such that $\mathscr{F}$ is transverse to ∂N and that $\mathscr{F}|_N$ is a trivial foliation by meridean disks. Consider mutually disjoint, properly embedded vertical annuli $C_1, \cdots, C_{n-1}$ in $V \setminus \mathrm{Int}\, N$ such that

(3.1) C_i is transverse to ∂N,

(3.2) ∂C_i is transverse to $\mathscr{F}|_{\partial N}$,

(3.3) each component $V_1', \cdots, V_n'$ of $V \setminus (N \cup \bigcup_i C_i)$ contains exactly one exceptional fiber.

By Théorème 2 of Roussarie [13], one can further assume that each C_i is transverse to $\mathscr{F}$. Let V_i be the smooth solid torus obtained from $\mathrm{Cl}\,(V_i')$ by cutting off corners.

First of all suppose that for each V_i, $\mathscr{F}|_{\partial V_i}$ has no Reeb component. Then $\mathscr{F}|_{\partial V_i}$ is a trivial foliation by meridean circles. If not, one would obtain a null transversal. This shows $\mathscr{F}|_{V_i}$ is a trivial foliation by meridean disks. It is easy to show that in this case $\mathscr{F}$ can be made to be transverse to all the fibers of p.

Suppose to the contrary that $\mathscr{F}|_{\partial V_i}$ has Reeb components for some V_i. Then the leaf F of $\mathscr{F}|_{V_i}$ containing a compact leaf L of ∂V_i is compact. In fact F is a vertical annulus. To show this, suppose in way of contradiction that F is noncompact. Then there exists a closed transversal γ of $\mathscr{F}|_{V_i}$ passing through L. Consider the classes $[\gamma]$, $[L] \in \pi_1(V_i) \cong \boldsymbol{Z}$. Passing to an iterate of γ, if necessary, one may assume $[\gamma]=n[L]$ for some integer n. By adding a leaf curve $-nL$ to γ, we obtain a null transversal. Recall that we always assume throughout that the foliation we consider is transversely oriented. Thus F is a compact leaf. By Theorem 6 (2.1), a leaf of $\mathscr{F}$ is incompressible. This implies F is an annulus. Further by our construction, $\mathscr{F}|_{\partial V_i}$ is transverse to a certain ordinary fiber. Thus L is isotopic to an ordinary fiber. This shows F is isotopic to a vertical torus.

Consider $W = V\backslash \mathrm{Int}\, N$. W is a Seifert fibered space over D^2 and contains a horizontal surface Σ. Let K be the union of compact leaves of $\mathscr{F}|_{V_i}$. K is a compact saturated set of $\mathscr{F}$, contained in Int W. By a suitable isotopy one can assume that it is a union of fibers of p. To show this, first by a preliminary isotopy make $K \cap (\bigcup_i \partial V_i)$ to be a union of fibers. Define an equivalence relation among compact leaves of $\mathscr{F}|_{\partial V_i}$ by the following; two are equivalent if they belong to the same component of the complement of all the open Reeb components. Next define an equivalence relation among compact leaves of $\mathscr{F}|_{V_i}$; two are equivalent in case their boundary curves are one by one equivalent. There are finitely many equivalence classes. We can make by an isotopy all the compact leaves of a class to be vertical. After successive steps all of $K \cap V_i$ are made to be vertical.

By a further isotopy we may also assume that Σ is in a general position w.r.t. $\mathscr{F}$. Then we can find a minimal set of $\mathscr{F}|_\Sigma$ in $\Sigma \cap K$. By Schwarz [14], it is a closed curve (because $\mathscr{F}|_\Sigma$ is C^2). Saturating the curve along Seifert fibration, we obtain a compact leaf of $\mathscr{F}$ in Int W, which is vertical. This implies the existence of a dead end component, contrary to the hypothesis. q.e.d.

§ 4. Special case

In this section we prove Theorem 1 in the special case where there exist exactly 4 exceptional fibers, all with Seifert index (2.1).

To start with let us notice the reason why the argument of the previous sections fails in the present case. It is due to the fact that Lemma 5 (1.11) does not hold and thus the argument of case (2.14) loses its base. But the other parts of the previous argument remain valid.

Thus we can take two tori T_1 and T_2 so that they satisfy (2.10), (2.11) and (2.12). Let $a_1, a_2, \cdots, a_s$ be the points of tangency of $T_1 \cap T_2$ with $\mathscr{F}$. Let I_1^i (resp. I_2^i) be the component containing a_i, of the intersection of $T_1 \cap T_2$ with the open Reeb component containing a_i of $\mathscr{F}|_{T_1}$ (resp. $\mathscr{F}|_{T_2}$). Because the argument of case (2.13) works also in the present case, we have only to consider the case where (2.14) holds for any point a_i, that is, $I_1^i = I_2^i$. Let R be the union of all the boundary leaves of Reeb components of $\mathscr{F}|_{T_1}$ and $\mathscr{F}|_{T_2}$. It is a saturated set of $\mathscr{F}|_{T_1 \cup T_2}$. Let K be an arbitrary component of R. In what follows, we shall show the following.

(4.1) The leaf F of $\mathscr{F}$ containing K is a compact leaf.

This suffices for our purpose. For if K is vertical, then as we have shown before there exist dead end components of $\mathscr{F}$. Suppose K is horizontal. Then we can easily find another component K' of R which is transversely oriented in the opposite direction to K. This also implies the existence of dead end components.

Proof of (4.1). Notice first that in the present case $\mathscr{F}$ has a compact leaf, say C, because $\pi_1(V)$ has polynomial growth and the Betti number of V is at most 1 ([13]). As remarked before, C is horizontal. As it is incompressible, C is a torus. If not, $\pi_1(V)$ has an exponential growth.

Cut V along C. The resultant manifold $\tilde{V}$ is diffeomorphic to $T^2 \times I$. Components of $T_1 \cap \tilde{V}$ and $T_2 \cap \tilde{V}$ are respectively parallel annuli. A component of $T_1 \cap \tilde{V}$ intersects (nontrivially) a component of $T_2 \cap \tilde{V}$ transversely. This shows $V \setminus (C \cup T_1 \cup T_2)$ consists of open balls $B_1, B_2, \cdots, B_r$.

If the leaf F we are considering coincides with C, there are nothing to prove. So assume they are distinct. Consider an (angular) tubular neighbourhood N of K in F. Each boundary component $\partial_j N$ of N is contained in some ball B_i. So $\partial_j N$ is a null homotopic simple closed curve. By the incompressibility of F, $\partial_j N$ bounds a disk D_j in F. But D_j does not contain N, in which there is contained a homotopically nontrivial curve (a boundary leaf of a Reeb component). Thus F consists of N and a finite number of disks D_j. This shows F is closed. q.e.d.

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Department of Mathematics
College of Science and Technology
Nihon University
Kanda-Surugadai, Chiyoda-ku, Tokyo 101
Japan