Advanced Studies in Pure Mathematics 5, 1985
Foliations
pp. 121–134

Foliated Cobordisms of PA Foliations

Tadayoshi Mizutani

Introduction

Let M be an n-dimensional closed oriented C^∞ manifold and $\mathscr{F}$ a transversely oriented codimension one, C^∞ foliation of M. The pair $(M, \mathscr{F})$ determines an n-cycle of $B\bar{\Gamma}_1$, where $B\bar{\Gamma}_1$ is Haefliger's classifying space for C^∞, $\bar{\Gamma}_1$-structures. Two such foliations are said to be *homologous* if the corresponding n-cycles in $B\bar{\Gamma}_1$ are homologous. In [6], we proved $(M, \mathscr{F})$ is homologous to a finite union of foliated circle bundles over the $(n-1)$-torus if $\mathscr{F}$ is almost without holonomy. In this paper, we show that a PA foliation is also homologous to a finite union of foliated circle bundles over the $(n-1)$-torus. A PA *foliation* $\mathscr{F}$ is a foliation of a closed manifold which satisfies the following two conditions; (P) Each leaf of $\mathscr{F}$ has polynomial growth, and (A) The holonomy group of each leaf is abelian (see [11]). An almost without holonomy foliation of a closed manifold is a PA foliation ([4]). The Godbillon-Vey class of a PA foliation is known to be zero ([2], [11]).

In the case $\dim M=3$, by a result of Thurston ([9]), our result says that $(M, \mathscr{F})$ is *foliated cobordant* to a finite union of S^1-bundles over T^2 (see also [6]). Thus, it is an important and interesting problem to prove that every foliated S^1-bundle over T^2 is foliated cobordant to zero. A partial answer to this problem is found in a paper of Tsuboi ([10]).

The main tools we use are (1) The Nishimori-Tsuchiya decomposition (the N-T decomposition for short) of PA foliations due to Tsuchiya ([11], [12]) and (2) Foliated J-bundles for a certain type of $\bar{\Gamma}_1$-structures which were developed in [5, 6]. In Sections 1 and 2, we review the definitions of foliated J-bundles and the N-T decomposition for PA foliations. We refer the reader to [5], [6], [7], [11] and [12] for further details and the related results of these sections.

In Section 3, we introduce auxiliary foliated spaces, pillars and counter-staircases, and using them we prove our theorem in Section 4.

The author wishes to express his gratitude to N. Tsuchiya and T. Tsuboi for stimulating conversations and also to all other members of TIT Saturday Seminar for their interest to this problem.

Received November 1, 1983.

§ 1. Foliated J-bundles associated with $\bar{\Gamma}_1$-structures

Let M be a connected CW-complex and let L^+, L^- be subcomplexes of M.

Definition. A family of foliated bundles and foliated bundle isomorphisms (E, E^+, E^-, b^+, b^-) is called a *foliated J-bundle* over (M, L^+, L^-) if

(1) E is a foliated $\mathring{I}$-bundle over M and E^+ (resp. E^-) is a foliated I_+- (resp. I_--) bundle over L^+ (resp. L^-), where $\mathring{I}=(-1, +1)$, $I_+=(-1, +1]$ and $I_-=[-1, +1)$, and

(2) b^+ and b^- are isomorphisms of foliated $\mathring{I}$-bundles;

$$b^+ : \mathring{E}^+ \longrightarrow E|_{L^+}, \qquad b^- : \mathring{E}^- \longrightarrow E|_{L^-},$$

where $\mathring{E}^+$ (resp. $\mathring{E}^-$) is the foliated $\mathring{I}$-bundle associated with E^+ (resp. E^-).

Let $\boldsymbol{E}$ be the space obtained from the disjoint union of E, E^+ and E^- by identifying $\mathring{E}^+$ (resp. $\mathring{E}^-$) with $E|_{L^+}$ (resp. $E|_{L^-}$) by the isomorphism b^+ (resp. b^-). There is a canonical projection π: $\boldsymbol{E} \to M$ and $\boldsymbol{E}$ has a codimension one 'foliation' $\mathscr{F}_{\boldsymbol{E}}$ defined by the foliated bundle structure. E is naturally a subspace of $\boldsymbol{E}$ and called the principal part of the foliated J-bundle. Sometimes, we call the triple $(\boldsymbol{E}, \pi, M)$ a foliated J-bundle. The total holonomy of E is called the principal holonomy of $\boldsymbol{E}$.

Let $(M, \mathscr{F})$ be a $\bar{\Gamma}_1$-structure and $(\boldsymbol{E}, \pi, M)$ be a foliated J-bundle over (M, L^+, L^-). If there exists a section $s : M \to \boldsymbol{E}$ such that $\mathscr{F}=s^*\mathscr{F}_{\boldsymbol{E}}$, then $(\boldsymbol{E}, \pi, M)$ is called a *foliated J-bundle associated with* $(M, \mathscr{F})$. A foliation $(M, \mathscr{F})$ which is a foliated I-bundle over K has a canonical associated foliated J-bundle. In fact, let $p : M \to K$ be the projection and $\boldsymbol{E}=p^*M$ the pull-back of $p : M \to K$ by p. $\boldsymbol{E}$ is given by $\{(x, y) \in M \times M;\ p(x)=p(y)\}$. Let $\mathscr{F}_{\boldsymbol{E}}$ denote $p^*\mathscr{F}$ and define the section $s : M \to \boldsymbol{E}$ by $s(x) =(x, x)$. Then we have $s^*\mathscr{F}_{\boldsymbol{E}}=\mathscr{F}$.

Also, if $(M, \mathscr{F})$ is a foliation of compact manifold whose boundary components are leaves and if each interior leaf of $\mathscr{F}$ has no holonomy, it has an associated foliated J-bundle ([5], [11] appendix).

Here we recall the definition of the modification of a foliated interval bundle and quote a lemma from [6].

We call a foliated I-bundle over L *layered* if in the total space, the foliation on some neighbourhoods of $L\times\{-1\}$ and $L\times\{+1\}$ is trivial (that is isomorphic to $\{L\times\{\text{const.}\}\}$). If the foliation is trivial on some neighbourhood of either $L\times\{-1\}$ or $L\times\{+1\}$, we call it *half-layered.* Let $E \to L$ be a foliated $[-1, +1)$-bundle. A *modified foliated I-bundle* $\hat{E}$ of E is a half-layered foliated I-bundle over L which is unchanged outside a small neighbourhood of $L\times\{+1\}$ in E. More precisely, $\hat{E} \to L$ satisfies (i) the underlying space of $\hat{E}$ is homeomorphic to the associated I-bundle of E, (ii) the foliation of $\hat{E}$ coincides with that of E outside a small neighbourhood of $L\times\{+1\}$ and (iii) the foliation of $\hat{E}$ is isomorphic to $\{L\times\{\text{const.}\}\}$ in a smaller neighbourhood of $L\times\{+1\}$. One can also define a modified foliated I-bundle for a foliated $(-1, +1]$-bundle and for a foliated $(-1, +1)$-bundle. For the latter, the modified foliated I-bundle should be layered.

In [6], we proved the following

Lemma 1. *If the total holonomy of a foliated half open interval bundle $E\to L$ is abelian, then there exists a modified foliated I-bundle $\hat{E}\to L$ which is half layered. If the total holonomy of a foliated $\mathring{I}$-bundle is contained in a 1-parameter subgroup generated by a smooth vector field on $\mathring{I}$, then there exists a modified foliated I-bundle which is layered.*

§ 2. Units, decomposition and a theorem of Tsuchiya

Let $(M, \mathscr{F})$ be a $\bar{\Gamma}_1$-structure of a compact connected manifold possibly with corner. We assume that ∂M is divided by the corner into two parts, $\partial_{\tan}M$ and $\partial_{\mathrm{tr}}M$, and $\mathscr{F}$ is a non-singular foliation on a neighbourhood of ∂M, which is tangent to $\partial_{\tan}M$ and transverse to $\partial_{\mathrm{tr}}M$. If $\mathscr{F}$ is trivial (product foliation) near $\partial_{\mathrm{tr}}M$, $(M, \mathscr{F})$ is called a *unit*. In the case $\mathscr{F}$ has no singularity (that is, if $\mathscr{F}$ is a foliation of M), $(M, \mathscr{F})$ is called a *non-singular* unit.

Definition. Let $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ be $\bar{\Gamma}_1$-structures on manifolds. A smooth immersion $\varphi : M'\to M$ is called a *$\bar{\Gamma}_1$-structure preserving* if the pull-back $\varphi^*\mathscr{F}$ coincides with $\mathscr{F}'$.

Now, we define a decomposition of a $\bar{\Gamma}_1$-structure.

Definition. Let $(M, \mathscr{F})$ be a $\bar{\Gamma}_1$-structure of an n-dimensional closed manifold. A pair (Δ, φ) is called a *decomposition* of $(M, \mathscr{F})$ if $\Delta=\{(M_i, \mathscr{F}_i), i=1, \cdots, m\}$ is a finite family of n-dimensional units and φ is a $\bar{\Gamma}_1$-structure preserving immersion of the disjoint union $\bigcup_{i=1}^m (M_i, \mathscr{F}_i)$ to $(M, \mathscr{F})$ such that

(1) for each i, $\varphi|_{M_i-\partial_{\tan}M_i}$ is an embedding,

(2) if $i\neq i'$, then $\varphi(\mathrm{Int}\,(M_i))\cap\varphi(\mathrm{Int}\,(M_{i'}))=\emptyset$,

(3) $\bigcup_{i=1}^m \varphi(M_i)=M$.

Now, we recall the definition of three kinds of units which appear in the N-T decomposition (see [11], [12]).

I. *Regular staircase.* Let K be a compact manifold (possibly with boundary) and N a codimension one closed submanifold of Int K with trivial normal bundle, which does not separate K. Let $C(K, N)$ denote the manifold with boundary obtained from $K-N$ by attaching two copies N_1 and N_2 of N as boundary. $C(K, N)$ is called the *cut of K along N*. Let $f:[0, 1]\to[0, \delta]$ be a diffeomorphism such that $f(t)<t$ for $t\neq 0$, where $0<\delta<1$. Let $X(K, N, f)$ denote the manifold with corner, which is the quotient space of $C(K, N)\times[0, 1]$ by the equivalence relation $\sim$ which is defined by $(x_1, t)\sim(x_2, f(t))$, where $x_1\in N_1$ and $x_2\in N_2$ are the same point in N. $X(K, N, f)$ naturally has a foliation $\mathscr{F}(K, N, f)$ which is induced from the product foliation $\{C(K, N)\times\{t\}; t\in[0, 1]\}$.

Definition. A non-singular unit $(S, \mathscr{F})$ is called a *regular staircase* if it is diffeomorphic to some $(X(K, N, f), \mathscr{F}(K, N, f))$. Let $h:(X(K, N, f), \mathscr{F}(K, N, f))\to(S, \mathscr{F})$ be a foliation preserving diffeomorphism; then $C(S)=h(C(K, N)\times\{1\})$, $W(S)=h(N_2\times[\delta, 1])$, $F(S)=h(C(K, N)\times\{0\})$ and $D(S)=h(\partial K\times[0, 1])$ are called the *ceiling*, the *wall*, the *floor* and the *door* of $(S, \mathscr{F})$, respectively. f is called the *total holonomy* of $(S, \mathscr{F})$.

In this paper, we deal only with *regular* staircases.

II. *Abelian room.*

Definition. A non-singular unit $(R, \mathscr{G})$ is called an *abelian room* if it has a structure of foliated I-bundle with abelian total holonomy. $D(R)=\partial_{\mathrm{tr}}R$ is called the *door* of $(R, \mathscr{G})$.

III. *Hall.*

Definition. A non-singular unit $(H, \mathscr{K})$ is called a *hall* if the following are satisfied;

(1) the corners of H are all convex,

(2) each connected component D of $\partial_{\mathrm{tr}}H$ is diffeomorphic to $C\times I$, where C is a connected component of ∂D, and

(3) all leaves except the boundary leaves have trivial holonomy. $D(H)=\partial_{\mathrm{tr}}H$ is again called the *door* of $(H, \mathscr{K})$.

Let $(M, \mathscr{F})$ be a foliation of a closed manifold. A decomposition $(\Delta=\{(M_i, \mathscr{F}_i), i=1, \cdots, m\}, \varphi)$ is called an *N-T decomposition* if the following conditions (1) $\sim$ (5) are satisfied;

(1) each $(M_i, \mathscr{F}_i)$ is either a regular staircase, an abelian room or a hall,

(2) for each i, and for each connected component D of the door of $(M_i, \mathscr{F}_i)$, there is a staircase $(M_j, \mathscr{F}_j)$ of Δ such that $\varphi(D)$ is contained in $\varphi(W(M_j))$,

(3) if $(M_i, \mathscr{F}_i)$ and $(M_j, \mathscr{F}_j)$ are distinct regular staircases, then the images $\varphi(W(M_i))$ and $\varphi(W(M_j))$ are disjoint.

Let $\mathscr{S}(\Delta)$ be the set of staircases in Δ. If there is a sequence $M_i=$

$M_{i_0}, M_{i_1}, \cdots, M_{i_a}=M_j$ of elements of $\mathscr{S}(\Delta)$ such that $\varphi(W(M_{i_k}))\cap \varphi(D(M_{i_{k+1}}))\neq\emptyset$, for $k=0, \cdots, a-1$, we write $M_i<M_j$.

(4) the relation $<$ is a partial order in $\mathscr{S}(\Delta)$,

(5) if $(M_i, \mathscr{F}_i)$ is a staircase in Δ, then the image $\varphi(C(M_i))$ of the ceiling of M_i has the trivial holonomy in $(M, \mathscr{F})$.

One of the main theorems of [11] is the following decomposition theorem for a PA foliation.

Theorem 1 (Tsuchiya [11]). *Let $(M, \mathscr{F})$ be a codimension one foliation of a closed manifold. Then $\mathscr{F}$ is PA, that is, each leaf of $\mathscr{F}$ has polynomial growth and the holonomy group of each leaf is abelian, if and only if $(M, \mathscr{F})$ admits an N-T decomposition.*

Remark. We often identify a unit in the N-T decomposition with its image by φ, although it may not be a one-to-one map on $\partial_{\tan}$.

An N-T decomposition was called a Nishimori decomposition in [11]. In [12], Tsuchiya uses 'N-T decomposition' in a little wider sense.

§ 3. Pillars and counter-staircases

In this section, we define two kinds of new units which we will use to modify a given PA foliation. They are (crooked) pillars and counter-staircases.

Let K be a compact connected oriented manifold with boundary, of dimension greater than 1. If L is a component of ∂K, let $K(L)$ denote the space which is obtained from K by attaching the cone $C(L)$ of L along L. We call $K(L)$ the *manifold with cone singularity* obtained from K by taking a cone at L. Let $\bar{K}$ denote the space obtained from K by taking a cone at each connected component of ∂K. Such $\bar{K}$ is called a *closed manifold with cone singularity.* Note that $\bar{K}$ has a fundamental cycle.

Let N be a closed codimension one submanifold of K which is contained in Int K. Suppose that N has a collar neighbourhood and does not separate K. Let $C(\bar{K}, N)$ be the cut of $\bar{K}$ along the submanifold N. The 'boundary' of $C(\bar{K}, N)$ is a disjoint union of two copies of N; N_0 and N_1. Consider the space $\bar{C}$ obtained from $C(\bar{K}, N)$ by taking cones at N_0 and N_1 separately. Inflating the collar of N_0, we regard $\bar{C}$ as the union

$$(\bar{C}-C(N_0))\cup(N\times[0, 1])\bigcup_{N\times\{0\}=N_0} C(N_0).$$

Suppose there is given a function g of $\bar{C}$ to $[0, 1]$ which is the projection to $[0, 1]$ on $N\times[0, 1]$, $g(C(N_0))=0$ and $g(\bar{C}-C(N_0))=1$. g defines a trivial $\bar{\Gamma}_1$-structure on $\bar{C}$. We want to have an explicit $\bar{\Gamma}_1$-structure which bounds the one on $\bar{C}$. For this, it is convenient to make the following definition of a pillar.

Let N be an $(n-2)$-dimensional closed manifold $(n>2)$. Consider the product foliation of $N\times[0, 1]$, which is defined by the projection $p: N\times[0, 1]\to[0, 1]$.

Definition. A $\bar{\Gamma}_1$-structure on an n-dimensional compact simplicial complex W, which is defined by a map $\tilde{p}: W\to[0, 1]$, is called a *pillar* for p, if the following conditions are satisfied;

(1) $N\times[0, 1]$ is a subspace of W and $\partial W=\tilde{p}^{-1}(0)\cup\tilde{p}^{-1}(1)\cup N\times[0, 1]$, (homologically),

(2) $\tilde{p}$ is an extension of p, and

(3) except a finite number of points in $(0, 1)$, each inverse image $\tilde{p}^{-1}(x)$, $x\in[0, 1]$ is an $(n-1)$-dimensional compact manifold with cone singularity and has a productly foliated neighbourhood.

Remarks. A typical example of a pillar is the mapping cylinder W of $p: N\times[0, 1]\to[0, 1]$ with the natural projection $\tilde{p}: W\to[0, 1]$.

For a generic interval $J\subset[0, 1]$, $\tilde{p}^{-1}(J)$ is also a pillar for the projection $N\times J\to J$.

The 'boundary' of a pillar is a closed manifold with cone singularity.

Now, we return to the problem of constructing a $\bar{\Gamma}_1$-structure on W which bounds $p: \bar{C}\to[0, 1]$, satisfying $\partial W=\bar{C}$.

Consider the space $Y=C(\bar{K}, N)\times[0, 1]$. We are going to define W to be the union of Y and a mapping cylinder M_h of some map $h: C(\bar{K}, N)\to[0, 1]$, in which $C(\bar{K}, N)\times\{0\}$ in Y and $\partial M_h\approx C(\bar{K}, N)$ are identified. Let $C(K, N)$ be the cut of K along N. Then the space $Z=C(K, N)\times[0, 1]$ is a (non-singular) manifold with corner. Note that Y is obtained from Z by attaching a suitable mapping cylinder along each $L\times[0, 1]$, where L is a component of ∂K. Let $(N_0\times[0, 1])\times[0, \varepsilon]$, $(0<\varepsilon)$ be a closed collar neighbourhood of $N_0\times[0, 1]$ in Z. Choose a map $r: Z\to[0, 1]$, which satisfies the following conditions;

(1) on $(N_0\times[0, 1])\times[0, \varepsilon/2]$, r is the projection to the second factor,

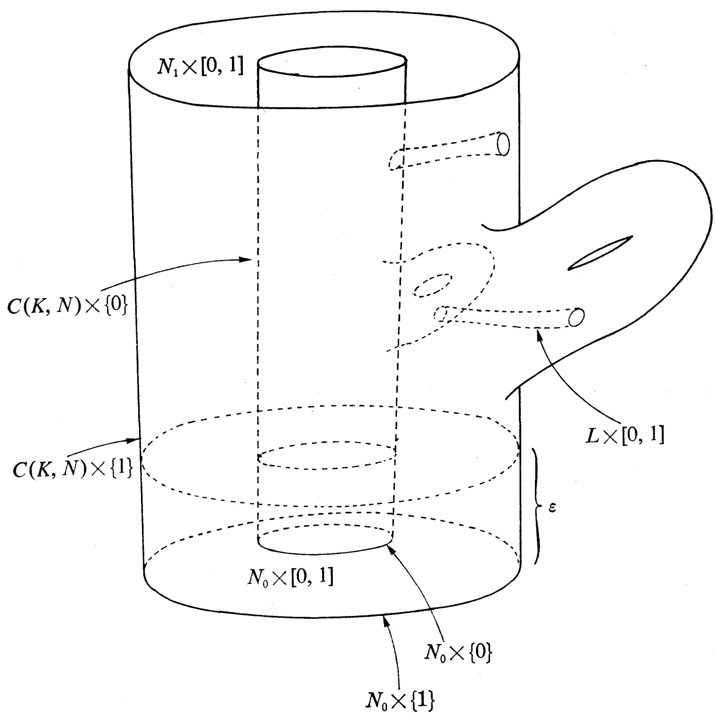
(2) on $N_0\times\{1\}\times[0, \varepsilon]$, r is the projection to $[0, \varepsilon]$ followed by an identification of $[0, \varepsilon]$ with $[0, 1]$,

(3) on $C(K, N)\times\{0\}$, r is a Morse function such that $r(N_0)=0$, $r(N_1)=1$ and $r(L)$ is a constant for each component of ∂K,

(4) for each $L\times[0, 1]$, $L\subset\partial K$, r is a non-singular Morse function, and

(5) on $\partial Z-(\partial K\times[0, 1]\cup N_0\times\{1\}\times[0, \varepsilon]\cup C(K, N)\times\{0\})$, r is identically equal to 1. (See the figure below).

Such a map clearly exists. We define W to be the disjoint union of Y and the mapping cylinder of $r|C(K, N)\times\{0\}$, in which $C(\bar{K}, N)\times\{0\}$'s in both spaces are identified in an obvious way. Then it is easy to see that there exists a map $\tilde{p}: W\to[0, 1]$, which is an extension of r and a pillar for $p=r|_{N_0\times\{1\}\times[0,\varepsilon]}$.



$Z=C(K, N)\times[0, 1]$

We remark here that the definition of a staircase can be extended to the case where the floor is a manifold with cone singularity, as long as the singular points are in the interior and finitely many. A similar extension is also possible in the case of an abelian room or a hall. Hereafter, we use these words in these extended meanings. Hence, in general, the boundary of a units is a manifold with cone singularity. Now, let S be a staircase and D a component of the door of S. D is homeomorphic to a product manifold $N\times[0, 1]$, for some closed manifold N. If we attach a pillar W for $N\times[0, 1]\to[0, 1]$, for each D, we obtain another space with $\bar{\Gamma}_1$-structure. We call such a space with $\bar{\Gamma}_1$-structure a *closed staircase.* Correspondingly, we define the floor, the ceiling and the wall of a closed staircase in a natural way. A *closed room* and a *closed hall* are also defined in the same way. Although these closed units may contain singularity much more complicated than that of cone singularity, the 'boundary' of these spaces contain only cone singularity, if any.

Let C_w denote the union of the ceiling and the wall of a closed staircase. As remarked above, it is a closed manifold with cone singularity. Also, it is easily seen that the $\bar{\Gamma}_1$-structure restricted to C_w is defined by a projection $p: C_w \to S^1$.

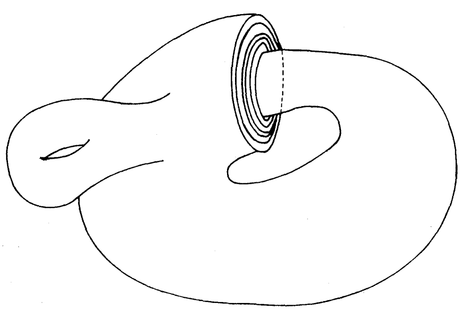
Definition. Let $n>2$. A $\bar{\Gamma}_1$-structure on an n-dimensional compact simplicial complex Q, which is defined by $\tilde{p}: Q \to S^1$ is called a *crooked pillar* for $p: C_w \to S^1$, if the following conditions are satisfied;

(1) $\partial Q = C_w$, (homologically),

(2) $\tilde{p}$ is an extension of p, and

(3) except for a finite number of points in S^1, each inverse image $\tilde{p}^{-1}(x)$, $x \in S^1$, is an $(n-1)$-dimensional compact manifold with cone singularity and has a productly foliated neighbourhood in Q.

Given a closed staircase S whose (ceiling) $\cup$ (wall) is C_w, one can construct a crooked pillar for C_w roughly as in the following way. Let U be a closed neighbourhood of C_w in S. On U, the $\bar{\Gamma}_1$-structure is given by a projection to S^1. We choose U so that $\partial' U = \partial U - C_w$ is in general position with respect to the $\bar{\Gamma}_1$-structure, in the sense that each level set is of codimension one and the number of the critical points of $\partial'' U = \partial' U -$ (*singular points*) $\to S^1$ is finite. Attaching the mapping cylinder of $\partial' U \to S^1$



A Crooked Pillar

to U, one obtains a crooked pillar Q for C_w. Observe that in the above construction, $\tilde{p}^{-1}(p(\text{ceiling}))$ is taken as an $(n-1)$-dimensional compact manifold with cone-singularity which has a productly foliated neighbourhood. Thus, if we cut Q along this space, we obtain a pillar similar to what we constructed before.

Let S be a staircase whose total holonomy is $f: [0, 1] \to [0, \delta]$, $0<\delta<1$. Extend f to a diffeomorphism $\bar{f}: [0, 2] \to [0, 2]$, which satisfies;

(1) $\bar{f}=f$ on $[0, 1]$,

(2) $\bar{f}(t)<t$ on $(0, 3/2)$,

(3) $\bar{f}(t)=t$ on $[3/2, 2]$.

$\bar{f}$ defines a foliated $[0, 2]$-bundle over K, where K is the floor of S. Let T denote the total space. S is naturally a subspace of T.

Definition. The complement of Int S in T is called a *counter-staircase* of S and denoted by $\hat{S}$.

Note that $\hat{S}$ has a structure of a staircase except that the floor is thickened.

Let $\bar{S}$ be a closed staircase obtained by attaching some pillars to S. If we attach suitable trivial pillars to $\hat{S}$, we obtain another closed staircase $\bar{\hat{S}}$ such that the union $\hat{S} \cup \bar{\hat{S}}$ is a closed room. We also call $\bar{\hat{S}}$ the *counter-staircase* of $\bar{S}$. Observe that the counter-staircases are constructed if the $\bar{\Gamma}_1$-structure near $C_w=(ceiling) \cup (wall)$ of S (or $\bar{S}$) is given.

We end this section with the following

Lemma 2. *Let $\bar{S}$ be an n-dimensional closed staircase. Choose any crooked pillar Q whose $\bar{\Gamma}_1$-structure near the boundary coincides with that of $\bar{S}$ near $C_w=(ceiling) \cup (wall)$. If $\bar{\hat{S}}$ is a counter-staircase of $\bar{S}$, then $\bar{\hat{S}} \bigcup_{C_w} Q$ represents an n-cycle of $B\bar{\Gamma}_1$ and it is homologous to zero.*

Proof. Since the boundary of $Z=\bar{\hat{S}} \bigcup_{C_w} Q$ is the floor of $\bar{\hat{S}}$ and the $\bar{\Gamma}_1$-structure in its neighbourhood is trivial, Z represents an n-cycle of $B\bar{\Gamma}_1$. Decompose Z into two parts; $Z=U \cup X$, where U is the maximal collar neighbourhood of the boundary of Z, in which the $\bar{\Gamma}_1$-structure is productly foliated and X is the complement of Int U. Then $L=U \cap X$ is the unique 'leaf' in Z with a non-trivial holonomy. Since both $\bar{\Gamma}_1$-structures on Q and on $\bar{\hat{S}}-U$ are defined by a projection onto S^1 and they coincide in their intresection, the $\bar{\Gamma}_1$-structure on $\mathrm{Int}\, X=Z-U$ is also defined by a projection to S^1. Let $q: \widetilde{\mathrm{Int}\, X} \to \mathrm{Int}\, X$ be the infinite cyclic covering associated with this projection. Consider the pull-back $\bar{\Gamma}_1$-structure on $\widetilde{\mathrm{Int}\, X}$ which is induced by q from that of Int X. It is a trivial $\bar{\Gamma}_1$-structure and its 'leaf space' is identified with $\boldsymbol{R}$. Since an element of the covering transformation group of $\widetilde{\mathrm{Int}\, X}$ $(\simeq \boldsymbol{Z})$ maps each 'leaf' onto another 'leaf', it induces a diffeomorphism of $\boldsymbol{R}$. Choose an identification $\boldsymbol{R}$ with $(0, 3/2)$ and let $[\varepsilon, 2]$ $(1<\varepsilon<3/2)$ be a transverse arc in Z, where 2 is on ∂Z and $3/2$ is on L. If ε is sufficiently near $3/2$, the holonomy map is defined on

$[\varepsilon, 2]$. Since this action of the holonomy map on $(\varepsilon, 3/2)$ is conjugate to the above $\boldsymbol{Z}$-action on $\boldsymbol{R}\approx(0, 3/2)$ near $3/2$, we have a well-defined $\boldsymbol{Z}$-action on $(0, 2]$.

Since U is homeomorphic to $L\times I$, U has an infinite cyclic covering $\tilde{U}$ corresponding to the generator of the holonomy of L. Let $\tilde{Z}$ be the canonical infinite cyclic covering of Z, which is the union of $\tilde{U}$ and $\widetilde{\operatorname{Int} X}$. Put $\boldsymbol{E}=\tilde{Z}\times_{\boldsymbol{Z}}(0, 2]$; the quotient space of $\tilde{Z}\times(0, 2]$ by the diagonal $\boldsymbol{Z}$-action. $\boldsymbol{E}$ is the total space of a foliated bundle over Z. In this situation, we have a section $s: Z\to\boldsymbol{E}$ such that the pull-back of the 'foliation' of $\boldsymbol{E}$ by s coincides with the original Γ_1-structure of Z. More precisely, let $\beta:\widetilde{\operatorname{Int} X}\to(0, 3/2)$ and $\gamma:\tilde{U}\to[3/2, 2]$ be the maps to the leaf spaces. Define $\tilde{s}:\tilde{Z}=\tilde{U}\cup\tilde{X}\to\tilde{Z}\times[0, 2]$ by

$$\tilde{s}(x)=\begin{cases}(x, \beta(x)), & x\in\widetilde{\operatorname{Int} X},\\ (x, \gamma(x)), & x\in\tilde{U}.\end{cases}$$

$\tilde{s}$ is equivariant and gives a desired section $s: Z\to\boldsymbol{E}$. Let s_0 be the section $Z\to Z\times\{2\}\subset\boldsymbol{E}$, then s is homotopic to s_0, keeping the boundary fixed and the Γ_1-structure on $s_0(Z)$ is obviously trivial. This completes the proof.

§ 4. Proof of the Theorem.

In this section, we shall prove our theorem:

Theorem 2. *Let $(M, \mathscr{F})$ be a smooth, transversely oriented foliation of codimension one of an n-dimensional smooth closed manifold. Suppose that each leaf of $\mathscr{F}$ has polynomial growth and the holonomy of each leaf is abelian. Then $(M, \mathscr{F})$ is homologous to a finite union of foliated S^1-bundles over T^{n-1} as an n-cycle of $B\bar{\Gamma}_1$.*

By a theorem of J. Mather, $B\bar{\Gamma}_1$ is known to be 2-connected and nothing is left to prove if $n\leqq 2$. Thus, hereafter, we assume $n>2$.

We divide the proof into three steps.

Step 1. By the theorem of Tsuchiya (Theorem 1), we have an N-T decomposition of $(M, \mathscr{F})$ into a union of units. The set of staircases of this decomposition has a partial ordering. Suppose S_0 is a minimal staircase (with respect to this ordering). Let K_0 denote the floor of S_0. K_0 is an $(n-1)$-dimensional closed manifold. Let C_w be the union of the ceiling and the wall of S_0. C_w is a submanifold of M homeomorphic to K_0. There exists a crooked pillar $(Q, \mathscr{F}_Q)$ such that $\mathscr{F}_Q$ restricted to a neighbourhood of ∂Q is isomorphic to $\mathscr{F}$ restricted near C_w in S_0. From the definition of a crooked pillar, we can assume if necessary, the saturations in Q of the components of the door in C_w have product structures except one. Let $\hat{S}_0$ be the counter staircase of S_0. Then by Lemma 2 in Section 3, $\hat{S}_0 \bigcup_{C_w} Q$ represents a null homologous n-cycle. Let $X(M, C_w)$ denote the cut of M along C_w. Its boundary $C_0 \cup C_1$ is a disjoint union of two copies of C_w. For definiteness, suppose the ceiling of S_0 in $X(M, C_w)$ is C_0. To $X(M, C_w)$, attach $\hat{S}_0$ along C_0 and Q along C_1. In this way, we obtain a space which is a manifold with singularity equipped with a $\bar{\Gamma}_1$-structure. It clearly represents an n-cycle in $B\bar{\Gamma}_1$ which is homologous to $(M, \mathscr{F})$. Note that if we regard $S_0 \cup \hat{S}_0$ as a closed room, this new space has a decomposition by units with one less staircases than the original decomposition of $(M, \mathscr{F})$.

Repeating the above modifications for all the minimal staircases $\{S_0, S_1, \cdots, S_{k_0}\}$, we obtain a space $M^{(1)}$ and a $\bar{\Gamma}_1$-structure $\mathscr{F}^{(1)}$ on it such that $(M^{(1)}, \mathscr{F}^{(1)})$ is homologous to $(M, \mathscr{F})$. Next, take a minimal staircase in $\mathscr{S}-\{S_0, \cdots, S_{k_0}\}$, where $\mathscr{S}$ is the set of staircases of the decomposition of $(M, \mathscr{F})$. If $S_0^{(1)}$ is such one, then in $M^{(1)}$, the door of $S_0^{(1)}$ is already filled up by some (crooked) pillars. The union of $S_0^{(1)}$ and these (crooked) pillars form a closed staircase. Then, there is another crooked pillar extending the $\bar{\Gamma}_1$-structure of a neighbourhood of (*ceiling*) $\cup$ (*wall*) is $S_0^{(1)}$ and one can continue a similar modification as in the case of minimal staircases.

Inductively, we can accomplish these modifications for all the staircases in $\mathscr{S}$. After these modifications, we finally obtain a space M^z and a $\bar{\Gamma}_1$-structure $\mathscr{F}^z$ on M^z which represents an n-cycle of $B\bar{\Gamma}_1$, homologous to $(M, \mathscr{F})$. $(M^z, \mathscr{F}^z)$ has a decomposition by closed rooms and closed halls and without any (closed) staircases.

Since a pillar has an associated foliated J-bundle, it is easy to see that these two kinds of closed units have associated foliated J-bundles.

Step 2. In this step, we will make modifications in foliated J-bundles.

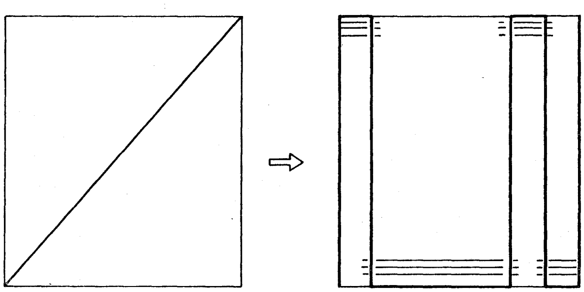
First, note that the union of the associated foliated J-bundles of the units in $(M^z, \mathscr{F}^z)$ gives a foliated microbundle for $(M^z, \mathscr{F}^z)$. Let Ξ denote the total space of the foliated microbundle obtained in this way. There is a section $\sigma : M^z \to \Xi$ such that the pull-back of the 'foliation' of Ξ by σ is $\mathscr{F}^z$, or in other words, the foliation of Ξ restricted to $\sigma(M^z)$ is identified with $\mathscr{F}^z$. We change σ by a homotopy in Ξ. Then, we clearly have a $\bar{\Gamma}_1$-structure homotopic to $(M^z, \mathscr{F}^z)$. This is done by makng homotopies over each unit (a closed room or a closed hall) of $(M^z, \mathscr{F}^z)$. See also [6] for a similar argument.

We begin by indicating the modifications over the closed rooms.

(1) *Modification over rooms.* First, consider the case when $(R, \mathscr{G})$ is a foliated I-bundle (with abelian holonomy) over a closed manifold L. R is diffeomorphic to $L\times I$, $I=[0, 1]$, and $\mathscr{G}$ is transverse to the I-factor. The total space E of the associated foliated J-bundle of $(R, \mathscr{G})$ is diffeomorphic to $R\times I=L\times I\times I$ and the section is given by $\sigma: L\times I\rightarrow L\times I\times I$, $\sigma(x, t)=(x, t, t)$, (see section 1). Let $L\times N(+1)$ (resp. $L\times N(-1)$) denote a small collar neighbourhood of $L\times\{+1\}$ (resp. $L\times\{-1\}$) in $R=L\times I$. Since the total holonomies of E, $E|_{L\times N(+1)}$ and $E|_{L\times N(-1)}$ are abelian, we make modifications of these foliated bundles (Lemma 1), obtaining three foliations $\mathscr{F}_0$, $\mathscr{F}_+$ and $\mathscr{F}_-$ on each modified space, respectively. These modifications should be made so that $\mathscr{F}_0$ is layered and $\mathscr{F}_+$ and $\mathscr{F}_-$ are half-layered. More precisely, we leave the foliation on $E|_{L\times N(+1)}$ (resp. $E|_{L\times N(-1)}$) unchanged near $L\times N(+1)\times\{+1\}$ (resp. $L\times N(-1)\times\{-1\}$). On $E|_{L\times N(+1)}$ and $E|_{L\times N(-1)}$, we replace the foliation $\mathscr{F}_0$ of E by $\mathscr{F}_+$ and $\mathscr{F}_-$, respectively. Note that $\mathscr{F}_+$ (resp. $\mathscr{F}_-$) is compatible with $\mathscr{F}_0$ except on a small neighbourhood of $L\times N(+1)\times\{-1\}$ (resp. $L\times N(-1)\times\{+1\}$) and we may assume the foliation of the neighbourhood of the image of σ coincides with the original one.

By a homotopy which fixes the boundary of $\sigma(R)$ and is indicated in the figure below, we obtain a union of foliated I-bundles over L layered or half-layered (with abelian total holonomies), which is homologous relative to the boundary, to $(R, \mathscr{G})$.

In the general case, $(R, \mathscr{G})$ is a closed room with pillars. Since $\bar{\Gamma}_1$-structures on pillars are defined by maps to $[0, 1]$ and have canonical associated foliated J-bundles, one sees that there is no problem to proceed as above. Thus, in this case too, we obtain a union of foliated I-bundles over $(n-1)$-dimensional closed manifolds with cone singularity, which is homologous relative to the boundary, to $(R, \mathscr{G})$.



Modification in a room.

(2) *Modification over Halls.* Let $(H, \mathscr{H})$ be a closed hall and $\pi: E\rightarrow H$ be the associated foliated J-bundle. H is a union of a manifold

(with corner) and pillars; $H=H_0\cup P_1\cup\cdots\cup P_k$. Thus $(H,\mathscr{K})$ is decomposed as follows;

$$(H,\mathscr{K})=(H_0,\mathscr{K}_0)\cup(P_1,\mathscr{K}_1)\cup\cdots\cup(P_k,\mathscr{K}_k),$$

where $\mathscr{K}_0$ is a foliation of a non-singular hall and $\mathscr{K}_i$ $(i=1,\cdots,k)$ is a $\bar{\Gamma}_1$-structure on a pillar P_i. Let $(\boldsymbol{E}_0,\boldsymbol{F}_0)$ be the foliated J-bundle associated to $(H_0,\mathscr{K}_0)$. As in the case of a closed room, the associated foliated J-bundle of $(H,\mathscr{K})$ is a union of $(\boldsymbol{E}_0,\boldsymbol{F}_0)$ and those of P_i's and it is written by $(\boldsymbol{E},\boldsymbol{F})$. We consider two cases separately.

Case 1. The rank of the total holonomy of each $\boldsymbol{E}_0|_L$ is 1, where L is a connected component of $\partial_{\tan}H_0$.

Case 2. There exists a component L of $\partial_{\tan}H_0$ such that the rank of the total holonomy of $\boldsymbol{E}_0|_L$ is greater than 1.

In Case 2, as we proved in [5, Lemma 3], the total holonomies of $\boldsymbol{E}_0|_{\mathrm{Int}\,H_0}$ and $\mathring{\boldsymbol{E}}_0|_L$ $(L\subset\partial_{\tan}H_0)$ are contained in 1-parameter subgroups generated by smooth vector fields on $(-1,+1)$.

Thus one can make modified foliated I-bundles. Consequently, proceeding as in the case of a closed room, we have a union of (half) layered foliated I-bundles over $(n-1)$-dimensional closed manifolds with cone singularity, to which the original $\bar{\Gamma}_1$-structure $(H,\mathscr{K})$ is homologous relative to the boundary.

In Case 1, the total holonomy of $\boldsymbol{E}_0|_{\mathrm{Int}\,H_0}$ may not be contained in a 1-parameter subgroup of a vector field. But, in this case, the argument of [6, Lemma 4.2] is valid and we have that $(H,\mathscr{K})$ is homologous (relative to the boundary) to a union of foliated S^1-bundles over T^{n-1} and half-layered foliated I-bundles. Moreover, these foliated I-bundles are obtained by modifying the foliated $[-1,+1)$ or $(-1,+1]$-bundles over $L\subset\partial H$ and have abelian total holonomies.

Step 3. End of the Proof. Let $\sigma: M^z\to\Xi$ be the section in the foliated microbundle of $(M^z,\mathscr{F}^z)$ in Step 2. Over each 'immersed' closed room or closed hall, $\sigma(M^z)$ can be homotoped in Ξ as we have seen in Step 2. Consequently, $(M^z,\mathscr{F}^z)$ is homologous to a finite union of layered foliated I-bundles over $(n-1)$-complexes (which are closed manifolds with cone singularity) and a finite number of foliated S^1-bundles over T^{n-1}. Since these layered foliated I-bundles have abelian total holonomies, by a standard classifying space argument, they are homologous to a union of foliated S^1-bundles over T^{n-1}. Since $(M,\mathscr{F})$ is homologous to $(M^z,\mathscr{F}^z)$, by Step 1, the proof is completed.

References

[1] J. Cantwell and L. Conlon, The dynamics of open, foliated manifolds and a vanishing theorem for the Godbillon-Vey class, Adv. in Math., **53**, 1 (1984), 1–27.

[2] G. Duminy, L'invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort, preprint, 1982.

[3] A. Haefliger, Feuilletages sur les variétés ouvertes, Topology, **9** (1970), 183–194.

[4] G. Hector, Croissance des feuilletages presque sans holonomie, Lecture Notes in Mathematics, **652**, Springer Verlag, New York, 1978, 141–182.

[5] T. Mizutani, S. Morita and T. Tsuboi, The Godbillon-Vey classes of codimension one foliations which are almost without holonomy, Ann. of Math., **113** (1981), 515–527.

[6] T. Mizutani, S. Morita and T. Tsuboi, On the cobordism classes of codimension one foliations which are almost without holonomy, Topology, **22** (1983), 325–343.

[7] T. Nishimori, SRH-decomposition of codimension one foliations and the Godbillon-Vey classes, Tôhoku Math. J., **32** (1980), 9–34.

[8] F. Sergeraert, Feuilletages et difféomorphismes infiniment tangents à l'identité, Invent. Math., **39** (1977), 253–275.

[9] W. Thurston, Existence of codimension-one foliations, Ann. of Math., **104** (1976), 249–268.

[10] T. Tsuboi, On 2-cycles of $B\mathrm{Diff}(S^1)$ which are represented by foliated S^1-bundles over T^2, Ann. Inst. Fourier, **31**, 2 (1981), 1–59.

[11] N. Tsuchiya, The Nishimori decomposition of codimension-one foliations and the Godbillon-Vey classes, Tôhoku Math. J., **34** (1982), 343–365.

[12] ——, On decompositions and approximations of foliated manifolds, in this volume.

Department of Mathematics
Faculty of Science
Saitama University
Urawa, 338 Japan