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On the Homology of Classifying Spaces for Foliated Products

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Contents

Introduction

Let $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be the group of C^r-diffeomorphisms $(1 \leqq r \leqq \infty)$ of $\boldsymbol{R}^n$ with compact support. $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ is a topological group with respect to the C^r-topology. Let $\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta$ denote the same group $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with the discrete topology. The identity map $\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta \to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ induces a continuous map between their classifying spaces; $B\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta \to B\mathrm{Diff}_c^r(\boldsymbol{R}^n)$. Let $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ denote the homotopy theoretic fiber of this map. $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ is the classifying space for C^r-foliated $\boldsymbol{R}^n$-products with compact support.

The topology of the space $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ is closely related to that of Haefliger's classifying space $B\bar{\Gamma}_n^r$ for Γ_n-structures of class C^r with trivialized normal bundles. In fact, Mather [19] and Thurston [33] (see

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also Mather [22]) proved that

$$H_*(\Omega^n B\bar{\Gamma}_n^r; Z) \cong H_*(B\overline{\mathrm{Diff}}_c^r(R^n); Z),$$

where Ω^n denotes the n-fold loop space. Hence, if $B\overline{\mathrm{Diff}}_c^r(R^n)$ is m-acyclic, then $B\bar{\Gamma}_n^r$ is $(n+m)$-connected.

It is known by using the characteristic classes of foliations (e.g. the Godbillon-Vey class [8, 33]) that $H_{n+1}(B\overline{\mathrm{Diff}}_c^r(R^n); Z)$ is highly nontrivial for $r \geqq 2$. It is natural to ask whether $H_m(B\overline{\mathrm{Diff}}_c^r(R^n); Z)=0$ for $r \geqq 2$, $1 \leqq m \leqq n$ or for $r \leqq 1$, m whatever (Thurston [33], Mather [22], etc.). In the case where $r=0$, Mather [18] proved that $B\overline{\mathrm{Homeo}_c}(R^n)$ $(\simeq B\mathrm{Homeo}_c(R^n)^\delta)$ is acyclic. For $m=1$, Mather [21] and Thurston [33] proved that $H_1(B\overline{\mathrm{Diff}}_c^r(R^n); Z)=0$ if $1 \leqq r \leqq \infty$, $r \neq n+1$.

Now our main theorem of this paper is the following.

Theorem. $H_2(B\overline{\mathrm{Diff}}_c^r(R^n); Z)=0$ *if* $1 \leqq r < [n/2]$.

$H_m(B\overline{\mathrm{Diff}}_c^r(R^n); Z)=0$ *if* $1 \leqq r < [(n+1)/m]-1$ $(m \geqq 1)$.

Hence $B\bar{\Gamma}_n^r$ is $(n+2)$-connected if $r<[n/2]$ and $B\bar{\Gamma}_n^r$ is $(n+m)$-connected if $r<[(n+1)/m]-1$. For example, for $r=2$, we see that $B\bar{\Gamma}_n^2$ $(n \geqq 6)$ is $(n+2)$-connected and that $B\bar{\Gamma}_n^2$ $(n \geqq 4m-1)$ is $(n+m)$-connected. Note that, since $H_{n+1}(B\overline{\mathrm{Diff}}_c^2(R^n); Z) \neq 0$, $B\bar{\Gamma}_n^2$ is not $(2n+1)$-connected.

This paper is organized as follows. In Section 1, we give the definition and the generalities of foliated products. We discuss their classifying spaces in Section 2. There, we briefly review the theorem of Mather [19] and Thurston [33] which says that $H_*(\Omega^n B\bar{\Gamma}_n^r; Z) \cong H_*(B\overline{\mathrm{Diff}}_c^r(R^n); Z)$.

We explain the idea of the proof of our main theorem in Section 3. The proof of our theorem is in some sense similar to Mather's proof ([18]) of the acyclicity of the group of homeomorphisms of R^n with compact support. For a homeomorphism f, Mather constructed an infinite composition of conjugates $g^i f g^{-i}$ $(i \geqq 0)$. This construction does not work in the differentiable case. To gain the differentiability, we replace the semigroup $\{g^i; i \geqq 0\}$ by a semigroup with bigger growth (e.g., a free semigroup $* Z_+$ with 2^n generators). For a foliated product over a cube, we define its subdivision. By conjugating the resulted foliated products by elements of the semigroup, we construct a foliated product which plays a role similar to the infinite composition $\prod g^i f g^{-i}$.

For a precise construction of this foliated product, we study the topology of the group of diffeomorphisms in Section 4 and we give a required semigroup in Section 5.

For our construction, it is more convenient to work with the cubic homology theory (Serre [31]), because it behaves nicer with respect to the subdivisions of foliated products and the conjugations by the elements of our semigroup. We review the cubic homology theory in Sections 6 and 7. It is worth noticing that for the homological study of foliated products, the cycles represented by tori are of special importance. The cubic homology theory is better adapted, because the tori are written simpler.

In Section 8 we give our construction. There, we reprove a theorem of Mather [21] which says that $H_1(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ if $1\leqq r<n+1$.

In Section 9 we prove that $H_2(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ for $r<[n/2]$. There, we give diagrams which represent the face relations of chains. In Section 10 we prove the other part of our main theorem;

$$H_m(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0 \qquad \text{for } 1\leqq r<[(n+1)/m]-1.$$

In Appendix, we describe several operations on foliated products and give a proof of the other part of Mather's perfectness theorem ([21]), i.e., $H_1(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ if $n+1<r<\infty$. The proof is obtained by reversing the construction for the case $1\leqq r<n+1$ and using a fixed point theorem.

By a method similar to that of this paper, the author proved that $H_2(B\overline{\mathrm{Diff}}{}_c^1(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$ $(n\geqq 1)$. The proof uses the Denjoy-Pixton C^1-action [26]. The proof of this will appear elsewhere.

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§ 1. Foliated products

In this section, we discuss the generalities on foliated products. See [5, 9, 12, 33]. Let G be a group of C^r-diffeomorphisms $(1\leqq r\leqq\infty)$ of a manifold M with the C^r-topology. Let G^δ denote the group G with the discrete topology.

G-foliated M-product. A G-foliated M-product over a topological space Y is an equivalence class of the triples $(\{U_\lambda\}_{\lambda\in\Lambda}, f_\lambda, g_{\lambda\mu})$ $(=(\{U_\lambda\}_{\lambda\in\Lambda}, \{f_\lambda\}_{\lambda\in\Lambda}, \{g_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}))$, where

(i) $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open covering of Y,

(ii) $f_\lambda\colon U_\lambda\to G$ is a continuous map, and

(iii) $g_{\lambda\mu}\colon U_\lambda\cap U_\mu\to G$ is a locally constant map such that

$$f_\mu(y)=f_\lambda(y)g_{\lambda\mu}(y) \quad (y\in U_\lambda\cap U_\mu).$$

Here, two such triples $(\{U^i_\lambda\}_{\lambda\in\Lambda_i}, f^i_\lambda, g^i_{\lambda\mu})$ $(i=1, 2)$ are equivalent if there exists a triple $(\{\bar{U}_\lambda\}_{\lambda\in\bar{\Lambda}}, \bar{f}_\lambda, \bar{g}_{\lambda\mu})$ which contains $(\{U^i_\lambda\}_{\lambda\in\Lambda_i}, f^i_\lambda, g^i_{\lambda\mu})$ $(i=1, 2)$. Note that for $g \in G$, $(\{U_\lambda\}_{\lambda\in\Lambda}, f_\lambda g, g^{-1}g_{\lambda\mu}g)$ is equivalent to $(\{U_\lambda\}_{\lambda\in\Lambda}, f_\lambda, g_{\lambda\mu})$.

An M-bundle with the structural group G over Y is given by a cocycle $(\{U_\lambda\}_{\lambda\in\Lambda}, g_{\lambda\mu})$, where $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open covering of Y and $g_{\lambda\mu}: U_\lambda\cap U_\mu\to G$ is a continuous map satisfying $g_{\lambda\mu}g_{\mu\nu}=g_{\lambda\nu}$ on $U_\lambda\cap U_\mu\cap U_\nu$. The total space of this bundle is obtained from the disjoint union of $U_\lambda\times M$ $(\lambda\in\Lambda)$, by identifying $(y_\lambda, x_\lambda)\in U_\lambda\times M$ and $(y_\mu, x_\mu)\in U_\mu\times M$ if $y_\lambda=y_\mu$ and $x_\lambda=g_{\lambda\mu}(y_\mu)(x_\mu)$. If $g_{\lambda\mu}$ is a locally constant map, that is, if $g_{\lambda\mu}: U_\lambda\cap U_\mu\to G^\delta$, a cocycle $(\{U_\lambda\}_{\lambda\in\Lambda}, g_{\lambda\mu})$ defines a G-foliated M-bundle. We have a foliation of the total space of this M-bundle transverse to the fibers. The leaves of this foliation are locally given as the level sets $U_\lambda\times\{x\}$ $(x\in M)$ in $U_\lambda\times M$.

G-foliated M-products are exactly those G-foliated M-bundles which are trivial(ized) M-bundles. For, for a triple $(\{U_\lambda\}, f_\lambda, g_{\lambda\mu})$, by the conditions (ii) and (iii), $(\{U_\lambda\}, g_{\lambda\mu})$ is a G^δ-valued cocycle which defines a G-foliated M-bundle; and by (ii), this cocycle is a coboundary as a G-valued cocycle. A trivialization is induced from the map which sends $(y_\lambda, x_\lambda)\in U_\lambda\times M$ to $(y_\lambda, f_\lambda(y_\lambda)(x_\lambda))\in Y\times M$. Hence the foliation of the product $Y\times M$ is described as follows: the leaf passing through the point $(y, x)\in Y\times M$ is locally of the form $\{(y_\lambda, f_\lambda(y_\lambda)f_\lambda(y)^{-1}(x)); y_\lambda\in U_\lambda\}$, where y is contained in an open set U_λ belonging to the covering $\{U_\lambda\}$ of Y (We may assume that U_λ is connected).

Holonomy. Let Y be a path connected space with the base point y_0. Put

$$PY=\{a: [0, 1]\to Y;\ a \text{ is continuous, } a(0)=y_0\}.$$

We also consider PG, where the base point of G is the identity;

$$PG=\{\alpha: [0, 1]\to G;\ \alpha \text{ is continuous, } \alpha(0)=\mathrm{id}\}.$$

PG is a topological group.

Let $\mathcal{F}$ be a G-foliated M-product over Y. For an element $a\in PY$ and a point $x\in M$, there uniquely exists a lift $\tilde{a}(x): [0, 1]\to Y\times M$ such that $\tilde{a}(x)(0)=(y_0, x)$ and $\tilde{a}(x)([0, 1])$ is contained in a leaf of $\mathcal{F}$. Put

$$\tilde{a}(x)(t)=(a(t), \hat{a}(t)(x)),\quad t\in[0, 1];$$

then $\hat{a}$ is an element of PG. Hence we obtain a map $PY\to PG$. If two paths $a_1, a_2\in PY$ are homotopic relative to $\{0, 1\}$, then we have $\hat{a}_1(1)=\hat{a}_2(1)\in G$. Suppose that Y has the universal cover $\pi: \tilde{Y}\to Y$. Then the map $PY\to PG$ induces a map $h: \tilde{Y}\to G$. This map h up to the right action of G does not depend on the choice of the base point y_0 of Y. We call h the holonomy of $\mathcal{F}$, which is defined up to the right action of G.

Let $\tilde{y}_0$ be the base point of $\tilde{Y}$ which is a lift of y_0; $\pi(\tilde{y}_0)=y_0$. For an element $l \in \pi_1(Y, y_0)$, define $\boldsymbol{h}(l)$ by $\boldsymbol{h}(l)=h(\tilde{y}_0 l) \in G$. (We are assuming here that $h(\tilde{y}_0)=\mathrm{id}$; otherwise we define $\boldsymbol{h}(l)$ by $\boldsymbol{h}(l)=h(\tilde{y}_0)^{-1}h(\tilde{y}_0 l)$.) Then $\boldsymbol{h}$ is a homomorphism $\pi_1(Y, y_0)\to G$ and h is a $\pi_1(Y, y_0)$-equivariant map with respect to $\boldsymbol{h}$, i.e.,

$$h(\tilde{y}l)=h(\tilde{y})\boldsymbol{h}(l) \qquad \text{for } \tilde{y} \in \tilde{Y} \text{ and } l \in \pi_1(Y, y_0).$$

Conversely, suppose that we have a $\pi_1(Y, y_0)$-equivariant map $h\colon \tilde{Y}\to G$ with respect to $\boldsymbol{h}\colon \pi_1(Y, y_0)\to G$. (Since $\boldsymbol{h}(l)=h(\tilde{y}_0)^{-1}h(\tilde{y}_0 l)$, $\boldsymbol{h}$ is determined by h.) Then we obtain a G-foliated M-product $\mathcal{F}$ over Y with holonomy h. For, take a covering $\{U_\lambda\}$ of Y which is evenly covered by $\pi\colon \tilde{Y}\to Y$. Let $V_\lambda \subset \tilde{Y}$ be an open set such that $\pi \mid V_\lambda;\ V_\lambda \to U_\lambda$ is a homeomorphism. The triple representing $\mathcal{F}$ is obtained as follows: The map $f_\lambda\colon U_\lambda\to G$ is given as the composition of $h \mid V_\lambda$ and $(\pi \mid V_\lambda)^{-1}\colon U_\lambda\to V_\lambda$; the map $g_{\lambda\mu}$ is obtained from $\boldsymbol{h}\colon \pi_1(Y, y_0)\to G$. In other words, h defines a G-foliated M-product over $\tilde{Y}$ whose leaf passing through $(\tilde{y}, x) \in \tilde{Y}\times M$ is $\{(z, h(z)h(\tilde{y})^{-1}(x));\ z \in \tilde{Y}\}$. This foliation is invariant under the action of $\pi_1(Y, y_0)$ in the direction of $\tilde{Y}$. Hence we obtain a G-foliated M-product over Y.

It is easy to see that two $\pi_1(Y, y_0)$-equivariant maps $h_i\colon \tilde{Y}\to G$ $(i=1, 2)$ with respect to $\boldsymbol{h}_i\colon \pi_1(Y, y_0)\to G$ define the same G-foliated M-product over Y if there exists an element $g \in G$ such that $h_2=h_1 g$ and $\boldsymbol{h}_2=g^{-1}\boldsymbol{h}_1 g$.

If we have a map $f\colon Y\to G$, by composing with $\pi\colon \tilde{Y}\to Y$, we obtain a G-foliated M-product $\mathcal{F}_f$ whose holonomy is $f\pi$. In this case, the open covering of Y of the defining triple of the foliated product consists of only one open set Y. For $f_i\colon Y\to G$ $(i=1, 2)$, if $\mathcal{F}_{f_1}=\mathcal{F}_{f_2}$, then there exists an element g of G such that $f_2=f_1 g$.

If Y is simply connected, then $\tilde{Y}=Y$ and the holonomy is a map $Y\to G$. That is, any G-foliated M-product $\mathcal{F}$ over Y is obtained as $\mathcal{F}_f$ with $f\colon Y\to G$.

For a G-foliated M-bundle over Y, we have its holonomy $\boldsymbol{h}\colon \pi_1(Y, y_0)\to G$. Conversely, if we have a homomorphism $\boldsymbol{h}\colon \pi_1(Y, y_0)\to G$, we obtain a G-foliated M-bundle over Y with holonomy $\boldsymbol{h}$ from the product foliation $\tilde{Y}\times\{x\}$ $(x \in M)$ of $\tilde{Y}\times M$ by the identification under the action of $\pi_1(Y, y_0)$. Here the action of $\pi_1(Y, y_0)$ is given by $(\tilde{y}, x)l=(\tilde{y}l, \boldsymbol{h}(l)^{-1}(x))$ for $l \in \pi_1(Y, y_0)$.

For a G-foliated M-product over Y, we obtain the holonomy h and the homomorphism $\boldsymbol{h}$. The homomorphism $\boldsymbol{h}$ defines a G-foliated M-bundle in the way described above and h gives a trivialization of this M-bundle.

Homotopy. Two G-foliated M-products over a topological space Y are said to be homotopic if they are the restrictions to $Y\times\{0\}$ and to $Y\times\{1\}$ of a G-foliated M-product over $Y\times[0, 1]$. In terms of holonomy, a homotopy is given by a continuous family $\{(h_t, \boldsymbol{h}_t); t\in[0, 1]\}$ of continuous maps $h_t: \tilde{Y}\to G$ and homomorphisms $\boldsymbol{h}_t: \pi_1(Y, y_0)\to G$ such that $h_t(\tilde{y}l)=h_t(\tilde{y})\boldsymbol{h}_t(l)$ and $\boldsymbol{h}_t(l)=\alpha(t)\boldsymbol{h}_0(l)\alpha(t)^{-1}$, for $\tilde{y}\in\tilde{Y}$, $l\in\pi_1(Y, y_0)$ and $t\in[0, 1]$, where α is an element of PG.

Conjugation. For an element $g\in G$ and a G-foliated M-product $\mathscr{F}$ over Y, we define a G-foliated M-product $g\mathscr{F}$ over Y. Let $\mathscr{F}$ be given by $(\{U_\lambda\}_{\lambda\in\Lambda}, f_\lambda, g_{\lambda\mu})$. Then $g\mathscr{F}$ is given by $(\{U_\lambda\}_{\lambda\in\Lambda}, gf_\lambda, g_{\lambda\mu})$. Note that this triple is equivalent to $(\{U_\lambda\}_{\lambda\in\Lambda}, gf_\lambda g^{-1}, gg_{\lambda\mu}g^{-1})$. In terms of the holonomy, if $\mathscr{F}$ is given by the holonomy $h: \tilde{Y}\to G$ and the homomorphism $\boldsymbol{h}: \pi_1(Y, y_0)\to G$, $g\mathscr{F}$ is given by gh and $\boldsymbol{h}$ (or ghg^{-1} and $g\boldsymbol{h}g^{-1}$).

Let α be an element of PG. Then $\alpha(1)\mathscr{F}$ is homotopic to $\mathscr{F}$. For, we have a G-foliated M-product $(\{U_\lambda\times[0, 1]\}_{\lambda\in\Lambda}, \alpha f_\lambda, g_{\lambda\mu})$ over $Y\times[0, 1]$ whose restrictions to $Y\times\{0\}$ and to $Y\times\{1\}$ coincide with $\mathscr{F}$ and $\alpha(1)\mathscr{F}$, respectively.

Foliated products with smooth leaves. Now suppose that G has a C^∞-manifold structure such that $G\times G\longrightarrow G$ given by $(g_1, g_2)\to g_1g_2$ is smooth with respect to g_1. We can consider a G-foliated M-product $(\{U_\lambda\}_{\lambda\in\Lambda}, f_\lambda, g_{\lambda\mu})$ over a smooth manifold Y such that $f_\lambda: U_\lambda\to G$ is a smooth map. In other words, such a G-foliated M-product is given by a holonomy $h: \tilde{Y}\to G$ which is smooth. When G is the group $\mathrm{Diff}^r(M)$ [resp. $\mathrm{Diff}^r_c(M)$] of the C^r-diffeomorphisms of M [resp. with compact support], any G-foliated M-product over a manifold Y is homotopic to such a G-foliated M-product.

Let $\mathscr{F}$ be a G-foliated M-product over a smooth manifold Y such that $f_\lambda: U_\lambda\to G$ is smooth. Since the leaves of $\mathscr{F}$ are locally of the form

$$\{(y_\lambda, f_\lambda(y_\lambda)f_\lambda(y)^{-1}(x)); y_\lambda\in U_\lambda\}\subset U_\lambda\times M,$$

they are of class C^∞. We have a fibre-wise linear map X from the tangent bundle of Y to the space $\mathscr{X}^r(M)$ of C^r-vectorfields on M. For $y\in U_\lambda$ $(\subset Y)$ and $x\in M$, this linear map $X_y(x)$ is given as the Jacobian at y of the map $y_\lambda\to f_\lambda(y_\lambda)f_\lambda(y)^{-1}(x)$. Note that, if $1\leqq r<\infty$, this map may not be smooth with respect to y.

If we fix Riemannian metrics on Y and on M, we can define the norm (seminorm) of $\mathscr{F}$ by

$$|\mathscr{F}|=\sup_{y\in Y}|X_y|,$$

where $|X_y|$ is the C^r-norm of the vectorfield X_y.

§ 2. Classifying spaces for foliated products

Let G be a group of C^r-diffeomorphisms $(1 \leqq r \leqq \infty)$ of a manifold M with the C^r-topology. Let G^δ denote the group G with the discrete topology as before.

Classifying spaces. We have a classifying space $B\bar{G}$ for G-foliated M-products. For a reasonable space Y, the homotopy classes of G-foliated M-products correspond bijectively to the homotopy classes of continuous maps $Y \to B\bar{G}$. This classifying space is denoted by $B\bar{G}$ ([33]). For, this space can be identified with the classifying space for the topological group $\bar{G}$ which is the fiber product of the identity $G^\delta \to G$ and $PG \to G$, where

$$PG = \{\alpha\colon [0, 1] \to G;\ \alpha \text{ is continuous}, \alpha(0) = \mathrm{id}\}$$

and the latter map is given by $\alpha \to \alpha(1)$. $B\bar{G}$ is the homotopy theoretic fiber of the map $BG^\delta \to BG$ induced from the identity map $G^\delta \to G$, where the space BG classifies M-bundles with the structural group G and the space BG^δ classifies M-bundles with the structural group G^δ, i.e., G-foliated M-bundles. Keep in mind the following fibration sequence:

$$\bar{G} \longrightarrow G^\delta \longrightarrow G \longrightarrow B\bar{G} \longrightarrow BG^\delta \longrightarrow BG.$$

We have the following construction of $B\bar{G}$ (see Mather [22]). Let $S_*(G)$ be the singular simplicial complex of G. The group G acts on $S_*(G)$ freely on the right. Let

$$\Delta^m = \{(t_1, \cdots, t_m) \in \boldsymbol{R}^m;\ 1 \geqq t_1 \geqq \cdots \geqq t_m \geqq 0\}$$

be the standard m-simplex. For a singular simplex $\sigma\colon \Delta^m \to G$ and an element g of G, $\sigma g\colon \Delta^m \to G$ is defined by

$$(\sigma g)(t) = \sigma(t) g, \quad t \in \Delta^m.$$

This action commutes with the face operators, so $S_*(G)/G$ is a semi-simplicial complex. $B\bar{G}$ is given as the geometric realization $|S_*(G)/G|$ of $S_*(G)/G$. We always consider $B\bar{G}$ given in this way.

A singular m-simplex $\sigma\colon \Delta^m \to G$ determines a G-foliated M-product $\mathscr{F}_\sigma$ over Δ^m. Conversely, any G-foliated M-product $\mathscr{F}$ over Δ^m is written as $\mathscr{F}_\sigma$ with a map $\sigma\colon \Delta^m \to G$. This map σ is unique up to the right G-action. Thus, we have a bijective correspondence between the m-simplices of $B\bar{G}$ and the G-foliated M-products over Δ^m. Roughly speaking, $B\bar{G}$ classifies foliated products because it contains all foliated products over simplices. Note that $B\bar{G}$ has only one 0-simplex. Hence every 1-chain of $S_*(G)/G$ is a 1-cycle.

Now suppose that G has a C^∞-manifold structure such that $G\times G\to G$ given by $(g_1, g_2)\to g_1g_2$ is smooth with respect to g_1. Then we can replace $S_*(G)$ above by $S_*^\infty(G)$, the smooth singular simplicial complex of G. If $\sigma\colon \Delta^m\to G$ is smooth, then $\sigma g\colon \Delta^m\to G$ is also smooth. Hence we obtain a subcomplex $S_*^\infty(G)/G$ of $S_*(G)/G$. In many cases, for example when $G=\mathrm{Diff}_c^r(M)$ $(r\geqq 1)$, the inclusion $|S_*^\infty(G)/G|\to|S_*(G)/G|$ is a homotopy equivalence.

Now we consider the case when the group G is the group $\mathrm{Diff}^r(M)$ [resp. $\mathrm{Diff}_c^r(M)$] of C^r-diffeomorphisms $(1\leqq r\leqq\infty)$ of M [resp. with compact support]. We call a G-foliated M-product a (C^r-) foliated M-product [resp. with compact support]. The classifying space $B\overline{\mathrm{Diff}}_c^r(M)$ for C^r-foliated M-products with compact support is closely related to the classifying space $B\Gamma_n^r$ for Haefliger's Γ_n^r-structures ([10, 11]).

$B\mathrm{Diff}(M^n)$ *and* $B\Gamma_n$. (See Mather [22].) Let Γ_n^r denote the groupoid of germs of C^r-diffeomorphisms of $\boldsymbol{R}^n$ $(1\leqq r\leqq\infty)$ with the sheaf topology. A Γ_n^r-structure on a topological space X is given by a Γ_n^r-valued 1-cocycle $(\{U_\lambda\}, \gamma_{\lambda\mu})$ (Haefliger [9]). We have the classifying space $B\Gamma_n^r$ for Γ_n^r-structures (Haefliger [10, 11]). Let $\nu\colon B\Gamma_n^r\to BO_n$ be the classifying map of the normal bundle of $B\Gamma_n^r$. Let $B\bar{\Gamma}_n^r$ denote the homotopy theoretic fiber of this map. For a smooth manifold M of dimension n, since its manifold structure is a Γ_n^r-structure, we have a classifying map $M\to B\Gamma_n^r$. The classifying map $\tau_M\colon M\to BO_n$ of the tangent bundle of M is the composition of the classifying map $M\to B\Gamma_n^r$ and $\nu\colon B\Gamma_n^r\to BO_n$. Then we obtain a fiber space $\tau_M^*\nu$ over M with fiber $B\bar{\Gamma}_n^r$ (the pull-back of ν). This fiber space $\tau_M^*\nu$ admits a section given by $M\to B\Gamma_n^r$.

$$\begin{tikzcd}
\tau_M^*(B\Gamma_n^r) \arrow[r] \arrow[d, "\tau_M^*\nu"'] & B\Gamma_n^r \arrow[d, "\nu"] \\
M \arrow[r, "\tau_M"'] \arrow[ur] & BO_n.
\end{tikzcd}$$

Let $\Gamma_c(\tau_M^*\nu)$ denote the space of sections which differ from the section given by $M\to B\Gamma_n^r$ only on a compact set. We have the following theorem due to Mather [19] and Thurston [33].

Theorem (Mather-Thurston). *Suppose that $\partial M=\phi$. Then there exists a mapping $B\overline{\mathrm{Diff}}_c^r(M)\to\Gamma_c(\tau_M^*\nu)$ which induces an isomorphism in integer homology. In particular, there exists a mapping $B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)\to \Omega^n B\bar{\Gamma}_n^r$ which induces an isomorphism in integer homology, where Ω^n denotes the n-fold loop space.*

This theorem gives a motivation to studying the homology of

$B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)$. The followings have been known.

$$H_i(B\overline{\mathrm{Homeo}_c}(\boldsymbol{R}^n); \boldsymbol{Z})=0 \quad (i>0) \qquad \text{(Mather [18]).}$$

$$H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0 \quad (1\leqq r\leqq\infty,\ r\neq n+1)$$
(Mather [21], Thurston [33]).

$$H_{n+1}(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})\neq 0 \quad (r\geqq 2)$$
(Godbillon-Vey [8], Thurston [33], etc.).

There are many other results which says that $H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})\neq 0$ for many m greater than n and $r\geqq 2$. These results are obtained by using the characteristic classes of foliations and these homology groups often have the dimension equal to the continuum (see [6,14, 32, 33]).

By the theorem above $B\overline{\Gamma}_n^r$ is $(n+1)$-connected if $r\neq n+1$ (at least n-connected if $r=n+1$). Also, if $r\geqq 2$, then $B\overline{\Gamma}_n^r$ is not $(2n+1)$-connected, and $B\overline{\Gamma}_n^0$ is contractible. (A similar theorem is true in the case when $r=0$ ([33]).)

The main theorem of this paper says that $H_2(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ if $r<[n/2]$ and $H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ if $r<[(n+1)/m]-1$. Hence $B\overline{\Gamma}_n^r$ is $(n+2)$-connected if $r<[n/2]$ and $(n+m)$-connected if $r<[(n+1)/m]-1$.

§ 3. The idea of the proof of the main theorem

The proof of our theorem is in some sense similar to that of the following theorem.

Theorem (3.1) (Mather [18]).

$$H_i(B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta; \boldsymbol{Z})=\begin{cases}\boldsymbol{Z} & (i=0)\\ 0 & (i>0).\end{cases}$$

Note that $\mathrm{Homeo}_c(\boldsymbol{R}^n)$ is contractible. For, define h: $\mathrm{Homeo}_c(\boldsymbol{R}^n)\times[0,1]\to\mathrm{Homeo}_c(\boldsymbol{R}^n)$ by

$$h(f,t)(x)=tf(x/t) \quad (t>0) \quad \text{and} \quad h(f,0)=\mathrm{id};$$

then h is continuous. Hence $B\overline{\mathrm{Homeo}_c}(\boldsymbol{R}^n)\simeq B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta$. First we explain the proof of Theorem (3.1).

Proof of the acyclicity of $\mathrm{Homeo}_c(\boldsymbol{R}^n)$. The homology of $B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta$ is the homology of the following complex.

$$0\xleftarrow{\partial}\boldsymbol{Z}\xleftarrow{\partial}\boldsymbol{Z}[G]\xleftarrow{\partial}\boldsymbol{Z}[G\times G]\xleftarrow{\partial}\boldsymbol{Z}[G\times G\times G]\xleftarrow{\partial}\cdots,$$

where $G=\mathrm{Homeo}_c(\boldsymbol{R}^n)$ and $\boldsymbol{Z}[G^m]=\boldsymbol{Z}[G\times\cdots\times G]$ is the free abelian group generated by m-tuples $(f_1,\cdots,f_m)$ $(f_i\in G;\, i=1,\cdots,m)$. The boundary ∂ on the generators is given by

$$\begin{aligned}\partial(f_1,\cdots,f_m)=&(f_2,\cdots,f_m)\\&+\sum_{i=1}^{m-1}(-1)^i(f_1,\cdots,f_{i-1},f_if_{i+1},f_{i+2},\cdots,f_m)\\&+(-1)^m(f_1,\cdots,f_{m-1})\quad(m>1)\quad\text{and}\\ \partial(f)=0\quad(m=1).&\end{aligned}$$

For an element $f\in\mathrm{Homeo}_c(\boldsymbol{R}^n)$ consider the following construction. First choose an open ball U such that

$$\mathrm{Supp}\,(f)=\mathrm{Cl}\,\{x;f(x)\neq x\}\subset U.$$

For U, choose an element g, such that $g^i(U)$ $(i\geqq 0)$ are disjoint and $\mathrm{diam}\,g^i(U)\to 0$ as $i\to\infty$. We may assume that there is an open ball V such that $\mathrm{Cl}\,(\bigcup_{i\geqq 1}g^i(U))\subset V$ and $U\cap V=\phi$. Since $\mathrm{Supp}\,(g^ifg^{-i})\subset g^i(U)$ and $|g^ifg^{-i}-\mathrm{id}|_0\leqq\mathrm{diam}\,g^i(U)$, the infinite composition $\prod_{i=0}^{\infty}g^ifg^{-i}$ is a well-defined homeomorphism of $\boldsymbol{R}^n$ with support in $U\cup V$. Put

$$I(f)=\prod_{i=0}^{\infty}g^ifg^{-i}.$$

Note that $\prod_{i=1}^{\infty}g^ifg^{-i}$ can be written as $I(gfg^{-1})$. Then we have

$$I(f)=fI(gfg^{-1})\quad\text{and}\quad gI(f)g^{-1}=I(gfg^{-1}).$$

We prove Theorem (3.1) by an induction on i. Theorem (3.1) is obvious for $i=0$. Suppose that Theorem (3.1) is true in the dimensions smaller than i. Let

$$c=\sum_j a_j(f_1^{(j)},\cdots,f_i^{(j)})\quad(a_j\in\boldsymbol{Z})$$

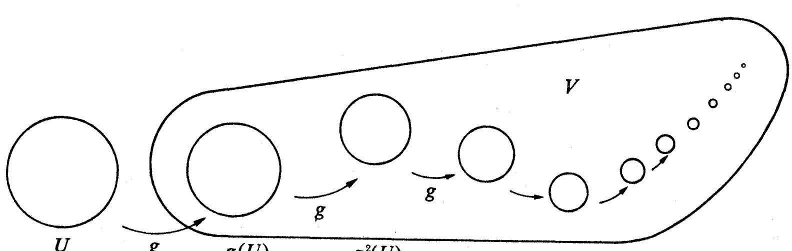

Figure (3.1)

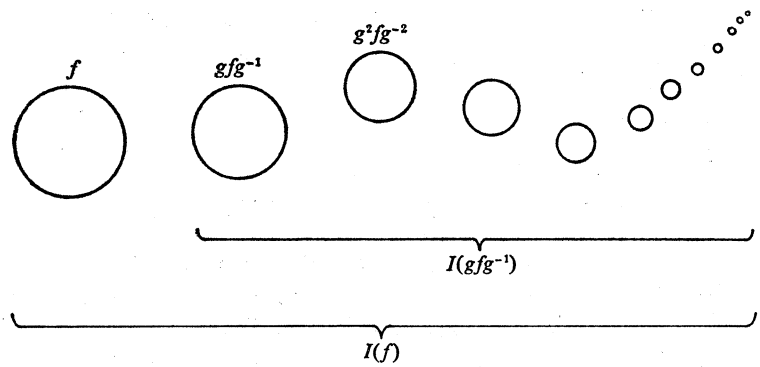

Figure (3.2)

be an i-cycle. We may assume that $\operatorname{Supp}(f_k^{(j)})\subset U$ for any j and k appearing in the i-cycle c. Put

$$I(c)=\sum_j a_j(I(f_1^{(j)}), \cdots, I(f_i^{(j)})) \quad \text{and}$$
$$I(gcg^{-1})=\sum_j a_j(I(gf_1^{(j)}g^{-1}), \cdots, I(gf_i^{(j)}g^{-1})).$$

Since $f\to I(f)$ is a homomorphism, these two i-chains are i-cycles. Moreover, since $gI(c)g^{-1}=I(gcg^{-1})$ and the inner automorphisms act as the identity on the homology of groups, $I(c)$ is homologous to $I(gcg^{-1})$ in $B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta$.

On the other hand we have the following commutative diagram.

Hence we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Homeo}_{U\cup V}(\boldsymbol{R}^n) \arrow[d, "\cong"] \arrow[dr] & \\
\mathrm{Homeo}_U(\boldsymbol{R}^n)\times\mathrm{Homeo}_V(\boldsymbol{R}^n) \arrow[r] & \mathrm{Homeo}_c(\boldsymbol{R}^n)\,,
\end{tikzcd}$$

where

$$B\mathrm{Homeo}_{U\cup V}(\boldsymbol{R}^n)^\delta \longrightarrow B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta\times B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta$$

is a homotopy equivalence. Since U and V are open balls, we have

$$\begin{tikzcd}
B\mathrm{Homeo}_{U\cup V}(\boldsymbol{R}^n)^\delta \arrow[d, "\simeq"] \arrow[dr] & \\
B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta\times B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta \arrow[r] & B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta\,.
\end{tikzcd}$$

$$H_*(B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}) \cong H_*(B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}) \\ \cong H_*(B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}).$$

By the induction hypothesis and the Künneth formula, we have

$$H_i(B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta \times B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}) \\ \cong H_i(B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}) \oplus H_i(B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}).$$

The homology class of the i-chain $I(c)$ of $B\mathrm{Homeo}_{U\cup V}(\boldsymbol{R}^n)^\delta$ is mapped to that of $c+I(gcg^{-1})$ in $H_i(B\mathrm{Homeo}_U(\boldsymbol{R}^n)^\delta; \boldsymbol{Z}) \oplus H_i(B\mathrm{Homeo}_V(\boldsymbol{R}^n)^\delta; \boldsymbol{Z})$. Hence $c+I(gcg^{-1})$ and $I(c)$ are homologous in $B\mathrm{Homeo}_c(\boldsymbol{R}^n)^\delta$. Since $I(gcg^{-1})=gI(c)g^{-1}$ is homologous to $I(c)$, the i-cycle c is homologous to zero.

The idea of the proof of the main theorem. In the proof of the acyclicity of $\mathrm{Homeo}_c(\boldsymbol{R}^n)$, the construction of $I(f)$ is essential. This construction cannot be applied to the group of diffeomorphisms. For, the C^1-norm of $g^j f g^{-j}-\mathrm{id}$ does not converge to zero as j tends to the infinity; hence $I(f)$ is not an element of $\mathrm{Diff}_c^1(\boldsymbol{R}^n)$. In order to gain the differentiability, we use a free semigroup $\Lambda = \overset{N}{*}\boldsymbol{Z}_+ = \boldsymbol{Z}_+ * \cdots * \boldsymbol{Z}_+$ generated by N elements instead of the semigroup $\{\mathrm{id}, g, g^2, \cdots\} \cong \boldsymbol{Z}_+$. Precisely, suppose now that there is a homomorphism $\Phi\colon \Lambda \to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with an open ball U such that $\Phi(\lambda)(U)$ $(\lambda \in \Lambda)$ are disjoint. To explain our idea, we take a one-parameter subgroup $\{f^t;\ t \in \boldsymbol{R}\}$ of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with support in U. Then $\Phi(\lambda) f^{N^{-l(\lambda)}} \Phi(\lambda)^{-1}$ has support in $\Phi(\lambda)(U)$, where $l(\lambda)$ denotes the word length of $\lambda \in \Lambda$ $(l(\mathrm{id})=0)$. If the C^r-norm of $\Phi(\lambda) f^{N^{-l(\lambda)}} \Phi(\lambda)^{-1} - \mathrm{id}$ converges to zero as $l(\lambda)$ tends to the infinity,

$$F = \prod_{\lambda \in \Lambda} \Phi(\lambda) f^{N^{-l(\lambda)}} \Phi(\lambda)^{-1}$$

is a diffeomorphism of class C^r. (We see later that we can make F to be of class C^r provided that $r<n+1$.) For this F, we have

$$F^{1/N} = \prod_{\lambda \in \Lambda} \Phi(\lambda) f^{N^{-l(\lambda)-1}} \Phi(\lambda)^{-1}.$$

Let i $(i=1, \cdots, N)$ denote the generator of Λ. Then we have

$$\Phi(i) F^{1/N} \Phi(i)^{-1} = \prod_{\lambda \in i\Lambda} \Phi(\lambda) f^{N^{-l(\lambda)}} \Phi(\lambda)^{-1}.$$

Thus we have

$$F = f^1 \prod_{i=1}^{N} \Phi(i) F^{1/N} \Phi(i)^{-1}.$$

These formulae imply that the 1-cycle (f^1) is homologous to zero.

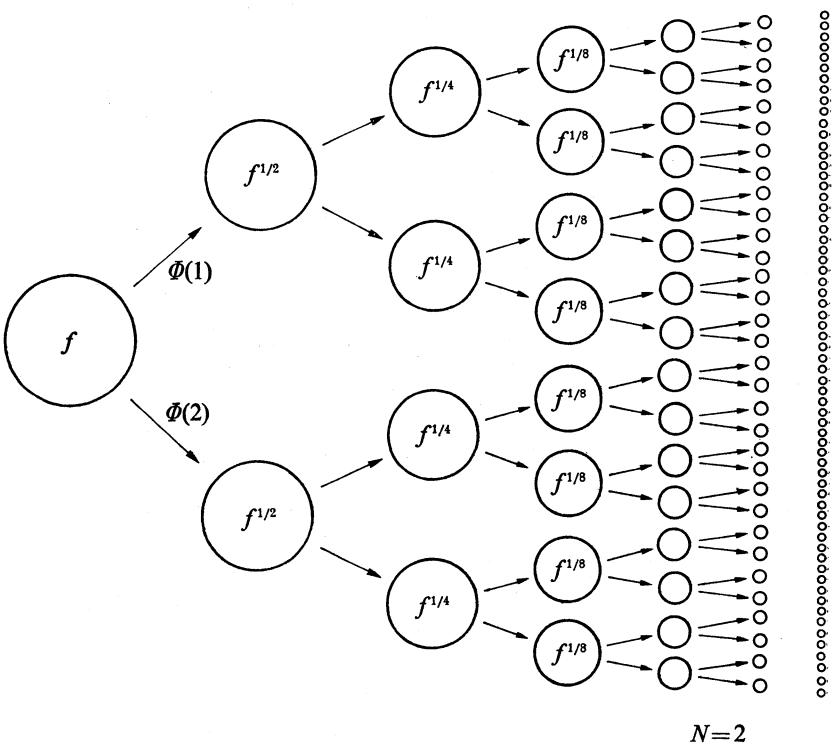

Figure (3.3)

We can apply this construction to the m-cycles represented by tori whose holonomy are contained in a one-parameter subgroup. In fact, suppose that we have a homomorphism $\Phi\colon \Lambda=\overset{N^m}{*}\boldsymbol{Z}_+\to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with an open ball U such that $\Phi(\lambda)(U)$ ($\lambda\in\Lambda$) are disjoint. Suppose also that U and $\mathrm{Cl}\,(\bigcup_{\lambda\in i\Lambda}\Phi(\lambda)(U))$ ($i=1,\cdots,N^m$) are contained in disjoint open balls. Let $\{f^t; t\in\boldsymbol{R}\}$ be a one-parameter subgroup of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with support in U. Take a homomorphism $\boldsymbol{Z}^m\to\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ such that the image of the standard basis of $\boldsymbol{Z}^m$ is $f^{t(1)},\cdots,f^{t(m)}$. This defines a foliated $\boldsymbol{R}^n$-bundle over T^m with compact support and determines an m-dimensional homology class of $B\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta$, which is denoted by $\{f^{t(1)},\cdots,f^{t(m)}\}$. Suppose that

$$F^{t(i)}=\prod_{\lambda\in\Lambda}\Phi(\lambda)f^{t(i)N^{-l(\lambda)}}\Phi(\lambda)^{-1}\quad(i=1,\cdots,m)$$

are C^r-diffeomorphisms. (We see later that we can make $F^{t(i)}$ to be of class C^r if $r<[n/m]+1$.) Then we have

$$\{F^{t(1)}, \cdots, F^{t(m)}\}=N^m\{F^{t(1)/N}, \cdots, F^{t(m)/N}\}.$$

We also observe that $\Phi(i)\{F^{t(1)/N}, \cdots, F^{t(m)/N}\}\Phi(i)^{-1}$ is the restriction of $\{F^{t(1)}, \cdots, F^{t(m)}\}$ to $\mathrm{Cl}(\bigcup_{\lambda \in i\Lambda} \Phi(\lambda)(U))$ $(i=1, \cdots, N^m)$. If we proved inductively that the homology classes of the form $\{f^{t(1)}, \cdots, f^{t(j)}\}$ $(j<m)$ are zero, by the Künneth formula, we would have

$$\{F^{t(1)}, \cdots, F^{t(m)}\}=\{f^{t(1)}, \cdots, f^{t(m)}\} + \sum_{1=1}^{N^m} \Phi(i)\{F^{t(1)/N}, \cdots, F^{t(m)/N}\}\Phi(i)^{-1}.$$

On the other hand, the inner automorphisms induce the identity in $H_*(B\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta; \boldsymbol{Z})$. Hence we have

$$\Phi(i)\{F^{t(1)/N}, \cdots, F^{t(m)/N}\}\Phi(i)^{-1}=\{F^{t(1)/N}, \cdots, F^{t(m)/N}\}.$$

Thus we can show that $\{f^{t(1)}, \cdots, f^{t(m)}\}=0$.

There are several problems to be solved for applying this idea to our problem. First, a diffeomorphism f may not belong to a one-parameter subgroup $\{f^t; t \in \boldsymbol{R}\}$. Secondly, we cannot expect the cycles to be represented by the tori. As to the first problem, it is necessary that we work with $B\overline{\mathrm{Diff}_c^r}(\boldsymbol{R}^n)$ but not with $B\mathrm{Diff}_c^r(\boldsymbol{R}^n)^\delta$. We define the subdivision of foliated products with which we can perform a construction similar to that of F. Because of the second problem, our proof contains a long computation of chains of $B\overline{\mathrm{Diff}_c^r}(\boldsymbol{R}^n)$. We have to treat the chains of $B\overline{\mathrm{Diff}_c^r}(\boldsymbol{R}^n)$ systematically using the property of the semigroup Λ.

§ 4. Group of diffeomorphisms

We review several properties of the group of diffeomorphisms of $\boldsymbol{R}^n$ with compact support.

Group of diffeomorphisms of $\boldsymbol{R}^n$ with compact support. Let K be a relatively compact subset of $\boldsymbol{R}^n$. Let $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ denote the group of C^r-diffeomorphisms of $\boldsymbol{R}^n$ which have support in K $(1 \leqq r \leqq \infty)$;

$$\mathrm{Diff}_K^r(\boldsymbol{R}^n)=\{f \in \mathrm{Diff}^r(\boldsymbol{R}^n); \mathrm{Cl}\{x; f(x)\neq x\}\subset K\}.$$

For K compact, $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ has the natural C^r-topology Let $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ denote the group of diffeomorphisms of $\boldsymbol{R}^n$ with compact support, that is, put

$$\mathrm{Diff}_c^r(\boldsymbol{R}^n)=\varinjlim_K \mathrm{Diff}_K^r(\boldsymbol{R}^n),$$

where the limit is taken over all compact subsets of $\boldsymbol{R}^n$. Thus $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ has the direct limit topology. For a bounded open set K, $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ has a similar topology.

Moreover, for a compact set K of $\boldsymbol{R}^n$ and r with $1 \leqq r < \infty$, $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ is a C^∞-Banach manifold modelled on $\mathscr{X}_K^r(\boldsymbol{R}^n)$, the space of C^r-vectorfields on $\boldsymbol{R}^n$ with support in K. $\mathscr{X}_K^r(\boldsymbol{R}^n)$ is isomorphic to $C_K^r(\boldsymbol{R}^n, \boldsymbol{R}^n)$, the space of C^r-functions $\boldsymbol{R}^n \to \boldsymbol{R}^n$ with support in K. A local coordinate neighborhood around $f \in \mathrm{Diff}_K^r(\boldsymbol{R}^n)$ is of the form $f + U_f$, where U_f is a neighborhood of 0 in $\mathscr{X}_K^r(\boldsymbol{R}^n)$ and the addition is that in $C_K^r(\boldsymbol{R}^n, \boldsymbol{R}^n)$. If

$$(f_1 + U_{f_1}) \cap (f_2 + U_{f_2}) \neq \phi,$$

the coordinate transformation

$$U_{f_1} \cap ((f_2 + U_{f_2}) - f_1) \longrightarrow ((f_1 + U_{f_1}) - f_2) \cap U_{f_2}$$

is the translation by $f_1 - f_2$.

Since $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ has the direct limit topology, it also has a smooth structure (modelled on $\mathscr{X}_c^r(\boldsymbol{R}^n)$).

It is easy to see that for $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ or $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ $(1 \leqq r < \infty)$, the composition $(g_1, g_2) \to g_1 g_2$ is smooth with respect to g_1. (It is in fact affine with respect to g_1. It is not smooth with respect to g_2. This is the reason why $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ or $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ is not a Banach group. However, $\mathrm{Diff}_K^\infty(\boldsymbol{R}^n)$ or $\mathrm{Diff}_c^\infty(\boldsymbol{R}^n)$ has the structure of a Fréchet group.)

For the group of diffeomorphisms of a manifold M with compact support, we have a similar topology. If we fix a Riemannian metric on M, for a compact subset K of M, we obtain a manifold structure of $\mathrm{Diff}_K^r(M)$ modelled on $\mathscr{X}_K^r(M)$, the space of C^r-vectorfields on M with support in K.

Let G denote $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$. The fact that $|S_*^\infty(G)| \to |S_*(G)|$ is a homotopy equivalence is a consequence of the following straightening.

Straightening. Let σ: $\Delta^m \to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be a singular m-simplex, where $\Delta^m = \{(t_1, \cdots, t_m) \in \boldsymbol{R}^m;\ 1 \geqq t_1 \geqq \cdots \geqq t_m \geqq 0\}$. Δ^m is the closed convex hull of $\{0, e_1, e_1 + e_2, \cdots, \sum_{i=1}^m e_i\}$, where $\{e_1, \cdots, e_m\}$ is the standard basis of $\boldsymbol{R}^m$. Suppose that $\sigma(\Delta^m)$ is contained in a coordinate neighborhood of $\sigma(0)$ in the Banach manifold $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ for some compact subset K. This neighborhood is of the form $\sigma(0) + U_{\sigma(0)}$, where $U_{\sigma(0)}$ is a neighborhood of zero in $\mathscr{X}_K^r(\boldsymbol{R}^n)$, the space of C^r-vectorfields on $\boldsymbol{R}^n$ with support in K. We may assume that $U_{\sigma(0)}$ is convex. Then, we define the straightening $L\sigma$: $\Delta^m \to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ of σ by

$$(L\sigma)(t_1, \cdots, t_m)(x) = \sigma(0)(x) + \sum_{j=1}^m t_j \left(\sigma\Big(\sum_{i=1}^j e_i\Big)(x) - \sigma\Big(\sum_{i=1}^{j-1} e_i\Big)(x) \right).$$

For an element g of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$, we have

$$L(\sigma g)=(L\sigma)g.$$

$L\sigma$ is canonically homotopic to σ. For, we have a homotopy $\tilde{L}\colon \Delta^m \times [0, 1] \to \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ given by

$$\tilde{L}(t, s)(x)=(1-s)(\sigma(t)(x))+s((L\sigma)(t)(x))$$

for $(t, s) \in \Delta^m \times [0, 1]$ and $x \in \boldsymbol{R}^n$. Note that this homotopy commutes with the face operators. The foliated product $\mathscr{F}_{L\sigma}$ corresponding to $L\sigma$ is described as follows. The leaf of $\mathscr{F}_{L\sigma}$ passing through $(0, x) \in \Delta^m \times \boldsymbol{R}^n$ is the convex hull of

$$\{(0, x)\} \cup \left\{\left(\sum_{i=1}^{j} e_i, \sigma\left(\sum_{i=1}^{j} e_i\right)\sigma(0)^{-1}(x)\right); j=1, \cdots, m\right\}.$$

Then we see that $L\sigma$ is a smooth singular simplex.

For a positive integer r, there are positive real numbers c_r and C_r such that

$$|X_{L\sigma}|_r \leqq C_r \sup\left\{\left|\sigma\left(\sum_{i=1}^{j} e_i\right)\sigma(0)^{-1}-\mathrm{id}\right|_r; j=1, \cdots, m\right\}$$

provided $|\sigma(\sum_{i=1}^{j} e_i)\sigma(0)^{-1}-\mathrm{id}|_r \leqq c_r$ $(j=1, \cdots, m)$. Here, $X_{L\sigma}$ denotes the map $\Delta^m \times \boldsymbol{R}^n \to L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ associated to $\mathscr{F}_{L\sigma}$ and $|\ \ |_r$ is the C^r-norm $(|X|_r=\sup_{0\leqq k\leqq r} \sup |D^k X|)$.

For the group $\mathrm{Diff}^r(M)$ of the diffeomorphisms of M, the notion of the support of elements is important. We also define the support of C^r-foliated M-products as follows. Let G denote the group $\mathrm{Diff}^r(M)$.

Support of foliated products. Let $\mathscr{F}$ be a C^r-foliated M-product over Y. A point x of M does not belong to the support of $\mathscr{F}$ if there is a neighborhood V of x in M such that the foliation $\mathscr{F}$ restricted to $Y \times V$ is the product foliation $(Y \times \{v\}, v \in V)$. If $\mathscr{F}$ is given as $\mathscr{F}_f$ with a map $f\colon Y \to G$, then

$$\mathrm{Supp}\,(\mathscr{F}_f)=\mathrm{Cl}\,\{x \in M; f(z)f(y)^{-1}(x) \neq x \text{ for some } z \text{ and } y \in Y\}.$$

For a relatively compact set K of M, let G_K denote the subgroup of G consisting of the elements which have support in K. The support of $\mathscr{F}_f$ is contained in K if and only if $f(z)f(y)^{-1} \in G_K$ for any z and $y \in Y$.

For an element $g \in G$ and a C^r-foliated M-product $\mathscr{F}$ over Y, $\mathrm{Supp}\,(g\mathscr{F})=g\,\mathrm{Supp}\,(\mathscr{F})$, where $g\mathscr{F}$ is the C^r-foliated product $\mathscr{F}$ conjugated by g (see § 1).

Let U be a bounded open ball of $\boldsymbol{R}^n$. For the classifying spaces

$B\overline{\mathrm{Diff}}{}_U^r(\boldsymbol{R}^n)$ and $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$, we have the following proposition (see Mather [18]). For a chain c of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$, the support of c is defined to be the union of the supports of the simplices appearing in c.

Proposition (4.1). *Let U be a bounded open ball in $\boldsymbol{R}^n$. Then the inclusion i: $\mathrm{Diff}_U^r(\boldsymbol{R}^n)\to\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ induces an isomorphism*

$$(Bi)_*: H_*(B\overline{\mathrm{Diff}}{}_U^r(\boldsymbol{R}^n); Z)\longrightarrow H_*(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); Z).$$

Proof. Surjectivity. Let c be an m-cycle of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$. We can find a bounded open ball U' such that all simplices appearing in c have support in U'. Let α: $[0, 1]\to\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be an element of $P\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ such that $\alpha(1)(U')\subset U$. Then $\alpha(1)c$ (the cycle c conjugated by $\alpha(1)$) is homologous (homotopic) in $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ to c and $\alpha(1)c$ lies in the image of Bi.

Injectivity. Let c be an m-cycle of $B\overline{\mathrm{Diff}}{}_U^r(\boldsymbol{R}^n)$ such that $(Bi)_*c$ is homologous to zero in $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$; $(Bi)_*c=\partial d$ for some $(m+1)$-chain d of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$. Let U_1 be an open ball such that $\mathrm{Supp}(c)\subset\overline{U}_1\subset U$, and U_2 a bounded open ball such that $\mathrm{Supp}(d)\subset U_2$. Then we can find an element α of $P\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ such that $\alpha(t)\,|\,U_1=\mathrm{id}_{U_1}$ ($t\in[0, 1]$) and $\alpha(1)(U_2)\subset U$. Since $\mathrm{Supp}(\alpha(1)d)\subset U$, $\alpha(1)d$ (the chain d conjugated by $\alpha(1)$) can be considered as an $(m+1)$-chain of $B\overline{\mathrm{Diff}}{}_U^r(\boldsymbol{R}^n)$. It is easy to see that $c=\partial(\alpha(1)d)$.

Since we have the notion of the support of foliated products, we can restrict or take the union of foliated products.

Union and restriction. Let $\mathscr{F}$ be a C^r-foliated M-product with compact support over Y. Suppose that there is a compact subset K of M such that $Y\times K(\subset Y\times M)$ is a union of leaves of $\mathscr{F}$. In terms of holonomy, K is invariant under the holonomy; $h(\tilde{y})(K)=K$ for $\tilde{y}\in\tilde{Y}$. Here we are assuming that $h(\tilde{y}_0)=\mathrm{id}$. Suppose also that for any $\tilde{y}\in\tilde{Y}$, $h(\tilde{y})$ is r-flat along the frontier of K; $j_x^r(h(\tilde{y}))=j_x^r(\mathrm{id}_M)$ ($x\in\mathrm{Fr}(K)$).

Let $h\,|\,K$: $\tilde{Y}\to\mathrm{Diff}_c^r(M)$ be the map defined by

$$(h\,|\,K)(\tilde{y})(x)=\begin{cases}h(\tilde{y})(x), & x\in K\\ x, & x\in M-K\end{cases}\qquad\text{for } \tilde{y}\in\tilde{Y}$$

and $(\boldsymbol{h}\,|\,K)$: $\pi_1(Y, y_0)\to\mathrm{Diff}_c^r(M)$ the homomorphism defined by

$$(\boldsymbol{h}\,|\,K)(l)(x)=\begin{cases}\boldsymbol{h}(l)(x), & x\in K\\ x, & x\in M-K\end{cases}\qquad\text{for } l\in\pi_1(Y, y_0).$$

Then $h\,|\,K$ and $\boldsymbol{h}\,|\,K$ are well-defined and $h\,|\,K$ is $\pi_1(Y, y_0)$-equivariant with respect to $\boldsymbol{h}|K$. These define a C^r-foliated M-product $\mathscr{F}|K$ over M with compact support which we call the restriction of $\mathscr{F}$ to K.

Let $\mathscr{F}_i$ $(i=1, \cdots, k)$ be C^r-foliated M-products with compact support over Y such that

$$\text{Int Supp}(\mathscr{F}_i) \cap \text{Int Supp}(\mathscr{F}_j)=\phi \ (i \neq j).$$

Let h_i be the holonomy of $\mathscr{F}_i$ $(i=1, \cdots, k)$ which is equivariant with respect to $\boldsymbol{h}_i: \pi_1(Y, y_0) \to \mathrm{Diff}_c^r(M)$ and satisfies $h_i(\tilde{y}_0)=\mathrm{id}$. Let h: $\tilde{Y} \to \mathrm{Diff}_c^r(M)$ be the map given by $h(\tilde{y})=\prod_i h_i(\tilde{y})$, i.e.,

$$h(\tilde{y})(x)=\begin{cases} h_i(\tilde{y})(x), & x \in \text{Supp}(\mathscr{F}_i) \quad (i=1, \cdots, k) \\ x, & x \in M-\bigcup_i \text{Supp}(\mathscr{F}_i) \end{cases}$$

and $\boldsymbol{h}: \pi_1(Y, y_0) \to \mathrm{Diff}_c^r(M)$ the homomorphism defined by

$$\boldsymbol{h}(l)=\prod_i \boldsymbol{h}_i(l) \quad (l \in \pi_1(Y, y_0)).$$

Then h is $\pi_1(Y, y_0)$-equivariant with respect to $\boldsymbol{h}$ and defines a C^r-foliated M-product $\bigcup_{i=1}^k \mathscr{F}_i$ with compact support, which we call the union of $\mathscr{F}_i$ $(i=1, \cdots, k)$.

For infinitely many foliated products $\mathscr{F}_i$ $(i \in \boldsymbol{N})$ with

$$\text{Int Supp}(\mathscr{F}_i) \cap \text{Int Supp}(\mathscr{F}_j)=\phi \ (i \neq j),$$

if h: $\tilde{Y} \to \mathrm{Diff}_c^r(M)$ (constructed in a similar way) is well-defined, we can define the union $\bigcup_{i \in \boldsymbol{N}} \mathscr{F}_i$.

Smoothness of foliated products. Let $\mathrm{Diff}_K^r(M)$ be the group of C^r-diffeomorphisms $(1 \leqq r < \infty)$ of a manifold M with support in a compact subset K of M. As we noted before, $\mathrm{Diff}_K^r(M)$ is a C^∞-Banach manifold. For a smooth (C^∞) map f: $Y \to \mathrm{Diff}_K^r(M)$, we obtain a map X: $TY \to \mathscr{X}_K^r(M)$ which is the Jacobian at y of $z \to f(z)f(y)^{-1}$. This $\mathscr{X}_K^r(M)$ valued 1-form X is continuous but may not be smooth. (If $r=\infty$, it is smooth.) However, the associated map $TY \times M \to TM$ is of class C^r. It is worth considering the foliated products with this property. For a C^r-map $TY \times M \to TM$ satisfying the integrability condition, we have a C^r-foliated M-product $\mathscr{F}$ over Y. The holonomy h: $\tilde{Y} \to \mathrm{Diff}_K^r(M)$ of $\mathscr{F}$ is of class C^1 in general. As we see below, it is hard to expect higher differentiability.

We are going to study the differentiability of f: $Y \to \mathrm{Diff}_K^r(M)$ and that of the associated map $TY \times M \to TM$ $(1 \leqq r < \infty)$. It is enough to consider the group $\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ and foliated $\boldsymbol{R}^n$-products over the cubes $[0, 1]^m$. Note again that the C^r-norm $|\ \ |_r$ on $\mathscr{X}_K^r(\boldsymbol{R}^n)$ is given by

$$|X|_r = \sup_{0 \leqq k \leqq r} \sup_{x \in \boldsymbol{R}^n} |D^k X(x)|.$$

Since the map $T[0, 1]^m \times \boldsymbol{R}^n \to T\boldsymbol{R}^n$ is fiber-wise linear, we obtain a matrix ($L(\boldsymbol{R}^m, \boldsymbol{R}^n)$) valued function on $[0, 1]^m \times \boldsymbol{R}^n$ and the differentiability of the map $T[0, 1]^m \times \boldsymbol{R}^n \to T\boldsymbol{R}^n$ is equal to that of the map $[0, 1]^m \times \boldsymbol{R}^n \to L(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

Let K be a compact subset of $\boldsymbol{R}^n$. Let $Q: [0, 1] \to \mathrm{Diff}^r_K(\boldsymbol{R}^n)$ $(1 \leqq r < \infty)$ be a C^1-map. That is, for $t_0 \in [0, 1]$, there exists an element

$$(\partial Q/\partial t)|_{t_0} \in L(T_{t_0}[0, 1], \mathscr{X}^r_K(\boldsymbol{R}^n)) \cong \mathscr{X}^r_K(\boldsymbol{R}^n)$$

such that $|Q(t) - Q(t_0) - (\partial Q/\partial t)|_{t_0}(t - t_0)|_r / |t - t_0| \to 0$ as $t \to t_0$, and $(\partial Q/\partial t)|_{t_0}$ depends continuously on $t_0 \in [0, 1]$. Then, $t_0 \mapsto (\partial Q/\partial t)|_{t_0}(Q(t_0)^{-1})$ is a continuous map from $[0, 1]$ to $\mathscr{X}^r_K(\boldsymbol{R}^n)$.

Conversely, for a continuous map $X: [0, 1] \to \mathscr{X}^r_K(\boldsymbol{R}^n)$ $(1 \leqq r < \infty)$, we have a differential equation

$$(d\varphi/dt)(t, x) = X(t, \varphi(t, x)), \qquad \varphi(0, x) = x.$$

By the fundamental theorem of ordinary differential equations, there uniquely exists a solution of class C^1 which is of class C^r with respect to x. Then we obtain a map $\varphi: [0, 1] \to \mathrm{Diff}^r_K(\boldsymbol{R}^n)$ which sends t to φ_t, where $\varphi_t(x) = \varphi(t, x)$. First, this map is continuous. For,

$$\begin{aligned} |\varphi_t - \varphi_{t_0}|_r &\leqq \int_{t_0}^{t} |X(s, \varphi(s, x))|_r ds \\ &\leqq |t - t_0| \sup_s |X(s)|_r. \end{aligned}$$

Secondly, $t \mapsto X_t \varphi_t \in \mathscr{X}^r_K(\boldsymbol{R}^n)$ is continuous, where $X_t \varphi_t(x) = X(t, \varphi(t, x))$. For,

$$|X_t \varphi_t - X_{t_0} \varphi_{t_0}|_r \leqq |X_t \varphi_t - X_{t_0} \varphi_t|_r + |X_{t_0} \varphi_t - X_{t_0} \varphi_{t_0}|_r.$$

Here, since $X: [0, 1] \to \mathscr{X}^r_K(\boldsymbol{R}^n)$ is continuous, the first term tends to zero as $t \to t_0$. Since $X(t, x)$ is of class C^r with respect to x and φ_t is continuous with respect to the C^r norm, the second term tends to zero as $t \to t_0$. From these, it follows that $t \mapsto \varphi_t \in \mathrm{Diff}^r_K(\boldsymbol{R}^n)$ is a C^1-map. For,

$$|\varphi_t - \varphi_{t_0} - (X_{t_0} \varphi_{t_0})(t - t_0)|_r \leqq \int_{t_0}^{t} |X_s \varphi_s - X_{t_0} \varphi_{t_0}|_r ds,$$

where the integrand tends to zero as $t \to t_0$. Thus, $X_t \varphi_t$ is the derivative at t of φ. Since $X_t \varphi_t$ is continuous with respect to t, $t \mapsto \varphi_t$ is of class C^1.

Suppose that Q: $[0, 1]\to\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ is a C^k-map. This implies that the map $[0, 1]\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ given by $(t, x)\mapsto Q(t)(x)$ is of class $C^{\min\{r,k\}}$. Then, although $t\mapsto((\partial Q/\partial t)|_t)(Q(t)^{-1})\in\mathscr{X}_K^r(\boldsymbol{R}^n)$ is only continuous, $(t, x)\mapsto(\partial Q/\partial t)|_t(Q(t)^{-1}(x))\in T_x(\boldsymbol{R}^n)$ is of class $C^{\min\{r,k-1\}}$. For, $(\partial Q/\partial t)$: $[0, 1]\times\boldsymbol{R}^n\to L(\boldsymbol{R}^1, \boldsymbol{R}^n)$ is C^r with respect to $x\in\boldsymbol{R}^n$ and C^{k-1} with respect to $t\in[0, 1]$. On the other hand, $(t, x)\mapsto Q(t)^{-1}(x)=\bar{Q}(t, x)$ is of class C^r with respect to $x\in\boldsymbol{R}^n$ and of class $C^{\min\{r,k\}}$ with respect to $t\in[0, 1]$ because we have

$$(\partial\bar{Q}/\partial t)(t, x)=-[(\partial Q/\partial x)(t, \bar{Q}(t, x))]^{-1}(\partial Q/\partial t)(t, \bar{Q}(t, x)).$$

We consider the converse. Suppose that we have a C^r-map X: $[0, 1]\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ such that $X(t, x)=0$ for $x\in\boldsymbol{R}^n-K$. Then the solution φ of the differential equation

$$(d\varphi/dt)\,(t, x)=X(t, \varphi(t, x))$$

is of class C^r and of class C^{r+1} with respect to t. By the previous argument, the map $[0, 1]\to\mathrm{Diff}_K^r(\boldsymbol{R}^n)$ given by $t\mapsto\varphi_t$ is of class C^1, and the derivative $\partial\varphi_t/\partial t$ is equal to the vectorfield $X_t\varphi_t$, where $X_t\varphi_t(x)=X(t, \varphi(t, x))$. It is not differentiable with respect to t as a function to $\mathscr{X}_K^r(\boldsymbol{R}^n)$.

To summarize the above, we have the following inclusions.

$$\begin{aligned}&\{C^k\text{-maps }([0, 1], 0)\to(\mathrm{Diff}_K^r(\boldsymbol{R}^n), \mathrm{id})\ (k\geqq r+1)\}\\&\quad\subset\{C^r\text{-maps }[0, 1]\times\boldsymbol{R}^n\to\boldsymbol{R}^n\text{ with support in }[0, 1]\times K\}\\&\quad\subset\{C^1\text{-maps }([0, 1], 0)\to(\mathrm{Diff}_K^r(\boldsymbol{R}^n), \mathrm{id})\}\\&\quad=\{C^0\text{-maps }[0, 1]\to\mathscr{X}_K^r(\boldsymbol{R}^n)\}.\end{aligned}$$

For a C^{r+1}-map $[0, 1]^m\to\mathrm{Diff}_K^r(\boldsymbol{R}^n)$, we also have a map $[0, 1]^m\times\boldsymbol{R}^n\to L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ which is of class C^r. Suppose that we have a C^r-map X: $[0, 1]^m\times\boldsymbol{R}^n\to L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ $(r\geqq 1)$ satisfying the integrability condition

$$\sum_k X_i^k(\partial X_j^l/\partial x_k)-\sum_k X_j^k(\partial X_i^l/\partial x_k)+\partial X_j^l/\partial t_i-\partial X_i^l/\partial t_j=0$$

and the condition on the support

$$X(t, x)=0 \qquad \text{for } x\in\boldsymbol{R}^n-K.$$

Then we obtain a C^1-map $[0, 1]^m\to\mathrm{Diff}_K^r(\boldsymbol{R}^n)$.

§ 5. Semigroup actions

In this section we construct a homomorphism

$$\Phi\colon (Z_+ * Z_+)^n \longrightarrow P\,\mathrm{Diff}_c^\infty(R^n)$$

with an open ball U such that $\Phi(\lambda)^{(1)}(U)$, $\lambda \in (Z_+ * Z_+)^n$ are disjoint, where

$$P\,\mathrm{Diff}_c^\infty(R^n) = \{f\colon [0,1] \to \mathrm{Diff}_c^\infty(R^n);\, f^{(0)} = \mathrm{id}\}.$$

We use this homomorphism to prove our main theorem.

First we consider the case when $n=1$. Let the symbols $-$ and $+$ denote the two generators of the free semigroup $Z_+ * Z_+$. Let $(\xi_-)(\partial/\partial x)$ be a C^∞-vectorfield on R such that

$$\xi_-(x) = -(x+1), \quad x \in [-1, 1],$$
$$\xi_-(x) = 0, \quad x \in (-\infty, -2] \cup [2, \infty).$$

Let f_-^t be the time t map of $(\xi_-)(\partial/\partial x)$. Then,

$$f_-^t(x) = (x+1)\exp(-t) - 1 \qquad \text{for } t \geqq 0,\ x \in [-1, 1].$$

Let $(\xi_+)(\partial/\partial x)$ be a C^∞-vectorfield on R given by

$$\xi_+(x) = -\xi_-(-x)$$

and f_+^t the time t map of $(\xi_+)(\partial/\partial x)$. Then we have

$$f_+^t(x) = (x-1)\exp(-t) + 1 \qquad \text{for } t \geqq 0,\ x \in [-1, 1].$$

Take a positive real number ε and put

$$\Phi(\sigma)^{(t)}(x) = f_\sigma^{t\log(2+\varepsilon)}(x) \quad (\sigma \in \{-, +\}).$$

This defines a homomorphism $\Phi\colon Z_+ * Z_+ \to P\,\mathrm{Diff}_c^\infty(R^n)$ and we see that, for $x \in [-1, 1]$,

$$\Phi(-)^{(1)}(x) = (x+1)/(2+\varepsilon) - 1,$$
$$\Phi(+)^{(1)}(x) = (x-1)/(2+\varepsilon) + 1.$$

Put $U = (-\varepsilon/(2+\varepsilon), \varepsilon/(2+\varepsilon))$; then $\Phi(\lambda)^{(1)}(U)$, $\lambda \in Z_+ * Z_+$ are disjoint open intervals on $[-1, 1]$. ($[-1,1] - \bigcup_\lambda \Phi(\lambda)^{(1)}(U)$ is a Cantor set.)

Now we construct the homomorphism Φ for $n \geqq 2$. Let $-_i$, $+_i$ $(1 \leqq i \leqq n)$ denote the generators of $(Z_+ * Z_+)^n$, where $\sigma_i \tau_j = \tau_j \sigma_i$ $(\sigma, \tau \in \{-, +\},\ 1 \leqq i, j \leqq n,\ i \neq j)$. Let ρ be a function on R such that

$$0 \leqq \rho(x) \leqq 1 \quad (x \in R),$$
$$\rho(x) = 1 \ (|x| \leqq 3) \quad \text{and} \quad \rho(x) = 0 \ (|x| \geqq 4).$$

Put $\xi_{\sigma i}=\prod_{k=1}^{n}\rho(x_k)\xi_\sigma(x_i)(\partial/\partial x_i)$ ($\sigma\in\{-,+\}, i=1,\cdots,n$). Then, since $(\partial\rho(x_i)/\partial x_i)\xi_\sigma(x_i)=0$ ($\sigma\in\{-,+\}, i=1,\cdots,n$), we have, for $\sigma,\tau\in\{-,+\}$, $i\neq j$,

$$\begin{aligned}&[\xi_{\sigma i},\xi_{\tau j}]\\ &=\left(\prod_{k=1}^{n}\rho(x_k)\right)\xi_\sigma(x_i)\rho(x_1)\cdots\widehat{\rho(x_i)}\cdots\rho(x_n)(\partial\rho(x_i)/\partial x_i)\xi_\tau(x_j)(\partial/\partial x_j)\\ &\quad-\left(\prod_{k=1}^{n}\rho(x_k)\right)\xi_\tau(x_j)\rho(x_1)\cdots\widehat{\rho(x_j)}\cdots\rho(x_n)(\partial\rho(x_j)/\partial x_j)\xi_\sigma(x_i)(\partial/\partial x_i)\\ &=0.\end{aligned}$$

Let $\Phi(\sigma_i)^{(t)}$ be the time $t\log(2+\varepsilon)$ map of ξ_{σ_i}. Then we have

$$\Phi(\sigma_i)^{(t)}\Phi(\tau_j)^{(t)}=\Phi(\tau_j)^{(t)}\Phi(\sigma_i)^{(t)}$$

for $\sigma,\tau\in\{-,+\}$, $1\leqq i,j\leqq n$, $i\neq j$. Thus we have a homomorphism

$$\Phi\colon(\boldsymbol{Z}_+*\boldsymbol{Z}_+)^n\longrightarrow P\operatorname{Diff}_c^\infty(\boldsymbol{R}^n).$$

It is easy to see that

$$\Phi(-_i)^{(1)}(x_1,\cdots,x_n)=(x_1,\cdots,x_{i-1},(x_i+1)/(2+\varepsilon)-1,x_{i+1},\cdots,x_n),$$
$$\Phi(+_i)^{(1)}(x_1,\cdots,x_n)=(x_1,\cdots,x_{i-1},(x_i-1)/(2+\varepsilon)+1,x_{i+1},\cdots,x_n),$$

for $(x_1,\cdots,x_n)$ with $|x_i|\leqq 1$ ($i=1,\cdots,n$). Put $U=(-\varepsilon/(2+\varepsilon),\ \varepsilon/(2+\varepsilon))^n$; then $\Phi(\lambda)^{(1)}(U)$, $\lambda\in(\boldsymbol{Z}_+*\boldsymbol{Z}_+)^n$ are disjoint. Moreover, we have the following lemmas.

Lemma (5.1). *$\Phi(\lambda)^{(1)}|U$ is the restriction to U of an affine map $x\to Ax+b$ where A is a diagonal matrix*

$$\operatorname{diag}((2+\varepsilon)^{-l_1(\lambda)},\cdots,(2+\varepsilon)^{-l_n(\lambda)}).$$

Here, $l_i(\lambda)=l(\mathrm{pr}_i(\lambda))$, and $\mathrm{pr}_i\colon(\boldsymbol{Z}_+*\boldsymbol{Z}_+)^n\to\boldsymbol{Z}_+*\boldsymbol{Z}_+$ is the projection to the i-th factor ($i=1,\cdots,n$) and l denotes the word length $l\colon\boldsymbol{Z}_+*\boldsymbol{Z}_+\to\boldsymbol{Z}_+$.

Lemma (5.2). *Let η be a C^r-vectorfield ($r\geqq 0$) on $\boldsymbol{R}^n$ with support in U. Then*

$$|(\Phi(\lambda)^{(1)})_*\eta|_r\leqq(2+\varepsilon)^{-\min\{l_i(\lambda)\}+r\max\{l_i(\lambda)\}}|\eta|_r,$$

where $|\ \ |_r$ denotes the C^r-norm.

Remark (5.3). We have actually constructed a homomorphism $\Psi\colon(\boldsymbol{R}_+*\boldsymbol{R}_+)^n\to\operatorname{Diff}_c^\infty(\boldsymbol{R}^n)$ and $\Phi\colon(\boldsymbol{Z}_+*\boldsymbol{Z}_+)^n\to P\operatorname{Diff}_c^\infty(\boldsymbol{R}^n)$ is written as $\Phi(\lambda)^{(t)}=\Psi(t\lambda)$.

The semigroup $(\boldsymbol{Z}_+*\boldsymbol{Z}_+)^n$ has several nice subsemigroups. We will use the following subsemigroups later.

First we have the diagonal subsemigroup

$$\{\lambda \in (Z_+ * Z_+)^n;\ l_1(\lambda) = \cdots = l_n(\lambda)\}$$

which is isomorphic to $\overset{2^n}{*} Z_+$.

Secondly, for $m<n$, we have the following subsemigroup.

$$\{\lambda \in (Z_+ * Z_+)^n;\ l_{k[n/m]+1}(\lambda) = \cdots = l_{(k+1)[n/m]}(\lambda)\ (k=0, \cdots, m-2),$$
$$l_{(m-1)[n/m]+1}(\lambda) = \cdots = l_n(\lambda)\}$$

which is isomorphic to $(\overset{2^{[n/m]}}{*} Z_+)^{(m-1)} \times (\overset{2^{n-(m-1)[n/m]}}{*} Z_+)$. By choosing $2^{[n/m]}$ generators of $\overset{2^{n-(m-1)[n/m]}}{*} Z_+$, we obtain a subsemigroup which is isomorphic to $(\overset{2^{[n/m]}}{*} Z_+)^m$.

In general, for integers k, k' such that $km+k'=n$, we have a subsemigroup isomorphic to $(\overset{2^k}{*} Z_+)^m \times (Z_+ * Z_+)^{k'}$ or $(\overset{2^k}{*} Z_+)^m \times Z_+^{k'}$.

Here are the figures of $\Phi(\lambda)^{(1)}(U)$, $\lambda \in (Z_+ * Z_+)^2$ and $\Phi(\lambda)^{(1)}(U)$, $\lambda \in \overset{4}{*} Z_+$, the diagonal subsemigroup of $(Z_+ * Z_+)^2$.

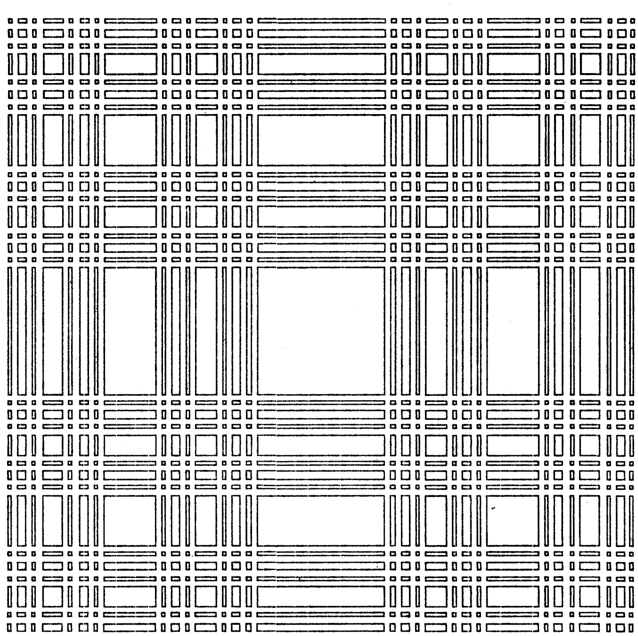

Figure (5.1)

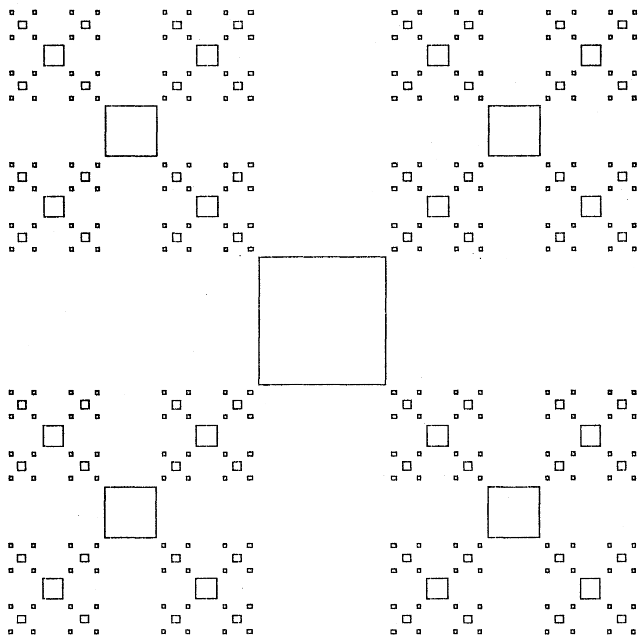

Figure (5.2)

§ 6. Cubic homology

Let G denote $\mathrm{Diff}_c^r(R^n)$. The homology of $B\bar{G}$ is of course the homology of the complex $S_*(G)/G$. Since the subcomplex $DS_*(G)$ of degenerate chains is invariant under the action of G, we may use the normalized complex $S'_*(G)/G$. We will use, however, the normalized cubic complex $Q'_*(G)/G$. First we review the singular cubic complex (see Serre [31]).

Cubic complex. Let $[0, 1]^m$ be the standard m-dimensional cube. A continuous map Q: $[0, 1]^m \to G$ is called a singular m-cube. Let $Q_m(G)$ be the free abelian group generated by singular m-cubes. For a singular m-cube Q, let $F_i^\varepsilon Q$: $[0, 1]^{m-1} \to G$ ($\varepsilon = 0, 1$; $i = 1, \cdots, m$) be the map given by

$$(F_i^\varepsilon Q)(t_1, \cdots, t_{m-1}) = Q(t_1, \cdots, t_{i-1}, \varepsilon, t_i, \cdots, t_{m-1}).$$

Put $\partial_i Q = F_i^0 Q - F_i^1 Q$ ($i = 1, \cdots, m$). The boundary homomorphism ∂: $Q_m(G) \to Q_{m-1}(G)$ is defined by

$$\partial Q = \sum_{i=1}^{m} (-1)^i \partial_i Q.$$

Thus we obtain a chain complex $(Q_*(G), \partial)$, the cubic complex of G. To represent the homology of G, we have to consider the normalized complex.

Let $D_iQ\colon [0, 1]^{m+1} \to G$ $(i=1, \cdots, m+1)$ be the map given by

$$(D_iQ)(t_1, \cdots, t_{m+1}) = Q(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{m+1}).$$

Let $DQ_*(G)$ denote the subcomplex of $Q_*(G)$ generated by the chains of the form D_iQ. Then the normalized cubic complex $Q'_*(G)$ is given by $Q'_*(G) = Q_*(G)/DQ_*(G)$.

It is easy to see that $DQ_*(G)$ is G-invariant. Hence we can define $Q'_*(G)/G$ which is denoted by $Q'_*(B\bar{G})$.

We have a chain equivalence $S'_*(G) \to Q'_*(G)$ which is induced from the map $[0, 1]^m \to \Delta^m$ given by

$$(t_1, t_2, \cdots, t_m) \longrightarrow (t_1, t_1t_2, \cdots, t_1t_2\cdots t_m).$$

Since this chain equivalence is G-equivariant, this induces a chain equivalence $S'_*(G)/G \to Q'_*(G)/G$.

We may consider the smooth singular cubic complex $Q^\infty_*(G)$. We obtain $Q^{\infty\prime}_*(G) = Q^\infty_*(G)/DQ^\infty_*(G)$ and $Q^{\infty\prime}_*(G)/G$. Then we have a chain equivalence $Q^{\infty\prime}_*(G)/G \to Q'_*(G)/G$.

Let $\{e_1, \cdots, e_m\}$ be the standard basis for $\boldsymbol{R}^m$. Let $\langle v_0, \cdots, v_m \rangle$ $(v_i \in \boldsymbol{R}^m, i=0, \cdots, m)$ denote the affine map $\Delta^m \to \boldsymbol{R}^n$ which sends 0 to v_0 and $\sum_{j=1}^{i} e_j$ to v_i $(i=1, \cdots, m)$. We have another chain map $\iota\colon Q'_*(G) \to S'_*(G)$ which sends a singular cube $Q\colon [0, 1]^m \to G$ to

$$\sum \det(\varepsilon_1, \cdots, \varepsilon_i, \cdots, \varepsilon_m) Q\left\langle 0, \varepsilon_1, \cdots, \sum_{j=1}^{i} \varepsilon_j, \cdots, \sum_{j=1}^{m} \varepsilon_j \right\rangle,$$

where the sum is taken over $\{\varepsilon_1, \cdots, \varepsilon_m\} = \{e_1, \cdots, e_m\}$.

Let bsd: $S'_*(G) \to S'_*(G)$ denote the barycentric subdivision. We have a map $\kappa\colon S'_*(G) \to Q'_*(G)$ which makes the following diagram commute:

$$\begin{tikzcd} S'_*(G) \arrow[rr, "\mathrm{bsd}"] \arrow[dr, "\kappa"'] & & S'_*(G) \\ & Q'_*(G) \arrow[ur, "\iota"'] & \end{tikzcd},$$

where $\iota\colon Q'_*(G) \to S'_*(G)$ is the map defined above. We are going to describe κ explicitly. Let $[b_0, \cdots, b_m]$ $(b_i \geqq 0, \sum_{i=0}^{m} b_i = 1)$ be the barycentric coordinate on $\Delta^m = \{(u_1, \cdots, u_m);\ 1 \geqq u_1 \geqq \cdots \geqq u_m \geqq 0\}$. Here $[b_0, \cdots, b_m]$ corresponds to $(b_1 + \cdots + b_m, b_2 + \cdots + b_m, \cdots, b_m)$ in Δ^m. Let $\alpha_j\colon [0, 1]^m \to \Delta^m$ $(j=0, \cdots, m)$ be the map given by

$$\alpha_j(t_1, \cdots, t_m)=[b_0, \cdots, b_m],$$

where
$$b_i=t_{i+1}\Big/\left(1+\sum_{l=1}^{m} t_l\right) \quad (i<j),$$
$$b_j=1\Big/\left(1+\sum_{l=1}^{m} t_l\right) \quad \text{and}$$
$$b_k=t_k\Big/\left(1+\sum_{l=1}^{m} t_l\right) \quad (k>j).$$

It might be helpful to look at the m-simplex in $\boldsymbol{R}^{m+1}$ spanned by $e_0, \cdots, e_m$ and the union X of the faces of the $(m+1)$-cube $[0, 1]^{m+1}$ which do not contain the origin. κ can be identified with the radial projection from X to this simplex. Then $\kappa\sigma$ is given by

$$\kappa\sigma=\sum_{j=0}^{m}(-1)^j\sigma\alpha_j.$$

It is easy to check that $\kappa\partial\sigma=\partial\kappa\sigma$.

Since the maps bsd, κ and ι are G-equivariant, we obtain the following commutative diagram:

$$\begin{tikzcd} S'_*(G)/G \arrow[rr, "\mathrm{bsd}"] \arrow[dr, "\kappa"'] & & S'_*(G)/G \\ & Q'_*(G)/G \arrow[ur, "\iota"'] & \end{tikzcd}$$

Since bsd is a chain equivalence, this commutative diagram implies the following proposition.

Proposition (6.1). *The chain map* ι: $Q'_*(G)/G \to S'_*(G)/G$ *induces a surjective map in homology.*

For the normalized singular cubic complex $Q'_*(B\bar{G})=Q'_*(G)/G$, we formulate several operations which are used later. As we explained before, a singular m-cube Q of $G=\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ corresponds to a C^r-foliated $\boldsymbol{R}^n$-product over $[0, 1]^m$. The support of Q is defined to be the support of this foliated product.

In the singular cubic homology theory, homotopies are written easily. We will use the following homotopies.

Conjugations. Let ψ: $[0, 1]\to G=\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be a map such that $\psi(0)=\mathrm{id}$. For a singular m-cube Q: $[0, 1]^m\to G$, define

$$\bar{C}_\psi Q\colon [0,1]^m\times[0, 1]\longrightarrow G$$

by $\overline{C}_\psi Q(t, s)=\psi(s)Q(t)$. $\overline{C}_\psi Q$ gives a homotopy between Q and $\psi(1)Q$. It is easy to see that $\mathrm{Supp}\,(\psi(1)Q)=\psi(1)(\mathrm{Supp}\,(Q))$. If Q is smooth and $\psi(1)$ is a C^∞-diffeomorphism with compact support, $\psi(1)Q$ is also smooth. The action of $\psi(1)$ on the left induces a chain map $\psi(1)\colon Q'_*(B\overline{G})\to Q'_*(B\overline{G})$ and $\overline{C}_\psi$ gives a chain homotopy C_ψ between the identity and $\psi(1)$;

$$\partial C_\psi Q+C_\psi\partial Q=Q-\psi(1)Q.$$

For, we have $\partial_i\overline{C}_\psi Q=\overline{C}_\psi\partial_i Q$ $(i=1, \cdots, m)$ and $\partial_{m+1}\overline{C}_\psi Q=Q-\psi(1)Q$. Hence we have

$$\partial\overline{C}_\psi Q=\overline{C}_\psi\partial Q+(-1)^{m+1}(Q-\psi(1)Q).$$

We put $C_\psi Q=(-1)^{m+1}\overline{C}_\psi Q$.

Subdivisions. A natural subdivision for the singular simplicial homology theory is the barycentric subdivision. For the singular cubic homology theory, a natural subdivision is obtained by cutting along hyperplanes parallel to the coordinate hyperplanes.

Let N be a positive integer. Let $\tau_{(i_1,\dots,i_m)}$ denote the translation by $(i_1, \cdots, i_m)\in \boldsymbol{Z}^m$;

$$\tau_{(i_1,\dots,i_m)}(t_1, \cdots, t_m)=(t_1+i_1, \cdots, t_m+i_m).$$

Let $a^{(N)}\colon [0, N]^m\to[0, 1]^m$ be the homothety by $1/N$. For $Q\colon [0, 1]^m\to G$, define the N-subdivision $s^{(N)}Q$ of Q by

$$s^{(N)}Q=\sum_{0\leqq i_1,\dots,i_m\leqq N-1} Qa^{(N)}\tau_{(i_1,\dots,i_m)}.$$

Let $b^{(N)}\colon [0, N]^m\to[0, 1]^m$ be the map given by

$$b^{(N)}(t_1, \cdots, t_m)=(\min\{t_1, 1\}, \cdots, \min\{t_m, 1\}).$$

Note that

$$\sum_{0\leqq i_1,\dots,i_m\leqq N-1} Qb^{(N)}\tau_{(i_1,\dots,i_m)}=Q \quad \text{in } Q'_m(G).$$

The maps $a^{(N)}$ and $b^{(N)}$ are homotopic. For, there is a homotopy $A^{(N)}\colon [0, N]^m\times[0, 1]\to[0, 1]^m$ given by

$$A^{(N)}(t, s)=(1-s)b^{(N)}(t)+sa^{(N)}(t).$$

Now define $\overline{S}^{(N)}\colon Q'_m(G)\to Q'_{m+1}(G)$ by

$$\overline{S}^{(N)}Q=\sum_{0\leqq i_1,\dots,i_m\leqq N-1} QA^{(N)}\overline{\tau}_{(i_1,\dots,i_m)},$$

where $\bar{\tau}_{(i_1,\dots,i_m)}(t,s)=(\tau_{(i_1,\dots,i_m)}(t),s)$. Then

$$\partial_i \bar{S}^{(N)}Q=\bar{S}^{(N)}\partial_i Q \quad (i=1,\cdots,m) \quad \text{and}$$
$$\partial_{m+1}\bar{S}^{(N)}Q=Q-s^{(N)}Q \quad \text{in } Q'_m(G).$$

Thus we have

$$\partial\bar{S}^{(N)}Q=\bar{S}^{(N)}\partial Q+(-1)^{m+1}(Q-s^{(N)}Q) \quad \text{in } Q'_m(G).$$

By putting $S^{(N)}Q=(-1)^{m+1}\bar{S}^{(N)}Q$, we have a chain homotopy $S^{(N)}$ between $s^{(N)}$ and the identity;

$$\partial S^{(N)}Q+S^{(N)}\partial Q=Q-s^{(N)}Q.$$

This chain homotopy is G-equivariant. Hence $s^{(N)}$ and $S^{(N)}$ are well-defined in $Q'_*(G)/G$.

Partitions. The following homotopy is similar to those in Banyaga [2] and Mather [22] (see also § 11). Let K_j $(j=1,\cdots,N)$ be compact subsets of $\boldsymbol{R}^n$ such that $\operatorname{Int} K_i\cap\operatorname{Int} K_j=\phi(i\neq j)$. Let Q be a singular m-cube written as $\bigcup_{j=1}^N Q_{(j)}$, where $Q_{(j)}\colon[0,1]^m\to G=\mathrm{Diff}_c^r(\boldsymbol{R}^n)$, $Q_{(j)}(0)=\mathrm{id}$ and $\operatorname{Supp}(\mathscr{F}_{Q_{(j)}})\subset K_j$ $(j=1,\cdots,N)$, where $\mathscr{F}_{Q_{(j)}}$ denotes the C^r-foliated $\boldsymbol{R}^n$-product corresponding to $Q_{(j)}$. Let $b_j\colon[0,N]\to[0,1]$ be the map given by

$$b_j(t)=\begin{cases}0 & [t]<j-1\\ t-[t], & [t]=j-1\\ 1, & [t]>j-1.\end{cases}$$

Let $w_j\colon[0,N]^m\to[0,1]^m$ be the map defined by

$$w_j(t_1,\cdots,t_m)=(b_j(t_1),\cdots,b_j(t_m)),$$

$(i=1,\cdots,m)$. Note that

$$b^{(N)}(t_1,\cdots,t_m)=(b_1(t_1),\cdots,b_1(t_m)).$$

The map w_j is homotopic to $b^{(N)}$. For, there is a homotopy $W_j\colon[0,N]^m\times[0,1]\to[0,1]^m$ given by

$$W_j(t,s)=(1-s)b^{(N)}(t)+sw_j(t).$$

Now consider $\bigcup_{j=1}^N Q_{(j)}w_j$. If $Q=\bigcup_{j=1}^N Q_{(j)}$ is smooth, then $\bigcup_{j=1}^N Q_{(j)}w_j$ is smooth except along the $m(N-1)$ hypersurfaces where $t_i=j$ $(i=1,\cdots,m; j=1,\cdots,N-1)$. This is homotopic to $\bigcup_{j=1}^N Q_{(j)}b^{(N)}$. Note that

$$\sum_{0\leq i_1,\cdots,i_m\leq N-1}\left(\bigcup_{j=1}^{N} Q_{(j)}b^{(N)}\right)\tau_{(i_1,\cdots,i_m)}=\bigcup_{j=1}^{N} Q_{(j)}$$

in $Q'_m(G)/G$. Put

$$\bar{p}Q=\sum_{0\leq i_1,\cdots,i_m\leq N-1}\left(\bigcup_{j=1}^{N} Q_{(j)}w_j\right)\tau_{(i_1,\cdots,i_m)} \quad\text{and}$$

$$\bar{P}Q=\sum_{0\leq i_1,\cdots,i_m\leq N-1}\left(\bigcup_{j=1}^{N} Q_{(j)}W_j\right)\bar{\tau}_{(i_1,\cdots,i_m)},$$

where $\bar{\tau}_{(i_1,\cdots,i_m)}(t, s)=(\tau_{(i_1,\cdots,i_m)}(t), s)$. We call $\bar{p}Q$ the partition of Q. $\bar{p}Q$ is a sum of N^m cubes. If Q is smooth, $\bar{p}Q$ as well as $\bar{P}Q$ is a sum of smooth cubes. Put $pQ=\sum_{i=1}^{N} Q_{(i)}$; then pQ is contained in $\bar{p}Q$ as

$$\sum_{i=1}^{N}\left(\bigcup_{j=1}^{N} Q_{(j)}w_j\right)\tau_{(i-1,\cdots,i-1)}.$$

Put $\bar{p}Q=pQ+rQ$. The support of the holonomy of the foliated product corresponding to

$$\left(\bigcup_{j=1}^{N} Q_{(j)}w_j\right)\tau_{(i_1,\cdots,i_m)}$$

in the direction of e_l lies in K_{i_l+1}. Thus rQ is a sum of m-cubes which are "decomposable".

Let $Q'_*(B\overline{\mathrm{Diff}}^r_{K_1,\cdots,K_N}(\boldsymbol{R}^n))$ be the normalized complex generated by the foliated $\boldsymbol{R}^n$-products $\mathscr{F}_Q$ such that $Q=\bigcup_{j=1}^{N} Q_{(j)}$, where $Q_{(j)}: [0, 1]^m \to \mathrm{Diff}^r_c(\boldsymbol{R}^n)$, $Q_{(j)}(0)=\mathrm{id}$ and $\mathrm{Supp}\,(\mathscr{F}_{Q_{(j)}})\subset K_j$. The partition $\bar{p}$ is a chain map from $Q'_*(B\overline{\mathrm{Diff}}^r_{K_1,\cdots,K_N}(\boldsymbol{R}^n))$ to $Q'_*(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n))$. Since the homotopy $\bar{P}$ commutes with the face operators, we have a chain homotopy P between the natural map

$$Q'_*(B\overline{\mathrm{Diff}}^r_{K_1,\cdots,K_N}(\boldsymbol{R}^n))\longrightarrow Q'_*(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n))$$

and the partition $\bar{p}$;

$$\partial PQ+P\partial Q=Q-\bar{p}Q.$$

Note that, by the definition of P, the chain PQ depends only on Q and $\bar{p}Q$.

For cubes of dimensions one and two, $\bar{p}Q$ is written as follows.

For $\quad Q=\bigcup_{i=1}^{N} Q_{(i)}: [0, 1]\to G, \qquad \bar{p}Q=pQ=\sum_{i=1}^{N} Q_{(i)}.$

For $\quad Q=\bigcup_{i=1}^{N} Q_{(i)}: [0, 1]^2\to G,$

$$\bar{p}Q=\sum_{i=1}^{N} Q_{(i)}+\sum_{i>j} F_2^0 Q_{(i)}\times F_1^1 Q_{(j)}+\sum_{i<j} F_2^1 Q_{(i)}\times F_1^0 Q_{(j)}.$$

Here for σ_1, σ_2: $[0, 1]\to G$ satisfying Int Supp $(\mathscr{F}_{\sigma_1})\cap$ Int Supp $(\mathscr{F}_{\sigma_2})=\phi$, $\sigma_1\times\sigma_2$: $[0, 1]^2\to G$ is defined by $(\sigma_1\times\sigma_2)(t_1, t_2)=\sigma_1(t_1)\sigma_1(0)^{-1}\sigma_2(t_2)\sigma_2(0)^{-1}$.

§ 7. A cubic complex $\hat{Q}$

Let G be a topological group. In this section we define a cubic complex $\hat{Q}'_*(G)/G$ which we use later.

$\hat{Q}_*(G)$. Let $e_1, e_2, \cdots$ be the standard basis of $\boldsymbol{R}^\infty$. For positive integers $j_1, \cdots, j_m$ with $j_1<\cdots<j_m$, let $Q(j_1, \cdots, j_m)$ denote the cube spanned by $e_{j_1}, \cdots, e_{j_m}$. Let $Q_{(j_1,\cdots,j_m)}(G)$ denote the free abelian group whose basis is the set of maps from $Q(j_1, \cdots, j_m)$ to G. We have the face maps

$$\partial_{j_k}: Q_{(j_1,\cdots,j_m)}(G)\longrightarrow Q_{(j_1,\cdots,\hat{j}_k,\cdots,j_m)}(G)$$

given by $\partial_{j_k}Q=F^0_{j_k}Q-F^1_{j_k}Q$ provided that $j_k\in\{j_1, \cdots, j_m\}$. Here,

$$F^\varepsilon_{j_k}Q(t_{j_1}, \cdots, \widehat{t_{j_k}}, \cdots, t_{j_m})=Q(t_{j_1}, \cdots, t_{j_{k-1}}, \varepsilon, t_{j_{k+1}}, \cdots, t_{j_m}).$$

Put

$$\hat{Q}_m(G)=\bigoplus_{j_1<\cdots<j_m} Q_{(j_1,\cdots,j_m)}(G).$$

We define the boundary $\hat{\partial}$: $\hat{Q}_m(G)\to\hat{Q}_{m-1}(G)$ by

$$\hat{\partial}=\sum(-1)^k\partial_{j_k}.$$

Then we have $\hat{\partial}^2=0$ and $\hat{Q}_*(G)=\oplus\hat{Q}_m(G)$ becomes a chain complex. For $j\notin\{j_1, \cdots, j_m\}$, say $j_l<j<j_{l+1}$, we define the degenerate cube D_jQ by

$$D_jQ(t_{j_1}, \cdots, t_{j_l}, t_j, t_{j_{l+1}}, \cdots, t_{j_m})=Q(t_{j_1}, \cdots, t_{j_m}).$$

We obtain the subcomplex $D\hat{Q}_*(G)$ of degenerate chains and we can define the normalization $\hat{Q}'_*(G)$ by $\hat{Q}'_*(G)=\hat{Q}_*(G)/D\hat{Q}_*(G)$. We have a chain map ϕ: $\hat{Q}'_*(G)\to Q'_*(G)$ induced from the map $[0, 1]^m\to Q(j_1, \cdots, j_m)$ which sends $(t_1, \cdots, t_m)$ to $\sum_{i=1}^m t_i e_{j_i}$. Since the face operators are G-equivariant, we obtain $\hat{Q}'_*(B\bar{G})=\hat{Q}'_*(G)/G$. We also have the smooth version $\hat{Q}^{\infty\prime}_*(B\bar{G})=\hat{Q}^{\infty\prime}_*(G)/G$.

Partial subdivisions. In $\hat{Q}'_*(G)/G$, we have the following partial subdivision. Let $e_1, \cdots, e_m$ be the standard basis of $\boldsymbol{R}^m(\subset\boldsymbol{R}^\infty)$. For $k=1, \cdots, m$, define $a_k^{(N)}$: $[0, 1]^{k-1}\times[0, N]\times[0, 1]^{m-k}\to[0, 1]^m$ by

$$a_k^{(N)}(t, t', t'')=(t, t'/N, t'').$$

For $Q\colon [0, 1]^m \to G$, put

$$s_k^{(N)}Q=\sum_{i=0}^{N-1} Qa_k^{(N)}\tau_{ie_k},$$

where τ_{ie_k} is the translation by ie_k. Let $b_k^{(N)}\colon [0, 1]^{k-1}\times[0, N]\times[0, 1]^{m-k} \to[0, 1]^m$ be the map defined by

$$b_k^{(N)}(t, t', t'')=(t, \min\{t', 1\}, t'').$$

Let $A_k^{(N)}\colon ([0, 1]^{k-1}\times[0, N]\times[0, 1]^{m-k})\times[0, 1]\to[0, 1]^m$ be the homotopy between $a_k^{(N)}$ and $b_k^{(N)}$ defined by

$$A_k^{(N)}(t, s)=(1-s)b_k^{(N)}(t)+sa_k^{(N)}(t).$$

Put

$$\bar{S}_k^{(N)}Q=\sum_{i=0}^{N-1} QA_k^{(N)}\tau_{ie_k}.$$

Then we have

$$\begin{aligned}&\partial_i\bar{S}_k^{(N)}Q=\bar{S}_k^{(N)}\partial_iQ \quad (i=1, \cdots, m; i\neq k)\\ &\partial_k\bar{S}_k^{(N)}Q=0 \quad\text{and}\quad \partial_{m+1}\bar{S}_k^{(N)}Q=Q-s_k^{(N)}Q \quad\text{in } Q'_m(G),\end{aligned}$$

where we consider Q as an element of $Q_{(1,2,\dots,m)}(G)$ $(\subset\hat{Q}_m(G))$ and $s_k^{(N)}$ and $\bar{S}_k^{(N)}$ are maps from $\hat{Q}_m(G)$ to $Q'_m(G)$ and to $Q'_{m+1}(G)$, respectively. We obtain $S_k^{(N)}$ such that

$$\begin{aligned}&\partial S_k^{(N)}Q+S_k^{(N)}\hat{\partial}Q=Q-s_k^{(N)}Q\\ &S_k^{(N)}\partial_kQ=0.\end{aligned}$$

In order to deduce the implication of our construction given in the next section, it is easier to work with $\hat{Q}'_*(G)/G=\hat{Q}'_*(B\bar{G})$ than with $Q'_*(G)/G=Q'_*(B\bar{G})$. We have the following proposition which tells us the triviality of the homology of $B\bar{G}$. We look at the chain maps $\phi\colon \hat{Q}'_*(G)/G \to Q'_*(G)/G$ and $\iota\colon Q'_*(G)/G\to S'_*(G)/G$.

Proposition (7.1). *Suppose that*

$$(\iota\phi)_i\colon H_i(\hat{Q}'_*(G)/G)\longrightarrow H_i(S'_*(G)/G)$$

is the zero map for $i=1, \cdots, m$. Then,

$$H_i(S'_*(G)/G)=H_i(B\bar{G}; Z)=0 \qquad \textit{for } i=1, \cdots, m.$$

Proposition (7.1) follows from Proposition (6.1) and the following lemma.

Lemma (7.2). *Suppose that*

$$(\iota\phi)_i\colon H_i(\hat{Q}'_*(G)/G) \longrightarrow H_i(S'_*(G)/G)$$

is the zero map for $i=1, \cdots, m$. Then,

$$\iota_i\colon H_i(Q'_*(G)/G) \longrightarrow H_i(S'_*(G)/G)$$

is the zero map for $i=1, \cdots, m$.

To prove this lemma, we look at those singular cubes which are invariant under the reflection with respect to the hyperplane $\{t_i=t_{i+1}\}\subset [0, 1]^{m+1}$ $(i=1, \cdots, m)$. Let Q be a singular m-cube of G. For $i=1, \cdots, m$, define $\nabla_i Q\colon [0, 1]^{m+1} \to G$ by

$$\nabla_i Q(t_1, \cdots, t_{m+1}) = Q(t_1, \cdots, t_{i-1}, \max\{t_i, t_{i+1}\}, t_{i+2}, \cdots, t_{m+1}).$$

Note that $\iota\nabla_i Q=0$ in $S'_*(G)$. We have the following formula in $Q'_*(G)$:

$$\partial_j \nabla_i Q = \begin{cases} \nabla_{i-1}\partial_j Q & (1\leqq j<i) \\ Q & (j=i, i+1) \\ \nabla_i \partial_{j-1} Q & (i+1<j\leqq m). \end{cases}$$

Then we have the following formulae for $\partial_j(1-\nabla_i\partial_i)$:

For $j<i$, $\quad \partial_j(1-\nabla_i\partial_i)=\partial_j-\nabla_{i-1}\partial_j\partial_i=(1-\nabla_{i-1}\partial_{i-1})\partial_j$.

For $j=i$, $\quad \partial_i(1-\nabla_i\partial_i)=\partial_i-\partial_i=0$.

For $j=i+1$, $\quad \partial_{i+1}(1-\nabla_i\partial_i)=\partial_{i+1}-\partial_i=(1-\nabla_i\partial_i)(\partial_{i+1}-\partial_i)$.

For $j>i+1$, $\quad \partial_j(1-\nabla_i\partial_i)=\partial_j-\nabla_i\partial_{j-1}\partial_i=(1-\nabla_i\partial_i)\partial_j$.

Put $B_j=(1-\nabla_{j-1}\partial_{j-1})\cdots(1-\nabla_1\partial_1)$ $(j=1, \cdots, m)$. (B_1 is understood to be 1.) Then we have

$$\partial_i B_j = 0 \quad (i=1, \cdots, j-1) \quad \text{and}$$

$$(-1)^j \partial_j B_j = B_{j-1}\left(\sum_{l=1}^{j}(-1)^l \partial_l\right) \quad (B_0=1).$$

The latter is obtained by an induction on j. It holds for $j=1$. First we note that

$$
\begin{aligned}
(-1)^j \partial_j B_j &= ((-1)^j \partial_j + (-1)^{j-1} \partial_{j-1}) B_{j-1} \\
&= (1 - \nabla_{j-1} \partial_{j-1})((-1)^j \partial_j + (-1)^{j-1} \partial_{j-1}) B_{j-1}. \\
&= (1 - \nabla_{j-1} \partial_{j-1})(-1)^j \partial_j B_j.
\end{aligned}
$$

We have

$$
\begin{aligned}
(-1)^j \partial_j B_j &= ((-1)^j \partial_j + (-1)^{j-1} \partial_{j-1}) B_{j-1} \\
&= B_{j-1}(-1)^j \partial_j + (-1)^{j-1} \partial_{j-1} B_{j-1} \\
&= B_{j-1}(-1)^j \partial_j + (1 - \nabla_{j-2} \partial_{j-2})(-1)^{j-1} \partial_{j-1} B_{j-1} \\
&= B_{j-1}(-1)^j \partial_j + (1 - \nabla_{j-2} \partial_{j-2}) B_{j-2} \Big(\sum_{l=1}^{j-1} (-1)^l \partial_l \Big) \\
&= B_{j-1} \Big(\sum_{l=1}^{j} (-1)^l \partial_l \Big).
\end{aligned}
$$

Proof of Lemma (7.2). By the assumption of Lemma (7.2), we have a chain homotopy A between $\iota\phi$: $\hat{Q}'_*(G)/G \to S'_*(G)/G$ and 0 in dimensions between 1 and m.

$$
\begin{tikzcd}
Z & \hat{Q}'_1(G)/G \arrow[l, "0"'] \arrow[d, "\iota\phi_1"] \arrow[dr, "A"] & \hat{Q}'_2(G)/G \arrow[l, "\hat{\partial}"'] \arrow[d, "\iota\phi_2"] \cdots & \hat{Q}'_{m-1}(G)/G \arrow[d, "\iota\phi_{m-1}"] & \hat{Q}'_m(G)/G \arrow[l, "\hat{\partial}"'] \arrow[d, "\iota\phi_m"] \arrow[dl, phantom] \arrow[dr, "A"] & \hat{Q}'_{m+1}(G)/G \arrow[l, "\hat{\partial}"'] \arrow[d, "\iota\phi_{m+1}"] \\
Z & S'_1(G)/G \arrow[l, "0"] & S'_2(G)/G \arrow[l, "\partial"] \cdots & S'_{m-1}(G)/G & S'_m(G)/G \arrow[l, "\partial"] & S'_{m+1}(G)/G. \arrow[l, "\partial"]
\end{tikzcd}
$$

We write $A \mid Q_{(j_1, \ldots, j_i)}$ by $A_{j_1 \ldots j_i}$. We construct a chain homotopy D between ι: $Q'_*(G)/G \to S'_*(G)/G$ and 0 by using A.

$$
\begin{tikzcd}
Z & Q'_1(G)/G \arrow[l, "0"'] \arrow[d, "\iota_1"] \arrow[dr, "D"] & Q'_2(G)/G \arrow[l, "\partial"'] \arrow[d, "\iota_2"] \cdots & Q'_{m-1}(G)/G \arrow[d, "\iota_{m-1}"] \arrow[dr, "D"] & Q'_m(G)/G \arrow[l, "\partial"'] \arrow[d, "\iota_m"] \arrow[dr, "D"] & Q'_{m+1}(G)/G \arrow[l, "\partial"'] \arrow[d, "\iota_{m+1}"] \\
Z & S'_1(G)/G \arrow[l, "0"] & S'_2(G)/G \arrow[l, "\partial"] \cdots & S'_{m-1}(G)/G & S'_m(G)/G \arrow[l, "\partial"] & S'_{m+1}(G)/G. \arrow[l, "\partial"]
\end{tikzcd}
$$

The chain homotopy is given by $D_j = A_{12 \ldots j} B_j$ $(j = 1, \cdots, m)$. In fact, for $Q \in Q'_j(G)/G$, by using the above formulae on B_j, we have

$$
\begin{aligned}
(\partial D_j + D_{j-1} \partial) Q &= \iota B_j Q - \sum_{i=1}^{j} A_{1 \ldots \hat{i} \ldots j} (-1)^i \partial_i B_j Q + A_{1 \ldots (j-1)} B_{j-1} \partial Q \\
&= \iota B_j Q = \iota Q.
\end{aligned}
$$

Here, in order to apply $A_{1 \ldots j}$, we identify $Q'_j(G)/G$ with $Q'_{(1, \ldots, j)}(G)/G$. Therefore, ι induces the zero map in homology in dimensions between 1 and m.

Proposition (7.3). *Let U be a bounded open ball in $\boldsymbol{R}^n$. Let $\bar{\phi}$ denote the composition*

$$\hat{Q}'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n)) \longrightarrow Q'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n)) \longrightarrow Q'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)).$$

Suppose that $\iota\bar{\phi}_i$: $H_i(\hat{Q}'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) \to H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)))$ is the zero map for $i=1, \cdots, m$. Then

$$H_i(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0 \qquad \textit{for } i=1, \cdots, m.$$

Proof. Since $(\iota\bar{\phi})_i$ is the zero map for $i=1, \cdots, m$, the proof of Lemma (7.2) implies that

$$\bar{\iota}_i\colon H_i(Q'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) \longrightarrow H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)))$$

is the zero map $(i=1, \cdots, m)$. On the other hand, $\bar{\iota}_i$ factors through $H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n)))$;

$$\begin{array}{ccc} H_i(Q'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) & \xrightarrow{\iota_i} & H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) \\ & \searrow^{\bar{\iota}_i} & \downarrow \\ & & H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n))). \end{array}$$

Since ι_i is surjective by Proposition (6.1) and $H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) \to H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)))$ is an isomorphism by Proposition (4.1), $\bar{\iota}_i$ is surjective. Thus $H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n))) = H_i(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0$ for $i=1, \cdots, m$.

Let $\hat{Q}_{*,m}(G)$ be the subcomplex of $\hat{Q}_*(G)$ given by

$$\hat{Q}_{i,m}(G) = \bigoplus_{1 \leqq j_1 < \cdots < j_i \leqq m} \hat{Q}_{(j_1, \cdots, j_i)}(G).$$

We also have its normalization $\hat{Q}'_{*,m}(G)$ and $\hat{Q}'_{*,m}(B\bar{G}) = \hat{Q}'_{*,m}(G)/G$. It would be convenient to consider the restriction $\bar{\phi}_{,m}$ of $\bar{\phi}$ to $\hat{Q}'_{*,m}(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))$. Since the proof of Lemma (7.2) as well as that of Proposition (7.3) uses the existence of a chain homotopy defined on this subcomplex $\hat{Q}'_{*,m}(G)/G$, we have the following proposition.

Proposition (7.4). *Suppose that*

$$\iota\bar{\phi}_{i,m}\colon H_i(\hat{Q}'_{*,m}(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n))) \longrightarrow H_i(S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)))$$

is the zero map for $i=1, \cdots, m$. Then $H_i(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0$ for $i=1, \cdots, m$.

§ 8. A construction

In this section, first we consider the 1-dimensional chains and show how our construction works. Using this construction, we prove a theorem of Mather [21] (Theorem (8.1)).

We take the homomorphism Φ: $(Z_+ * Z_+)^n \to \mathrm{Diff}_c^\infty(R^n)$ with an open ball U which we constructed in Section 5. Let $B=\{\beta\}$ denote the generating set of the subsemigroup $\overset{2^n}{*} Z_+$ which is given at the end of Section 5. Suppose that we have a smooth path Q: $[0, 1] \to G$ which has support in U; $\mathrm{Supp}(\mathscr{F}_Q) \subset U$. Let s denote the 2^n-subdivision. The subdivision sQ is a sum of 2^n paths (1-simplices) which are naturally ordered; hence by giving an arbitrary order to B, we can index the paths (1-simplices) of sQ using B as the index set;

$$sQ = \sum_{\beta \in B} s_\beta Q.$$

The simplex $\Phi(\beta)^{(1)} s_\beta Q$ has support in $\Phi(\beta)^{(1)}(U)$. We subdivide it again;

$$s\Phi(\beta(1))^{(1)} s_{\beta(1)} Q = \sum_{\beta(2) \in B} s_{\beta(2)} \Phi(\beta(1))^{(1)} s_{\beta(1)} Q.$$

Then we have $\Phi(\beta(2))^{(1)} s_{\beta(2)} \Phi(\beta(1))^{(1)} s_{\beta(1)} Q$ which has support in $\Phi(\beta(2)\beta(1))^{(1)}(U)$. Inductively, we obtain a path

$$Q_{\beta(k)\cdots\beta(1)} = \Phi(\beta(k))^{(1)} s_{\beta(k)} \cdots \Phi(\beta(1))^{(1)} s_{\beta(1)} Q$$

which has support in $\Phi(\beta(k)\cdots\beta(1))^{(1)}(U)$. Now put

$$IQ = \bigcup_{\lambda \in \Lambda} Q_\lambda, \qquad \text{where } \Lambda = \overset{2^n}{*} Z_+.$$

If $r-1<n$, we can take a sufficiently small positive real number ε in Section 5 so that IQ is a path in $\mathrm{Diff}_c^r(R^n)$. For, for $\beta \in B$, we have

$$|s_\beta Q| \leqq 2^{-n} |Q|,$$

where $|Q|$ denotes the C^r-norm $|\mathscr{F}_Q|$ of the foliated product $\mathscr{F}_Q$ over $[0, 1]$ corresponding to Q. By Lemmas (5.1) and (5.2),

$$|\Phi(\beta)^{(1)} s_\beta Q| \leqq (2+\varepsilon)^{r-1} 2^{-n} |Q|.$$

Hence, if $r-1<n$, then for sufficiently small ε, we have

$$|Q_\lambda| \longrightarrow 0 \quad \text{as} \quad l(\lambda) \longrightarrow \infty.$$

We have the following formulae for IQ.

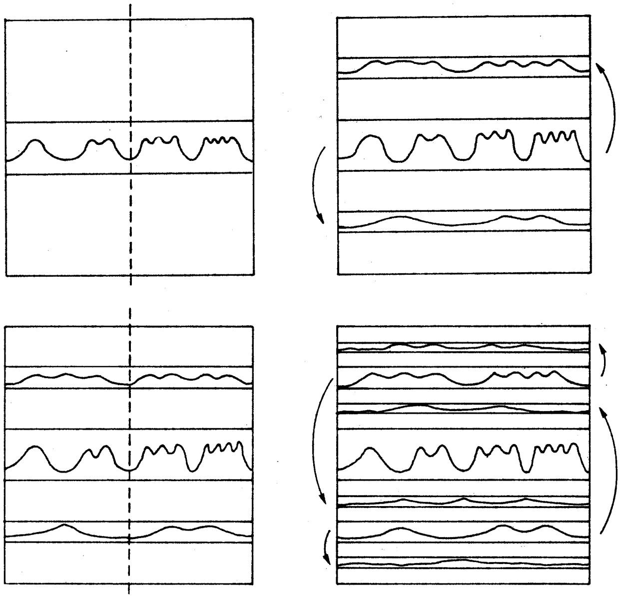

Figure (8.1)

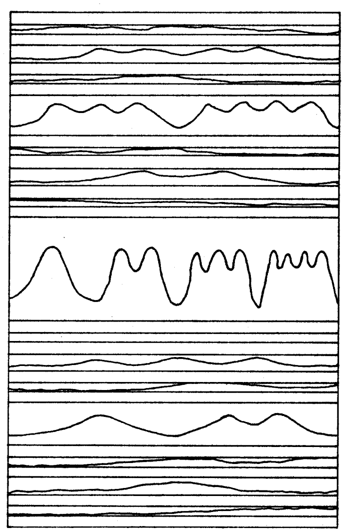

Figure (8.2)

$$sIQ = \sum_{\beta \in B} s_\beta IQ,$$

$$\Phi(\beta)^{(1)} s_\beta IQ = \bigcup_{\lambda \in \beta\Lambda} Q_\lambda \quad (\beta \in B),$$

$$IQ = Q \cup (\bigcup_{\beta \in B} (\bigcup_{\lambda \in \beta\Lambda} Q_\lambda)).$$

If we take the partition $\bar{p}$ with respect to the compact sets $K_0 = \mathrm{Cl}\, U$, $K_\beta = \Phi(\beta)^{(1)}([-1, 1]^n)$ $(\beta \in B)$, we have

$$\bar{p}IQ = Q + \sum_{\beta \in B} \Phi(\beta)^{(1)} s_\beta IQ.$$

Thus we have

$$Q = \partial(SIQ + \sum_{\beta \in B} C_{\Phi(\beta)} s_\beta IQ - PIQ).$$

Since every 1-cycle of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ is homologous to a sum of smooth 1-cycles with support in U (Proposition (4.1)), we have proved the following theorem which has been obtained by Mather [21] in a little different way.

Theorem (8.1). *$H_1(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$ if $1 \leqq r < n+1$.*

Now we consider a similar construction for m-dimensional chains. Here, it seems better to work with the cubic homology. We take Φ with U in Section 5 again. Let $\Lambda_1 \times \cdots \times \Lambda_m$, $\Lambda_i \cong \overset{2^{[n/m]}}{*} \boldsymbol{Z}_+ (i = 1, \cdots, m)$ denote the subsemigroup of $(\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$ given at the end of Section 5. Let $B_i = \{\beta_i\}$ denote the generating set of Λ_i $(i = 1, \cdots, m)$. Let $Q: [0, 1]^m \to G$ be a smooth singular m-cube which has support in U. Here the support of Q is the support of the foliated $\boldsymbol{R}^n$-product $\mathscr{F}_Q$. Let s denote the $2^{[n/m]}$-subdivision. Since sQ is a sum of $(2^{[n/m]})^m$ cubes which are lexicographically ordered, these cubes are indexed by $B_1 \times \cdots \times B_m$;

$$sQ = \sum_{(\beta_1, \cdots, \beta_m) \in B_1 \times \cdots \times B_m} s_{\beta_1 \cdots \beta_m} Q.$$

As before, we make $\Phi(\beta_1, \cdots, \beta_m)^{(1)}$ operate on $s_{\beta_1 \cdots \beta_m} Q$ and subdivide the resulted cube. We repeat this procedure and obtain an m-cube

$$Q_{\beta_1(k) \cdots \beta_1(1) \cdots \beta_m(k) \cdots \beta_m(1)}$$
$$= \Phi(\beta_1(k), \cdots, \beta_m(k))^{(1)} s_{\beta_1(k) \cdots \beta_m(k)} \cdots \Phi(\beta_1(1), \cdots, \beta_m(1))^{(1)} s_{\beta_1(1) \cdots \beta_m(1)} Q$$

which has support in $\Phi(\beta_1(k) \cdots \beta_1(1), \cdots, \beta_m(k) \cdots \beta_m(1))^{(1)}(U)$. Put

$$I_{12 \cdots m} Q = \bigcup_{l(\lambda_1) = \cdots = l(\lambda_m),\ \lambda_i \in \Lambda_i\ (i = 1, \cdots, m)} Q_{\lambda_1 \lambda_2 \cdots \lambda_m},$$

where l denotes the word length. We have

$$|s_{\beta_1\cdots\beta_m}Q|\leqq 2^{-[n/m]}|Q|,$$

and by Lemmas (5.1) and (5.2),

$$|\Phi(\beta_1, \cdots, \beta_m)^{(1)}s_{\beta_1\cdots\beta_m}Q|\leqq(2+\varepsilon)^{r-1}2^{-[n/m]}|Q|.$$

Hence for sufficiently small ε,

$$|Q_{\lambda_1\cdots\lambda_m}|\longrightarrow 0 \quad \text{as } l(\lambda_1)=\cdots=l(\lambda_m)\longrightarrow\infty$$

provided $r-1<[n/m]$. Thus, if $r-1<[n/m]$, $I_{12\cdots m}Q$ is a singular m-cube of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$.

We have the following formulae for $I_{12\cdots m}Q$.

$$sI_{12\cdots m}Q=\sum_{(\beta_1,\cdots,\beta_m)\in B_1\times\cdots\times B_m} s_{\beta_1\cdots\beta_m}I_{12\cdots m}Q,$$

$$\Phi(\beta_1, \cdots, \beta_m)^{(1)}s_{\beta_1\cdots\beta_m}I_{12\cdots m}Q=\bigcup_{\lambda_i\in\beta_i\Lambda_i\,(i=1,\cdots,m),\ l(\lambda_1)=\cdots=l(\lambda_m)} Q_{\lambda_1\lambda_2\cdots\lambda_m},$$

$$I_{12\cdots m}Q=Q\cup\Big(\bigcup_{(\beta_1,\cdots,\beta_m)\in B_1\times\cdots\times B_m}\Phi(\beta_1, \cdots, \beta_m)^{(1)}s_{\beta_1\cdots\beta_m}I_{12\cdots m}Q\Big).$$

Unfortunately we cannot conclude that $H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ directly from these formulae. For, $Q\rightarrow I_{12\cdots m}Q$ is not a chain map. We can, however, prove the following theorem.

Theorem (8.2). *Let $Q=\sigma_1\times\cdots\times\sigma_m$: $[0, 1]^m\rightarrow\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be the map given by*

$$(\sigma_1\times\cdots\times\sigma_m)(t_1, \cdots, t_m)=\sigma_1(t_1)\sigma_1(0)^{-1}\cdots\sigma_m(t_m)\sigma_m(0)^{-1},$$

where σ_i: $[0, 1]\rightarrow\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ $(i=1, \cdots, m)$ satisfy

$$\mathrm{Int\,Supp}\,(\mathscr{F}_{\sigma_i})\cap\mathrm{Int\,Supp}\,(\mathscr{F}_{\sigma_j})=\phi \quad (i\neq j).$$

Then Q is an m-cycle of $B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)$ and, if $r-1<[n/m]$, Q is homologous to zero.

Proof. We prove the theorem by an induction on m. The case when $m=1$ is true by Theorem (8.1). Suppose that the cycles of the form $\sigma_1\times\cdots\times\sigma_j$ $(1\leqq j<m)$ are homologous to zero in $B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)$. We may assume that $\mathscr{F}_{\sigma_i}$ $(i=1, \cdots, m)$ has support in U. Since sQ is a sum of cubes which are of the form $\sigma_1\times\cdots\times\sigma_m$, $I_{12\cdots m}Q$ is an m-cycle. We take the partition $\bar{p}$ with respect to the compact sets $K_0=\mathrm{Cl}\,U$, $K_{\beta_1\cdots\beta_m}=\Phi(\beta_1, \cdots, \beta_m)^{(1)}([-1, 1]^n)$ $((\beta_1, \cdots, \beta_m)\in B_1\times\cdots\times B_m)$. Note that these compact sets are closed balls. Then we have

$$\bar{p}I_{12\cdots m}Q = Q + \sum_{(\beta_1,\cdots,\beta_m)\in B_1\times\cdots\times B_m} \Phi(\beta_1, \cdots, \beta_m)^{(1)} s_{\beta_1\cdots\beta_m} I_{12\cdots m}Q + \sum{}'.$$

Here $\sum'$ is a sum of the cubes of the form $\omega_1 \times \omega_2$, where ω_1 and ω_2 are of the form $\sigma_1 \times \cdots \times \sigma_j$ with $1 \leqq j < m$ and $\mathrm{Supp}\,(\mathscr{F}_{\omega_1})$ and $\mathrm{Supp}\,(\mathscr{F}_{\omega_2})$ are contained in the unions of sets belonging to $\{U\} \cup \{K_{\beta_1\cdots\beta_m}, (\beta_1, \cdots, \beta_m) \in B_1 \times \cdots \times B_m\}$ which are disjoint. By the formula above, we have

$$\begin{aligned} &\partial(SI_{12\cdots m}Q - PI_{12\cdots m}Q + \sum_{(\beta_1,\cdots,\beta_m)\in B_1\times\cdots\times B_m} C_{\Phi(\beta_1,\cdots,\beta_m)} s_{\beta_1\cdots\beta_m} I_{12\cdots m}Q) \\ &\quad = Q + \sum{}'. \end{aligned}$$

Since $r-1 < [n/m]$ and $[n/m] \leqq [n/j]$ for j with $1 \leqq j < m$, by the induction hypothesis, the cubes of the form $\omega_1 \times \omega_2$ in $\sum'$ are homologous to zero. Thus Q is homologous to zero.

We will show that $\iota\bar{\phi}_{,m}\colon \hat{Q}'_{*,m}(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n)) \to S'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n))$ given in Section 7 induces the trivial map in homology of low dimensions. Then, by Proposition (7.4), we conclude that the homology of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ is trivial in the corresponding dimensions. In the next section we consider the 2-dimensional case.

§ 9. The second homology of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ $(1 \leqq r < [n/2])$

In this section, we show the first part of our main theorem;

$$H_2(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 \qquad \text{for } 1 \leqq r < [n/2].$$

Let $\Phi\colon (\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n \to P\,\mathrm{Diff}^\infty_c(\boldsymbol{R}^n)$ and U be those given in Section 5. Let $\Lambda_1 \times \Lambda_2$ $(\Lambda_i \cong \overset{2[n/2]}{*} \boldsymbol{Z}_+; i=1, 2)$ denote the subsemigroup which is also given in Section 5. Let G and G_U denote $\mathrm{Diff}^r_c(\boldsymbol{R}^n)$ and $\mathrm{Diff}^r_U(\boldsymbol{R}^n)$, respectively. We will construct a map $A\colon \hat{Q}'_{*,2}(B\overline{\mathrm{Diff}}{}^r_U(\boldsymbol{R}^n)) \to Q'_*(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n))$ such that $\bar{\phi} = \partial A + A\hat{\partial}$ (in dimensions 1 and 2);

$$\begin{tikzcd} \boldsymbol{Z} & \hat{Q}'_{1,2}(B\bar{G}_U) \arrow[l, "0"'] \arrow[d, "\bar{\phi}_1"] \arrow[dr, "A"] & \hat{Q}'_{2,2}(B\bar{G}_U) \arrow[l, "\hat{\partial}"'] \arrow[d, "\bar{\phi}_2"] \arrow[dr, "A"] & \\ \boldsymbol{Z} & Q'_1(B\bar{G}) \arrow[l, "0"'] & Q'_2(B\bar{G}) \arrow[l, "\partial"'] & Q'_3(B\bar{G}). \arrow[l, "\partial"'] \end{tikzcd}$$

Then, by applying Proposition (7.4) to $\iota\bar{\phi}$, $H_i(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$ for $i = 1, 2$.

Let A_j $(j=1, 2)$ and A_{12} denote $A\,|\,Q'_{(j)}(B\bar{G}_U)$ and $A\,|\,Q'_{(1,2)}(B\bar{G}_U)$, respectively. We would like to use $I_{12}Q$ given in Section 8 in order to construct A_1, A_2 and A_{12} such that $\bar{\phi} = \partial A_{12} + A_2(-\partial_1) + A_1(+\partial_2)$.

Let $Q\colon [0, 1]^2 \to G = \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be a smooth singular 2-cube which has support in U. We take $I_{12}Q$ given in Section 8. This $I_{12}Q$ has support in

$$K = \mathrm{Cl} \bigcup_{l(\lambda_1)=l(\lambda_2)} \Phi(\lambda_1, \lambda_2)^{(1)}(U).$$

Hence, $\partial_1 I_{12}Q$ and $\partial_2 I_{12}Q$ also have support in K. We are going to construct a 2-cycle from $I_{12}Q$. To make a 2-chain which bounds $\partial_1 I_{12}Q$ and $\partial_2 I_{12}Q$, we consider a construction similar to that in Theorem (8.1). Note that Int $\Phi(\lambda_1, e)^{(1)}(K)$ $(\lambda_1 \in \Lambda_1)$ are disjoint, so are Int $\Phi(e, \lambda_2)^{(1)}(K)$ $(\lambda_2 \in \Lambda_2)$. Hence, for a 1-simplex σ which has support in K, we can construct $I\sigma$ by using the semigroup Λ_1 or Λ_2.

More precisely, let $\sigma\colon [0, 1] \to G$ be a smooth path such that

$$\mathrm{Supp}\,(\mathscr{F}_\sigma) \subset K.$$

We consider σ as an element of $Q'_{(1)}(B\bar{G}) = Q'_{(1)}(G)/G$. The 1-simplices of the $2^{[n/2]}$-subdivision $s_1\sigma$ of $\sigma \in Q'_{(1)}(B\bar{G})$ are indexed by Λ_1;

$$s_1\sigma = \sum_{\beta_1 \in B_1} s_{\beta_1}\sigma.$$

Put

$$I_1\sigma = \bigcup_{\lambda_1 \in \Lambda_1} \sigma_{\lambda_1},$$

where, for $\lambda_1 = \beta_1(k) \cdots \beta_1(1)$,

$$\sigma_{\lambda_1} = \Phi(\beta_1(k), e)^{(1)} s_{\beta_1(k)} \cdots \Phi(\beta_1(1), e)^{(1)} s_{\beta_1(1)}\sigma.$$

For the C^r-norm $|\ \ |$, we have

$$|s_{\beta_1}\sigma| \leqq 2^{-[n/2]}|\sigma|.$$

By Lemmas (5.1) and (5.2), we have the following estimate as before.

$$|\Phi(\beta_1, e)^{(1)} s_{\beta_1}\sigma| \leqq (2+\varepsilon)^r |s_{\beta_1}\sigma|.$$

Hence, by the assumption that $r < [n/2]$, for sufficiently small ε, $I_1\sigma$ is a path in $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$.

We need several partitions $\bar{p}$ to give a 2-chain which bounds σ. Let $\bar{p}_{12}^d$ denote the partition with respect to the compact sets K and $[0, 1]^n -$ Int(K). Let $\bar{p}_1$ denote the partition with respect to the compact sets

$$K_{\beta_1} = \mathrm{Cl} \bigcup_{\lambda_1 \in \beta_1\Lambda_1, \lambda_2 \in \Lambda_2} \Phi(\lambda_1, \lambda_2)^{(1)}(U)\ (\subseteqq \Phi(\beta_1, e)^{(1)}([-1, 1]^n)) \quad (\beta_1 \in B_1).$$

Put

$$t_1\sigma = \bigcup_{\beta_1 \in B_1} \sigma_{\beta_1} \quad \text{and} \quad t_{11}\sigma = \bigcup_{l(\lambda_1)=2} \sigma_{\lambda_1}.$$

Then, by using $\bar{p}^d_{12}I_1\sigma = p^d_{12}I_1\sigma = \sigma + I_1t_1\sigma$ and

$$\bar{p}_1I_1t_1\sigma = p_1I_1t_1\sigma = \sum_{\beta_1 \in B_1} \Phi(\beta_1, e)^{(1)}s_{\beta_1}I_1\sigma,$$

we have

$$\phi\sigma = \partial(-P^d_{12}I_1\sigma + S_1I_1\sigma + \sum_{\beta_1 \in B_1} C_{\Phi(\beta_1,e)}s_{\beta_1}I_1\sigma - P_1I_1t_1\sigma).$$

Let $A_1\sigma$ denote the chain in this parenthesis;

$$\phi\sigma = \partial A_1\sigma.$$

In the same way, by using Λ_2, we define $A_2\sigma$ for $\sigma \in Q'_{(2)}(B\bar{G})$ with support in K;

$$\phi\sigma = \partial A_2\sigma.$$

A_1 and A_2 are defined for elements of $Q'_{(1)}(B\bar{G})$ and $Q'_{(2)}(B\bar{G})$ with support in K, respectively. Hence A_j $(j=1, 2)$ are defined on $Q'_{(j)}(B\bar{G}_U)$ and they are, in fact, the desired maps

$$A_j\colon Q'_{(j)}(B\bar{G}_U) \longrightarrow Q'_2(B\bar{G}) \quad (j=1, 2).$$

We are going to construct $A_{12}\colon Q'_{(1,2)}(B\bar{G}_U) \to Q'_3(B\bar{G})$ such that

$$\partial A_{12} + A_2(-\partial_1) + A_1(+\partial_2) = \bar{\phi}.$$

For $Q\colon [0, 1]^2 \to G$ with support in U, we have constructed two 2-cycles $\phi Q - A_2(-\partial_1)Q - A_1(+\partial_2)Q$ and $\phi I_{12}Q - A_2(-\partial_1)I_{12}Q - A_1(+\partial_2)I_{12}Q$ (we are considering Q and $I_{12}Q$ as elements of $Q'_{(1,2)}(B\bar{G})$). Put

$$t_{12}Q = \bigcup_{(\beta_1,\beta_2) \in B_1 \times B_2} \Phi(\beta_1, \beta_2)^{(1)}s_{\beta_1\beta_2}Q.$$

Then, we can see that $I_{12}Q = Q \cup I_{12}t_{12}Q$. To construct A_{12}, we will show that $\phi I_{12}Q - A_2(-\partial_1)I_{12}Q - A_1(+\partial_2)I_{12}Q$ is homologous to

$$\phi t_{12}I_{12}Q - A_2(-\partial_1)t_{12}I_{12}Q - A_1(+\partial_2)t_{12}I_{12}Q.$$

On the other hand, we will see that $\phi I_{12}Q - A_2(-\partial_1)I_{12}Q - A_1(+\partial_2)I_{12}Q$ is homologous to the sum of $\phi Q - A_2(-\partial_1)Q - A_1(+\partial_2)Q$ and

$$\phi t_{12}I_{12}Q - A_2(-\partial_1)t_{12}I_{12}Q - A_1(+\partial_2)t_{12}I_{12}Q.$$

These imply that $\phi Q - A_2(-\partial_1)Q - A_1(+\partial_2)Q$ is homologous to zero. Therefore, there exists A_{12}.

Before verifying these, we give here a list of partitions $\bar{p}$ with respect to several finite families of compact sets $\{K_i\}$.

$\bar{p}_{12}^e$: $K_1 = \mathrm{Cl}\, U$, $K_2 = \mathrm{Cl} \bigcup_{(\lambda_1, \lambda_2) \neq (e, e)} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$ $(\subseteqq [-1, 1]^n - U)$.

$\bar{p}_1^e$: $K_1 = \mathrm{Cl} \bigcup_{\lambda_2 \in \Lambda_2} \Phi(e, \lambda_2)^{(1)}(U)$,

$K_2 = \mathrm{Cl} \bigcup_{\lambda_1 \neq e} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$ $(\subseteqq \bigcup_{\beta_1 \in B_1} \Phi(\beta_1, e)^{(1)}([-1, 1]^n))$.

$\bar{p}_2^e$: $K_1 = \mathrm{Cl} \bigcup_{\lambda_1 \in \Lambda_1} \Phi(\lambda_1, e)^{(1)}(U)$,

$K_2 = \mathrm{Cl} \bigcup_{\lambda_2 \neq e} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$ $(\subseteqq \bigcup_{\beta_2 \in B_2} \Phi(e, \beta_2)^{(1)}([-1, 1]^n))$.

$\bar{p}_{12}^d$: $K_1 (= K) = \mathrm{Cl} \bigcup_{l(\lambda_1) = l(\lambda_2)} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$,

$K_2 = \mathrm{Cl} \bigcup_{l(\lambda_1) \neq l(\lambda_2)} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$.

$\bar{p}_{12}^{d1}$: $K_1 (= K^1) = \mathrm{Cl} \bigcup_{l(\lambda_1) = l(\lambda_2)+1} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$,

$K_2 = \mathrm{Cl} \bigcup_{l(\lambda_1) \neq l(\lambda_2)+1} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$.

$\bar{p}_1$: $K_{\beta_1} = \mathrm{Cl} \bigcup_{\lambda_1 \in \beta_1 \Lambda_1, \lambda_2 \in \Lambda_2} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$ $(\subseteqq \Phi(\beta_1, e)^{(1)}([-1, 1]^n))$ $(\beta_1 \in B_1)$.

$\bar{p}_2$: $K_{\beta_2} = \mathrm{Cl} \bigcup_{\lambda_1 \in \Lambda_1, \lambda_2 \in \beta_2 \Lambda_2} \Phi(\lambda_1, \lambda_2)^{(1)}(U)$ $(\subseteqq \Phi(e, \beta_2)^{(1)}([-1, 1]^n))$ $(\beta_2 \in B_2)$.

Let $Q: [0, 1]^2 \to G$ be a smooth singular 2-cube which has support in U. We consider Q as an element of $Q'_{(1,2)}(B\bar{G}_U)$; then we have its partial subdivisions (§ 7). Put

$$s_1 Q = \sum_{\beta_1 \in B_1} s_{\beta_1} Q \quad \text{and} \quad t_1 Q = \bigcup_{\beta_1 \in B_1} \Phi(\beta_1, e)^{(1)} s_{\beta_1} Q.$$

We also have

$$s_2 Q = \sum_{\beta_2 \in B_2} s_{\beta_2} Q \quad \text{and} \quad t_2 Q = \bigcup_{\beta_2 \in B_2} \Phi(e, \beta_2)^{(1)} s_{\beta_2} Q.$$

Then we have

$$s_2 \Phi(\beta_1, e)^{(1)} s_{\beta_1} Q = \sum_{\beta_2 \in B_2} s_{\beta_2} \Phi(\beta_1, e)^{(1)} s_{\beta_1} Q$$

and put

$$t_{12} Q = \bigcup_{(\beta_1, \beta_2) \in B_1 \times B_2} \Phi(e, \beta_2)^{(1)} s_{\beta_2} \Phi(\beta_1, e)^{(1)} s_{\beta_1} Q.$$

Note that

$$s_{\beta_1\beta_2}Q = s_{\beta_2}s_{\beta_1}Q = s_{\beta_1}s_{\beta_2}Q \quad \text{and}$$
$$\Phi(\beta_1, \beta_2)^{(1)}s_{\beta_1\beta_2}Q = \Phi(e, \beta_2)^{(1)}s_{\beta_2}\Phi(\beta_1, e)^{(1)}s_{\beta_1}Q.$$

We consider a 2-cycle which is obtained from $I_{12}t_1Q$. For a path $\sigma: [0, 1] \to G$ with support in

$$K^1 = \mathrm{Cl} \bigcup_{l(\lambda_1)=l(\lambda_2)+1} \Phi(\lambda_1, \lambda_2)^{(1)}(U) = \bigcup_{\beta_1 \in B_1} \Phi(\beta_1, e)(K),$$

we define $A_1^1\sigma$ and $A_2^1\sigma$ by using Λ_1 and Λ_2 in such a way as we defined A_1 and A_2, respectively;

$$\phi\sigma = \partial A_1^1\sigma \quad \text{and} \quad \phi\sigma = \partial A_2^1\sigma.$$

We are going to show that $\phi I_{12}Q - A_2(-\partial_1)I_{12}Q - A_1(+\partial_2)I_{12}Q$ is homologous to $\phi I_{12}t_1Q - A_2^1(-\partial_1)I_{12}t_1Q - A_1^1(+\partial_2)I_{12}t_1Q$. We have to compare A_1 with A_1^1 and A_2 with A_2^1. This comparison is done as follows.

Let $\sigma: [0, 1] \to G = \mathrm{Diff}_c^r(\boldsymbol{R}^n)$ be a smooth path with support in K. For σ, we note the following ladder (we are assuming that $r < [n/2]$). In this diagram (Diagram (9.1)), we consider σ as an element of $Q'_{(1)}(B\bar{G})$. $I_1\sigma$, $t_1\sigma$ and $I_1t_1\sigma$ are also considered as elements of $Q'_{(1)}(B\bar{G})$. Here, for $X, Y \in Q'_{(1)}(G)/G$, $X \xrightarrow{Z} Y$ means $\phi(X - Y) = \partial Z$; C_{β_1} stands for $C_{\Phi(\beta_1, e)}$.

First, note that A_1 is obtained by using the first three arrows of the first column and the first row. Similarly, using the second row and the three subsequent arrows of the first column, we obtain A_1^s such that

$$s_1\sigma = \partial A_1^s s_1\sigma.$$

We also have A_1^p such that

$$p_1t_1\sigma = \partial A_1^p p_1t_1\sigma.$$

We observe that A_1^1 satisfying

$$t_1\sigma = \partial A_1^1 t_1\sigma$$

is also defined in this way. Here, in fact, A_1^1 is defined for any element of $Q'_{(1)}(B\bar{G})$ with support in $K^1 (= \bigcup_{\beta_1 \in B_1} \Phi(\beta_1, e)^{(1)}(K))$.

Secondly, the boundary of each rectangle is the boundary of some 3-chain:

$$S_1(1 - p_{12}^d)I_1\sigma - P_{12}^d(1 - s_1)I_1\sigma = \partial(P_{12}^d S_1 I_1\sigma + Br_{12}^d S_1 I_1\sigma),$$
$$\sum C_{\beta_1}(1 - p_{12}^d)s_{\beta_1}I_1\sigma - P_{12}^d s_1 I_1\sigma + P_{12}^{d1}p_1 I_1 t_1\sigma = \partial(-\sum C_{\beta_1}P_{12}^d s_{\beta_1}I_1\sigma),$$
$$-P_1(1 - p_{12}^{d1})I_1t_1\sigma + P_{12}^{d1}(1 - p_1)I_1t_1\sigma = \partial(P_1 P_{12}^{d1} I_1 t_1\sigma + Br_1 P_{12}^{d1} I_1 t_1\sigma).$$

$$
\begin{tikzcd}
I_1\sigma \arrow[r, "P^d_{12}I_1\sigma"] \arrow[d, "S_1I_1\sigma"] & p^d_{12}I_1\sigma=\sigma+I_1t_1\sigma \arrow[d, "S_1p^d_{12}I_1\sigma"] \\
s_1I_1\sigma \arrow[r, "P^d_{12}s_1I_1\sigma"] \arrow[d, "\sum C_{\beta_1}s_{\beta_1}I_1\sigma"] & p^d_{12}s_1I_1\sigma=s_1\sigma+s_1I_1t_1\sigma \arrow[d, "\sum C_{\beta_1}p^d_{12}s_{\beta_1}I_1\sigma"] \\
p_1I_1t_1\sigma \arrow[r, "P^{d1}_{12}p_1I_1t_1\sigma"] \arrow[d, "-P_1I_1t_1\sigma"] & p^{d1}_{12}p_1I_1t_1\sigma=p_1t_1\sigma+p_1I_1t_{11}\sigma \arrow[d, "-P_1p^{d1}_{12}I_1t_1\sigma"] \\
I_1t_1\sigma \arrow[r, "P^{d1}_{12}I_1t_1\sigma"] \arrow[d, "S_1I_1t_1\sigma"] & p^{d1}_{12}I_1t_1\sigma=t_1\sigma+I_1t_{11}\sigma \arrow[d, "S_1p^{d1}_{12}I_1t_1\sigma"] \\
s_1I_1t_1\sigma \arrow[r, "P^{d1}_{12}s_1I_1t_1\sigma"] \arrow[d, "\sum C_{\beta_1}s_{\beta_1}I_1t_1\sigma"] & p^{d1}_{12}s_1I_1t_1\sigma=s_1t_1\sigma+s_1I_1t_{11}\sigma \arrow[d, dash] \\
p_1I_1t_{11}\sigma \arrow[d, "-P_1I_1t_{11}\sigma"] & \vdots \\
I_1t_{11}\sigma \arrow[d, dash] & \\
\vdots &
\end{tikzcd}
$$

Diagram (9.1)

Here, $Br_1P_{12}^{d1}I_1t_1\sigma$ is a 3-chain bounding $r_1P_{12}^{d1}I_1t_1\sigma$ obtained by Theorem (8.1) (or (8.2)); $r_{12}^dS_1I_1\sigma$ is a sum of 2-cycles satisfying the condition of Theorem (8.2) and $Br_{12}^dS_1I_1\sigma$ denotes the 3-chain bounding $r_{12}^dS_1I_1\sigma$ given by Theorem (8.2).

The part lower than $I_1t_1\sigma$ of the first column is a summand of the whole second column. Hence the following 2-cycles are homologous to zero by the three formulae above.

$$\begin{aligned}
&S_1\sigma-A_1\sigma+A_1^ss_1\sigma=-\partial(P_{12}^dS_1I_1\sigma+Br_{12}^dS_1I_1\sigma),\\
&\sum C_{\beta_1}s_{\beta_1}\sigma-A_1^ss_1\sigma+A_1^pp_1t_1\sigma=-\partial(-\sum C_{\beta_1}P_{12}^ds_{\beta_1}I_1\sigma),\\
&-P_1t_1\sigma-A_1^pp_1t_1\sigma+A_1^1t_1\sigma=-\partial(P_1P_{12}^{d1}I_1t_1\sigma+Br_1P_{12}^{d1}I_1t_1\sigma).
\end{aligned}$$

By using Λ_2 we also obtain a similar ladder for $\sigma\in Q'_{(2)}(B\overline{G})$ with support in K, and we have A_2, A_2^s, A_2^p, etc., satisfying similar formulae. If $\sigma(\in Q'_{(2)}(B\overline{G}))$ has support in K^1 instead of K, we use P_{12}^{d1} instead of P_{12}^d and we define A_2^1, A_2^{1s}, A_2^{1p} (by using Λ_2). Note that we have

$$\sum \Phi(\beta_1, e)^{(1)}A_2(-\partial_1)s_{\beta_1}I_{12}Q=A_2^1(-\partial_1)p_1I_{12}t_1Q.$$

Now for the 2-cycle $\phi I_{12}Q-A_2(-\partial_1)I_{12}Q-A_1(+\partial_2)I_{12}Q$, using the above formulae, we have the following diagram (Diagram (9.2)).

Here, the chain in the parenthesis { } at the third arrow is obtained by looking at the expression

$$(P_1+Br_1)A_2^1(-\partial_1)I_{12}t_1Q.$$

This expression does not make sense. For, $A_2^1(-\partial_1)I_{12}t_1Q$ has the term $C_{\beta_2}s_{\beta_2}I_2(-\partial_1)I_{12}t_1Q$. The support of this term contains $[-1,1]^n$ and is not contained in $\bigcup_{\beta_1\in B_1}K_{\beta_1}$ (nor in $\bigcup_{\beta_1\in B_1}\Phi(\beta_1, e)^{(1)}([-1,1]^n)$). Hence we cannot apply the homotopy P_1 to this term.

However, by changing the order of P_1 and C_{β_2} in this expression, we obtain the above parenthesis which gives the desired boundary. (See also Diagrams (9.4)–(9.7).)

By Diagram (9.2), $\phi I_{12}Q-A_2(-\partial_1)I_{12}Q-A_1(+\partial_2)I_{12}Q$ is homologous to $\phi I_{12}t_1Q-A_2^1(-\partial_1)I_{12}t_1Q-A_1^1(+\partial_2)I_{12}t_1Q$. In a similar way, $\phi I_{12}t_1Q-A_2^1(-\partial_1)I_{12}t_1Q-A_1^1(+\partial_2)I_{12}t_1Q$ is homologous to $\phi I_{12}t_{12}Q-A_2(-\partial_1)I_{12}t_{12}Q-A_1(+\partial_2)I_{12}t_{12}Q$.

On the other hand, we have the following diagram (Diagram (9.3)). To obtain this diagram, we apply the partitions $\bar{p}_{12}^e$, $\bar{p}_1^e$ and $\bar{p}_2^e$ to $I_{12}Q$, $A_2(-\partial_1)I_{12}Q$ and $A_1(+\partial_2)I_{12}Q$, respectively. Note that, to obtain the correct 3-chains, we have to change the order of P_1^e and C_{β_2}, or P_2^e and C_{β_1} as in Diagram (9.2). Since $p_{12}^e\partial_2I_{12}Q=p_2^e\partial_2I_{12}Q$, we see that $P_1^e\partial_2I_{12}Q=$

$$
\begin{array}{l}
\phi I_{12}Q-A_2(-\partial_1)I_{12}Q-A_1(+\partial_2)I_{12}Q \\
\Bigg\downarrow \begin{array}{l} S_1I_{12}Q \\ \quad -(P^d_{12}+Br^d_{12})S_1I_1(+\partial_2)I_{12}Q \end{array} \\
\phi s_1I_{12}Q-A_2(-\partial_1)s_1I_{12}Q-A^s_1(+\partial_2)s_1I_{12}Q \\
\Bigg\downarrow \begin{array}{l} \sum C_{\beta_1}s_{\beta_1}I_{12}Q \\ \quad -\sum C_{\beta_1}A_2(-\partial_1)s_{\beta_1}I_{12}Q \\ \quad +\sum C_{\beta_1}P^d_{12}s_{\beta_1}I_1(+\partial_2)I_{12}Q \end{array} \\
\phi p_1I_{12}t_1Q-A^1_2(-\partial_1)p_1I_{12}t_1Q-A^p_1(+\partial_2)p_1I_{12}t_1Q \\
\Bigg\downarrow \begin{array}{l} -(P_1+Br_1)I_{12}t_1Q \\ \quad +\{(P_1+Br_1)((-P^{d1}_{12}+S_2)I_2(-\partial_1)I_{12}t_1Q-P_2I_2(-\partial_1)I_{12}t_{12}Q) \\ \qquad +\sum C_{\beta_2}P_1s_{\beta_2}I_2(-\partial_1)I_{12}t_1Q\} \\ \quad -(P_1+Br_1)P^{d1}_{12}I_1(+\partial_2)I_{12}t_1Q \end{array} \\
\phi I_{12}t_1Q-A^1_2(-\partial_1)I_{12}t_1Q-A^1_1(+\partial_2)I_{12}t_1Q
\end{array}
$$

Diagram (9.2)

$P^e_2\partial_2I_{12}Q$. We also have $P^e_{12}\partial_1I_{12}Q=P^e_1\partial_1I_{12}Q$. (See also Diagrams (9.4)–(9.7).)

By Diagrams (9.2) and (9.3), $\phi Q-A_2(-\partial_1)Q-A_1(+\partial_2)Q$ is homologous to zero. By Proposition (7.4), this completes the proof of

$$H_2(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n);\boldsymbol{Z})=0 \qquad \text{for } 1\leqq r<[n/2].$$

In order to treat the homology groups of higher dimensions of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ in the next section, we devote the rest of this section to examining the construction of 3-chains in the above proof.

The 3-chain used in this proof is constructed from Diagram (9.4)

$$
\begin{array}{l}
\phi I_{12}Q - A_2(-\partial_1)I_{12}Q - A_1(+\partial_2)I_{12}Q \\
\Bigg\downarrow \;
\begin{array}{l}
(P^e_{12}+Br^e_{12})I_{12}Q \\
\quad +(P^e_1+Br^e_1)((P^d_{12}-S_2)I_2(-\partial_1)I_{12}Q+P_2I_2(-\partial_1)I_{12}t_2Q) \\
\quad +\sum C_{\beta_2}P^e_1 s_{\beta_2}I_2(-\partial_1)I_{12}Q \\
\quad +(P^e_2+Br^e_2)((P^d_{12}-S_1)I_1(+\partial_2)I_{12}Q+P_1I_1(+\partial_2)I_{12}t_1Q) \\
\quad +\sum C_{\beta_1}P^e_2 s_{\beta_1}I_1(+\partial_2)I_{12}Q
\end{array} \\
\phi p^e_{12}I_{12}Q - A_2(-\partial_1)p^e_{12}I_{12}Q - A_1(+\partial_2)p^e_{12}I_{12}Q \\
\quad = \phi Q - A_2(-\partial_1)Q - A_1(+\partial_2)Q \\
\qquad + \phi I_{12}t_{12}Q - A_2(-\partial_1)I_{12}t_{12}Q - A_1(+\partial_2)I_{12}t_{12}Q.
\end{array}
$$

Diagram (9.3)

which describes the face relations. Note again that $P^e_{12}\partial_2 I_{12}Q=P^e_2\partial_2 I_{12}Q$ and $P^e_{12}\partial_1 I_{12}Q=P^e_1\partial_1 I_{12}Q$. Diagram (9.4) represents the 3-chain which bounds

$$
\begin{aligned}
(\phi-A_2(-\partial_1)-A_1(+\partial_2))p^e_{12}I_{12}Q-(\phi-A_2(-\partial_1)-A_1(+\partial_2))I_{12}t_{12}Q \\
=(\phi-A_2(-\partial_1)-A_1(+\partial_2))Q.
\end{aligned}
$$

For each face of Diagram (9.4), we have a 3-chain bounding the 2-cycle on its boundary. Thus the upper half of the front faces of Diagram (9.4) corresponds to Diagram (9.2), and the three top faces of Diagram (9.4) correspond to Diagram (9.3).

The 3-chains corresponding to the faces with (*) in Diagram (9.4) are obtained easily. These are the top face and the three front faces of Diagram (9.5). In Diagram (9.5), we do not write the orientations of edges. The face relations such as

$$\partial P^e_{12}I_{12}Q=I_{12}Q-\bar{p}^e_{12}I_{12}Q-P^e_{12}\partial I_{12}Q$$

are written in Diagram (9.5) modulo the terms such as $r^e_{12}I_{12}Q$, $r^e_{12}S_1\partial_2 I_{12}Q$, etc. For, these terms are 2-cycles homologous to zero by Theorem (8.1) or (8.2). To check this diagram, note that we have $\Phi(\beta_1)^{(1)}P^e_{12}=P^{e1}_{12}\Phi(\beta_1)^{(1)}$ by the definition of P^e_{12} and P^{e1}_{12}, and $\sum P^{e1}_{12}\Phi(\beta_1)^{(1)}\partial_1 s_{\beta_1}I_{12}Q=p_1P^{e1}_{12}\partial_1 I_{12}t_1Q$. Note also that Diagram (9.5) consists of the faces of 4-chains $P^e_{12}S_1I_{12}Q$, $\sum C_{\beta_1}P^e_{12}s_{\beta_1}I_{12}Q$ and $P_1P^{e1}_{12}I_{12}t_1Q$ modulo the terms containing r.

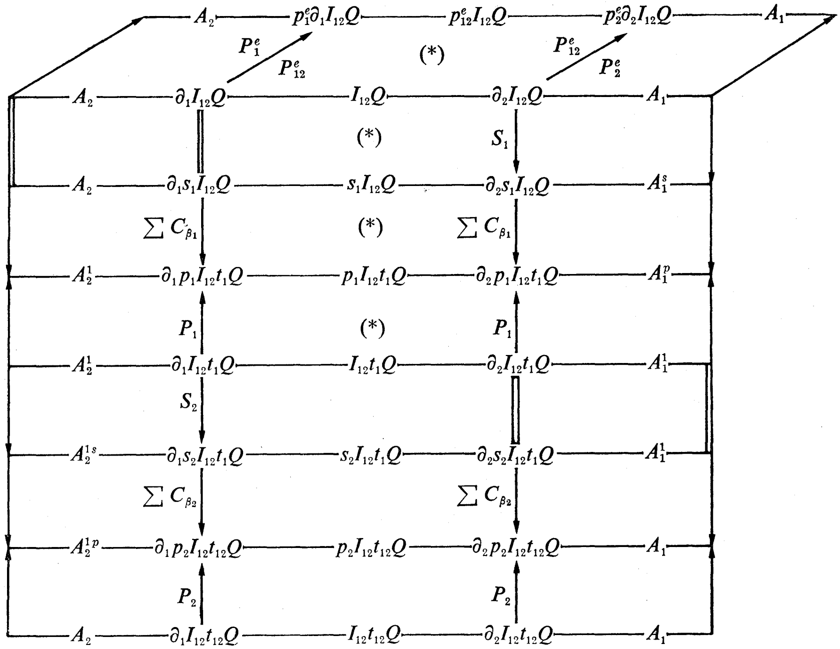

Diagram (9.4)

The 3-chains corresponding to the faces in the upper half of Diagram (9.4) which contain A_1, A_1^s, A_1^p and ∂_2 (the faces on the right of $(*)$) are obtained from Diagram (9.6) by substituting $\partial_2 I_{12} Q$ for Q. Note again that the face relations are written up to the terms such as $r_{12}^d S_1 I_1 Q$, $r_1 P_{12}^{d1} I_1 t_1 Q$. The front faces of Diagram (9.6) are the faces of Diagram (9.1). Diagram (9.6) is obtained from Diagram (9.1) by taking the product with "the edge P_2^e". Note that, to represent 3-chains, we are writing the symbols C_{β_1}, P_1, P_2^e, P_{12}^d, S_1 in this order. Thus the 3-chain corresponding to the top face in Diagram (9.4), bounding $P_2^e\partial_2 I_{12}Q - A_1\partial_2 I_{12}Q + A_1 p_2^e \partial_2 I_{12} Q$ is given by the top face and the faces on the left-hand side of Diagram (9.6). The 3-chains corresponding to the other three faces containing ∂_2 are obtained from the front faces of Diagram (9.6) (as are written down after Diagram (9.1)). Note that Diagram (9.6) consists of the faces of the 4-chains $P_2^e P_{12}^d S_1 I_1 Q$, $\sum C_{\beta_1} P_2^e P_{12}^d s_{\beta_1} I_1 Q$ and $P_1 P_2^e P_{12}^{d1} I_1 t_1 Q$ modulo the terms containing r.

On the other hand, the 3-chains corresponding to the faces in the upper half of Diagram (9.4) containing A_2, A_2^1 and ∂_1 (the faces on the left of $(*)$) are obtained from Diagram (9.7). Diagram (9.7) is obtained from $A_2\partial_1 s_1 I_{12} Q$ by taking the product with the edges $\sum C_{f_1}$ and P_1. Here note that $\Phi(\beta_1)^{(1)}C_{\beta_2}=C_{\beta_2}\Phi(\beta_1)^{(1)}$ by Remark (5.3). The 3-chain corresponding to the top face in Diagram (9.4) bounding $P_1^e\partial_1 I_{12}Q-A_2\partial_1 I_{12}Q+A_2 p_1^e\partial_1 I_{12}Q$

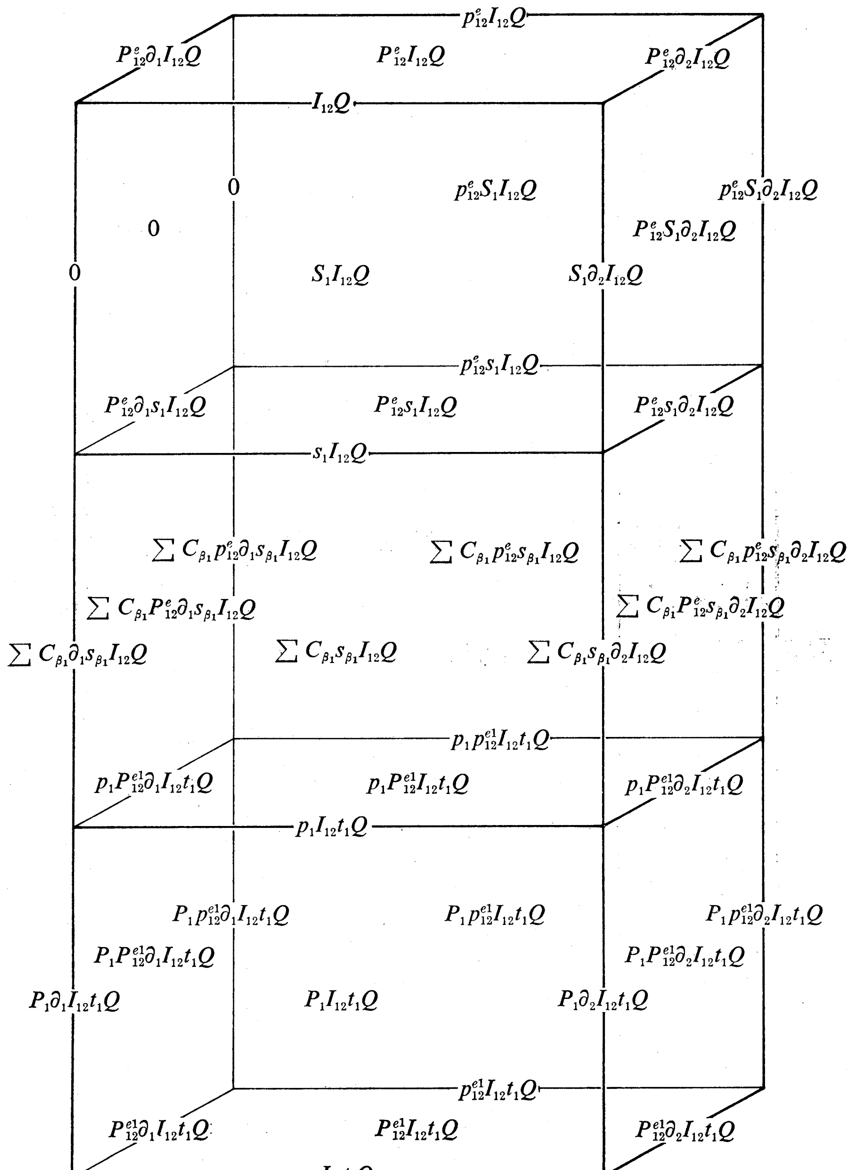

Diagram (9.5)

is obtained as the product of $A_2\partial_1 s_1 I_{12}Q = A_2\partial_1 I_{12}Q$ and "the edge P_1^e". These 3-chains containing ∂_1 are faces of the diagram which is the product

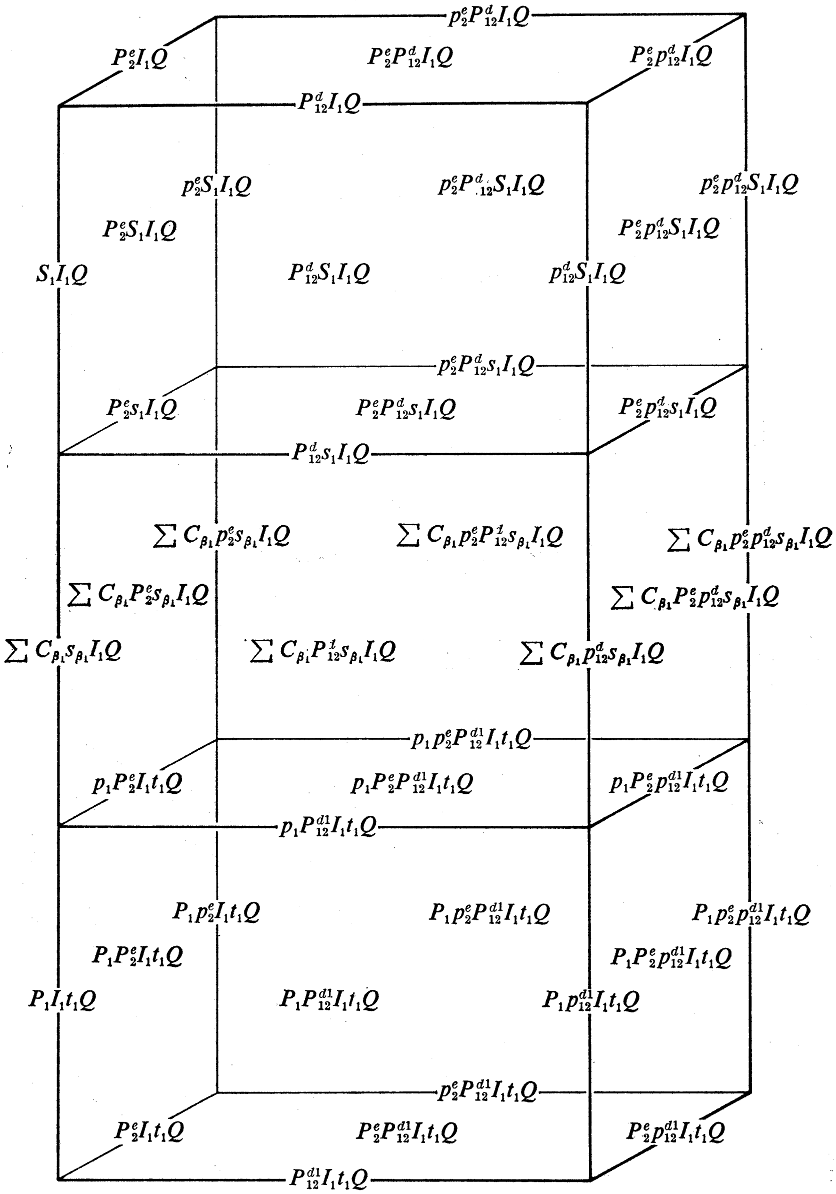

Diagram (9.6)

of Diagram (9.7) and "the edge P_1^{e}". Note again that, to represent 3-chains, we are writing the symbols C_{β_1}, C_{β_2}, P_1, P_2, P_{12}^{d} or P_{12}^{d1}, S_1 or S_2 in this order.

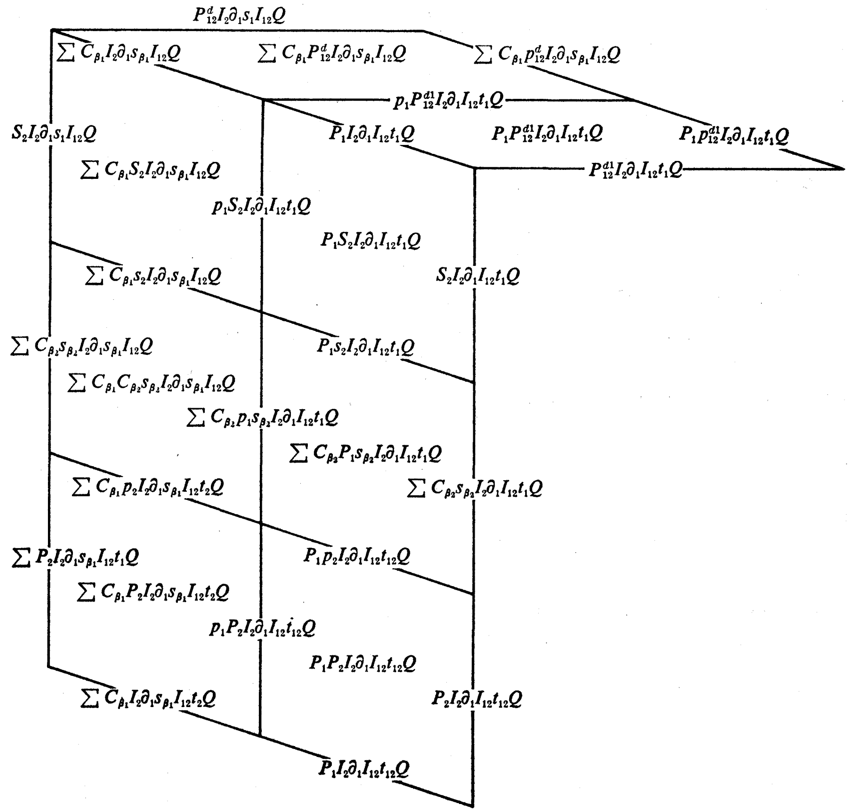

Diagram (9.7)

Now we consider the third homology of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$. We have the subsemigroup $\Lambda_1\times\Lambda_2\times\Lambda_3$ ($\Lambda_j\cong \overset{2[n/3]}{*} Z_+; j=1,2,3$) of $(Z_+ * Z_+)^n$ given in Section 5 and the construction of $I_{123}Q$ for a 3-cube Q with support in U. We would like to use $I_{123}Q$ to construct a map $A\colon \hat{Q}'_{*,3}(B\overline{G}_U)\to Q'_*(B\overline{G})$ such that $\phi=\partial A+A\hat{\partial}$. As we did in this section, by using Λ_j ($j=1,2,3$), we can define $A_j=A\,|\,Q'_{(j)}$. We can also define $A_{j_1j_2}=A\,|\,Q'_{(j_1,j_2)}(B\overline{G}_U)$ ($1\leqq j_1<j_2\leqq 3$) by using $\Lambda_{j_1}\times\Lambda_{j_2}$. Note that A_j or $A_{j_1j_2}$ are defined for 1- or 2-cubes with support in an appropriate compact subset of $\boldsymbol{R}^n$ such as K. Then, as we constructed Diagram (9.4) in the second homology

case, we try to construct Diagram (9.8), which would imply the existence of $A_{123}=A\,|\,Q'_{(1,2,3)}(B\bar{G})$.

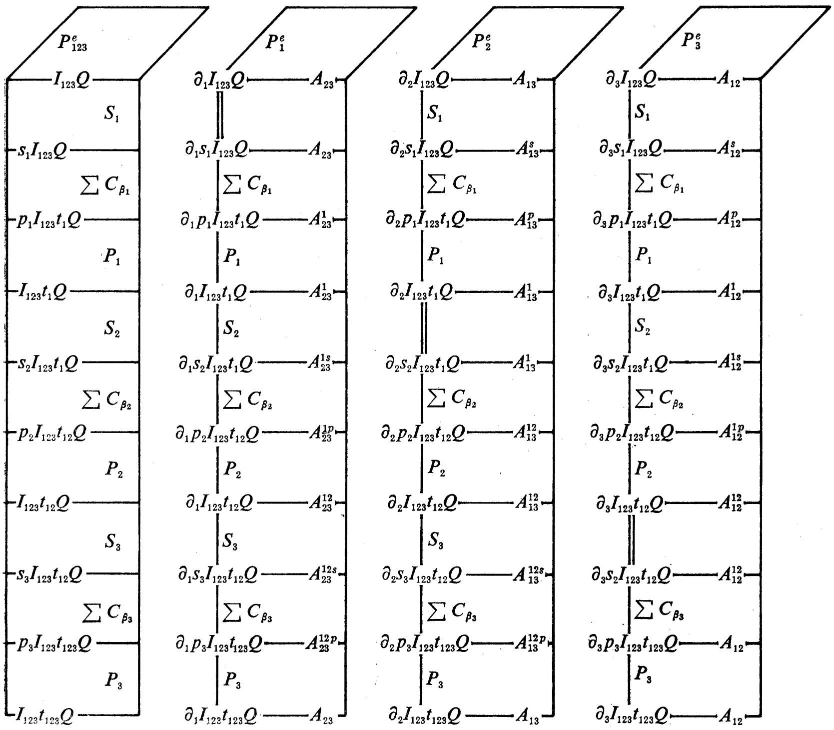

Diagram (9.8)

As we used Diagram (9.1) in order to define A_1, A^s_1, A^p_1, we try to use Diagram (9.9) in order to define A_{12}, A^s_{12}, A^p_{12}. Diagram (9.9) makes sense at least up to the terms containing r. For, Diagram (9.5) and Diagram (9.6) substituted $\partial_2I_{12}Q$ for Q are faces of 4-chains which match up modulo the terms containing r. The 4-chains we are looking for are those in Diagrams (9.5) and (9.6) with P^e_{12}, P^e_2 replaced by P^d_{123}, P^d_{23}. That is, the diagram consisting of $P^d_{123}S_1I_{12}Q$, $\sum C_{\beta_1}P^d_{123}s_{\beta_1}I_{12}Q$ and $P_1P^{d1}_{123}I_{12}t_1Q$, and that consisting of $P^d_{12}P^d_{23}S_1I_1Q$, $\sum C_{\beta_1}P^d_{12}P^d_{23}s_{\beta_1}I_1Q$ and $P_1P^{d1}_{12}P^d_{23}I_1t_1Q$.

By now, however, we have not been able to control the terms containing r. The difficulty is that the 3-chains bounding $r^d_{123}s_{\beta_1}I_{12}Q$, $r^{d1}_{123}I_{12}t_1Q$, $r^{d1}_{12}S_1I_1Q$, etc. are obtained by using Theorem (8.2) and these 3-chains usually have big support.

In the next section, we use a little more complicated construction to

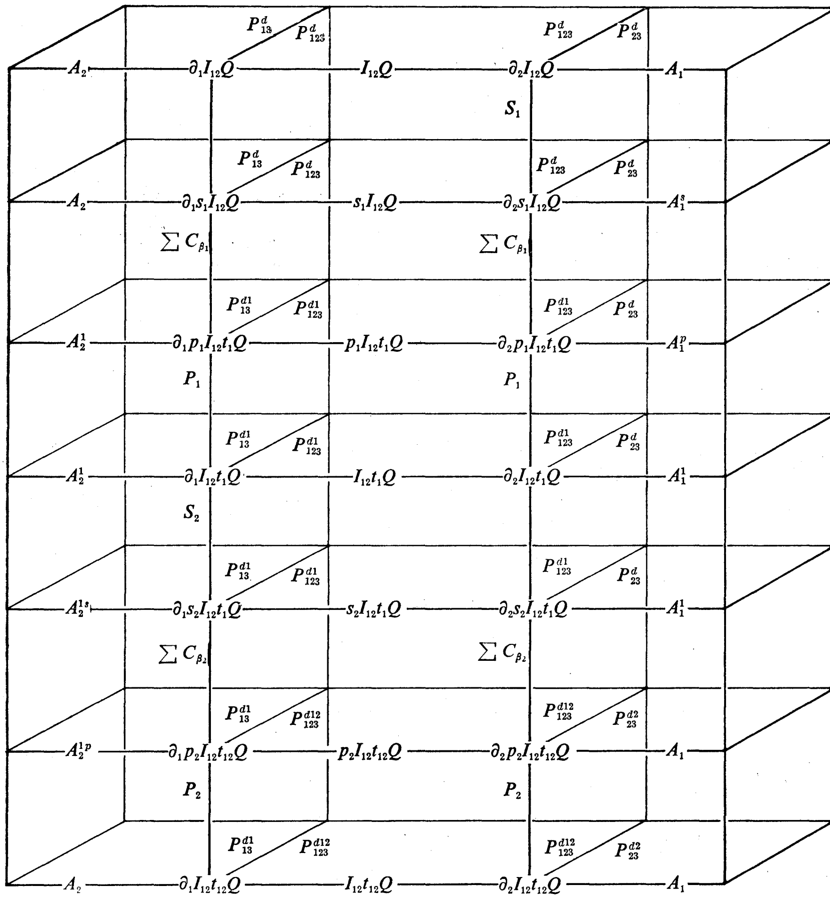

Diagram (9.9)

show that the higher homology of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ vanishes provided that n is a little larger. There, we prove the vanishing of the m-th homology group inductively on m by using the partitions with respect to families of closed balls.

§ 10. The *m*-th homology of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ $(1 \leqq r < [(n+1)/m]-1)$

In this section we prove the following theorem.

Theorem (10.1). *$H_m(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0$ if $1 \leqq r < [(n+1)/m]-1$.*

First we consider the case where $1 \leqq r < [n/m]-1$ and show that

$H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$. The proof is easily modified for $1 \leqq r < [(n+1)/m] - 1$.

For a k-cube Q $(k \leqq m)$ of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ with support in an appropriate subset of $\boldsymbol{R}^n$, we define a k-cube $I_{1\cdots k}^{(k)}Q$ and inductively construct a map $A\colon \hat{Q}'_{*,m}(B\overline{\mathrm{Diff}}_U^r(\boldsymbol{R}^n)) \to Q'_*(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n))$ such that $\iota\bar{\phi}_{,m} = (\iota A)\hat{\partial} + \partial(\iota A)$. Then by Proposition (7.4), we obtain the desired result.

Let $\Phi\colon (\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n \to P\mathrm{Diff}_c^\infty(\boldsymbol{R}^n)$ denote the homomorphism given in Section 5 with an open ball U such that $\Phi(\lambda)^{(1)}(U)$, $\lambda \in (\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$ are disjoint. Since $m([n/m]-1)+m \leqq n$, as we saw in Section 5, we have a subsemigroup $(\boldsymbol{Z}_+)^m \times \Lambda_1 \times \cdots \times \Lambda_m$ $(\Lambda_i \cong \overset{2^{[n/m]-1}}{*} \boldsymbol{Z}_+$ $(i=1, \cdots, m))$ of $(\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$. Let $\gamma', \gamma'', \cdots, \gamma^{(m)}$ be the generators of $(\boldsymbol{Z}_+)^m$ and $B_i = \{\beta_i\}$ the generating set of Λ_i $(i=1, \cdots, m)$.

By using $\Lambda_{j_1} \times \cdots \times \Lambda_{j_k}$ $(1 \leqq j_1 < \cdots < j_k \leqq m)$, we have the construction of $I_{j_1\cdots j_k}Q$ for a k-cube $Q \in Q'_{(j_1,\cdots,j_k)}(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n))$ with support in a suitable compact subset of $\boldsymbol{R}^n$ (see § 8). Here, the k-cubes of the $2^{[n/m]-1}$-subdivision of a k-cube $Q \in Q'_{(j_1,\cdots,j_k)}(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n))$ are indexed by $B_{j_1} \times \cdots \times B_{j_k}$;

$$sQ = \sum s_{\beta_{j_1}\cdots\beta_{j_k}}Q.$$

$I_{j_1\cdots j_k}\,Q$ is given by

$$I_{j_1\cdots j_k}Q = \bigcup_{l(\lambda_{j_1})=\cdots=l(\lambda_{j_k})} Q_{\lambda_{j_1}\cdots\lambda_{j_k}},$$

where, for $\lambda_{j_q} = \beta_{j_q}(p)\cdots\beta_{j_q}(1)$ $(q=1, \cdots, k)$,

$$Q_{\lambda_{j_1}\cdots\lambda_{j_k}} = \Phi(\beta_{j_1}(p)\cdots\beta_{j_k}(p))^{(1)} s_{\beta_{j_1}(p)\cdots\beta_{j_k}(p)} \cdots \Phi(\beta_{j_1}(1)\cdots\beta_{j_k}(1))^{(1)} s_{\beta_{j_1}(1)\cdots\beta_{j_k}(1)}Q.$$

Since the C^r-norm of $Q_{\lambda_{j_1}\cdots\lambda_{j_k}}$ is estimated by $(2^{-[n/m]+1}(2+\varepsilon)^r)^{l(\lambda_{j_1})}|Q|$ (Lemma (5.2)), $I_{j_1\cdots j_k}Q$ is a smooth k-cube of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ provided that $r < [n/m]-1$. $I_{j_1\cdots j_k}(Q)$ is defined if Int Supp $Q_{\lambda_{j_1}\cdots\lambda_{j_k}}$ are disjoint.

Define $I_{j_1\cdots j_k}^{(k)}Q$ by

$$I_{j_1\cdots j_k}^{(k)}Q = \bigcup_{l(\lambda_{j_1})=\cdots=l(\lambda_{j_k})} \Phi((\gamma^{(k)})^{l(\lambda_{j_1})})^{(1)} Q_{\lambda_{j_1}\cdots\lambda_{j_k}}.$$

If $r < [n/m]-1$, then $I_{j_1\cdots j_k}^{(k)}Q$ is also a smooth singular k-cube of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$.

Put $K_1^{(k)} = \Phi(\gamma^{(k)})^{(1)}([-1, 1]^n)$ and $K_0^{(k)} = \mathrm{Cl}([-1, 1]^n - K_1^{(k)})$. Note that $K_0^{(k)}$ and $K_1^{(k)}$ are closed balls. For a k-cube

$$Q \in Q'_{(j_1,\cdots,j_k)}(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n))$$

with support in $K_0^{(k)}$, we can define $I_{j_1\cdots j_k}^{(k)}Q$. Let $\bar{p}^{(k)}$ denote the partition with respect to $K_0^{(k)}$ and $K_1^{(k)}$.

We also use the partition $\bar{p}_i$ $(i=1, \cdots, m)$ with respect to the family of closed balls

$$K_{\beta_i}=\Phi(\beta_i)^{(1)}([-1, 1]^n) \quad (\beta_i \in B_i).$$

Consider the set of closed balls

$$\bar{U},\ K_{\beta_1}\cap\cdots\cap K_{\beta_m}\cap K'_{a_1}\cap\cdots\cap K_{a_m}^{(m)}$$
$$(\beta_i \in B_i,\ a_i=0, 1;\ i=1, \cdots, m).$$

The interiors of these balls are disjoint. Moreover, we can shrink $\bar{U}$ and $K_0^{(i)}$ and fatten $K_1^{(i)}$ and K_{β_i} so that these closed balls (intersections) become disjoint. Hereafter, $\bar{U}$ and $K_0^{(i)}$ shrinked or $K_1^{(i)}$ and K_{β_i} fattened are denoted by the same symbols as well as the partitions defined above. Let V denote the interior of the union of these balls (intersections). Note that, if Q is supported in $V\cap K'_0\cap\cdots\cap K_0^{(k)}$, then $I_{j_1\cdots j_k}^{(k)}Q$ is supported in $V\cap K'_0\cap\cdots\cap K_0^{(k-1)}$.

For a k-cube Q and $1\leqq l\leqq k$, put

$$t_{1\cdots l}Q=\bigcup_{(\beta_1,\cdots,\beta_l)\in B_1\times\cdots\times B_l}\Phi(\beta_l)^{(1)}s_{\beta_l}\cdots\Phi(\beta_1)^{(1)}s_{\beta_1}Q.$$

Here, $s_iQ=\sum s_{\beta_i}Q$ $(i=1, \cdots, m)$ is the partial subdivision of Q considered as an element of $Q'_{(1,\cdots,k)}(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n))$. Let $t^{(k)}Q$ denote $\Phi(\gamma^{(k)})^{(1)}Q$. Note that

$$I_{1\cdots k}^{(k)}Q=Q\cup I_{1\cdots k}^{(k)}t^{(k)}t_{1\cdots k}Q$$

for $Q\in Q'_{(1,\cdots,k)}(B\overline{\mathrm{Diff}}{}^r_{\mathrm{Int}(K'_0\cap\cdots\cap K_0^{(k)})}(\boldsymbol{R}^n))$ and

$$p^{(k)}I_{1\cdots k}^{(k)}Q=Q+I_{1\cdots k}^{(k)}t^{(k)}t_{1\cdots k}Q.$$

The construction of the map $A\colon \hat{Q}'_{k,m}(B\overline{\mathrm{Diff}}{}_U^r(\boldsymbol{R}^n))\to Q'_{k+1}(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n))$ is carried out inductively on k as follows. Let $A_{j_1\cdots j_k}$ denote

$$A\,|\,Q'_{(j_1,\cdots,j_k)}(B\overline{\mathrm{Diff}}{}^r_{\mathrm{Int}(K'_0\cap\cdots\cap K_0^{(k)})}(\boldsymbol{R}^n)).$$

By using $I_{1\cdots k}^{(k)}Q$, $A_{1\cdots\hat{i}\cdots k}$ $(1\leqq i\leqq k)$ and the action of $Z_+(\gamma')\times\cdots\times Z_+(\gamma^{(k)})\times\Lambda_1\times\cdots\times\Lambda_k$, we construct $A_{12\cdots k}$ such that

$$\iota\phi=\partial\iota A_{12\cdots k}+\sum_{i=1}^{k}\iota A_{1\cdots\hat{i}\cdots k}(-1)^i\partial_i.$$

Then we can construct $A_{j_1\cdots j_k}$ such that

$$\iota\phi=\partial\iota A_{j_1\cdots j_k}+\sum_{i=1}^{k}\iota A_{j_1\cdots\hat{j}_i\cdots j_k}(-1)^i\partial_{j_i}\quad(1\leqq j_1<\cdots<j_k\leqq m)$$

in a similar way by using $Z_+(\gamma')\times\cdots\times Z_+(\gamma^{(k)})\times\Lambda_{j_1}\times\cdots\times\Lambda_{j_k}$. Thus, hereafter we construct $A_{12\cdots m}$ assuming the existence of $A_{1\cdots\hat{i}\cdots m}$. By Proposition (7.4), the existence of $A_{12\cdots m}$ implies $H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$.

For the construction of $A_{12\cdots m}$, it is necessary to treat the locally degenerate chains.

In general, let V be a finite disjoint union of bounded open balls U_j $(j=1,\cdots,N)$ in $\boldsymbol{R}^n$. An m-cube Q of $B\overline{\mathrm{Diff}}_V^r(\boldsymbol{R}^n)$ is said to be *locally degenerate* if the following condition is satisfied: For any $j\in\{1,\cdots,N\}$, there exists $k\in\{1,\cdots,m\}$ such that the holonomy in the direction of e_k is the identity on U_j. A cubic m-chain is said to be locally degenerate if it is a sum of locally degenerate m-cubes.

There are no nontrivial 1-chains which are locally degenerate. Any locally degenerate 2-cube is a product of two 1-cubes with support in the unions of U_j which are disjoint. Hence any locally degenerate 2-chain is a 2-cycle. If $H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$, any locally degenerate 2-cycle is homologous to zero. Moreover it is the boundary of a locally degenerate 3-chain (which is in fact a sum of product 3-cubes).

We consider locally degenerate chains with respect to the union V of open balls given before. Note that $r^{(k)}Q$ and r_kQ $(k=1,\cdots,m)$ are "product chains"; hence are locally degenerate chains. Moreover, if Q is locally degenerate, S_kQ, $P^{(k)}Q$ and P_kQ $(k=1,\cdots,m)$ are also locally degenerate as well as s_kQ, $\bar{p}^{(k)}Q$ and $\bar{p}_kQ$.

First we look at the construction of $A_{12\cdots m}$ for $m=1,2,3$.

For a 1-cube $Q\in Q'_{(1)}(B\overline{\mathrm{Diff}}_{\mathrm{Int}(K'_0)}^r(\boldsymbol{R}^n))$, we have the following diagram (Diagram (10.1)). Hereafter, $C_{\gamma^{(k)}}$ denotes $C_{\Phi(\gamma^{(k)})}$. This diagram plays a role similar to Diagram (9.1).

From this diagram, A_1, A_1^s and A_1^p are defined. For example,

$$A_1Q=-P'I_1'Q+C_{\gamma'}I_1'Q+S_1I_1't'Q+\sum C_{\beta_1}s_{\beta_1}I_1't'Q-P_1I_1't't_1Q.$$

Hence, for r, n such that A_1Q is well-defined, we see that

$$H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0.$$

To construct A_{12}, we look at Diagram (10.2) for a 2-cube Q with support in $K_0'\cap K_0''$. Here, for a 1-cube $Q\in Q'_{(2)}(B\overline{\mathrm{Diff}}_{\mathrm{Int}(K'_0)}^r(\boldsymbol{R}^n))$. A_2 is obtained in a way similar to A_1, by using I_2' and the action of γ' and Λ_2.

Diagram (10.2) corresponds to Diagram (9.4). $A_{12}Q$ is obtained as the 3-chain which bounds the difference between the top edge and the

$$\begin{tikzcd}
I_1't'Q \arrow[d, "S_1I_1't'Q"] & I_1'Q \arrow[l, "C_{\gamma'}I_1'Q"'] \arrow[r, "P'I_1'Q"] \arrow[d, "S_1I_1'Q"] & p'I_1'Q=Q+I_1't't_1Q \arrow[d, "p'S_1I_1'Q"] \\
s_1I_1't'Q \arrow[d, "\sum C_{\beta_1}s_{\beta_1}I_1't'Q"] & s_1I_1'Q \arrow[l, "C_{\gamma'}s_1I_1'Q"'] \arrow[r, "P's_1I_1'Q"] \arrow[d, "\sum C_{\beta_1}s_{\beta_1}I_1'Q"] & p's_1I_1'Q \arrow[d, "\sum C_{\beta_1}p's_{\beta_1}I_1'Q"] \\
p_1I_1't't_1Q & p_1I_1't_1Q \arrow[l, "C_{\gamma'}p_1I_1't_1Q"'] \arrow[r, "P'p_1I_1't_1Q"] & p'p_1I_1't_1Q \\
I_1't't_1Q \arrow[u, "P_1I_1't't_1Q"] & I_1't_1Q \arrow[l, "C_{\gamma'}I_1't_1Q"'] \arrow[r, "P'I_1't_1Q"] \arrow[u, "P_1I_1't_1Q"] & p'I_1't_1Q \arrow[u, "P_1p'I_1't_1Q"]
\end{tikzcd}$$

Diagram (10.1)

bottom edge of Diagram (10.2).

Let $B_1[P'']=B_1[P''\partial_2I_{12}''Q]$, $B_1[C_{\gamma''}]=B_1[C_{\gamma''}\partial_2I_{12}''Q]$, etc., denote the 3-chains which correspond to the faces indicated in Diagram (10.2). For example, $B_1[S_1]=B_1[S_1\partial_2I_{12}''t''Q]$ is a 3-chain which bounds $S_1\partial_2I_{12}''t''Q-A_1\partial_2I_{12}''t''Q+A_1^s\partial_2s_1I_{12}''t''Q$.

The 3-chains $B_1[\ \ ]$ or $B_2[\ \ ]$ are obtained from Diagrams (10.3) and (10.4) up to the terms which bound 2-cycles of the form $r'Q$, $r''Q$, r_1Q, r_2Q. Since $\bar{p}'$, $\bar{p}''$, $\bar{p}_1$, $\bar{p}_2$ are partitions with respect to closed balls, these 2-cycles are locally degenerate. Since $H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n);\boldsymbol{Z})=0$, these terms are homologous to zero. Moreover, we can choose the bounding 3-chains which are also locally degenerate.

Diagrams (10.3) and (10.4) correspond to Diagrams (9.6) and (9.7). The method to write down the 3-chains is similar to that in Section 9.

Diagram (10.3) is obtained from Diagram (10.1) by putting the product of A_1Q and the two edges $C_{\gamma''}$ and P''.

Now, $B_1[\ \ ]$ are obtained as follows.

$B_1[S_1]=B_1[S_1\partial_2I_{12}''t''Q]$, $B_1[\sum C_{\beta_1}]=B_1[\sum C_{\beta_1}\partial_2s_{\beta_1}I_{12}''t''Q]$, $B_1[P_1]=B_1[P_1\partial_2I_{12}''t''t_1Q]$ are obtained from the front faces of Diagram (10.3) substituted $\partial_2I_{12}''Q$ for Q. In fact, they are

$$C_{\gamma'}S_1I_1'\partial_2I_{12}''t''Q-P'S_1I_1'\partial_2I_{12}''t''Q,$$

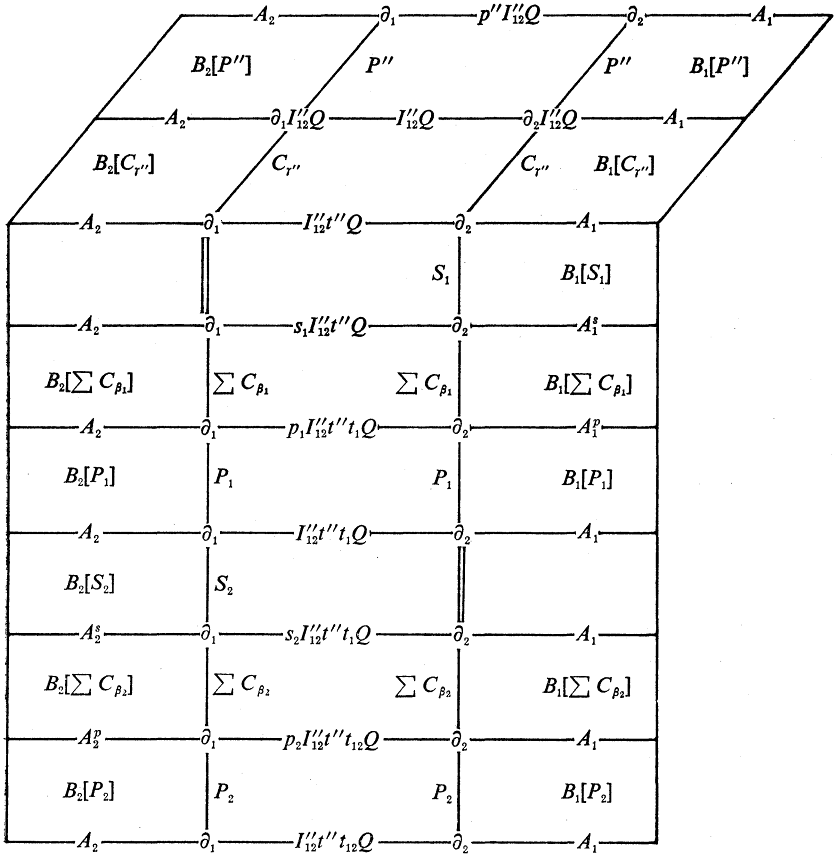

Diagram (10.2)

$$\sum C_{\gamma'} C_{\beta_1} s_{\beta_1} I'_1 \partial_2 I''_{12} t'' Q + \sum C_{\beta_1} P' s_{\beta_1} I'_1 \partial_2 I''_{12} t'' Q \quad \text{and}$$
$$C_{\gamma'} P_1 I'_1 \partial_2 I''_{12} t'' t_1 Q - P' P_1 I'_1 \partial_2 I''_{12} t'' t_1 Q$$

up to the 3-chains which bound the 2-cycles of the form $r'Q$, $r''Q$, r_1Q, r_2Q.

$B_1[P''] = B_1[P''\partial_2 I''_{12}Q]$ and $B_1[C_{\gamma''}] = B_1[C_{\gamma''}\partial_2 I''_{12}Q]$ are obtained from Diagram (10.3). They are the 3-chains corresponding to the top faces and the faces on the left-hand side of Diagram (10.3) substituted $\partial_2 I''_{12}Q$ for Q.

$B_2[P'']$ and $B_2[C_{\gamma''}]$ are obtained similarly to $B_1[P'']$ and $B_1[C_{\gamma''}]$.

On the other hand, $B_2[\sum C_{\beta_1}]$ and $B_2[P_1]$ are obtained from Diagram (10.4). Diagram (10.4) is obtained as a product of A_2 and two edges $\sum C_{\beta_1}$ and P_1.

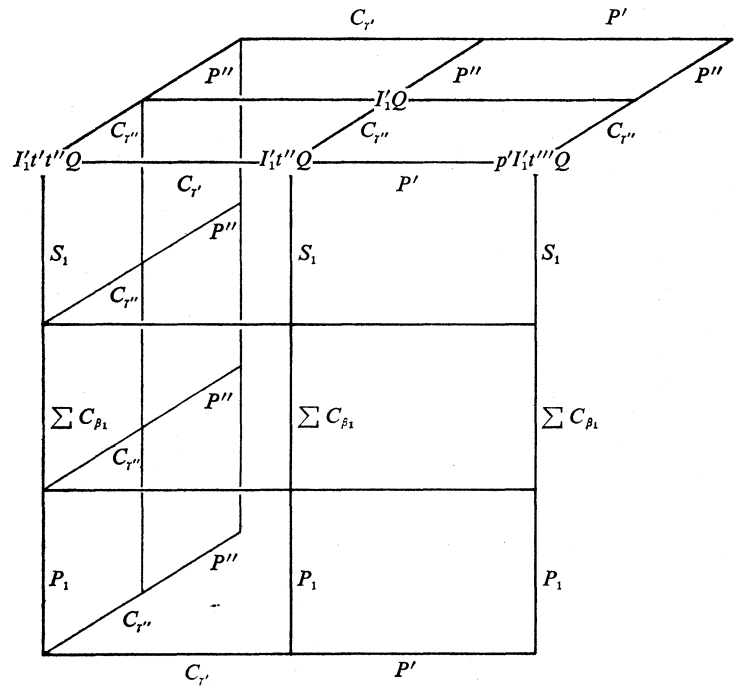

Diagram (10.3)

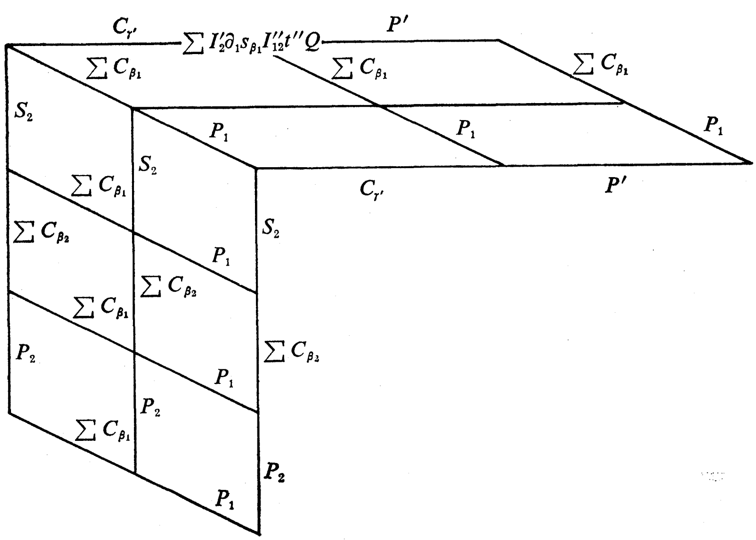

Diagram (10.4)

Note that, to obtain the 3-chains, we should write down the symbols $C_{\gamma'}$, $C_{\gamma''}$, C_{β_1}, C_{β_2}, P', P'', P_1, P_2, S_1 or S_2 always in this order.

Diagrams (10.3) and (10.4) concern the upper half of Diagram (10.2) and the lower half of Diagram (10.2) is obtained similarly. These diagrams represent face relations modulo the terms containing 2-cycles of the form $r'Q$, $r''Q$, r_1Q, r_2Q.

Thus we can complete Diagram (10.2) and define A_{12}. Therefore, by Proposition (7.4), $H_2(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0$.

To construct A_{123} for a 3-cube Q with support in $K_0' \cap K_0'' \cap K_0'''$, we need A_{12}^s, A_{12}^p, etc. These are obtained from Diagram (10.5), which is a product of the vertical part of Diagram (10.2) and the edges P'' and $C_{\gamma''}$. For example, the 3-chain A_{12}^sQ which bounds $s_1Q-A_2(-\partial_1)s_1Q-$

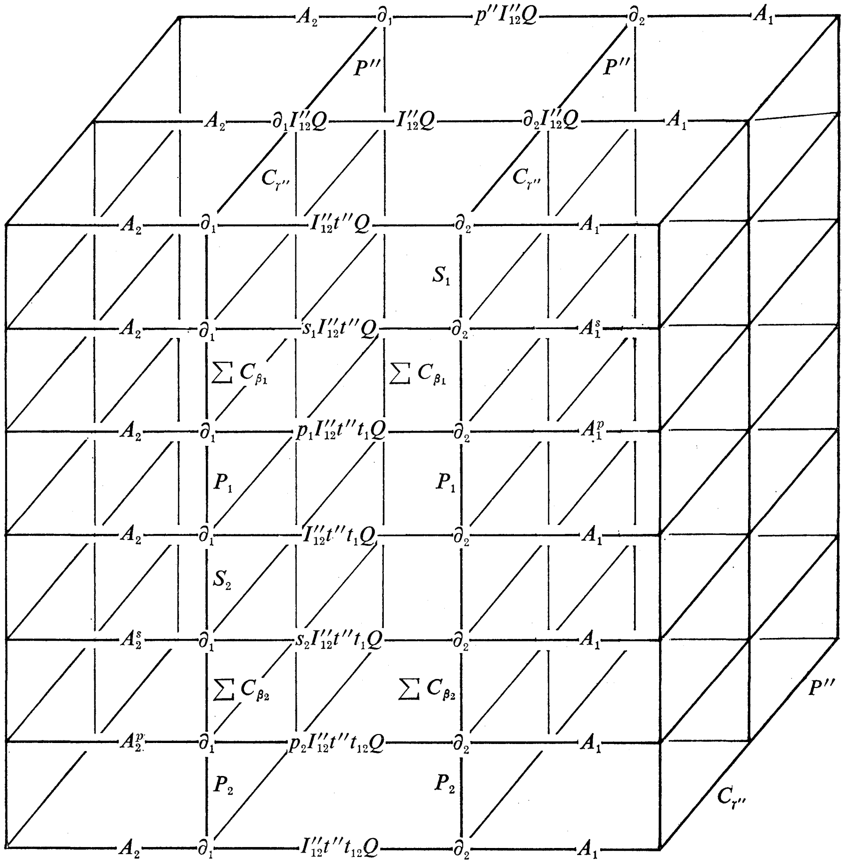

Diagram (10.5)

$A_1^s(+\partial_2)s_1Q$ is obtained as the 3-chain in Diagram (10.5) bounding

$$p''s_1I''_{12}Q - A_2(-\partial_1)p''s_1I''_{12}Q - A_1^s(+\partial_2)p''s_1I''_{12}Q$$
$$-(s_1I''_{12}t''t_{12}Q - A_2(-\partial_1)s_1I''_{12}t''t_{12}Q - A_1^s(+\partial_2)s_1I''_{12}t''t_{12}Q).$$

For a 2-cube Q, we can read from Diagrams (10.6) and (10.7), the 4-chains $B_{12}[S_1Q]$, $B_{12}[\sum C_{\beta_1}s_{\beta_1}Q]$, etc., bounding

$$S_1Q - A_{12}Q + A_{12}^s s_1Q + B_1[S_1\partial_2Q],$$
$$\sum C_{\beta_1}s_{\beta_1}Q - A_{12}^s s_1Q + A_{12}^p p_1t_1Q + B_1[\sum C_{\beta_1}\partial_2 s_{\beta_1}Q] + B_2[\sum C_{\beta_1}(-\partial_1)s_{\beta_1}Q],$$

etc., modulo the 3-chains of the form CrQ (for example, $C_{\gamma'}r''Q^2$, $C_{\beta_1}r''Q^2$) and the locally degenerate 3-chains.

Diagram (10.6) is obtained as a product of Diagram (10.1) and the edges P'' and $C_{\gamma''}$. The top faces, the front faces and the faces on the

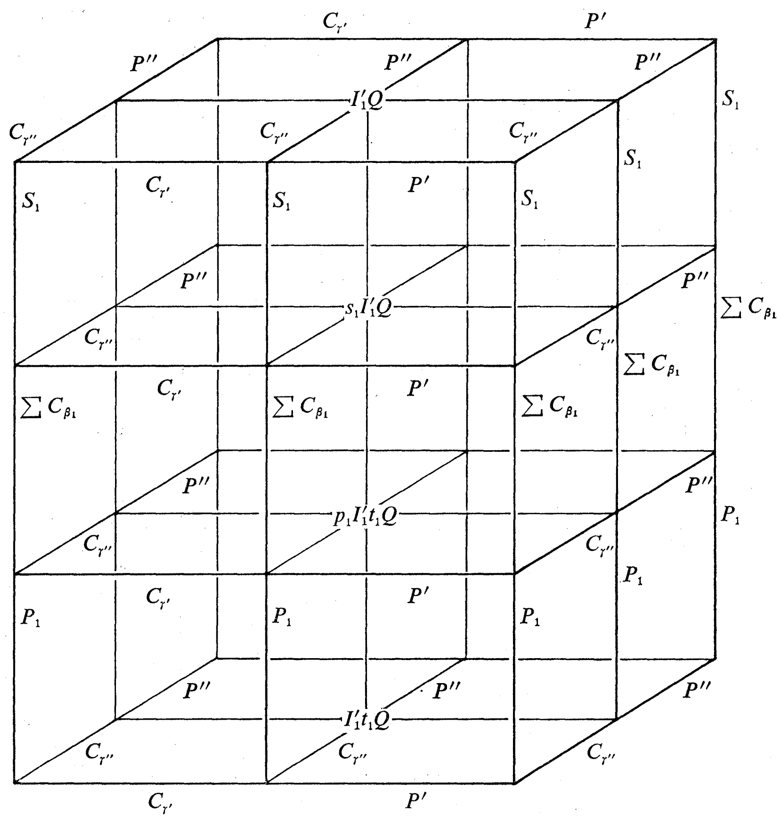

Diagram (10.6)

left-hand side of Diagram (10.6) is Diagram (10.3). The 4-chain $B_{12}[S_1Q]$ is the sum of $C_{\gamma''}S_1I''_{12}Q$, $-P''S_1I''_{12}Q$, and the four 4-cubes on the third floor of Diagram (10.6) (i.e., $C_{\gamma'}P''S_1I'_1\partial_2I''_{12}Q$, $-P'P''S_1I'_1\partial_2I''_{12}Q$, $-C_{\gamma'}C_{\gamma''}S_1I'_1\partial_2I''_{12}Q$, $-C_{\gamma''}P'S_1I'_1\partial_2I''_{12}Q$) up to the terms needed for the 3-chains of the form CrQ and the locally degenerate 3-chains.

Diagram (10.7) is also obtained as a product of Diagram (10.4) and the edges P'' and $C_{\gamma''}$. The 4-chain $B_{12}[\sum C_{\beta_1}s_{\beta_1}Q]$ is the sum of $\sum C_{\gamma''}C_{\beta_1}s_{\beta_1}I''_{12}Q$, $\sum C_{\beta_1}P''s_{\beta_1}I''_{12}Q$, the 4-cubes represented by the second floor of Diagram (10.7) and those represented by the second floor of Diagram (10.6) substituted $\partial_2I''_{12}Q$ for Q (up to the chains as above).

Now we can complete Diagram (10.8) which corresponds to Diagram (9.8). Note that the terms $B_i[\quad]$ $(i=1, 2, 3)$ cancel by themselves because $\hat{\partial}^2=0$.

From Diagram (10.8), we obtain a 4-chain which bounds $Q-A_{23}(-\partial_1)Q-A_{13}(+\partial_2)Q-A_{12}(-\partial_3)Q$ modulo the 3-chains of the form CrQ^2 and locally degenerate 3-chains. For the 3-chain of the form CrQ^2, since $H_1(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); \boldsymbol{Z})=0$, we obtain a locally degenerate 3-chain b such that $rQ^2=\partial b$ (b is in fact of the form $Q^2\times Q^1$, where Q^2 and Q^1 are supported in the unions of connected components of V which are disjoint). We have

$$\partial Cb=b-cb-CrQ^2.$$

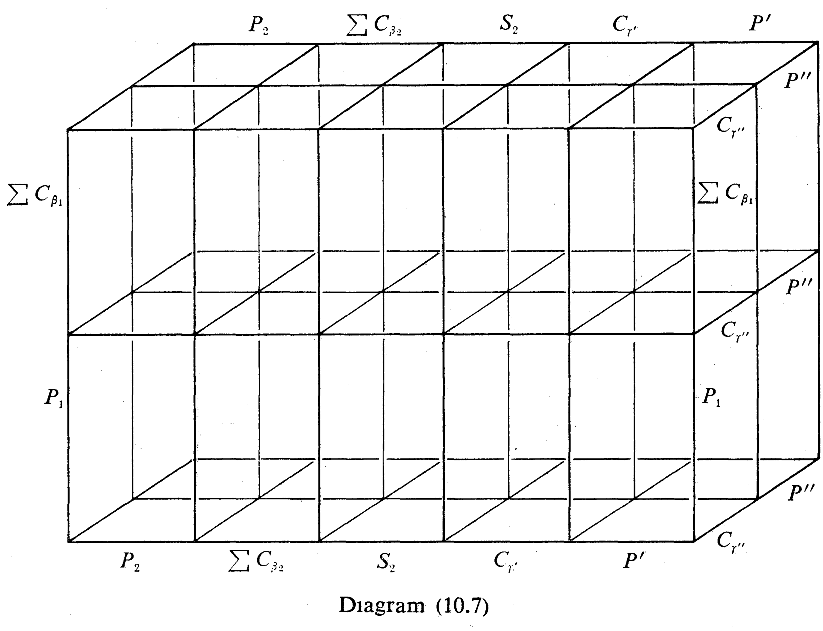

Diagram (10.7)

By adding the boundary of 4-chains of the form Cb, we obtain a 3-cycle homologous to $Q-A_{23}(-\partial_1)Q-A_{13}(+\partial_2)Q-A_{12}(-\partial_3)Q$, which is locally degenerate. Note that cb is again a locally degenerate 3-chain. Since $H_i(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n);\boldsymbol{Z})=0$ $(i=1,2)$, the resulted 3-cycle is homologous to zero (see Lemma (10.2)).

Therefore, by Proposition (7.4), $H_3(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n);\boldsymbol{Z})=0$.

Note that we can use P^e_{123}, P^e_1, P^e_2, P^e_3 instead of P'''. Hence, by using the semigroup $(\boldsymbol{Z}_+)^2\times\Lambda_1\times\Lambda_2\times\Lambda_3$ $(\Lambda_i\cong \overset{2[(n+1)/3]-1}{*}\ \boldsymbol{Z}_+$ $(i=1,2,3))$, we can prove that $H_3(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n);\boldsymbol{Z})=0$ $(r<[(n+1)/3]-1)$.

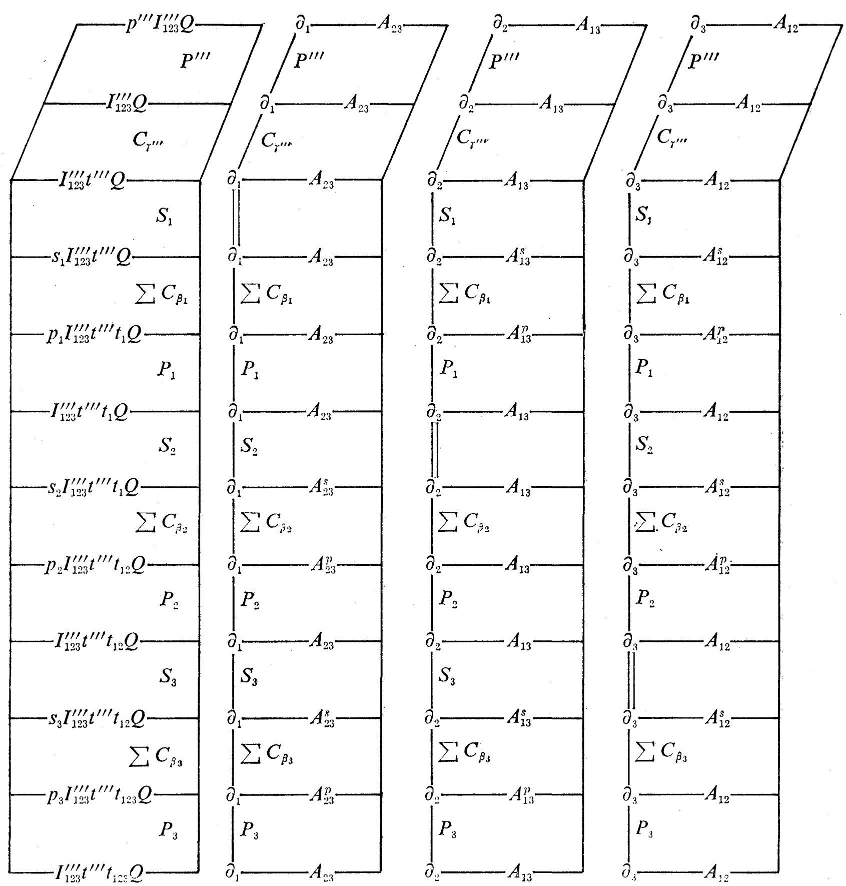

Diagram (10.8)

To construct $A_{12\cdots m}$ inductively on m, we first show the following lemma. Here we suppose that $H_i(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$ for $i=1, \cdots, m-1$.

Lemma (10.2). *Let $V=\bigcup_{j=1}^N U_j$ be a finite disjoint union of bounded open balls in $\boldsymbol{R}^n$. Let $a=\sum \pm Q_{(i)}$ be an m-cycle of $B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n)$ which is locally degenerate with respect to V. Then the cycle ιa is homologous to zero in $S'_*(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n))$. In fact, there exists a locally degenerate $(m+1)$-chain $b \in Q'_*(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n))$ such that $\iota a=\partial\iota b$ and the support of b is contained in a neighborhood of the support of a.*

To prove this lemma, consider the following action of the symmetric group $\mathscr{S}_m$ on $Q_m(B\bar{G})$. For $\pi \in \mathscr{S}_m$ and Q: $[0, 1]^m \to G$, define $Q\pi$: $[0, 1]^m \to G$ by

$$(Q\pi)(t_1, \cdots, t_m)=(\det \pi)Q(\pi(t_1, \cdots, t_m)).$$

Here $\pi \in \mathscr{S}_m$ is considered as a permutation matrix. Then the boundary homomorphism ∂: $Q_i(B\bar{G}) \to Q_{i-1}(B\bar{G})$ induces a map ∂: $Q_i(B\bar{G})/\mathscr{S}_i \to Q_{i-1}(B\bar{G})/\mathscr{S}_{i-1}$. Let $Q_*(B\bar{G})/\mathscr{S}$ denote the quotient complex. Then ι: $Q_*(B\bar{G}) \to S_*(B\bar{G})$ (§ 6) factors through $Q_*(B\bar{G})/\mathscr{S}$;

$$\begin{tikzcd}
Q_*(B\bar{G}) \arrow[rr, "\iota"] \arrow[dr] & & S_*(B\bar{G}) \\
& Q_*(B\bar{G})/\mathscr{S}. \arrow[ur] &
\end{tikzcd}$$

Proof of Lemma (10.2). We prove Lemma (10.2) by an induction on N. If $N=1$, the locally degenerate cycle a is in fact a degenerate cycle. Hence, Lemma (10.2) is true when $N=1$.

Let $\bar{p}$ denote the partition with respect to $\bigcup_{j=1}^{N-1} \bar{U}_j$ and $\bar{U}_N$. Then $\bar{p}a$ is an m-cycle homologous to a.

First, the part pa of $\bar{p}a$ is homologous to zero. For, the m-cycle a restricted to $\bar{U}_N$ is a degenerate cycle. On the other hand, the m-cycle a restricted to $\bigcup_{j=1}^{N-1} \bar{U}_j$ satisfies the assumption of Lemma (10.2) for $N-1$ bounded open balls $U_1, \cdots, U_{N-1}$. Hence, it is homologous to zero by the induction hypothesis (it is the boundary of a locally degenerate chain).

Thus, a is homologous to $\bar{p}a-pa$ which is written in $Q'_*(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n))/\mathscr{S}$ as

$$\sum \pm Q_{(i)}^{m-1} \times Q'^{1}_{(i)} + \sum \pm Q_{(i)}^{m-2} \times Q'^{2}_{(i)} + \cdots + \sum \pm Q^1_{(i)} \times Q'^{m-1}_{(i)},$$

where $Q_{(i)}^{m-k}$ is an $(m-k)$-cube with support in $\bigcup_{j=1}^{N-1} U_j$ and $Q'^{k}_{(i)}$ is a k-cube with support in U_N $(k=1, \cdots, m-1)$. Since $H_1(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})=0$,

$Q'^1_{(i)}$ is written as a boundary; $Q'^1_{(i)}=\partial BQ'^1_{(i)}$, where Supp $(BQ'^1_{(i)})\subset U_N$. Then we have

$$\begin{aligned}\bar{p}a-pa-(-1)^{m-1}\partial(\sum \pm Q^{m-1}_{(i)}\times BQ'^1_{(i)})\\ =\{-(-1)^{m-1}\sum\partial(\pm Q^{m-1}_{(i)})\times BQ'^1_{(i)}+\sum \pm Q^{m-2}_{(i)}\times Q'^2_{(i)}\}\\ +\cdots+\sum \pm Q^1_{(i)}\times Q'^{m-1}_{(i)}.\end{aligned}$$

Here, the terms in the parenthesis { } is rewritten as

$$\sum_i Q^{m-2}_{(i)}\times(\sum_j \pm Q'^2_{(i,j)}),$$

where $Q^{m-2}_{(i)}$ are distinct. Then, $\sum_j \pm Q'^2_{(i,j)}$ is a 2-cycle with support in U_N. For, the boundary of $\bar{p}a-pa-(-1)^{m-1}\partial(\sum_i \pm Q^{m-1}_{(i)}\times BQ'^1_{(i)})$ contains $\sum_i Q^{m-2}_{(i)}\times\partial(\sum_j \pm Q'^2_{(i,j)})$, which is zero.

Since $H_2(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n);\boldsymbol{Z})=0$, $\sum_j \pm Q'^2_{(i,j)}$ is written as a boundary: $\sum_j \pm Q'^2_{(i,j)}=\partial B(\sum_j \pm Q'^2_{(i,j)})$, where $\mathrm{Supp}(B(\sum_j \pm Q'^2_{(i,j)}))\subset U_N$. Then we have

$$\begin{aligned}\bar{p}a-pa-(-1)^{m-1}\partial(\sum_i \pm Q^{m-1}_{(i)}\times BQ'^1_{(i)})\\ -(-1)^{m-2}\partial(\sum_i Q^{m-2}_{(i)}\times B(\sum_j \pm Q'^2_{(i,j)}))\\ =\{-(-1)^{m-2}\sum_i(\partial Q^{m-2}_{(i)})\times B\sum_j(\pm Q'^2_{(i,j)})+\sum_i \pm Q^{m-3}_{(i)}\times Q'^3_{(i)}\}\\ +\cdots+\sum_i \pm Q^1_{(i)}\times Q'^{m-1}_{(i)}.\end{aligned}$$

Again the terms in the parenthesis { } is rewritten as

$$\sum_i Q^{m-3}_{(i)}\times(\sum_j \pm Q'^3_{(i,j)}),$$

where $Q^{m-3}_{(i)}$ are distinct and $\sum_j \pm Q'^3_{(i,j)}$ is a 3-cycle.

Inductively, by using $H_i(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n);\boldsymbol{Z})=0$ $(i=1,\cdots,m-1)$, we obtain

$$\begin{aligned}\bar{p}a-pa-(-1)^{m-1}\partial(\sum_j \pm Q^{m-1}_{(i)}\times BQ'^1_{(i)})\\ -(-1)^{m-2}\partial(\sum_i Q^{m-2}_{(i)}\times B(\sum_j \pm Q'^2_{(i,j)}))\\ -\cdots-(-1)\partial(\sum_i Q^1_{(i)}\times B(\sum_j \pm Q'^{m-1}_{(i,j)}))=0.\end{aligned}$$

Hence we have proved Lemma (10.2).

Now we construct $A_{12\cdots m}$. We assume that we defined $A_{12\cdots(m-1)}$. Hence, as we remarked, $A_{1\cdots\hat{i}\cdots m}$ $(i=1,\cdots,m)$ is also defined. We are going to show that

$$\iota\phi Q-\sum_{i=1}^{m}\iota A_{1\dots\hat{i}\dots m}(-1)^i\partial_i Q$$

is homologous to zero. We look at the cycle

$$\iota\phi I^{(m)}_{1\dots m}Q-\sum_{i=1}^{m}\iota A_{1\dots\hat{i}\dots m}(-1)^i\partial_i I^{(m)}_{1\dots m}Q$$

and compare it with

$$\iota\phi Q-\sum_{i=1}^{m}\iota A_{1\dots\hat{i}\dots m}(-1)^i\partial_i Q$$
$$+\iota\phi I^{(m)}_{1\dots m}t^{(m)}t_{1\dots m}Q-\sum_{i=1}^{m}\iota A_{1\dots\hat{i}\dots m}(-1)^i\partial_i I^{(m)}_{1\dots m}t^{(m)}t_{1\dots m}Q.$$

We would like to show that they are homologous on one hand, and that

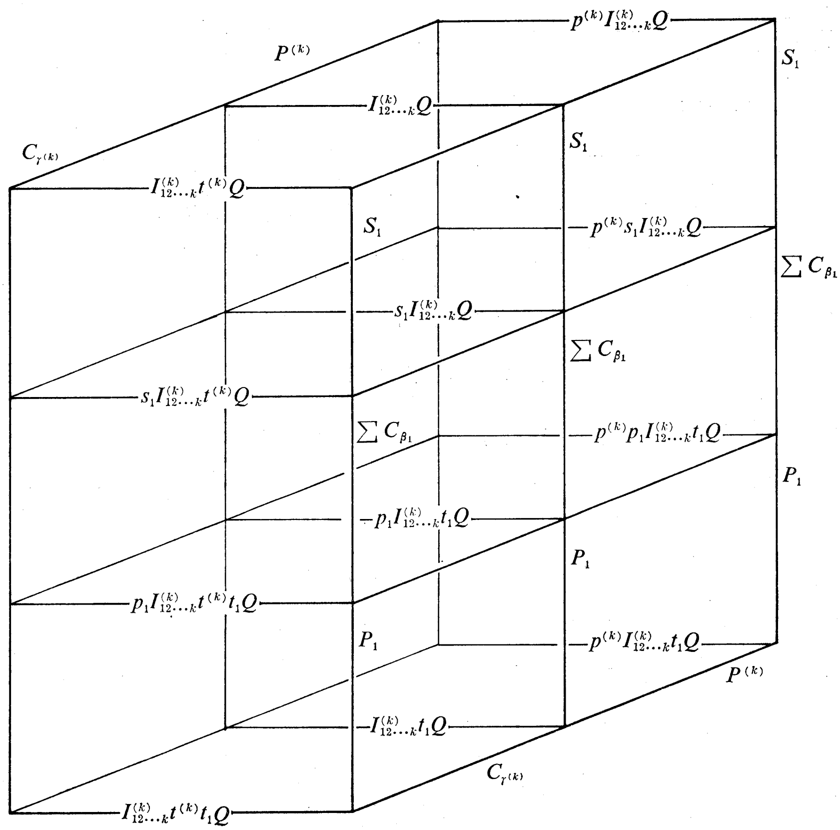

Diagram (10.9)

$$\iota\phi I^{(m)}_{1\ldots m}Q-\sum_{i=1}^{m}\iota A_{1\ldots\hat{i}\ldots m}(-1)^{i}\partial_{i}I^{(m)}_{1\ldots m}Q \quad \text{is homologous to}$$

$$\iota\phi I^{(m)}_{1\ldots m}t^{(m)}t_{1\ldots m}Q-\sum_{i=1}^{m}\iota A_{1\ldots\hat{i}\ldots m}(-1)^{i}\partial_{i}I^{(m)}_{1\ldots m}t^{(m)}t_{1\ldots m}Q$$

on the other hand.

The $(m+1)$-chains which bound the differences of these cycles are obtained from either the top faces or the front faces of Diagrams (10.9)–(10.11) for $k=m$. Diagram (10.9)–(10.11) correspond to the three top floors of Diagram (10.5).

Diagram (10.9) consists of

$$C_{\gamma^{(k)}}S_1I^{(k)}_{12\ldots k}Q,\quad P^{(k)}S_1I^{(k)}_{12\ldots k}Q,\quad \sum C_{\gamma^{(k)}}C_{\beta_1}s_{\beta_1}I^{(k)}_{12\ldots k}Q,$$
$$\sum C_{\beta_1}P^{(k)}s_{\beta_1}I^{(k)}_{12\ldots k}Q,\quad C_{\gamma^{(k)}}P_1I^{(k)}_{12\ldots k}t_1Q \quad\text{and}\quad P^{(k)}P_1I^{(k)}_{12\ldots k}t_1Q.$$

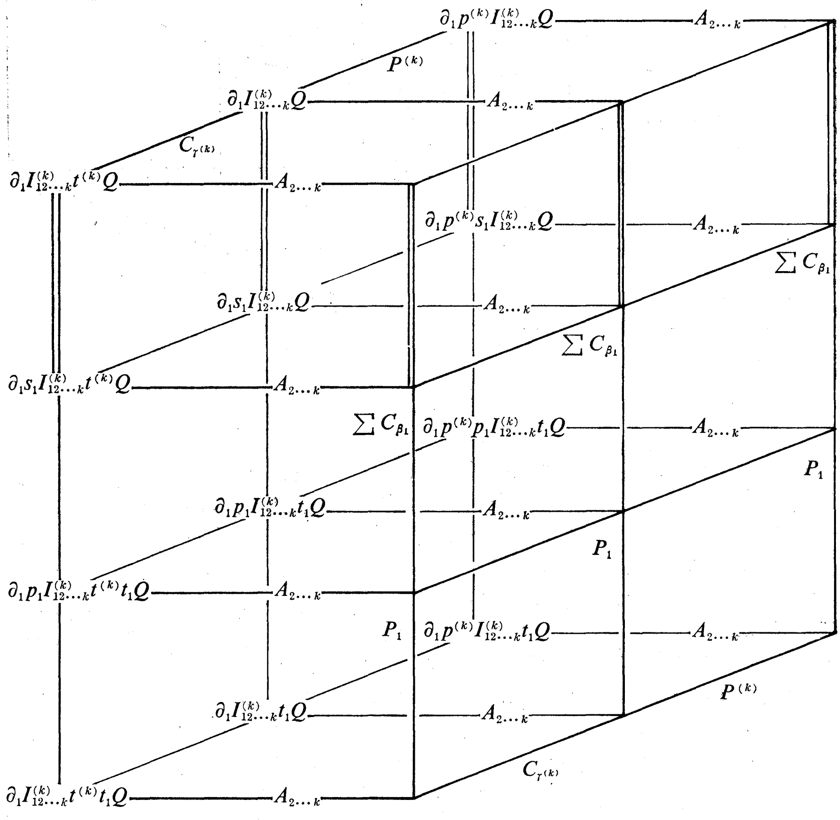

Diagram (10.10)

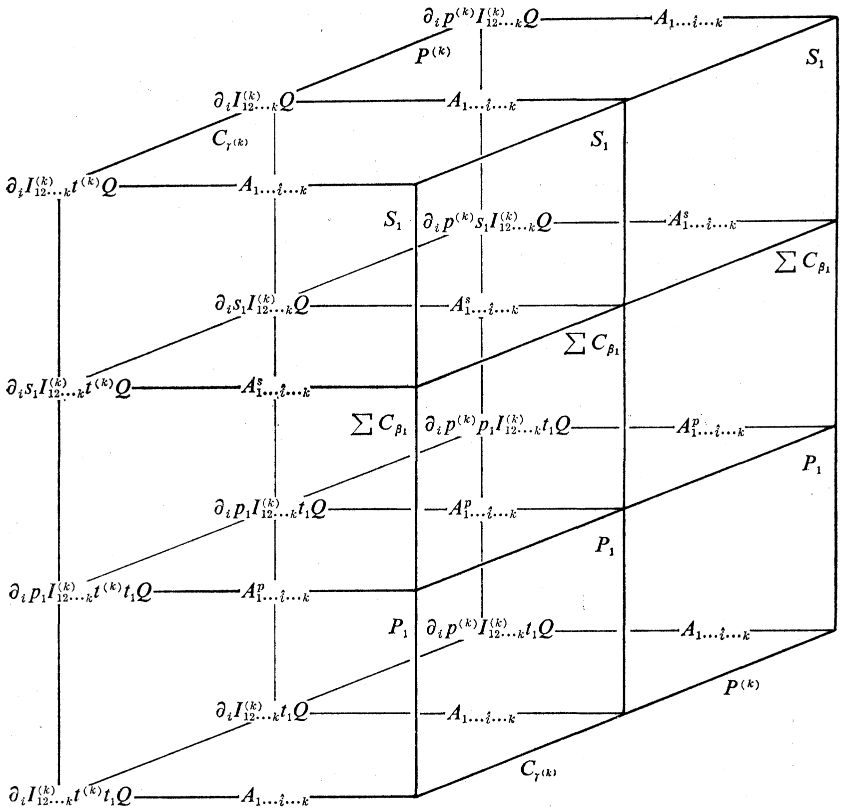

Diagram (10.11)

The top faces of Diagrams (10.10) and (10.11) for $k=m$ are obtained as the products of $A_{1\cdots\hat{i}\cdots m}$ and the edges $C_{\gamma(m)}$ and $P^{(m)}$. On the other hand, the front faces of Diagrams (10.10) and (10.11) for $k=m$ are obtained from Diagrams (10.9)–(10.11) for $k=m-1$ or from the diagram which is the product of $A_{1\cdots\hat{i}\cdots m}$ and the edges C_{β_i} and P_i. Note that $A^s_{1\cdots\hat{i}\cdots m}$ and $A^p_{1\cdots\hat{i}\cdots m}$ are defined from Diagrams (10.9)–(10.11) for $k=m-1$ which is the product of the vertical part of $A_{12\cdots(m-1)}$ and the edges $P^{(m-1)}$ and $C_{\gamma(m-1)}$. Inductively $A_{1\cdots\hat{i}\cdots m}$ is written by the symbols $C_{\gamma(k)}$, C_{β_k}, $P^{(k)}$, P_k, S_k ($1\leqq k\leqq m$, $k\neq i$, $\beta_k\in B_k$) and the locally degenerate chains.

Thus the $(m+1)$-chain which is obtained from the top faces and the front faces of Diagrams (10.9)–(10.11) for $k=m$ is also written by these symbols and the locally degenerate chains. Note again that we fixed an order

$$C_{\gamma'} < C_{\gamma''} < \cdots < C_{\gamma^{(m)}} < C_{\beta_1} < C_{\beta_2} < \cdots < C_{\beta_m} < P' < P''$$
$$< \cdots < P^{(m)} < P_1 < P_2 < \cdots < P_m < S_1, S_2, \cdots, S_m.$$

We write the symbols in the chains obtained by these diagrams always in this order. Then the terms which do not contain r or the locally degenerate chains coincide with those of

$$\phi Q - \sum_{i=1}^{m} A_{1\ldots\hat{i}\ldots m}(-1)^i \partial_i Q.$$

Hence the terms which contain r or the locally degenerate chains have the form described in Lemma (10.3). Note that the construction of $A_{1\ldots\hat{i}\ldots m}$ does not use $C_{\gamma^{(m)}}$, $P^{(m)}$, C_{β_i}, P_i. Hence the locally degenerate chains (with respect to V) which appear in $A_{1\ldots\hat{i}\ldots m}$ can be taken locally degenerate with respect to V_i, where V_i is the union of U and

$$\operatorname{Int} K_{\beta_1} \cap \cdots \cap K_{\beta_{i-1}} \cap K_{\beta_{i+1}} \cap \cdots \cap K_{\beta_m} \cap K'_{a_1} \cap \cdots \cap K^{(m-1)}_{a_{m-1}}$$
$$(\beta_j \in B_j, j \neq i;\ a_j = 0, 1, j \neq m).$$

Since the locally degenerate chains conjugated by $\gamma^{(m)}$ or β_i are again locally degenerate, the locally degenerate chains appearing in the construction of $A_{12\ldots m}$ satisfy the condition (2) of Lemma (10.3).

Lemma (10.3). *Let a be a cubic m-cycle which is given as a sum of the chains of the form $C_{(1)}\ldots C_{(i)}Q^{m-i}$ satisfying the following conditions.*

(1) *$C_{(s)}$ $(s=1, \cdots, i)$ are $C_{\gamma^{(k)}}$ or C_{β_k} $(k=1, \cdots, m)$ written in the fixed order.*

(2) *Q^{m-i} is a locally degenerate $(m-i)$-cube as well as the cube $c_{(s_1)}\ldots c_{(s_j)}Q^{m-i}$ $(1 \leqq s_1 < \cdots < s_j \leqq i)$, where $c_{(s)}$ denotes $\Phi(\gamma^{(k)})^{(1)}$ or $\Phi(\beta_k)^{(1)}$ when $C_{(s)}$ is $C_{\gamma^{(k)}}$ or C_{β_k}.*

Then the m-cycle ιa is homologous to zero. In fact, there exists a cubic $(m+1)$-chain b which is a sum of the chains satisfying (1) *and* (2) *above such that $\iota a = \partial \iota b$.*

Proof. First note the following formula.

$$\partial(C_{(1)}\cdots C_{(i)}Q) = \sum_{l=1}^{i}(-1)^{l-1}C_{(1)}\cdots C_{(l-1)}(1-c_{(l)})C_{(l+1)}\cdots C_{(i)}Q$$
$$+(-1)^i C_{(1)}\cdots C_{(i)}\partial Q.$$

Now we put together the terms in a which have the same terms of $C_{(s)}$'s.

$$a=\sum_k C^k_{(1)}\cdots C^k_{(i_k)}a_k,$$

where a_k is an $(m-i_k)$-chain without $C_{(s)}$. If i_k is maximum among the terms in the cycle a, by the above formula, a_k is an $(m-i_k)$-cycle satisfying the assumption of Lemma (10.2). Hence, if $i_k\geqq 1$, by Lemma (10.2), we have an $(m-i_k+1)$-chain Ba_k such that $\partial\iota Ba_k=\iota a_k$. We may suppose that Ba_k is also supported in V and it is also locally degenerate. Then by adding

$$-(-1)^{i_k}\partial(C^k_{(1)}\cdots C^k_{(i_k)}Ba_k),$$

we can replace $C^k_{(1)}\cdots C^k_{(i_k)}a_k$ by terms with shorter length in $C_{(s)}$.

Thus, by replacing successively the terms with maximal length in $C_{(s)}$ by shorter ones, we obtain a locally degenerate m-cycle which is homologous to the cycle a. Since this is homologous to zero by Lemma (10.2), the proof of Lemma (10.3) is completed.

By Lemma (10.3), we can construct $A_{12\cdots m}$ and Theorem (10.1) is proved for $r<[n/m]-1$.

For $r<[(n+1)/m]-1$, we use the subsemigroup

$$(Z_+)^{m-1}\times A_1\times\cdots\times A_m\quad (A_i\cong \overset{2[(n+1)/m]-1}{*} Z_+\ (i=1,\cdots,m))\text{ of }(Z_+*Z_+)^n.$$

The proof goes on without change except that we do not have $\gamma^{(m)}$ and $P^{(m)}$ in Diagrams (10.9)–(10.11) for $k=m$. There, we replace $P^{(m)}$ by $P^e_{1\cdots m}$ and P^e_i $(i=1,\cdots,m)$. Then we obtain a diagram similar to Diagram (9.8). By using $I_{12\cdots m}Q$ instead of $I^{(m)}_{12\cdots m}$, we see that

$$Q-\sum_{i=1}^m A_{1\cdots\hat{i}\cdots m}(-1)^i\partial_i Q$$

is homologous to zero modulo the terms containing $r^{(i)}$, r_i, r^e_i, $r^e_{12\cdots m}$ or locally degenerate chains. Hence Theorem (10.1) follows from Lemma (10.3).

Appendix. The first homology of $B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n)$ $(n+1<r<\infty)$

In this Appendix, we describe several operations on foliated products and give a proof of the following theorem due to Mather ([21]).

Theorem (A.1). *$H_1(B\overline{\mathrm{Diff}}{}^r_c(\boldsymbol{R}^n); Z)=0$ if $n+1<r<\infty$.*

Our proof is based on the study of foliated products and the techniques used here are rather simple. However, there is still one step in our proof where we need a non-elementary argument. More precisely, the proof of Lemma (A.2) uses Lemma (A.3) which is a consequence of the small denominators theory ([37]) and Thurston's technique ([33, 2]). It would be nice to find an elementary proof of Lemma (A.2). In our proof, we also use homothetic expansions and the Schauder-Tychonoff fixed point theorem as in Mather [21].

Our main tool is the following homotopy which is described in Banyaga [2] and Mather [22].

A homotopy (*Fragmentation*). Let $\{\nu_1, \cdots, \nu_N\}$ be a partition of unity on a manifold M, i.e., a family of smooth non-negative functions on M such that $\sum_{i=1}^N \nu_i = 1$. Put $\mu_j = \sum_{i=1}^j \nu_i$ ($j=1, \cdots, N$) ($\mu_0=0$). Define λ: $[0, N]\times M \to [0, 1]$ by

$$\lambda(t, x) = (t-[t])\nu_{[t]+1}(x) + \mu_{[t]}(x).$$

Let h^1: $[0, N]^m \times M \to [0, 1]^m \times M$ be the map given by

$$h^1(t, x) = (\lambda_m(t, x), x),$$

where $\lambda_m((t_1, \cdots, t_m), x) = (\lambda(t_1, x), \cdots, \lambda(t_m, x))$. Let b: $[0, N]^m \times M \to [0, 1]^m \times M$ be the map given by

$$b(t, x) = (b^{(N)}(t), x),$$

where $b^{(N)}(t_1, \cdots, t_m) = (\min\{t_1, 1\}, \cdots, \min\{t_m, 1\})$ as in Section 6. Then there is a homotopy h: $[0, N]^m \times M \times [0, 1] \to [0, 1]^m \times M$ between b and h^1 given by

$$h(t, x, s) = ((1-s)b^{(N)}(t) + s\lambda_m(t, x), x).$$

It is easy to see that this homotopy commutes with the face operators.

Let $\mathscr{F}$ be a C^r-foliated M-product over $[0, 1]^m$ such that the associated map $T[0, 1]^m \times M \to TM$ is of class C^r ($1 \leq r \leq \infty$). Suppose that every leaf of $\mathscr{F}$ is transverse to the family of submanifolds $\{(\lambda_m(t, x), x); x \in M\}$ ($t \in [0, N]^m$) of $[0, 1]^m \times M$. Note that, if the C^0-norm of $\mathscr{F}$ is sufficiently small, every leaf of $\mathscr{F}$ is transverse to these submanifolds.

We consider the induced foliation $h^*\mathscr{F}$. This $h^*\mathscr{F}$ may not be smooth along the $(N-1)^m$ hypersurfaces $t_i = j$ ($i=1, \cdots, m; j=1, \cdots, N-1$) in $[0, N]^m \times M \times [0, 1]$. We will see that $h^*\mathscr{F}$ is a C^r-foliated M-product such that the associated map $T([0, N]^m \times [0, 1]) \times M \to TM$ is of class C^r except along these hypersurfaces.

The induced foliation. We may assume that the manifold M is the Euclidean space $\boldsymbol{R}^n$. Put

$$h((t_1, \cdots, t_m), (x_1, \cdots, x_n), s)=((u_1, \cdots, u_m), (x_1, \cdots, x_n)),$$

where $u_i=(1-s)\min\{t_i, 1\}+s((t_i-[t_i])\nu_{[t_i]+1}(x)+\mu_{[t_i]}(x))$ $(i=1, \cdots, m)$. The tangent map (Th) of h is given by a matrix

$$\begin{pmatrix} \partial u_i/\partial t_j & \partial u_i/\partial x_l & \partial u_i/\partial s \\ \partial x_k/\partial t_j & \partial x_k/\partial x_l & \partial x_k/\partial s \end{pmatrix},$$

where
$$\begin{aligned} &\partial u_i/\partial t_j=\delta^i_j(\delta^0_{[t_j]}(1-s)+s\nu_{[t_j]+1}(x)), \\ &\partial u_i/\partial x_l=s((t_i-[t_i])(\partial\nu_{[t_i]+1}/\partial x_l)+(\partial\mu_{[t_i]}/\partial x_l)), \\ &\partial u_i/\partial s=-\min\{t_i, 1\}+((t_i-[t_i])\nu_{[t_i]+1}(x)+\mu_{[t_i]}(x)), \\ &\partial x_k/\partial t_j=0, \\ &\partial x_k/\partial x_l=\delta^k_l \quad \text{and} \quad \partial x_k/\partial s=0. \end{aligned}$$

Thus (Th) is of the form

$$\begin{pmatrix} \partial u/\partial t & \partial u/\partial x & \partial u/\partial s \\ 0 & 1_m & 0 \end{pmatrix},$$

where 1_m denotes the identity matrix. Let

$$\begin{pmatrix} 1_m & 0 \\ X & \boldsymbol{x} \\ 0 & 1 \end{pmatrix}$$

be the matrix whose column vectors span the tangent plane of $h^*\mathscr{F}$ at $(t, x, s) \in [0, N]^m \times \boldsymbol{R}^n \times [0, 1]$. Then we have

$$(Th)\begin{pmatrix} 1_m & 0 \\ X & \boldsymbol{x} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} Y & \boldsymbol{y} \\ X & \boldsymbol{x} \end{pmatrix},$$

where

$$Y^i_j=(\partial u_i/\partial t)+\sum_k(\partial u_i/\partial x_k)X^k_j \quad \text{and} \quad \boldsymbol{y}^i=\sum_k(\partial u_i/\partial x_k)\boldsymbol{x}^k+(\partial u_i/\partial s).$$

Let
$$\begin{pmatrix} 1_m \\ Z \end{pmatrix}$$

be the matrix whose column vectors span the tangent plane of $\mathscr{F}$, where Z is an $L(\boldsymbol{R}^m, \boldsymbol{R}^n)$-valued C^r-function on $[0, 1]^m \times \boldsymbol{R}^n$. By the choice of the matrix X and the vector $\boldsymbol{x}$, we have

$$ZY=X \quad \text{and} \quad Z\boldsymbol{y}=\boldsymbol{x}.$$

If Z is sufficiently small, we can find such a matrix X and such a vector $\boldsymbol{x}$. For, we have

$$X=Z((\partial u/\partial t)+(\partial u/\partial x)X) \quad \text{and} \quad \boldsymbol{x}=Z((\partial u/\partial x)\boldsymbol{x}+(\partial u/\partial s)).$$

Thus we obtain

$$(1_n-Z(\partial u/\partial x))X=Z(\partial u/\partial t) \quad \text{and} \quad (1_n-Z(\partial u/\partial x))\boldsymbol{x}=Z(\partial u/\partial s). \quad (*)$$

If the matrix $Z(\partial u/\partial x)$ is well-defined and small, $1_n-Z(\partial u/\partial x)$ is invertible; hence, X and $\boldsymbol{x}$ are written as

$$X=(1_n-Z(\partial u/\partial x))^{-1}Z(\partial u/\partial t) \quad \text{and} \quad \boldsymbol{x}=(1_n-Z(\partial u/\partial x))^{-1}Z(\partial u/\partial s).$$

We see that X and $\boldsymbol{x}$ are of class C^r as a function of t and x if t_i is not an integer ($i=1, \cdots, m$). We see also that if $|Z|_0$ is small, X and $\boldsymbol{x}$ depend continuously on Z. That is, if $|\mathscr{F}|_0=|Z|_0$ is small, $h^*\mathscr{F}$ depends continuously on $\mathscr{F}$. Moreover, there are constants c_ν and $C_{\nu,r}$ such that

$$|h^{1*}\mathscr{F}|_r \leqq C_{\nu,r}|\mathscr{F}|_r \quad \text{provided} \quad |\mathscr{F}|_0 \leqq c_\nu.$$

Here we use the C^r-norm of $\mathscr{F}$ as an $L(\boldsymbol{R}^m, \boldsymbol{R}^n)$-valued function on $[0, 1]^m \times \boldsymbol{R}^n$. This estimate is obtained by differentiating the formulae $(*)$. Note that Z in the formulae $(*)$ is $Z|_{h(t,x,s)}$, while other terms are functions of (t, x, s).

Let Q: $[0, 1]^m \to \mathrm{Diff}_c^r(M)$ be a singular m-cube such that the associated map $T[0, 1]^m \times M \to TM$ is of class C^r. Let $h^{**}Q$ denote the singular cubic chain defined by

$$h^{**}Q=\sum_{0\leqq i_1,\cdots,i_m\leqq N-1} Q'\tau_{(i_1,\cdots,i_m)},$$

where Q': $[0, N]^m \to \mathrm{Diff}_c^r(M)$ represents the C^r-foliated M-product $h^{1*}\mathscr{F}_Q$ and

$$\tau_{(i_1,\cdots,i_m)}(t_1, \cdots, t_m)=(t_1+i_1, \cdots, t_m+i_m) \quad (i_1, \cdots, i_m \in \boldsymbol{Z}).$$

Note that the support of the holonomy of $Q'\tau_{(i_1,\cdots,i_m)}$ in the direction of e_l lies in $\mathrm{Supp}(\nu_{i_l+1})$. Thus, the support of $Q'\tau_{(i_1,\cdots,i_m)}$ lies in

$$\bigcup_{l=1}^{m} \mathrm{Supp}(\nu_{i_l+1}).$$

We call h^{**} the fragmentation with respect to the partition of unity $\{\nu_i\}$.

In order to apply the above homotopy to singular cubes or simplices of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$, we can use the straightening given in Section 4 to homotope them to smooth simplices.

Smoothing along the boundary. We need the following smoothing along the boundary. A similar idea was used to obtain an expression of the Godbillon-Vey class in terms of holonomy (see [15], [23]). Let G denote $\mathrm{Diff}_c^r(M)$. Take a positive real number ε' smaller than 1/4. Let $\psi\colon [0, 1] \to [0, 1]$ be a smooth function such that

$$0 \leqq \partial\psi/\partial t \leqq 1/(1-4\varepsilon'),$$

$$\psi(t)=0 \quad (t \in [0, \varepsilon']) \quad \text{and} \quad \psi(t)=1 \quad (t \in [1-\varepsilon', 1]).$$

Let $\psi_m\colon [0, 1]^m \to [0, 1]^m$ denote the function given by

$$\psi_m(t_1, \cdots, t_m)=(\psi(t_1), \cdots, \psi(t_m)).$$

For a smooth singular cube $Q\colon [0, 1]^m \to G$, consider $Q\psi_m$. This $Q\psi_m$ is smooth and the leaves of $\mathscr{F}_{Q\psi_m}$ are perpendicular to $\partial[0, 1]^m \times M$. Since ψ_m is homotopic to the identity by the homotopy $\Psi_m\colon [0,1]^m \times [0, 1] \to [0, 1]^m$ given by

$$\Psi_m(t, s)=(1-s)t+s\psi_m(t),$$

$Q\psi_m$ is homotopic to Q by the homotopy $Q\Psi_m$. Note that this homotopy commutes with the face operators.

If $\mathscr{F}_Q$ is a C^r-foliated M-product such that $T[0, 1]^m \times M \to TM$ is of class C^r, then so is $\mathscr{F}_{Q\psi_m}$. To see this, we may assume that $M=\boldsymbol{R}^n$. Let

$$\begin{pmatrix} 1_m \\ X \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_m \\ Z \end{pmatrix}$$

be the matrices whose column vectors span the tangent plane of $\mathscr{F}_Q$ at $(\psi_m(t), x) \in [0, 1]^m \times \boldsymbol{R}^n$ and that of $\mathscr{F}_{Q\psi_m}$ at $(t, x) \in [0, 1]^m \times \boldsymbol{R}^n$, respectively. Then we have

$$X=Z(\delta_j^i(\partial\psi/\partial t)(t_i)).$$

From this formula we see also that

$$|\mathscr{F}_{Q\psi_m}|_r \leqq |\mathscr{F}_Q|_r/(1-4\varepsilon')$$

with respect to the norm as an $\mathscr{X}_c^r(\boldsymbol{R}^n)$-valued continuous function on $[0, 1]^m$.

Composition. Let $\mathscr{F}_i$ $(i=1, \cdots, N)$ be a C^r-foliated M-product over $[0, 1]$ corresponding to

$$\sigma_i\colon [0, 1] \longrightarrow \mathrm{Diff}_c^r(M) \quad (\sigma_i(0)=\mathrm{id}).$$

Then we can define the composition of $\mathscr{F}_i$ $(i=1, \cdots, N)$ to be the C^r-foliated M-product $\mathscr{F}$ over $[0, 1]$ such that

$$s^{(N)}\mathscr{F}=\sum_{i=1}^{N}\mathscr{F}_i,$$

where $s^{(N)}$ denotes the N-subdivision. More precisely, the corresponding 1-simplex σ of $\mathrm{Diff}_c^r(M)$ is given by

$$\sigma(t)=\sigma_i(Nt-(i-1))\sigma_{i-1}(1)\cdots\sigma_1(1)$$

for $(i-1)/N\leqq t\leqq i/N$ $(i=1, \cdots, N)$. The composition depends on the number N and the order of $\mathscr{F}_i$. It is obvious that the composition $\mathscr{F}$ is homologous to $\sum_{i=1}^{N}\mathscr{F}_i$.

If $\mathscr{F}_i$ $(i=1, \cdots, N)$ is given by an $\mathscr{X}_c^r(M)$-valued continuous function X_i on $[0, 1]$ which vanishes on a neighborhood of $\{0, 1\}$, then $\mathscr{F}$ is also given by such a function X;

$$X(t)=NX_i(Nt-(i-1)),\ (i-1)/N\leqq t\leqq i/N \quad (i=1, \cdots, N).$$

Thus we have an estimate

$$|\mathscr{F}|_r\leqq N\sup_i|\mathscr{F}_i|_r,$$

on the norm of $\mathscr{F}$ (as an $\mathscr{X}_c^r(M)$-valued continuous function on $[0, 1]$). To X_i which does not vanish on a neighborhood of $\{0, 1\}$, we can apply the above smoothing along the boundary before taking the composition.

To prove Theorem (A.1), we use the homotopies described above to construct a map from a certain space of foliated products to itself. Then we apply the Schauder-Tychonoff fixed point theorem. As in Mather [21], we have to consider the functions of class $C^{r,\alpha}$ with α being a modulus of continuity.

Moduli of continuity (See [21]). A modulus of continuity is a continuous strictly increasing real valued function α on $[0, \infty)$ such that $\alpha(0)=0$ and $\alpha(tx)\leqq t\alpha(x)$ for $x\in[0, \infty)$, $t\geqq 1$. An $\boldsymbol{R}^n$-valued function f on $\boldsymbol{R}^m$ is said to be α-continuous if

$$|f(x)-f(y)|/\alpha(|x-y|) \quad (x, y\in\boldsymbol{R}^m, x\neq y)$$

is locally bounded. We say that f is of class $C^{r,\alpha}$ $(1\leqq r<\infty)$ if f is of class C^r and its r-th derivative is α-continuous. According to Mather [21], the sums, the products and the compositions of functions of class $C^{r,\alpha}$ $(r\geqq 1)$ are of class $C^{r,\alpha}$, as well as the inverses of diffeomorphisms of class $C^{r,\alpha}$ $(r\geqq 1)$.

For a compact subset K of $\boldsymbol{R}^n$, the space $\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)$ of $C^{r,\alpha}$-vectorfields $(1\leqq r<\infty)$ on $\boldsymbol{R}^n$ with support in K is a Banach space with respect to the norm $|\ \ |_{r,\alpha}$ given by

$$|X|_{r,\alpha}=\sup\{|D^rX(x)-D^rX(y)|/\alpha(|x-y|);\ x,y\in\boldsymbol{R}^n,\ x\neq y\},$$

where $|x|=\sum_{i=1}^n|x_i|$ for $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$. On the Banach space $\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)$, we have another norm

$$\|X\|_{r,\alpha}=\sup\{|X|_{r,\alpha},|X|_r,\cdots,|X|_0\},$$

where $|X|_i=\sup\{|D^iX(x)|;\ x\in\boldsymbol{R}^n\}$ $(i=0,\cdots,r)$. These norms are equivalent;

$$|X|_{r,\alpha}\leqq\|X\|_{r,\alpha}\leqq C_K|X|_{r,\alpha}.$$

Note that the positive real number C_K depends on the compact subset K.

It is easy to see that

$$\mathcal{X}_K^r(\boldsymbol{R}^n)=\bigcup_\alpha\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)$$

where the union is taken over all moduli of continuity α. Note that a bounded subset of $\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)$ is relatively compact in $\mathcal{X}_K^r(\boldsymbol{R}^n)$.

For two moduli of continuity α and α', if there are positive numbers c and c' such that $c\alpha\leqq\alpha'\leqq c'\alpha$ on a neighborhood of 0, then $\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)=\mathcal{X}_K^{r,\alpha'}(\boldsymbol{R}^n)$ and the norms $|\ \ |_{r,\alpha}$, $|\ \ |_{r,\alpha'}$ are equivalent. Hence hereafter we use the moduli of continuity α such that $\alpha(x)\leqq 1$, $x\in[0,\infty)$. For such a modulus of continuity α, we have

$$|X|_r\leqq|X|_{r,\alpha}\qquad\text{for } X\in\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n).$$

Let $\mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ denote the group of diffeomorphisms of $\boldsymbol{R}^n$ of class $C^{r,\alpha}$ with support in K. $\mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ has a Banach manifold structure modelled on $\mathcal{X}_K^{r,\alpha}(\boldsymbol{R}^n)$. K. Masuda pointed out to me that $\mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ is not a topological group with respect to this $C^{r,\alpha}$-topology. Hence, we consider the C^s-topology $(1\leqq s\leqq r)$ for $\mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ and the direct limit topology for $\mathrm{Diff}_c^{r,\alpha}(\boldsymbol{R}^n)$, the group of $C^{r,\alpha}$-diffeomorphisms of $\boldsymbol{R}^n$ with compact support. Note that $\mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ with the C^s-topology $(1\leqq s\leqq r)$ is a topological group and $\mathrm{Diff}_c^{r,\alpha}(\boldsymbol{R}^n)$ with the C^s-topology $(1\leqq s\leqq r)$ has the homotopy type of $\mathrm{Diff}_c^1(\boldsymbol{R}^n)$ with the C^1-topology. For definiteness, hereafter we consider the direct limit C^r-topology for $\mathrm{Diff}_c^{r,\alpha}(\boldsymbol{R}^n)$ and construct $B\overline{\mathrm{Diff}}_c^{r,\alpha}(\boldsymbol{R}^n)$, the classifying space for $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-products with compact support.

For a singular m-simplex σ: $\Delta^m \to \mathrm{Diff}_K^{r,\alpha}(\boldsymbol{R}^n)$ (continuous with respect to the C^r-topology) with image in a small neighborhood of $\sigma(0)$, we have its straightening $L\sigma$. As in Section 4, $L\sigma$ is homotopic to σ by a homotopy which commutes with the face operators. Note that the map $\Delta^m \times \boldsymbol{R}^n \to L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ associated to $L\sigma$ is of class $C^{r,\alpha}$. For, the $C^{r,\alpha}$-norm $\|\mathscr{F}_{L\sigma}\|_{r,\alpha}$ as a function on $\Delta^m \times \boldsymbol{R}^n$, we have a positive real number $C_{r,\alpha}$ such that

$$\|\mathscr{F}_{L\sigma}\|_{r,\alpha} \leqq C_{r,\alpha} \sup_{1 \leqq j \leqq m} \left| \sigma\left(\sum_{i=1}^{j} e_i\right)\sigma(0)^{-1} - \mathrm{id}\right|_{r,\alpha}.$$

Hence, the map $\Delta^m \to \mathscr{X}_K^{r,\alpha}(\boldsymbol{R}^n)$ associated to $L\sigma$ is bounded with respect to the $C^{r,\alpha}$-norm $|\ \ |_{r,\alpha}$.

Let $\mathscr{B}$ denote the set

$$\{X\colon [0,1] \to \mathscr{X}_{[0,1]^n}^{r,\alpha}(\boldsymbol{R}^n);$$
$$X \text{ is continuous with respect to the } C^r\text{-topology}, \sup_t |X_t|_{r,\alpha} < \infty\}.$$

Then, $\mathscr{B}$ is a Banach space with respect to the norm $|\ \ |_{\mathscr{B}}$ given by

$$|X|_{\mathscr{B}} = \sup_{t \in [0,1]} |X_t|_{r,\alpha}.$$

When X is an $\boldsymbol{R}^n$-valued $C^{r,\alpha}$-function on $[0,1] \times \boldsymbol{R}^n$ with support in $[0,1] \times [0,1]^n$, we also have its $C^{r,\alpha}$-norms $|X|_{r,\alpha}$ and $\|X\|_{r,\alpha}$. Note that $|X|_{\mathscr{B}} \leqq |X|_{r,\alpha} \leqq \|X\|_{r,\alpha}$.

By an argument similar to that in Section 4, we see that there is a bijective correspondence between $\mathscr{B}$ and the set

$$\{\varphi\colon ([0,1], 0) \to (\mathrm{Diff}_{[0,1]^n}^{r,\alpha}(\boldsymbol{R}^n), \mathrm{id});\ \varphi \text{ is } C^1 \text{ as a path in } \mathrm{Diff}_{[0,1]^n}^{r}(\boldsymbol{R}^n),$$
$$(\partial\varphi_t/\partial t)|_s \in \mathscr{X}_{[0,1]^n}^{r,\alpha}(\boldsymbol{R}^n)\ (s \in [0,1]),\ \sup_s |(\partial\varphi_t/\partial t)|_s|_{r,\alpha} < \infty\}.$$

Here, $X\colon [0,1] \to \mathscr{X}_{[0,1]^n}^{r,\alpha}(\boldsymbol{R}^n)$ corresponds to $\varphi\colon ([0,1], 0) \to (\mathrm{Diff}_{[0,1]^n}^{r,\alpha}(\boldsymbol{R}^n), \mathrm{id})$ such that $(\partial\varphi_t/\partial t)(\varphi_t)^{-1} = X_t$. Hence, such X defines a $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-product $\mathscr{F}_\varphi$ over $[0,1]$.

For an $\mathscr{X}_c^{r,\alpha}(\boldsymbol{R}^n)$-valued function X on $[0,1]$ which is continuous with respect to the C^r-topology and bounded with respect to the $C^{r,\alpha}$-norm, let $\mathscr{F}(X)$ denote the $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-product over $[0,1]$ defined by X. The 1-simplex of $B\mathrm{Diff}_c^{r,\alpha}(\boldsymbol{R}^n)$ corresponding to X is also denoted by $\mathscr{F}(X)$.

We are going to construct a map κ from a neighborhood of 0 in $\mathscr{B}$ to $\mathscr{B}$ which is continuous with respect to the C^r-topology. The first step is to show the following lemma.

Lemma (A. 2). *There exist positive numbers c, C and β_0 such that, for any positive number $\beta \leqq \beta_0$, there exists a map*

$$\gamma_\beta\colon \{X \in \mathscr{B};\, |X|_{\mathscr{B}} \leqq c\} \longrightarrow \mathscr{B}$$

satisfying the following conditions:

(1) *γ_β is continuous with respect to the C^r-topology,*

(2) *$\mathscr{F}(\gamma_\beta X)$ is homologous to $\mathscr{F}(X)$ in $B\overline{\mathrm{Diff}}{}_c^{r,\alpha}(\boldsymbol{R}^n)$,*

(3) *$|\gamma_\beta X|_0 \leqq \beta C |X|_{\mathscr{B}}$, and*

(4) *$|\gamma_\beta X|_{\mathscr{B}} \leqq C |X|_{\mathscr{B}}$.*

To prove Lemma (A.2), one can use the fact that $H_1(B\overline{\mathrm{Diff}}{}_c^\infty(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$, which follows from the fact that $H_1(B\overline{\mathrm{Diff}}{}^\infty(T^n); \boldsymbol{Z}) = 0$ by using Thurston's technique ([33], [2]). The fact that $H_1(B\overline{\mathrm{Diff}}{}^\infty(T^n); \boldsymbol{Z}) = 0$ is proved by Herman [15] by using the small denominators theory. However, it is much easier to show the following statement ([37, 19]):

For any positive integer r, the natural homomorphism

$$H_1(B\overline{\mathrm{Diff}}{}^\infty(T^n); \boldsymbol{Z}) \longrightarrow H_1(B\overline{\mathrm{Diff}}{}^r(T^n); \boldsymbol{Z})$$

is the zero map.

By Thurston's technique, this implies the following lemma, which we use now to prove Lemma (A.2).

Lemma (A.3). *For any positive integer r, the natural homomorphism*

$$H_1(B\overline{\mathrm{Diff}}{}_c^\infty(\boldsymbol{R}^n); \boldsymbol{Z}) \longrightarrow H_1(B\overline{\mathrm{Diff}}{}_c^r(\boldsymbol{R}^n); \boldsymbol{Z})$$

is the zero map.

Proof of Lemma (A.2). The desired foliated product $\mathscr{F}(\gamma_\beta X)$ is obtained as the straightening of the composition of $-\mathscr{F}(X')$ and $\mathscr{F}(X)$, where X' is a C^∞-approximation of X.

Let ρ be a smooth function on $\boldsymbol{R}$ with support in $[-1, 1]$ such that $\int_{-\infty}^{\infty} \rho(x)dx = 1$. Put

$$\rho_\beta(x) = \beta^{-1}\rho(x/\beta) \quad (x \in \boldsymbol{R}).$$

For the vectorfield X_t, let $\rho_\beta * X_t$ be the vectorfield on $\boldsymbol{R}^n$ with support in $[-\beta, 1+\beta]^n$ given by

$$\begin{aligned}(\rho_\beta * X_t)&(x_1, \cdots, x_n)\\ &= \int \rho_\beta(x_1 - y_1) \cdots \rho_\beta(x_n - y_n) X_t(y_1, \cdots, y_n) dy_1 \cdots dy_n\end{aligned}$$

$$=\int \rho_\beta(z_1)\cdots\rho_\beta(z_n)X_t(x_1-z_1,\cdots,x_n-z_n)dz_1\cdots dz_n.$$

Note that $\rho_\beta * X_t$ is smooth. We have the following estimates.

$$|\rho_\beta * X_t|_{r,\alpha} \leqq |X_t|_{r,\alpha},$$
$$|\rho_\beta * X_t - X_t|_0 \leqq \beta C_0 |X_t|_{r,\alpha},$$

where the positive number C_0 does not depend on β.

Let k_β be the affine map from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ given by

$$k_\beta(x_1,\cdots,x_n)=(y_1,\cdots,y_n),$$

where $y_i=(1+2\beta)^{-1}(x_i-1/2)+1/2$ $(i=1,\cdots,n)$. Note that

$$k_\beta([-\beta, 1+\beta]^n)=[0,1]^n.$$

The C^∞-approximation X' is obtained as $(k_\beta)_*(\rho_\beta * X)$. We have the following estimates.

$$|(k_\beta)_*(\rho_\beta * X_t)|_{r,\alpha} \leqq (1+2\beta)^{r-1}|X_t|_{r,\alpha} \quad \text{and}$$
$$\begin{aligned}&|(k_\beta)_*(\rho_\beta * X_t)-X_t|_0\\ &\quad \leqq |(k_\beta)_*(\rho_\beta * X_t)-(\rho_\beta * X_t)|_0+|(\rho_\beta * X_t)-X_t|_0\\ &\quad \leqq \beta C_1 |X_t|_{r,\alpha},\end{aligned}$$

where the positive number C_1 does not depend on β.

Let φ and φ': $([0,1],0)\to(\mathrm{Diff}^{r,\alpha}_{[0,1]^n}(\boldsymbol{R}^n),\mathrm{id})$ be the 1-simplices corresponding to X and X', respectively. We consider a singular 1-simplex σ of $\mathrm{Diff}^{r,\alpha}_{[0,1]^n}(\boldsymbol{R}^n)$ given by

$$\sigma(t)=\begin{cases}\varphi'(1-2t), & 0\leqq t\leqq 1/2\\ \varphi(2t-1), & 1/2\leqq t\leqq 1.\end{cases}$$

Then, for $\sigma(1)\sigma(0)^{-1}=\varphi(1)\varphi'(1)^{-1}$, we have positive real numbers C_2 and c_2 independent of β, such that

$$|\sigma(1)\sigma(0)^{-1}-\mathrm{id}|_{r,\alpha}\leqq C_2|X|_{\mathscr{B}} \quad \text{and} \quad |\sigma(1)\sigma(0)^{-1}-\mathrm{id}|_0\leqq \beta C_2|X|_{\mathscr{B}}$$

provided $|X|_{\mathscr{B}}\leqq c_2$.

We put $\gamma_\beta X$ to be the straightening $L\sigma$ of σ. If $|X|_{\mathscr{B}}$ is small, $\gamma_\beta X$ is well defined, and depends continuously on X. The desired estimates follow from the above estimates. By Lemma (A.3) (for $B\overline{\mathrm{Diff}}{}^{r+1}_c(\boldsymbol{R}^n)$), $\mathscr{F}(X')$ is homologous to zero in $B\overline{\mathrm{Diff}}{}^{r,\alpha}_c(\boldsymbol{R}^n)$. Since $\mathscr{F}(\gamma_\beta X)$ is homologous to $-\mathscr{F}(X')+\mathscr{F}(X)$, $\mathscr{F}(\gamma_\beta X)$ is homologous to $\mathscr{F}(X)$.

Let N be a positive integer. We are going to construct the mapping κ depending on N which we fix later. We use the fragmentation with respect to the following partition of unity.

Let ε be a positive real number smaller than 1/2. Let η be a smooth function on $\boldsymbol{R}$ such that

$$0 \leqq \eta(x) \leqq 1,$$
$$\eta(x)=0 \quad \text{for } x \in (-\infty, 0] \quad \text{and} \quad \eta(x)=1 \quad \text{for } x \in [\varepsilon, \infty).$$

Put $\eta_1 = 1-\eta(x-(1-\varepsilon))$,

$$\eta_j = \eta(x-(j-1)(1-\varepsilon))(1-\eta(x-j(1-\varepsilon))) \quad (j=2, \cdots, N-1)$$

and $\eta_N = \eta(x-(N-1)(1-\varepsilon))$.

It is easy to see that $\{\eta_j\}$ is a partition of unity on $\boldsymbol{R}$;

$$\sum_{j=1}^{N} \eta_j = 1.$$

We have $\operatorname{Supp}(\eta_1) \subset (-\infty, 1]$,

$$\operatorname{Supp}(\eta_j) \subset [(j-1)(1-\varepsilon), j(1-\varepsilon)+\varepsilon] \quad (j=2, \cdots, N-1) \quad \text{and}$$
$$\operatorname{Supp}(\eta_N) \subset [(N-1)(1-\varepsilon), \infty).$$

We define $\nu_{i_1 \cdots i_n} \colon \boldsymbol{R}^n \to [0, 1]$ $(1 \leqq i_1, \cdots, i_n \leqq N)$ by

$$\nu_{i_1 \cdots i_n}(x_1, \cdots, x_n) = \eta_{i_1}(x_1) \cdots \eta_{i_n}(x_n).$$

Then we have

$$\sum \nu_{i_1 \cdots i_n} = 1 \quad \text{and}$$
$$\operatorname{Supp}(\nu_{i_1 \cdots i_n}) \subset \operatorname{Supp}(\eta_{i_1}) \times \cdots \times \operatorname{Supp}(\eta_{i_n}).$$

We obtain constants c_ν and $C_{\nu,r,\alpha}$ with respect to this partition of unity $\{\nu_{i_1 \cdots i_n}\}$, such that, for a $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-product $\mathscr{F}$ over [0, 1],

$$\|h^{1*}\mathscr{F}\|_{r,\alpha} \leqq C_{\nu,r,\alpha} \|\mathscr{F}\|_{r,\alpha} \text{ provided } |\mathscr{F}|_0 \leqq c_\nu.$$

Here, we use the $C^{r,\alpha}$-norm $\|\ \|_{r,\alpha}$ as an $\boldsymbol{R}^n$-valued function on $[0, 1] \times \boldsymbol{R}^n$. Note that c_ν and $C_{\nu,r,\alpha}$ depend on the function η and the differentiability r, α but they do not depend on the positive integer N.

Construction of κ. Let X be an element of $\mathscr{B}$ such that $|X|_{\mathscr{B}} \leqq c$, where c is given in Lemma (A.2). We put $\beta = N^{-r}$ and apply Lemma (A.2) to X. Then we obtain $\gamma_\beta X$ such that

$$|\gamma_\beta X|_0 \leqq C N^{-r} |X|_{\mathscr{B}} \quad \text{and} \quad |\gamma_\beta X|_{\mathscr{B}} \leqq C |X|_{\mathscr{B}}.$$

Let $\mathscr{F}(L\gamma_\beta X)$ be the straightening of $\mathscr{F}(\gamma_\beta X)$. If N is sufficiently large, $L\gamma_\beta X$ is well-defined. (The element $\gamma_\beta X$ obtained by our proof of Lemma (A.2) is already straightened.) Then there exists a positive real number C_L such that

$$|L\gamma_\beta X|_0 \leqq C_L N^{-r}|X|_{\mathscr{B}} \quad \text{and} \quad |L\gamma_\beta X|_{\mathscr{B}} \leqq C_L |X|_{\mathscr{B}}.$$

These imply, by changing C_L if necessary, the following estimates:

$$|L\gamma_\beta X|_i \leqq C_L N^{i-r}|X|_{\mathscr{B}} \quad (0 \leqq i \leqq r).$$

Now, the N-subdivision $s^{(N)}\mathscr{F}(L\gamma_\beta X)$ is a sum of N $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-products $\mathscr{F}(X_j)$ over $[0, 1]$ with support in $[0, 1]^n$ $(j=1, \cdots, N)$.

Let ξ_N be a smooth vectorfield on $\boldsymbol{R}^n$ with compact support which coincides with $\sum_{i=1}^n x_i(\partial/\partial x_i)$ on $[-1, N+1]^n$. Let f^t denote the time $t\log(N(1-\varepsilon)+\varepsilon)$ map of ξ_N. Then $f^1([0, 1]^n)=[0, N(1-\varepsilon)+\varepsilon]^n$. Consider the foliated $\boldsymbol{R}^n$-product $f^1\mathscr{F}(X_j)=\mathscr{F}(f_*^1 X_j)$ $(j=1, \cdots, N)$ which has support in $[0, N(1-\varepsilon)+\varepsilon]^n$. Since

$$f_*^1 X_j(t, x)=((N(1-\varepsilon)+\varepsilon)/N)(L\gamma_\beta X)((t+(j-1))/N, x/(N(1-\varepsilon)+\varepsilon)),$$

we have the following estimates on the norms of $f_*^1 X_j$ $(j=1, \cdots, N)$:

$$\begin{aligned} |f_*^1 X_j|_{r,\alpha} &\leqq (N(1-\varepsilon)+\varepsilon)^{1-r}N^{-1}|L\gamma_\beta X|_{r,\alpha} \\ &\leqq C_L(N(1-\varepsilon)+\varepsilon)^{1-r}N^{-1}|X|_{\mathscr{B}}, \quad \text{and} \\ |f_*^1 X_j|_i &\leqq (N(1-\varepsilon)+\varepsilon)^{1-i}N^{-1}|L\gamma_\beta X|_i \\ &\leqq C_L(N(1-\varepsilon)+\varepsilon)^{1-i}N^{i-r-1}|X|_{\mathscr{B}} \quad (0 \leqq i \leqq r). \end{aligned}$$

Hence we have

$$\|f_*^1 X_j\|_{r,\alpha} \leqq C_L(N(1-\varepsilon)+\varepsilon)^{1-r}N^{-1}|X|_{\mathscr{B}}.$$

By taking a sufficiently large positive integer N, we may assume that $|f_*^1 X_j|_0 \leqq c_\nu$. We apply the fragmenting homotopy h with respect to the partition of unity $\{\nu_{i_1\cdots i_n}\}$. Put

$$h^{**}\mathscr{F}(f_*^1 X_j)=\sum_{1\leqq i_1,\cdots,i_n\leqq N} \mathscr{F}(X_{j,i_1,\cdots,i_n}),$$

where $X_{j,i_1,\cdots,i_n}$ has support in $[(i_1-1)(1-\varepsilon), i_1(1-\varepsilon)+\varepsilon]\times\cdots\times[(i_n-1)(1-\varepsilon), i_n(1-\varepsilon)+\varepsilon]$. We have an estimate on the norm of $X_{j,i_1,\cdots,i_n}$;

$$|X_{j,i_1,\cdots,i_n}|_{r,\alpha} \leqq C_{\nu,r,\alpha}\|f_*^1 X_j\|_{r,\alpha} \leqq C_{\nu,r,\alpha}C_L(N(1-\varepsilon)+\varepsilon)^{1-r}N^{-1}|X|_{\mathscr{B}}$$

$(j=1, \cdots, N; 1 \leqq i_1, \cdots, i_n \leqq N)$. Let $\mathscr{F}(\psi^* X_{j,i_1,\dots,i_n})$ denote the $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-product over [0, 1] obtained from $\mathscr{F}(X_{j,i_1,\dots,i_n})$ by the smoothing along the boundary. Then we have

$$|\mathscr{F}(\psi^* X_{j,i_1,\dots,i_n})|_{\mathscr{B}} \leqq (1-4\varepsilon')^{-1} C_{\nu,r,\alpha} C_L (N(1-\varepsilon)+\varepsilon)^{1-r} N^{-1} |X|_{\mathscr{B}}.$$

Take a one-parameter subgroup $T^{(t)}_{i_1,\dots,i_n}$ of $\mathrm{Diff}^\infty_c(\boldsymbol{R}^n)$ such that $T^{(1)}_{i_1,\dots,i_n}$ restricted to

$$[(i_1-1)(1-\varepsilon), i_1(1-\varepsilon)+\varepsilon] \times \cdots \times [(i_n-1)(1-\varepsilon), i_n(1-\varepsilon)+\varepsilon]$$

is the translation to $[0, 1]^n$. Then $T^{(1)}_{i_1,\dots,i_n} \mathscr{F}(\psi^* X_{j,i_1,\dots,i_n})$ has support in $[0, 1]^n$ and it is an element of $\mathscr{B}$ whose norm is the same as that of $\psi^* X_{j,i_1,\dots,i_n}$.

Let $\mathscr{F}(\kappa X)$ denote a $C^{r,\alpha}$-foliated $\boldsymbol{R}^n$-product over [0, 1] which is the composition of the N^{n+1} foliated products $T^{(1)}_{i_1,\dots,i_n} \mathscr{F}(\psi^* X_{j,i_1,\dots,i_n})$;

$$s^{(N^n+1)} \mathscr{F}(\kappa X) = \sum_{1 \leqq j, i_1, \dots, i_n \leqq N} T^{(1)}_{i_1,\dots,i_n} \mathscr{F}(\psi^* X_{j,i_1,\dots,i_n}).$$

Then κX is an element of $\mathscr{B}$ and we have

$$\begin{aligned} |\kappa X|_{\mathscr{B}} &\leqq N^{n+1} \sup_{j,i_1,\dots,i_n} |\psi^* X_{j,i_1,\dots,i_n}|_{\mathscr{B}} \\ &\leqq (1-4\varepsilon')^{-1} C_{\nu,r,\alpha} C_L (N(1-\varepsilon)+\varepsilon)^{1-r} N^n |X|_{\mathscr{B}}. \end{aligned}$$

Now suppose that $r > n+1$. Since $C_{\nu,r,\alpha}$ and C_L do not depend on N, by chocsing a sufficiently large N, we have

$$|\kappa X|_{\mathscr{B}} \leqq ((1-4\varepsilon')/2) |X|_{\mathscr{B}} \qquad \text{for } X \in \mathscr{B} \text{ with } |X|_{\mathscr{B}} \leqq c.$$

Note that $\mathscr{F}(\kappa X)$ is homologous to $\mathscr{F}(X)$ in $B\overline{\mathrm{Diff}}{}^{r,\alpha}_c(\boldsymbol{R}^n)$.

Proof of Theorem (A.1). Let X_0 and X_1 be elements of $\mathscr{B}$ such that $|X_i|_{\mathscr{B}} \leqq c(1-4\varepsilon')/2$ $(i=0, 1)$. Then we have an element X of $\mathscr{B}$ such that

$$s^{(2)} \mathscr{F}(X) = \mathscr{F}(\psi^* X_0) + \mathscr{F}(\psi^* X_1).$$

It is obvious that $|X|_{\mathscr{B}} \leqq c$. Hence we obtain κX. By the estimate on the norm of κX, we have

$$|\kappa X|_{\mathscr{B}} \leqq c(1-4\varepsilon')/2.$$

Thus $X_1 \to \kappa X$ is a map from $\{X_1 \in \mathscr{B}; |X_1|_{\mathscr{B}} \leqq c(1-4\varepsilon')/2\}$ to itself which is continuous with respect to the C^r-topology. This subset of $\mathscr{B}$ is compact with respect to the C^r-topology and is convex. Hence by the Schauder-

Tychonoff fixed point theorem, we have a fixed point X_1 of this map. By the construction of κX, $\mathscr{F}(\kappa X)$ is homologous to $\mathscr{F}(X)$. Hence $\mathscr{F}(X_0)+\mathscr{F}(X_1)$ is homologous to $\mathscr{F}(X_1)$, that is, $\mathscr{F}(X_0)$ is homologous to zero.

Since every 1-cycle of $B\overline{\mathrm{Diff}}_c^{r,\alpha}(\boldsymbol{R}^n)$ is homologous to a sum of elements in $\{X_1 \in \mathscr{B}; |X_1|_{\mathscr{B}} \leqq c(1-4\varepsilon')/2\}$, we have proved that

$$H_1(B\overline{\mathrm{Diff}}_c^{r,\alpha}(\boldsymbol{R}^n); \boldsymbol{Z})=0.$$

Since this is true for any modulus of continuity α, Theorem (A.1) follows.

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