Advanced Studies in Pure Mathematics 5, 1985
Foliations
pp. 29–36

Lift Foliations in Flat Principal Bundles and Modular Functions

Haruo Suzuki

§ 1. Introduction

In the differential topology of foliations, we have an h_1-invariant for a leaf which is a kind of secondary characteristic class (cf. B. Reinhart [R] and R. Goldman [G]). On the other hand, in the operator theory of foliations, a modular function (i.e. module) of transverse measure on the holonomy groupoid of a foliation plays an important role (cf. A. Connes [C]).

The h_1-invariant is closely related to the modular function of the foliation and in particular, nontriviality of the h_1-invariant means that of the modular function corresponding to a nowhere vanishing C^∞-density on the foliated manifold (cf. H. Suzuki [S2]).

Let M be a Hausdorff C^∞-manifold with a countable open base and $(M, \mathscr{F})$ a C^∞-foliation of codimension q on M. Sometimes we denote $(M, \mathscr{F})$ by $\mathscr{F}$ simply. Let K be a connected Lie group, $K_0 \subset K$ a discrete subgroup and $p: E \rightarrow M$ a C^∞-principal K-bundle, the structural group of which has K_0-reduction. Then it is well known that $(M, \mathscr{F})$ is lifted to a C^∞-foliation $(E, \mathscr{F}_E)$ of codimension q. The main result of the present note is stated as follows.

Theorem 4.2. *Let K be a connected Lie group, $(M, \mathscr{F})$ a foliation of codimension $q > 0$ and $\pi_1(M)$ be finite. If $(M, \mathscr{F})$ has a nontrivial h_1-invariant, then the module δ of transverse measure on the holonomy groupoid of $(E, \mathscr{F}_E)$, corresponding to a nowhere vanishing C^∞-density of E is nontrivial.*

In Section 2, we introduce the h_1-invariant for a leaf of a foliation. In Section 3, we consider locally free, foliation preserving, transverse Lie group action and examine h_1-invariants of the extended foliation by this Lie group action. In the last section, we prove Theorem 4.2 by making use of the result of Section 3 and Theorem of [S2].

The author wishes to thank the referee for his comments.

Received October 17, 1983.

§ 2. h_1-invariants

Let $(M, \mathscr{F})$ be a C^∞-foliation of codimension q. Let $T(M)$ be the tangent bundle of M and F the subbundle of $T(M)$ consisting of tangent vectors of leaves of $\mathscr{F}$. The quotient bundle $\nu(\mathscr{F})=T(M)/F$ is called the *transverse vector* (i.e. *normal*) *bundle* of $(M, \mathscr{F})$. Let ∇^B be a Bott connection (that is the basic connection of [B]) of $\mathscr{F}$ on the bundle $\nu(\mathscr{F})$ and Ω_B its local curvature matrix.

Let c_1 denote the first Chern polynomial. It is well known that $c_1(\Omega_B)$ is a globally defined differential 2-form on M. (In fact $c_1(\Omega_B)=(1/2\pi)$ trace Ω_B.) Since each component of the matrix Ω_B is contained in the ideal generated by q linearly independent differential 1-forms defining $\mathscr{F}$(this is Bott vanishing theorem [B, Section 6]), $c_1(\Omega_B)$ restricted to a leaf L of $\mathscr{F}$ vanishes, i.e.,

$$c_1(\Omega_B)|_L=0.$$

Let ∇^R be a Riemannian connection on $\nu(\mathscr{F})$ and Ω_R its local curvature matrix. Let π: $M\times \boldsymbol{R}\to M$ be the first factor projection map and define a connection ∇ on $\pi^*\nu(\mathscr{F})=\nu(\mathscr{F})\times \boldsymbol{R}$ by an affine combination of ∇^B and ∇^R,

$$\nabla=t\nabla^B+(1-t)\nabla^R \qquad t\in \boldsymbol{R}.$$

Let Ω be the local curvature matrix of ∇. $c_1(\Omega)$ is a globally defined differential 2-form on $M\times \boldsymbol{R}$. We denote by π_* the integration over the fiber for the trivial bundle π: $M\times I\to M(I=[0, 1])$. Then since Ω_R is skew symmetric, it follows that,

$$d(\pi_*(c_1(\Omega)|_{M\times I}))=c_1(\Omega_B).$$

Therefore we have

$$d(\pi_*(c_1(\Omega)|_{M\times I})|_L)=c_1(\Omega_B)|_L=0.$$

It can be shown that the de Rham cohomology class

$$[\pi_*(c_1(\Omega)|_{M\times I})|_L]\in H^1_{DR}(L)$$

does not depend on the choice of the connections ∇^B and ∇^R, and is determined uniquely by the foliation $(M, \mathscr{F})$ and its leaf L. We denote this cohomology class by $h_1(\mathscr{F}, L)$ and call it the *first leaf invariant of $\mathscr{F}$ with respect to L.*

Remark. For any odd integer i such that $1\leqq i\leqq 2[(q+1)/2]-1$, a cohomology class

$$h_i(\mathscr{F}, L) \in H_{DR}^{2i-1}(L)$$

is defined in a similar way. (See [G] and [S2].)

§ 3. The extended foliation by some Lie group action

In this section, we formulate, in rather general situation, a relation between the h_1-invariant of foliation and that of the foliation extended by transverse frame field preserving given foliation. This result is applied, in the last section, to prove Theorem 4.2 on the module of transverse measure of holonomy groupoid for the lift foliation in principal bundle over foliated manifold.

Let $\Gamma(\xi)$ denote the module of sections of vector bundle ξ. For the foliation $(M, \mathscr{F})$, we say that a tangent vector field $Y \in \Gamma(T(M))$ *preserves* $\mathscr{F}$, if for each $Z \in \Gamma(F)$ we have $[Y, Z] \in \Gamma(F)$. A (linearly independent) k-frame field $\{X_1, \cdots, X_k\}$, $X_i \in \Gamma(T(M))$ is called *transverse* to $\mathscr{F}$, if the span of $X_1, \cdots, X_k$ at each point has 0 intersection with F. It is said that the vector fields $X_1, \cdots, X_k$ form a *Lie algebra* mod $\mathscr{F}$, if there exists C^∞-functions α_{ij}^r and vector fields $Y_{ij} \in \Gamma(F)$, $i, j, r=1, \cdots, k$ such that

$$[X_i, X_j] = \sum_{r=1}^{k} \alpha_{ij}^r X_r + Y_{ij}.$$

Let ξ_i be the trivial line bundle determined by X_i. If $X_1, \cdots, X_k$ form a Lie algebra mod $\mathscr{F}$ and each X_i preserves $\mathscr{F}$, then the subbundle

$$F' = \bigoplus_{i=1}^{k} \xi_i \oplus F \subset T(M)$$

is integrable and defines an extended foliation $\mathscr{F}'$ on M. Let $\tilde{\mathscr{F}}$ be a foliation of M and $\tilde{F} \subset T(M)$ the subbundle determined by $\tilde{\mathscr{F}}$. If F is a subbundle of $\tilde{F}$, then $\mathscr{F}$ is called a *subfoliation* of $\tilde{\mathscr{F}}$ which is denoted by $\mathscr{F} \subset \tilde{\mathscr{F}}$.

Lemma 3.1. *Let $\mathscr{F}$ and $\tilde{\mathscr{F}}$ be C^∞-foliations of a manifold M such that $\mathscr{F} \subset \tilde{\mathscr{F}}$,* codim $\mathscr{F} = q$ *and* codim $\tilde{\mathscr{F}} = k < q$. *Let $\{X_1, \cdots, X_k\}$ be a k-frame field transverse to $\tilde{\mathscr{F}}$. Suppose that each X_i preserves $\mathscr{F}$ and $X_1, \cdots, X_k$ form a Lie algebra* mod $\mathscr{F}$. *Let $\mathscr{F}'$ be the foliation determined by $\{X_1, \cdots, X_k\}$ and the subbundle F corresponding to $\mathscr{F}$. Let L be a leaf of $\mathscr{F}$ and L' the leaf of $\mathscr{F}'$ containing L. Then we have*

$$h_1(\mathscr{F}, L) = j^* h_1(\mathscr{F}', L'),$$

where j is the inclusion map $L \to L'$.

Remark. The result of Lemma 3.1 is generalized to h_i-invariants for $1 \leqq i \leqq q-k$ (cf. [S1, Section 2]).

Proof. If we split the subbundle $\tilde{F} \subset T(M)$ determined by $\tilde{\mathscr{F}}$ into a Whitney sum: $\tilde{F} = F \oplus \tilde{V}$, then $\tilde{V}$ is isomorphic to $\nu(\mathscr{F}')$ and the transverse vector bundle $\nu(\mathscr{F})$ can be identified with the Whitney sum

$$V = \bigoplus_{i=1}^{k} \xi_i \oplus \tilde{V}.$$

Let $\tilde{\nabla}$ be any Bott connection for $\mathscr{F}'$ on V and ∇' the trivial connection with respect to the global k-frame field $\{X_1, \cdots, X_k\}$ on the trivial bundle $\oplus_{i=1}^{k} \xi_i$. We define a connection ∇ on V by the Whitney sum $\nabla = \nabla' \oplus \tilde{\nabla}$. Since $\oplus_{i=1}^{k} \xi_i \oplus \tilde{V}$ is a subbundle of $T(M)$, one can regard $\Gamma(\tilde{V})$ as a subset of $\Gamma(T(M))$.

Let ρ: $T(M) \to V = T(M)/F$ be the natural projection map. By our assumption, X_i preserves F and hence it follows that for $X \in \Gamma(F)$

$$\nabla'_X(X_i) = 0 = \rho([X, X_i]) \qquad i = 1, \cdots, k.$$

Therefore, for $s' \in \Gamma(\oplus_{i=1}^{k} \xi_i) \subset \Gamma(T(M))$, we have

$$\nabla_X(s') = \nabla'_X(s') = \rho([X, s']) \qquad X \in \Gamma(F).$$

On the other hand, the subbundle $\tilde{F} = F \oplus \tilde{V} \subset T(M)$ is integrable by assumption, and hence for any $\tilde{s} \in \Gamma(V) \subset \Gamma(T(M))$, the ξ_i-component of the vector field $[X, \tilde{s}]$ at each point is 0. This means that

$$\rho([X, s]) = \tilde{\rho}([X, \tilde{s}]) \qquad X \in \Gamma(F),$$

where $\tilde{\rho}$ is the natural projection map $T(M) \to \tilde{V} = T(M)/F'$ and therefore we have

$$\begin{aligned} \nabla_X(\tilde{s}) &= \tilde{\nabla}_X(\tilde{s}) = \tilde{\rho}([X, \tilde{s}]) \\ &= \rho([X, \tilde{s}]). \end{aligned}$$

Since any section $s \in \Gamma(\oplus_{i=1}^{k} \xi_i \oplus \tilde{V})$ is of the form $s = s' \oplus \tilde{s}$, we have for $X \in \Gamma(F)$,

$$\begin{aligned} \nabla_X(s) &= \rho([X, s']) + \rho([X, \tilde{s}]) \\ &= \rho([X, s' + \tilde{s}]) \\ &= \rho([X, s]) \end{aligned}$$

which shows ∇ is a Bott connection for $\mathscr{F}$.

Let $\tilde{\Phi}_{q-k}$ be a local $(q-k)$-frame section of $\tilde{V}$ and denote by Φ_q the local q-frame section obtained by adding the vector fields $X_1, \cdots, X_k$ to $\tilde{\Phi}_{q-k}$. Let θ (resp. $\tilde{\theta}$) be the connection matrix of ∇ (resp. $\tilde{\nabla}$) with respect to Φ_q (resp. $\tilde{\Phi}_{q-k}$). Then we have

$$\theta = \begin{bmatrix} \tilde{\theta} & 0 \\ 0 & 0 \end{bmatrix}.$$

We fix a Riemannian connection $\tilde{\nabla}^0$ on $\tilde{V}$ and then the connection $\nabla^0 = \nabla' \oplus \tilde{\nabla}^0$ on $V = \oplus_{i=1}^k \xi_i \oplus \tilde{V}$ is also a Riemannian connection. For connection matrix θ^0 (resp. $\tilde{\theta}^0$) of ∇^0 (resp. $\tilde{\nabla}^0$) with respect to Φ_q (resp. $\tilde{\Phi}_{q-k}$), we have similar formula to that of θ and $\tilde{\theta}$.

We form the connection

$$\nabla^* = t\nabla + (1-t)\nabla^0 \qquad t \in \boldsymbol{R}$$

on the vector bundle $V \times \boldsymbol{R} \to M \times \boldsymbol{R}$. The connection matrix of ∇^* with respect to Φ_q is

$$\theta^* = t\theta + (1-t)\theta^0.$$

We denote its curvature matrix by Ω^*. Similarly one obtains the connection $\tilde{\nabla}$ on the vector bundle $\tilde{V} \times \boldsymbol{R} \to M \times \boldsymbol{R}$ and its curvature matrix $\tilde{\Omega}^*$. It follows immediately that $c_1(\Omega^*) = c_1(\tilde{\Omega}^*)$ and hence

$$\begin{aligned} h_1(\mathscr{F}, L) &= [\pi_*(c_1(\Omega^*)|_{M\times I})|_L] \\ &= [\pi_*(c_1(\tilde{\Omega}^*)|_{M\times I})|_L] \\ &= j^*[\pi_*(c_1(\tilde{\Omega}^*)|_{M\times I})|_{L'}] \\ &= j^* h_1(\mathscr{F}', L'). \end{aligned}$$

q.e.d.

Corollary 3.2. *Let $(M, \mathscr{F})$ be a C^∞-foliation of codimension q. Suppose that a connected Lie group K of dimension $k < q$ acts on M, and the action is locally free, foliation preserving and is transverse to a foliation $\tilde{\mathscr{F}}$ such that $\tilde{\mathscr{F}} \supset \mathscr{F}$. Let $\mathscr{F}'$ be the extended foliation of $\mathscr{F}$ by the action of K. Then we have*

$$h_1(\mathscr{F}, L) = j^* h_1(\mathscr{F}', L'),$$

where L is a leaf of $\mathscr{F}$, L' is the leaf containing L and j: $L \to L'$ is the inclusion map.

Proof. We take vector fields on M defined by basis vectors of the Lie algebra of K as $X_1, \cdots, X_k$. These satisfy conditions of Lemma 3.1. The result follows immediately from the lemma. q.e.d.

§ 4. Foliation with nontrivial module

Let G be the holonomy groupoid of a C^∞-foliation $(M, \mathscr{F})$ of codimension q and δ a module of transverse measure on G by A. Connes (cf. [C] and [K]), corresponding to a nowhere vanishing C^∞-density of M. Let $H^1(G; \boldsymbol{R})$ be the groupoid 1-cohomology group with real coefficients (cf. [RN]). Let $C_{\mathscr{F}}$ be the sheaf of germs of real valued continuous functions constant along leaves and $H^1(M: C_{\mathscr{F}})$ the 1-sheaf cohomology vector space of M with the coefficient sheaf $C_{\mathscr{F}}$.

Let ι: $H^1(G; \boldsymbol{R}) \to H^1(M; C_{\mathscr{F}})$ denote the natural injective homomorphism (cf. [Y]). $\log \delta$ is a 1-cocycle of G with real coefficients and defines a cohomology class $[\log \delta] \in H^1(G; \boldsymbol{R})$. The sheaf cohomology class, $\iota[\log \delta] \in H^1(M; C_{\mathscr{F}})$ does not depend on the choice of C^∞-density of M and is called the *modular cohomology class* of $\mathscr{F}$. Let L be a leaf of $\mathscr{F}$ and i: $L \to M$ the natural inclusion map. The pullback $i^* C_{\mathscr{F}}$ of $C_{\mathscr{F}}$ is the sheaf of constant functions and i induces a homomorphism

$$i^*: H^1(M; C_{\mathscr{F}}) \longrightarrow H^1(L; \boldsymbol{R}) = H^1_{DR}(L).$$

Now we have

Theorem 4.1. *Let $(M, \mathscr{F})$ be a C^∞-foliation and δ a module of transverse measure on the holonomy groupoid of $\mathscr{F}$, corresponding to a nowhere vanishing C^∞-density of M. Let L be a leaf of $\mathscr{F}$. Then δ and $h_1(\mathscr{F}, L)$ are related by the formula*

$$i^* \circ \iota[\log \delta] = -2\pi h_1(\mathscr{F}, L).$$

Proof. See [S2].

Let K be a connected Lie group, $K_0 \subset K$ a discrete subgroup and p: $E \to M$ a C^∞-principal K-bundle, the structural group of which has K_0-reduction. Then there exists a homomorphism h of the fundamental group $\pi_1(M)$ to K_0. Therefore $\pi_1(M)$ acts on K by the left multiplication via h and at the same time it acts on the universal covering manifold $\overline{M}$ of M by covering transformation. It is well known that $E \cong \overline{M} \times_{\pi_1(M)} K$. Since the diagonal action of $\pi_1(M)$ on $\overline{M} \times K$ preserves the foliation $\{M \times \{g\}\}_{g \in K}$, it gives rise to a foliation $\tilde{\mathscr{F}}$ of codimension $k = \dim K$ on E. Obviously $\tilde{\mathscr{F}}$ is invariant under the right action of K.

Moreover, $\mathscr{F}$ determines a foliation $\overline{\mathscr{F}} = \{\overline{L}\}$ of codimension q on $\overline{M}$, which is preserved by the covering transformation of $\pi_1(M)$. Hence the diagonal action of $\pi_1(M)$ on $\overline{M} \times K$ preserves the foliation $\{\overline{L} \times \{g\}\}_{\overline{L} \in \overline{\mathscr{F}}, g \in K}$ of codimension $q + \dim K = q + k$. Therefore one obtains a foliation $\mathscr{F}_E$ of codimension $q + k$ on E. $\mathscr{F}_E$ is also invariant under the right action of K. Since the subbundle F_E of $T(E)$ determined by $\mathcal{F}_E$ is a subbundle of $\tilde{F}$ which is the subbundle of $T(E)$ determined by $\tilde{\mathcal{F}}$, we have $\mathcal{F}_E \subset \tilde{\mathcal{F}}$. A leaf L_E of $\mathcal{F}_E$ is a covering space of a leaf L of $\mathcal{F}$ by the bundle projection map $p: E \to M$ and so $(E, \mathcal{F}_E)$ is called the *lift foliation* of $(M, \mathcal{F})$.

Theorem 4.2. *Let K be a connected Lie group, $(M, \mathcal{F})$ a foliation of codimension $q>0$ and $\pi_1(M)$ be finite. If $(M, \mathcal{F})$ has a nontrivial h_1-invariant, then the module δ of transverse measure on the holonomy groupoid of $(E, \mathcal{F}_E)$, corresponding to a nowhere vanishing C^∞-density of E is nontrivial.*

Proof. It is sufficient to show that $[\log \delta] \neq 0$. By the assumption of our theorem, one can find a leaf L of $\mathcal{F}$ such that

$$h_1(\mathcal{F}, L) \neq 0.$$

In Corollary 3.2, we take E for M and $\mathcal{F}_E$ for $\mathcal{F}$. Then $p^*\mathcal{F} = \mathcal{F}'_E$ is the extended foliation of $\mathcal{F}_E$ by the action of K and $p^{-1}(L)$ is a leaf of $\mathcal{F}'_E$. By the naturality of h_1-invariant for the transverse map to foliation and by Corollary 3.2, it follows that

$$\begin{aligned} h_1(\mathcal{F}_E, L_E) &= j^* h_1(\mathcal{F}'_E, p^{-1}(L)) \\ &= j^* p^* h_1(\mathcal{F}, L) \\ &= (p|_{L_E})^* h_1(\mathcal{F}, L). \end{aligned}$$

By the assumption $\pi_1(M)$ is finite, $(p|_{L_E})^*: H^1(L: \boldsymbol{R}) \to H^1(L_E: \boldsymbol{R})$ is injective and therefore one obtains

$$h_1(\mathcal{F}_E, L_E) \neq 0.$$

On the other hand, by Theorem 4.1, we have

$$\begin{aligned} i^* \circ \iota[\log \delta] &= -2\pi h_1(\mathcal{F}_E, L_E) \\ &\neq 0 \end{aligned}$$

and hence $[\log \delta] \neq 0$. q.e.d.

References

[B] R. Bott, Lectures on characteristic classes and foliations, Lecture Notes in Math., **279**, Springer-Verlag, Berlin (1972), 1–94.

[C] A. Connes, Sur la théorie non commutative de l'intégration, Lecture notes in Math., **725**, Springer-Verlag, Berlin (1979), 19–143.

[G] R. Goldman, Characteristic classes on the leaves of foliated manifiolds, Ph. D. Thesis, Johns Hopkins Univ. 1973.

[K] D. Kastler, On a Connes' noncommutative integration theory, Comm. Math. Phys., **85** (1982), 99–102.

[RI] B. Reinhart, Holonomy invariants for framed foliations, Technical Report No. 32, Univ. of Maryland 1972.

[RN] J. Renoult, A groupoid approach to C^*-algebras, Lecture Notes in Math., **793**, Springer-Verlag, Berlin (1980).

[S1] H. Suzuki, Foliation preserving Lie group actions and characteristic classes, Proc. Amer. Math. Soc., **85** (1982), 633–637.

[S2] ——, Modular cohomology classes from view point of characteristic class, to appear in Proceedings of U.S.-Japan Seminar on Geometric Methods in Operator Algebras.

[Y] S. Yamagami, Modular cohomology class of foliation and Takesaki's duality, RIMS-417 Kyoto Univ. 1982.

Department of Mathematics
Hokkaido University
Sapporo, 060
Japan