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CHAPTER 26

Topological Optimization for Photonic and Phononic crystals problems, *by M.B. Dia, I. Faye and A. Sy*

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Abstract. In this paper, we propose a numerical method to study phononic and photonic crystal problems. The proposed method is based on topological optimization tools. In fact, after modeling crystal photonics and phononic problems, we use topological optimization tools to build a numerical method for getting optimal design. Here the optimal design is the one in which all frequencies near the reference frequency corresponding to the reference wave length a can pass.

Keywords. Crystals phononic and photonic; Topological optimization ; topological gradient; eigenvalue; numerical simulations.
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1. Introduction and modeling

This paper deals about crystals photonic and phononic problems. The aims of the works is to use topological optimization tools in order to study crystal photonic and phononics problems.

Photonic crystals are remarkable structured materials in which light can not propagate freely. It can be blocked, allowed only in certain directions or even located in certain areas. These are composite materials which generally consist of a periodic network of dielectric or metallic inclusions whose characteristic size of the structuring is of the order of the incident order length. There are many works related to crystal photonics and phononics using few approaches Hammari *et al.* (2010, 2009) and Yablonovitch (1987). The topological optimization was introduced by Schumacher (1995), Sokolowski and Zochowski (1999) and Garreau *et al.* (2001). Actually, the topological sensitivity was used in various domain Faye *et al.* (2008); Guillaume and Sid Idris (2002); Masmoudi (2002); Ngom et al. (2011).

Throughout this paper, we propose a numerical method so called topological optimization which allows us to get optimal design of the structure. The topological sensitivity analysis aims to provide an asymptotic expansion of a shape functional on the neighborhood of a small hole created inside the domain. The reported analysis shall be based on the principle that follows.

For a criterion $j(\Omega) = J_\Omega(u_\Omega)$, $\Omega \subset \mathbb{R}^N$, with u_Ω as the solution of a boundary value problem defined over Ω. The asymptotic expansion of the cost function $j(\Omega)$ can generally be written in the form:

$$j(\Omega \setminus \overline{x_0 + \epsilon\omega}) - j(\Omega) = f(\epsilon)g(x_0) + o(f(\epsilon)).$$

In this expression, ϵ and x_0 denote respectively the radius and the center of the hole, ω is a reference domain inside Ω and containing x_0. The function $g(x_0)$ is called topological derivative and is used as descent direction in the optimization process. For more details topological optimization, we refer to: Masmoudi (2002), Nazarov and Sokolowski (2003).

The paper is organized as follows: In section two, we recall to the modeling of crystal problem (by using acoustic waves for phononic problem and electromagnetic waves for photonic ones): different cases ares presented. The section three deals topological optimization principals and so the principal result is given and its proof. In section four, we give some numerical results by using finite element method s.

2. Modeling of crystal problems

2.1. Modeling of crystals photonic problems. The modeling of photonic crystals is based on electromagnetic waves. We assume, in this work that the wave is strongly perturbed by the periodicity of the medium, a.e. the wave length depends substantially to the period of the stack. It is also assumed that the medium is isotropic and homogeneous. Under these hypotheses, Maxwell's equations, in a linear dielectric medium, non-magnetic without charge and current are written

$$\left\{\begin{array}{c} \nabla\cdot\big(\varepsilon(r)E(r,t)\big)=0 \\ \nabla\cdot H(r,t)=0 \\ \nabla\times E(r,t)=\mu_0\frac{\partial}{\partial t}H(r,t) \\ \nabla\times H(r,t)=\varepsilon_0\varepsilon(r)\frac{\partial}{\partial t}E(r,t) \end{array}\right. \tag{2.1}$$

where $\varepsilon(r)$ and μ_0 are the dielectric permittivity and the magnetic constant, E and H are the electric and magnetic fields, r and t the spacial and time coordinates.

Maxwell's equation for a field E on a domain Ω is given by :

$$div(E)=\nabla\cdot E=\frac{\rho}{\varepsilon}, \tag{2.2}$$

where ρ is the charge density. Thanks to Gauss' theorem, we have the following equation

$$\oint_{\sum} E\,dS = \frac{1}{\varepsilon_0}\int\int_V \rho\,d\,\tau.$$

and the Maxwell-Thompson's equation says that the flux of the magnetic field along a closed surface Σ_f is zero. Let B be the magnetic induction, then we have

$$\int\int_{\Sigma_f} B\,dS = 0,$$

since

$$\int\int_{\Sigma_f} BdS = \int\int\int_V \nabla BdV,$$

It follows that

$$divB = \nabla B = 0 \text{ on } \Omega.$$

Since the magnetic field H is proportional to the magnetic indiction B then we have $B = \mu\,H$. The Maxwell-Faraday equation gives:

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

The Maxwell - Ampère equation is given by

$$rotB = \nabla \times B = \mu(j + J_D),$$

where μ is the magnetic permittivity, j is the current density $J = \sigma_c\,E$ with σ_c being the electric conductivity and $J_D = \varepsilon\dfrac{\partial E}{\partial t}$ is the displacement current density. Then we finally get

$$
(2.3) \qquad \begin{cases} divE &= \dfrac{\rho}{\varepsilon} \\ \\ divB &= 0, \\ \\ \nabla \times E &= -\mu \dfrac{\partial H}{\partial t}, \\ \\ \nabla \times H &= \mu(j + \varepsilon \dfrac{\partial E}{\partial t}), \end{cases}
$$

We will consider several situations for the domain Ω.

2.1.1. *Domain Ω without charge and without a source presence.* We consider an homogeneous medium Ω, then $\rho = 0,\ j = 0,\ \mu = \mu_0$ and $\varepsilon = \varepsilon_0$. Then we obtain the following system

$$
(2.4) \qquad \begin{cases} divE &= 0, \\ \\ divH &= 0, \\ \\ \nabla \times E &= -\mu_0 \dfrac{\partial H}{\partial t}, \\ \\ \nabla \times H &= (\varepsilon_0 \dfrac{\partial^2 E}{\partial t^2}), \end{cases}
$$

From the properties of the vector computation, we have $rot(rotE) = \nabla(\nabla \cdot E) - \nabla^2 E$. Since $\nabla(\nabla \cdot E) = 0$ then $rot(rotE) = -\nabla^2 E$.

Therefore

$$\nabla^2 E = \nabla \times (\mu_0 \frac{\partial H}{\partial t}) = \mu_0 \frac{\partial(\nabla \times H)}{\partial t} = \mu_0(\varepsilon_0 \frac{\partial^2 E}{\partial t^2}).$$

and it follows that

$$\Delta E - \mu_0(\varepsilon_0 \frac{\partial^2 E}{\partial t^2}) = 0. \tag{2.5}$$

For a monochromatic wave, the electric field is given by

$$E(x,t) = Re(u(x)e^{ikt})$$

where u is plane wave solution to the Helmotz equation:

$$\Delta u(x) + k^2 u(x) = 0, \ x \in \Omega. \tag{2.6}$$

2.1.2. *Domain* Ω *without charge (* $\rho = 0$ *) and with a source presence (*$j \neq 0$ *).* This case corresponds to the propagation of a wave in a good conductor material. Then we have

$$\Delta E - \varepsilon\, \mu \frac{\partial^2 E}{\partial t^2} - \mu\, \sigma_c \frac{\partial E}{\partial t} = 0. \tag{2.7}$$

2.1.3. *Domain with charge and source.* In this case, we have $\rho \neq 0, \ \varepsilon \neq \varepsilon_0$ and μ is not constant.Then we have the following system

$$\begin{cases} divE &= \frac{\rho}{\varepsilon_0} \\ divH &= 0, \\ \nabla \times E &= -\frac{\partial H}{\partial t}, \\ \nabla \times H &= \mu(j + \varepsilon \frac{\partial E}{\partial t}), \end{cases} \tag{2.8}$$

from which we have, using

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \Delta E \tag{2.9}$$

the following system

$$\nabla \times (-\frac{\partial H}{\partial t}) = -\frac{\partial \nabla \times H}{\partial t} = -\frac{\partial \mu (j + \varepsilon \frac{\partial E}{\partial t})}{\partial t} = -\frac{\partial (\mu\, j)}{\partial t} - \frac{\partial (\mu\, \varepsilon \frac{\partial E}{\partial t})}{\partial t}, \tag{2.10}$$

It follows from this equality that

$$\Delta E - \frac{1}{\varepsilon} \nabla \rho - \frac{\partial \mu\, j}{\partial t} - \frac{\partial (\mu\, \varepsilon \frac{\partial\, E}{\partial\, t})}{\partial t} = 0. \tag{2.11}$$

Then we have

$$\Delta E - \frac{1}{\varepsilon} \nabla \rho - \varepsilon\, \mu \frac{\partial^2 E}{\partial t^2} - \mu\, \sigma_c \frac{\partial E}{\partial t} = 0. \tag{2.12}$$

2.2. Modeling of phononics problem. The modeling of phononic crystals is based on the acoustic waves. Let us suppose that the force f is negligible over the inertial force $(f = 0)$. It is also assumed that the fluid is irrational

$$rot(u) = 0. \tag{2.13}$$

From (2.13), there exists a function φ such that:

$$u = \nabla \varphi. \tag{2.14}$$

Then by using the mass conservation, we get

$$\frac{\partial \rho}{\partial t} + \rho_i\, \varphi_i + \rho\, \Delta \varphi = 0, \tag{2.15}$$

$$\frac{\partial}{\partial x_i} [\frac{\partial \varphi}{\partial t} + \frac{u^2}{2}] + \frac{1}{\rho} \frac{P}{x_i} = 0 \tag{2.16}$$

with $i = 1, 2, \ldots\;\; u^2 = |\nabla \varphi|^2,\;\; P$ is the pressure and ρ the density.

We shall consider in the following the classic hypotheses for small perturbations

* The velocity u and it's variations u_i and $\dfrac{\partial u_i}{\partial t}$ are too small.
* The variations of the pressure P and the density ρ are small around the initials P_0 and ρ_0

Linearizing equations (2.15) and (2.16) leads to

$$\frac{\partial \rho}{\partial t} + \rho\, \delta\, \varphi = 0, \tag{2.17}$$

and

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial t}\right) + \frac{1}{\rho}\frac{dP}{d\rho}\frac{\partial \rho}{\partial x_i} = 0. \tag{2.18}$$

$\dfrac{\partial \varphi}{\partial t}$ and $\dfrac{\partial \rho}{\partial x_i}$ are to be understood in the weak sense, then, we can replace ρ and $\dfrac{d\,P}{d\,\rho}$ by ρ_0 is c_0^2. It follows that

$$\frac{\partial \rho}{\partial t} + \rho_0\, \delta\, \varphi = 0, \tag{2.19}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial t} + c_0^2\, \frac{\rho}{\rho_0}\right) = 0. \tag{2.20}$$

We have to notice that $\dfrac{\partial \varphi}{\partial t} + c_0^2\, \dfrac{\rho}{\rho_0}$, is a function of t. Then, $\dfrac{\partial \varphi}{\partial t} + c_0^2\, \dfrac{\rho}{\rho_0} = k(t)$. The function φ can be modified by a time function without changing the value of the velocity u. Indeed, let $\Phi = \varphi - \int_0^t k(s)\, ds,\ \ u = \nabla\, \Phi$ and $\Delta\, \varphi = \Delta\, \Phi$, we have

$$\frac{1}{c_0^2}\,\frac{\partial^2 \Phi}{\partial t^2} - \Delta\, \Phi = 0. \tag{2.21}$$

Equations (2.21) need to be provided with the following initial conditions

$$\begin{cases} \Phi(x,0) &= 0 \text{ in } \Omega \\ \dfrac{\partial \Phi(x,0)}{\partial t} &= 0 \text{ in } \Omega. \end{cases} \tag{2.22}$$

Let $\Phi = \Phi_0 \, e^{i\,\omega\,t}$, then $\dfrac{\partial\Phi}{\partial t} = i\omega\,\Phi$ and $\dfrac{\partial^2\Phi}{\partial t^2} = -\omega^2\,\Phi$. In this way, equation (2.21) becomes

$$-\frac{\omega^2}{c_0^2}\,\Phi - \Delta\,\Phi = 0. \tag{2.23}$$

Taking also, $k_0 = \dfrac{\omega}{c_0}$, we have the following system

$$\begin{cases} -\Delta\,\Phi &= k_0^2\,\Phi \quad \text{in } \Omega \\ \Phi &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

3. Problem formulation and topological optimization

The goal of the topological optimization problem is to find an optimal design with an a priori poor information on the optimal shape of the initial domain. The shape optimization problem consists in minimizing a functional $j(\Omega) = J(\Omega, u_\Omega)$ where the function u_Ω is defined on an open and bounded subset Ω of $\mathbb{R}^N$. For $\epsilon > 0$, let $\Omega_\epsilon = \Omega \backslash \overline{(x_0 + \epsilon\omega)}$ be the set obtained by removing a small part $x_0 + \epsilon\omega$ from Ω, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^N$ is a fixed open and bounded subset containing the origin. Then, using general adjoint method, an asymptotic expansion of the function can be obtained in the following form

$$j(\Omega_\epsilon) = j(\Omega) + f(\epsilon)g(x_0) + o(f(\epsilon)), \; \lim_{\epsilon\to 0} f(\epsilon) = 0, \; f(\epsilon) > 0 \tag{3.1}$$

The topological sensitivity $g(x_0)$provides information when the creating a small hole located at x_0. Hence, the function g will be used as descent direction in the optimization process.

3.1. A generalized adjoint method. For the definition of the adjoint method, we refer to Murat and Simon (1976) and Masmoudi (2002). Let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ (resp $\mathcal{L}_2(\mathcal{V})$) denotes the spaces of liner (resp. bilinear) forms on $\mathcal{V}$. Let us suppose the following hypotheses

(1) **H-1** : There exists a real function f, a bilinear form $\delta_a \in \mathcal{L}_2(\mathcal{V})$ and a linear form δ_l such that

$$f(\epsilon) \longrightarrow 0, \quad \epsilon \to 0 \tag{3.2}$$

$$\|a_\epsilon - a_0 - f(\epsilon)\,\delta_a\|_{L_2(\mathcal{V})} = o(f(\epsilon)) \tag{3.3}$$

$$\|l_\epsilon - l_0 - f(\epsilon)\,\delta_l\|_{L(\mathcal{V})} = o(f(\epsilon)). \tag{3.4}$$

where $a_\epsilon,\ a_0$ (resp $l_\epsilon,\ l_0$) are the bilinear (resp linear) form.

(2) **H-2** : The bilinear form a_0 is coercive. There exists a constant $\alpha > 0$ such that

$$a_0(u,u) \geq \alpha||u||^2, \forall u \in \mathcal{V}, \tag{3.5}$$

According to (3.3) the bilinear form a_ϵ depends continuously on ϵ, hence there exists ε_0 and $\beta > 0$ such that for $\epsilon \in [0, \epsilon_0]$ the following uniform coercivity holds $a_\epsilon(u,u) \geq \beta\|u\|^2, \forall u \in \mathcal{V}$. Moreover, according to Lax-Milgram's theorem, for $\epsilon \in [0, \epsilon_0]$, the problem find $u_\epsilon \in \mathcal{V}$, such that

$$a_\epsilon(u_\epsilon, v) = l_\epsilon(v) \forall v \in \mathcal{V} \tag{3.6}$$

has a unique solution.

We have the following lemma,

Lemma 55. *If hypotheses **H-1** and **H-2** holds, there exists a unique solution u^ϵ to (3.12). Further more, the following estimate holds:*

$$\|u_{\Omega_\epsilon} - u\|_{H_\epsilon} = O(f(\epsilon). \tag{3.7}$$

Proof. Because of the coercitivity of a_ϵ, we get

$$\alpha\,\|u_\epsilon - u_0\|^2 \leq a_\epsilon(u_\varepsilon - u_0, u_\epsilon - u_0)$$

Or

$$\begin{aligned}
a_\epsilon(u_\epsilon - u_0, u_\epsilon - u_0) &= a_\epsilon(u_\epsilon, u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0).\\
&= l_\epsilon(u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0)\\
&= l_0(u_\epsilon - u_0) + (l_\epsilon - l_0)(u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0)\\
&= a_0(u_0, u_\epsilon - u_0) - a_\epsilon(u_0, u_\epsilon - u_0) + (l_\epsilon - l_0)(u_\epsilon - u_0)\\
\alpha\,||u_\epsilon - u_0||^2 &\leq [-\delta\, a(u_0, u_\epsilon - u_0) + \delta\, l(u_\epsilon - u_0)]\ f(\epsilon) + ||u_0||||u_\varepsilon - u_0||\, o(f(\epsilon))\\
+||u_\epsilon - u_0||\, o(f(\epsilon))
\end{aligned}$$

Thus,

$$\alpha\,||u_\epsilon - u_0||^2 \leq (-\delta\, a + \delta\, l)\, f(\epsilon) + (1 + ||u_\epsilon - u_0||)\, o(f(\epsilon)). \square$$

We consider also the following hypothesis:

(3) **H-3** Consider a functional j such that $j(\epsilon) = J(u_\epsilon)$ where J is differentiable. For any $u \in \mathcal{V}$, there exists a linear and continuous form $DJ(u) \in \mathcal{L}(\mathcal{V})$ and δ_J such that:

(3.8) $$J(u) - J(v) = DJ(u)(v - u) + f(\epsilon)\delta_J(u) + o(\|u - v\|_{\mathcal{V}}).$$

The lagragian $\mathcal{L}$ is given by

(3.9) $$\mathcal{L}(u, v) = a(u, v) - l(v) + J(u),\ \forall u,\ v \in \mathcal{V}.$$

It's variation is given by

(3.10) $$\mathcal{L}_\epsilon(u, v) = a_\epsilon(u, v) - l_\epsilon(v) + J(u),\ \forall u,\ v \in \mathcal{V}.$$

We have also the following generic theorem

THEOREM 74. *Under hypotheses **H-1**, **H-2** and **H-3**, we have the following expansion of j.*

(3.11) $$j(\epsilon) - j(0) = f(\epsilon)\delta\mathcal{L}(u_0, v_0) + o(\rho(\epsilon)),$$

where u_0 is the solution of (3.12) with $\varepsilon = 0$, v_0 solution to the adjoint problem, find v_0 such that:

(3.12) $$a_0(\omega, v_0) = -DJ(u_0)\omega,\ \forall \omega \in \mathcal{V}, l_\varepsilon(v)\forall v \in \mathcal{V}$$

and

$$\delta\mathcal{L}(u,v) = \delta_a(u,v) - \delta_l(v) + \delta_J(u). \tag{3.13}$$

PROOF. For the proof of this theorem, see Masmoudi (2002) □

4. Application to photonics crystals problems and main result

In this section, we are going to use the previous theory in oder to study the photonic crystals problem. We will in the sequel, consider, an $\mathbb{R}$ valued application J differentiable defined in $H^1(\Omega)$ by a function J in the form

$$J(\Omega) = J(\Omega, u_\Omega) = \alpha \int_\Omega |u_\Omega - u_0|^2 dx + \beta \int_\Omega |\nabla u_\Omega|^2 dx \tag{4.1}$$

where u_Ω is solution of

$$\begin{cases} \Delta u_\Omega + k^2 u_\Omega = 0 \text{ in } \Omega \\ B_\Omega u_\Omega = 0 \text{ in } \partial\Omega \end{cases} \tag{4.2}$$

with B_Ω a boundary operator defined in $\partial\Omega$, and α and β constants.

Let

$$V = \{u \in H^1(\Omega) \text{ such that } B_\Omega u = 0 \text{ on } \partial\Omega\}$$

The variational form of (4.2) is

$$\text{Find } u_\Omega \in V \text{ such that } \int_\Omega \nabla u_\Omega \nabla v dx = k^2 \int_\Omega u_\Omega v dx \ \forall v \in V. \tag{4.3}$$

Let us denote by a_0 the bilinear form defined by

$$a_0(u,v) = \int_\Omega \nabla u_\Omega \nabla v dx - k^2 \int_\Omega u_\Omega v dx \ \forall v \in H^1(\Omega). \tag{4.4}$$

In the perforated domain Ω_ϵ, for all $\epsilon > 0$ we consider the functional J_ϵ defined by

(4.5) $$j(\epsilon) = J(\Omega_\epsilon, u_{\Omega_\epsilon}) = \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx + \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx$$

where u_{Ω_ϵ} is solution to

(4.6) $$\begin{cases} \Delta u_{\Omega_\epsilon} + k^2 u_{\Omega_\epsilon} = 0 \text{ in } \Omega_\epsilon \\ B_\Omega u_{\Omega_\epsilon} = 0 \text{ in } \partial\Omega \\ B_\omega = \begin{cases} u_{\Omega_\epsilon} = 0 \text{ on } \partial\omega_\epsilon, \\ \text{or} \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 \text{ on } \partial\omega_\epsilon. \end{cases} \end{cases}$$

We seek the asymptotic behavior of the difference $j(\epsilon) - j(0)$ when ϵ tends to zero. In the boundary of ω_ϵ we will consider two cases: a Neumann condition and a Dirichlet one.

Let us defined the following spaces:

$$H_\epsilon = \{u \in H^1(\Omega_\epsilon),\ B_\Omega = 0\}$$

and

$$\tilde{H}_\epsilon = \{u \in H_\epsilon,\ B_\omega u = 0 \text{ on } \partial\omega_\epsilon\}.$$

The variational formula of the problem (4.6) writes: Find $u_{\Omega_\epsilon} \in \tilde{H}_\epsilon$ such that

(4.7) $$a_\epsilon(u, v) = l_\epsilon(v)\ \forall v \in \tilde{H}_\epsilon,\ \forall v \in \tilde{H}_\epsilon$$

with

(4.8) $$a_\epsilon(u, v) = \int_{\Omega_\epsilon} (\nabla u \nabla v - k^2 uv) dx\ \forall u,\ v \in \tilde{H}_\epsilon$$

(4.9) $$l_\epsilon(v) = 0\ \forall v \in \tilde{H}_\epsilon.$$

We have to notice that the boundary condition $\partial\Omega$ has no influence on the calculation of the topological sensitivity. What is most important for us is the condition at the boundary of the hole.

4.1. Topological derivative.

THEOREM 75 (Main result for Neumann Boundary condition). *Let $J(\Omega_\epsilon, u_{\Omega_\epsilon})$ be the objective function defined by (4.5) and u_{Ω_ε} be the solution of (4.6) with Neumann boundary conditions on $\partial\omega_\epsilon$. The function j has the following asymptotic expansion*

$$j(\varepsilon)-j(0)=\varepsilon^2\Big(-\nabla u(x_0)\mathcal{A}^\omega\nabla v_0(x_0)+k^2 meas(\Omega)u(x_0)v_0(x_0)+\delta_J\Big)+o(\varepsilon^2) \tag{4.10}$$

where $\mathcal{A}^\omega$ denotes the polarization matrix and v_0 denotes the adjoint state, solution to

$$\begin{cases} v_0 \in V, \\ a(u,v_0) = -DJ(v_0)v, \;\; \forall u \in H(\Omega) \end{cases} \tag{4.11}$$

In the case where ω is the units ball, then the polarization matrix takes the form $\mathcal{A}^\omega = 2\pi I$, where I is the identity matrix.

PROOF. Due to Theorem 74, the proof of this theorem amounts simply to calculating the forms δ_a, δl and δ_J and to determinate the value of $f(\varepsilon)$. We begin by calculating, the variational of the bilinear form.

Taking the difference between a_ϵ given by (4.11) and a_0 given by (4.4) we have

$$a_\epsilon(u_{\Omega_\epsilon},v)-a(u_\Omega,v)=\int_{\Omega_\epsilon}\left(\nabla\left(u_{\Omega_\epsilon}-u_\Omega\right)\right)\nabla\, v\, dx-$$
$$\lambda\int_{\Omega_\epsilon}\left(u_{\Omega_\epsilon}-u_\Omega\right)v\, dx-\int_{\omega_\epsilon}\nabla\, u_\Omega\nabla v dx+\lambda\int_{\omega_\epsilon}u_\Omega\, v dx$$

From lemma 55, we have $u_{\Omega_\epsilon}=u_\Omega\ \forall x\in\Omega_\epsilon$; then we get

$$a_\epsilon(u_{\Omega_\epsilon},v)-a(u_\Omega,v)=-\int_{\omega_\epsilon}\nabla\, u_\Omega\nabla v dx+\lambda\int_{\omega_\epsilon}u_\Omega\, v dx.$$

Therefore, we have

$$a_\epsilon(u_{\Omega_\epsilon}, v) - a(u_\Omega, v) = -\int_{\omega_\epsilon} \nabla\, u_\Omega \nabla v dx + \lambda \int_{\omega_\epsilon} u_\Omega\, v dx$$

Let v_0 be a solution to the following adjoint problem

(4.12) $$\Delta\, v_0 + \lambda\, v_0 = DJ(u_\Omega)\, v_0,$$

Taking $v = v_0$ we have:

$$a_\epsilon(u_{\Omega_\epsilon}, v) - a(u_\Omega, v) = -\int_{\omega_\epsilon} \nabla\, u_\Omega \nabla v_0\, dx + \lambda \int_{\omega_\epsilon} u_\Omega\, v_0 .dx$$

Using Green's formula and because of hypotheses (3.3) yields

$$\begin{aligned} f(\epsilon)\,\delta\, a(u_\Omega, v_0) &= -\int_{\omega_\epsilon} \nabla\, u_\Omega \nabla v_0\, dx + \lambda \int_{\omega_\epsilon} u_\Omega\, v_0 dx, \\ &= -\int_{\partial\omega_\epsilon} \partial_n v_0\, u_\Omega\, dS + \int_{\omega_\epsilon} \lambda\, v_0\, dx + \int_{\omega_\epsilon} \Delta\, v_0\, u_\Omega\, dx, \\ &= -\int_{\partial\omega_\epsilon} \partial_n v_0\, u_\Omega\, dS + \int_{\omega_\epsilon} (\lambda\, v_0 + \Delta\, v_0)\, u_\Omega\, dx, \\ &= -\int_{\partial\omega_\epsilon} \partial_n v_0\, u_\Omega\, dS + \int_{\omega_\epsilon} DJ(u_\Omega) v_0. \end{aligned}$$

Let $w_\epsilon = u_\epsilon - u_0$, where $u_0 = u_\Omega$, then we have

$$a_\epsilon(u_{\Omega_\epsilon}, w_\epsilon) = a_\epsilon(u_{\Omega_\epsilon}, v_\epsilon) - a_\epsilon(u_{\Omega_\epsilon}, v_0)$$

w_ϵ is the solution of:

(4.13) $$\begin{cases} \Delta w_\epsilon + k^2 w_\epsilon &= 0 \text{ in } \Omega_\epsilon, \\ w_\varepsilon &= 0, \text{ p, } \partial\Omega, \\ \dfrac{\partial w_\varepsilon}{\partial n} &= -\dfrac{\partial v_0}{\partial n}, \text{ on } \partial\omega_\epsilon \end{cases}$$

Let us defined, $\forall \epsilon > 0$ and $\varphi \in H^{\frac{1}{2}}(\partial\omega_\epsilon)$ the function l_ϵ^φ as the solution in $H^1(\omega)$ to

(4.14) $$\begin{cases} \Delta l_\epsilon^\varphi &= 0 \text{ in } \omega_\epsilon, \\ l_\epsilon^\varphi &= \varphi \quad \text{, on } \partial\omega_\epsilon \end{cases}$$

In the following, we will approximate w_ϵ by $P \in W^1(\mathbb{R}^2 \setminus \bar{\omega})$ defined by$P_\epsilon = \epsilon\, P(\frac{x}{\epsilon})$ where P_ϵ is solution to to the exterior problem

$$\left\{ \begin{array}{lcl} \Delta\, P & = & 0 \text{ dans } \mathbb{R}^2 \setminus \bar{\omega}, \\ P & = & O(\frac{1}{r}) \text{ à l' } \infty \\ \dfrac{\partial\, P}{\partial n} & = & -\nabla\, v_0(0).n, \text{ sur } \partial\omega. \end{array} \right. \tag{4.15}$$

For the existence of (4.15) we refer to Guillaume and Sid Idris (2002). Therefore, P can be written by potential theory as

$$P(x) = \int_{\partial\omega} \eta(y)\, E(x-y) dS(y), \forall x \in \mathbb{R}^2 \setminus \bar{\omega}, \tag{4.16}$$

where η is the unique solution of the integral equation

$$\frac{\eta(x)}{2} + \int_{\partial\omega} \eta(y)\, \partial_{n_x} E(x-y) dS(y) = -\nabla\, v(0)\dot{}n, \forall x \in \partial\omega \tag{4.17}$$

and E the fundamental solution of the laplacian

(4.18)

$$(a_\epsilon - a_0)(u_\Omega, v_\epsilon) = -\int_{\partial\omega_\epsilon} \partial_n v_0\, (u_\Omega(x) - u_\Omega(x_0))\, dS + \int_{\omega_\epsilon} DJ(u_\Omega(x) - u_\Omega(x_0)) v_0.$$

Thus,

$$(a_\epsilon - a_0)(u_\Omega(x) - u_\Omega(x_0), v_\epsilon) = -\int_{\partial\omega_\epsilon} \partial_n\, v_0\, (u_\Omega(x) - u_\Omega(x_0))\, ds +$$

$$\int_{\omega_\epsilon} DJ(u_\Omega(x) - u_\Omega(x_0)) v_0. + k^2\, u_\Omega(x_0) \int_{\omega_\epsilon} v_\Omega dx - \int_{\partial\omega_\epsilon} \partial_n\, l_\varepsilon^{w_\epsilon}\, u_\Omega(x) - u_\Omega(x_0) ds$$

$$+k^2 \int_{\omega_\epsilon} u_\Omega\, l_\epsilon^{w_\epsilon}\, dx, \tag{4.19}$$

Then, we have

$$(a_\epsilon - a_0)(u_\Omega(x) - u_\Omega(x_0), v_\epsilon) = -\int_{\partial\omega_\epsilon} \partial_n v_0\, (u_\Omega(x) - u_\Omega(x_0))\, ds +$$

(4.20) $$\epsilon^2 k^2 meas(\omega) u_\Omega(x_0) v_\Omega(x_0) - \int_{\partial\omega_\epsilon} \partial_n l_\epsilon^{w_\epsilon} u_\Omega(x) - u_\Omega(x_0) ds + \sum_{i=1}^{n} \chi_i(\epsilon)$$

with

$$\chi_1 = -\int_{\partial\omega_\epsilon} (\partial_n l_\epsilon^{w_\epsilon} - (\partial_n l_\epsilon^{P_\epsilon} (u_\Omega(x) - u_\Omega)(x_0) ds$$

$$\chi_3 = k^2 u_\Omega(x_0) \int_{\omega_\epsilon} v_\Omega(x_0) dx$$

$$\chi_4 = k^2 \int_{\omega_\epsilon} u_\Omega l_\epsilon^{P_\epsilon} dx$$

$$\forall x \in \partial\omega_\epsilon, l_\epsilon^{P_\epsilon}(x) = \epsilon\, l^P(\frac{x}{\epsilon})$$

$$\begin{aligned}(a_\epsilon - a_0)(u_\Omega, v_\epsilon) &= -\int_{\partial\omega_\epsilon} \partial_n v_\Omega \left[u_\Omega - u_\Omega(x_0)\right] ds + k^2 \epsilon^2 meas(\omega) u_\Omega(x_0) v_\Omega(x_0) \\ &\quad - \int_{\partial\omega_\epsilon} \partial_n l_\epsilon^{p_\epsilon} (u_\Omega - u_\Omega(x_0))\, ds + \sum_{i=1}^{4} \chi_i(\epsilon)\end{aligned}$$

with $l_\epsilon^{p_\epsilon}(x) = \epsilon l^p(\frac{x}{\epsilon})$ and $\partial_n l_\epsilon^{p_\epsilon}(x) = \partial_n l^p(\frac{x}{\epsilon})$.

According to the jumps relation, we have $\eta(y) = -\nabla v_\Omega(x_0).n - \partial_n l^p(y)$, $y \in \partial\omega$.

$$\begin{aligned}(a_\epsilon - a_0)(u_\Omega, v_\epsilon) &= \int_{\partial\omega_\epsilon} \eta(\frac{x}{\epsilon})(u_\Omega(x) - u_\Omega(x_0)) ds + k^2\epsilon^2\, meas(\omega) u_\Omega v_\Omega \\ &\quad - \int_{\partial\omega_\epsilon} \left[\partial_n v_\Omega - \nabla v_\Omega(x_0) n\right] (u_\Omega(x) - u_\Omega(x_0)) ds + \sum_{i=1}^{4} \chi_i\end{aligned}$$

Let

$$\chi_4 = \epsilon \int_{\partial\omega} \left[\partial_n v_\Omega - \nabla v_\Omega n\right] \left[u_\Omega(\epsilon x) - u_\Omega(x_0)\right] ds$$

$$\chi_5 = \epsilon \int_{\partial\omega} \eta(y)\, (u_\Omega(\epsilon y) - u(x_0) - \nabla u_\Omega(x_0)\, \epsilon y)\, dS(y)$$

$$(a_\epsilon - a_0)(u_\Omega, v_\epsilon) = -\epsilon \int_{\partial\omega} \eta(y) y \nabla u_\Omega(x_0) \epsilon ds(y) + k^2\epsilon^2\, meas(\omega) u_\Omega v_\Omega \sum_{i=1}^{5} \chi_i$$

$$(a_\epsilon - a_0)(u_\Omega, v_\epsilon) = -\epsilon^2\, \nabla u_\Omega(x_0) \int_{\partial\omega} \eta(y) y ds(y) + k^2\epsilon^2\, meas(\omega) u_\Omega v_\Omega + \sum_{i=1}^{5} \chi_i$$

$$\Rightarrow (a_\epsilon - a_0)(u_\Omega, v_\epsilon) = \epsilon^2 \nabla u_\Omega(x_0) \int_{\partial\omega} \eta(x) x ds(x) + k^2 \epsilon^2 meas(\omega) u_\Omega v_\Omega + \sum_{i=1}^{5} \chi_i$$

It is know that $\int_{\partial\omega} \eta(y) y ds(y) = -\mathcal{A}\nabla v_\Omega(x_0)$ and $\int_{\partial\omega} \eta(x) x ds(x) = -\mathcal{A}\nabla u_\Omega(x_0)$, where $\mathcal{A}$ is the polarization matrix and η is the unique solution of the integral equation

$$\frac{\eta(x)}{2} + \int_{\partial\omega} \eta(y) \partial_{nx} E(x-y) ds(y) = V.n \forall x \in \partial\omega.$$

Taking $f(\epsilon) = \epsilon^2$, we have

$$\begin{aligned} (a_\epsilon - a_0)(u_\Omega, v_\epsilon) &= \epsilon^2 \nabla u_\Omega(x_0) \mathcal{A}_\omega \nabla u_\Omega(x_0) + k^2\, \epsilon^2 meas(\omega) u_\Omega v_\Omega) + \sum_{i=1}^{6} \chi_i \\ &= f(\epsilon)(\nabla u_\Omega \mathcal{A}_\omega \nabla v_\Omega(x_0) + k^2\, meas(\omega)) u_\Omega v_\Omega + \sum_{i=1}^{5} \chi_i \\ &= f(\epsilon)(\nabla u_\Omega \mathcal{A}_\omega \nabla v_\Omega(x_0) + k^2\, meas(\omega) u_\Omega(x_0) v_\Omega(x_0)) + o(f(\epsilon)) \end{aligned}$$

giving $\delta_a = \nabla u_\Omega(x_0) \mathcal{A}_\omega(y) \nabla v_\Omega(x_0) + k^2 meas(\omega) u_\Omega(x_0) v_\Omega(x_0)$.

When $\omega = B(x_0, 1)$, then the solution of the integral equation is equal to $\eta = 2v\, x\ \forall x \partial\omega$ and $\forall\, v \in \mathbb{R}^2\ A_\omega = 2\pi I$, where I is the identity matrix. Thus, the following result holds

$$J(u_{\Omega_\varepsilon}) - J(u_\Omega) = \alpha \int_{\Omega_\varepsilon} (|u_{\Omega_\varepsilon} - u_0|^2 - |u_\Omega - u_0|^2)\, dx + \beta \int_{\Omega_\varepsilon} (|\nabla\, u_{\Omega_\varepsilon}|^2 - |\nabla\, u_\Omega|^2)\, dx$$

$$-\alpha \int_{\omega_\varepsilon} |u_\Omega - u_0|^2\, dx - \beta \int_{\omega_\varepsilon} |\nabla\, u_\Omega|^2\, dx.$$

The First and the second term after the following equality can be written as follows:

$$\alpha \int_{\Omega_\varepsilon} (|u_{\Omega_\varepsilon} - u_0|^2 - |u_\Omega - u_0|^2)\, dx = \alpha \int_{\Omega_\varepsilon} (u_{\Omega_\varepsilon} + u_\Omega)(u_{\Omega_\varepsilon} - u_\Omega) dx$$

$$-2\,\alpha \int_{\Omega_\varepsilon} u_0\, (u_{\Omega_\varepsilon} - u_\Omega)\, dx,$$

and

$$\beta \int_{\Omega_\varepsilon} (|\nabla u_{\Omega_\varepsilon}|^2 - |\nabla u_\Omega|^2)\, dx = \beta \int_{\Omega_\varepsilon} |\nabla (u_{\Omega_\varepsilon} - u_\Omega)|^2\, dx$$

$$+2\,\beta \int_{\Omega_\varepsilon} \nabla\, u_\Omega\, \nabla (u_{\Omega_\varepsilon} - u_\Omega)\, dx.$$

Therefore, because of lemma 55, and using theorem 74, we get

$$J(u_{\Omega_\varepsilon}) - J(u_\Omega) = DJ(u_\Omega)(u_\Omega - u_0) + f(\varepsilon)\,\delta\, J(u_\Omega) + 0(f(\varepsilon))$$

and

$$f(\varepsilon)\delta J(u_\Omega) + o(f(\varepsilon)) = -\alpha \int_{\omega_\varepsilon} |u_\Omega - u_0|^2 dx - \beta \int_{\omega_\varepsilon} |\nabla u_\Omega|^2 dx$$

By a change of variables $x = \varepsilon X$, and by applying mean value theorem, we get

$$-\alpha \int_{\omega_\varepsilon} |u_\Omega - u_0|^2 dx - \beta \int_{\omega_\varepsilon} |\nabla u_\Omega|^2 dx = -\alpha\varepsilon^2 \int_\omega |u_\Omega(x_0 + \varepsilon\, x) - u_0(x_0 + \varepsilon\, x)|^2 dx$$

$$-\beta\varepsilon^2 \int_\omega |\nabla u_\Omega(x_0 + \varepsilon x)|^2 dx.$$

Then, we have finally

$$J(u_{\Omega\varepsilon}) - J(u_\Omega) = \varepsilon^2 \left[-\alpha \int_\omega |u_\Omega(x_0) - u_0|^2 dx + \beta \int_\omega |\nabla u_\Omega(x_0)|^2 dx \right] + \xi_1(\varepsilon) + \xi_2(\varepsilon).$$

with $\xi = 0(f(\varepsilon))$ and $\xi_2 = 0(f(\varepsilon))$, where $f(\varepsilon) = \varepsilon^2$. Thus,

$$\Rightarrow \delta\, J(u_\Omega) = -\alpha \int_\omega |u_\Omega - u_0|^2 dx - \beta \int_\omega |\nabla u_\Omega|^2 dx$$

giving

$$\delta\, J(u_\Omega) = -\alpha\, meas(\omega)\, |\, u_\Omega(x_0) - u_0\,|^2 - \beta\, |\nabla u_\Omega(x_0)|^2 meas(\omega).$$

□

In the case where the boundary condition on $\partial\omega_\epsilon$ is given by dirichlet condition, we have the following theorem, when the functioonal j is given by (4.1) and $\beta = 1,\ \alpha = 1$

THEOREM 76 (Main result for Neumann Dirichlet condition). *Let* $J(\Omega_\epsilon, u_{\Omega_\epsilon})$ *be the objective function defined by (4.5) and* u_{Ω_ε} *be the solution of (4.6) with Dirichlet boundary conditions on* $\partial\omega_\epsilon$. *The function* j *has the following asymptotic expansion*

$$j(\epsilon) - j(0) = -2\pi(\Big(\nabla u_\Omega(x_0)\cdot\nabla v_0(x_0) - k^2 u_\Omega(x_0)v_0(x_0)\Big) + o(f(\epsilon)) \tag{4.21}$$

where v_0 *the solution to the adjoint problem*

$$\begin{cases} v_0 \in V, \\ a(u, v_0) = -DJ(v_0)v, \ \forall u \in H_0^1(\Omega) \end{cases} \tag{4.22}$$

PROOF. It suffices to verify that, in this case, δ_a is given by $\delta_a = -2\pi(\Big(\nabla u_\Omega(x_0)\cdot\nabla v_0(x_0) - k^2 u_\Omega(x_0)v_0(x_0)\Big)$ and $\delta_l = \delta_J = 0$. □

5. Numerical simulations

In this section, we present some numerical simulations related to the problem studied in the above section by using finite element method and Matlab. The direct state u_Ω is solution of (4.2) and the adjoint state is given by (4.11). The topological derivative is given by (4.1). In all numerical results, we plot of the solution of the direct and the adjoint state and the topological derivative. At each step, we create a small hole, where the topological derivative is too small. Before going further, we will present the algorithm that we use in order to get the optimal design. The algorithm is given as follows:

(1) Initialisation: Choose Ω the initial domain.
(2) Repeat

- Compute $u_{\Omega_l}, p_{\Omega_l}$ the direct and adjoint solutions in the perturbed domain Ω_l.

- Compute the topological gradient g_l in Ω_l
- Set $\Omega_{l+1} = \Omega_l \backslash \{x_k, g_l(x_k) \geq t_{l+1}$
- $l \to l+1$.

In all the simulations presented, the set is given by $\Omega = [-1, 1] \times [-1, 1]$ and $U_0 = \frac{1}{2} \cos x$ and we consider the solution of the initial (4.2) and adjoint problem (4.11).

5.1. Application to the photonic problem. In this subsection we present some simulations of photonics problem. We will consider several cases. In each case we will present the direct and the adjoint state and at the end we give the topological derivative. The boundary condition of (4.2) is given by $B_\Omega = g$, where g is given. We will consider the eigen-value equal to $\lambda = 13^2$.

5.1.1. *Neumann condition on $\partial\Omega$ and Neumann Condition on $\partial\omega$.*. One assumes, in this case that, the boundary condition is given by

$$B_\Omega = \frac{\partial u}{\partial n} = g = cos(x) \times cos(y) \text{ on } \partial\Omega.$$

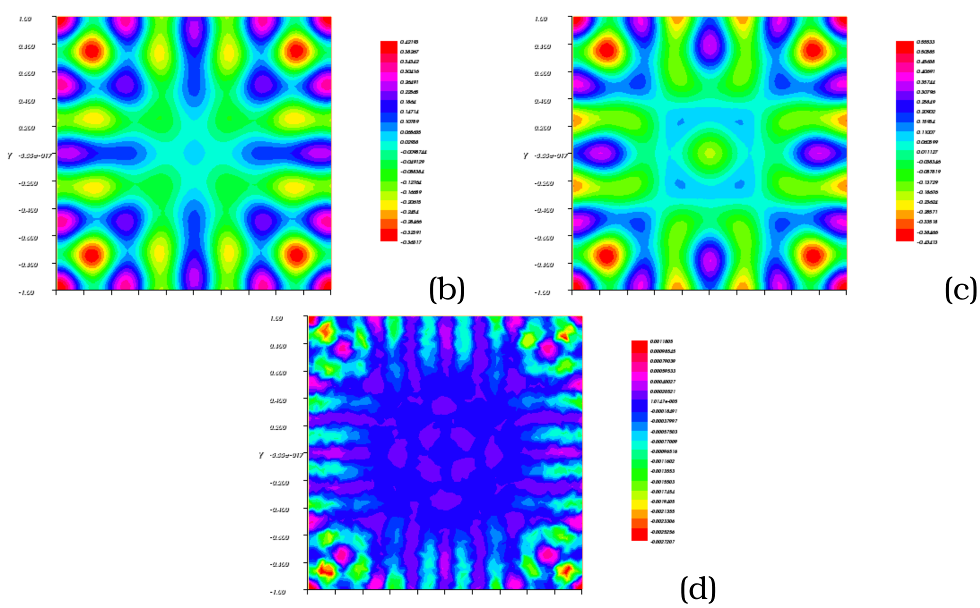

FIGURE 1. (B) THE DIRECT SOLUTION, (C) ADJOINT STATE, (D) TOPOLOGICAL DERIVATIVE. (b) the direct solution, (c) Adjoint State, (d) Topological derivative

After creating a small hole where the topological derivative is most negative and considering the direct problem, the adjoint problem and the topological derivative in the perfored domain, we have;

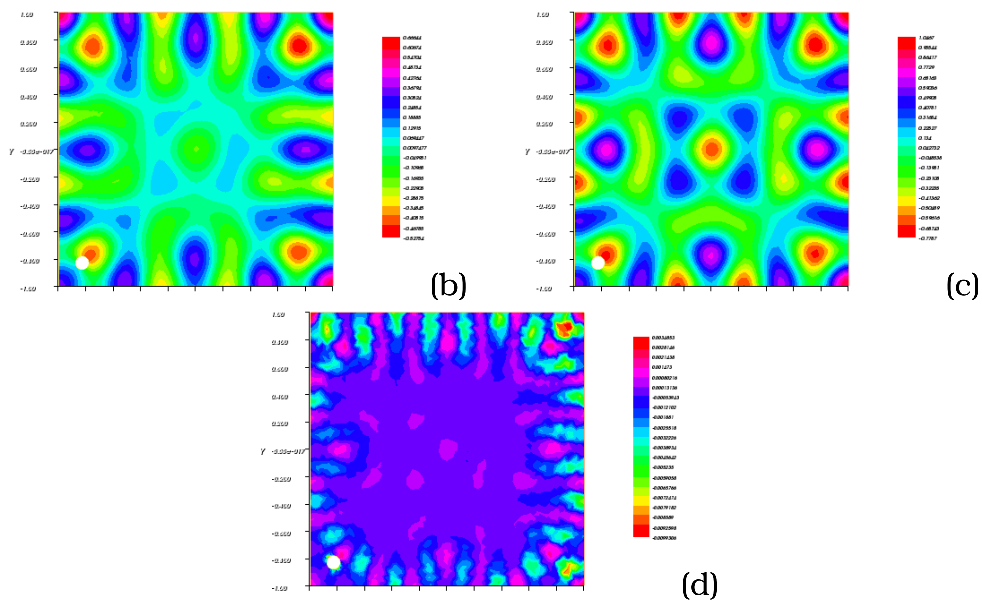

FIGURE 2. b)DIRECT STATE, c)ADJOINT STATE d)TOPOLOGICAL DERIVATIVE . b)Direct State, c)Adjoint state d)Topological derivative

5.1.2. *Dirichlet condition on* $\partial\,\Omega$ *and Neumann condition on* $\partial\,\omega$ *for* $k = 13$.

In this case, we consider equation (4.2) with initial condition given by $B_\Omega u = u = 0\ \partial\Omega$.

There is no possibilities to put a hole in the domain Ω given in Figure 7. The topological derivative is equal to zeros almost every where. Then we get the optimal design after putting one hole.

5.2. Application to Phononic problem.

In this section we simulate the problem of phononic crystal assuming the same hypothesis and for various value of k.

5.2.1. *Dirichlet Neumann condition for* $k = 10$. We consider the initial problem with the boundary condition $B_\Omega = g = 0$ on $\partial\Omega$ and in the perfored domain the following boundary condition $B_\omega = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. We have the following numerical results.

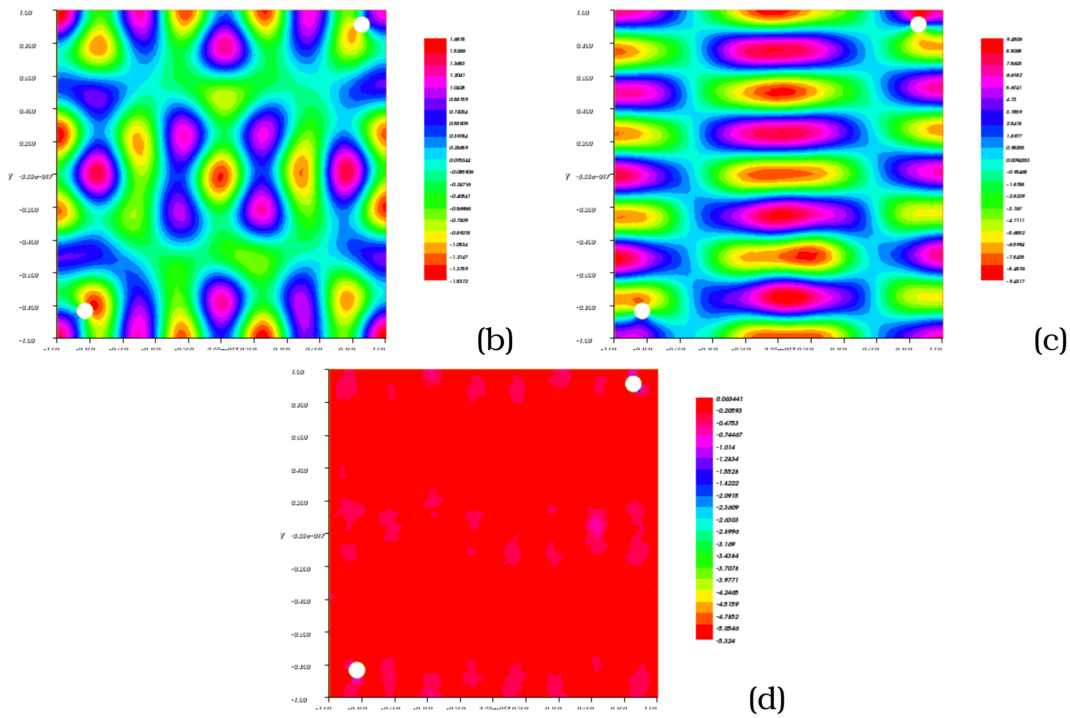

FIGURE 3. DOMAIN B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Domain b)Direct State, c)Adjoint state d)Topological derivative

5.2.2. ***Neumann Neumann condition for*** $k = 15$.

If the value of k vary, for the same hypotheses as in the following, we have the Figures 11-14

Neumann Dirichlet for $k = 13$. The results areillustrated in Fugures 15-22

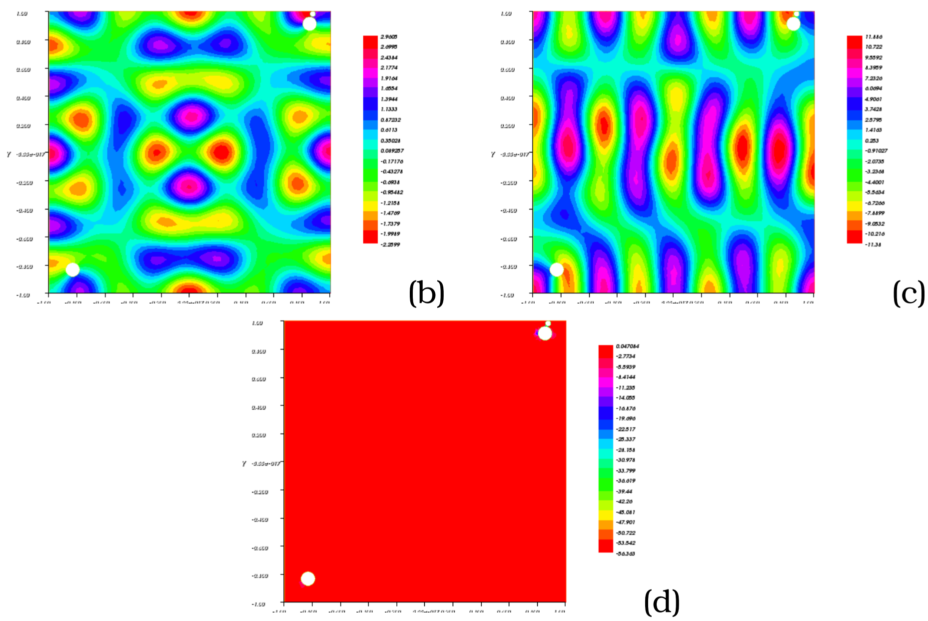

FIGURE 4. DOMAIN B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Domain b)Direct State, c)Adjoint state d)Topological derivative

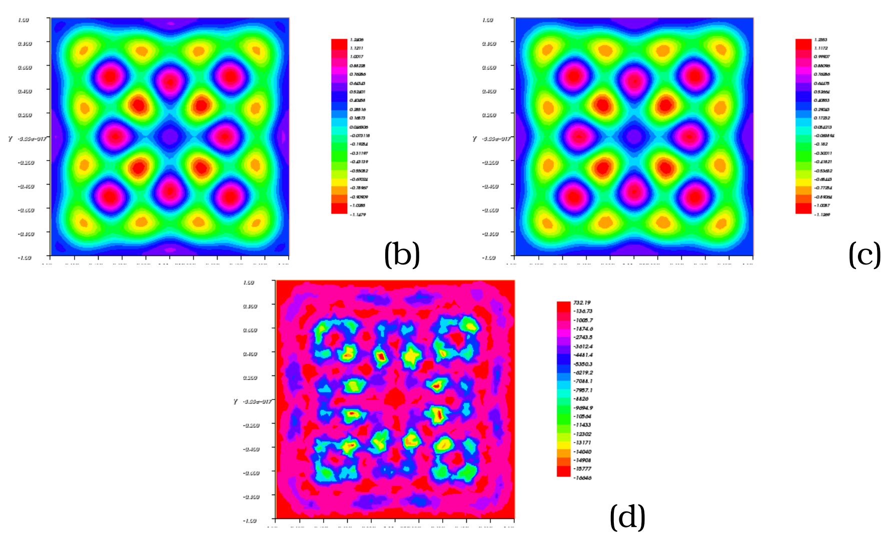

FIGURE 5. B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . b)Direct State, c)Adjoint state d)Topological derivative

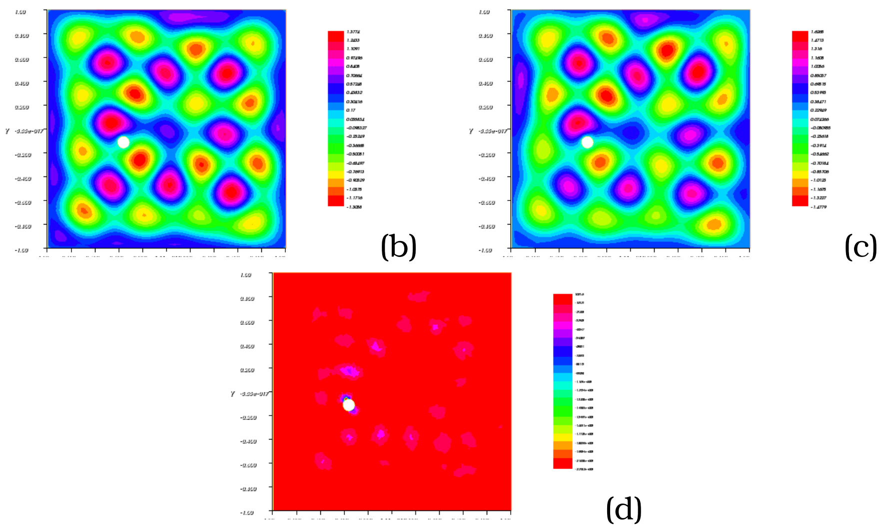

FIGURE 6. SECOND STEP : B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Second step : b)Direct State, c)Adjoint state d)Topological derivative

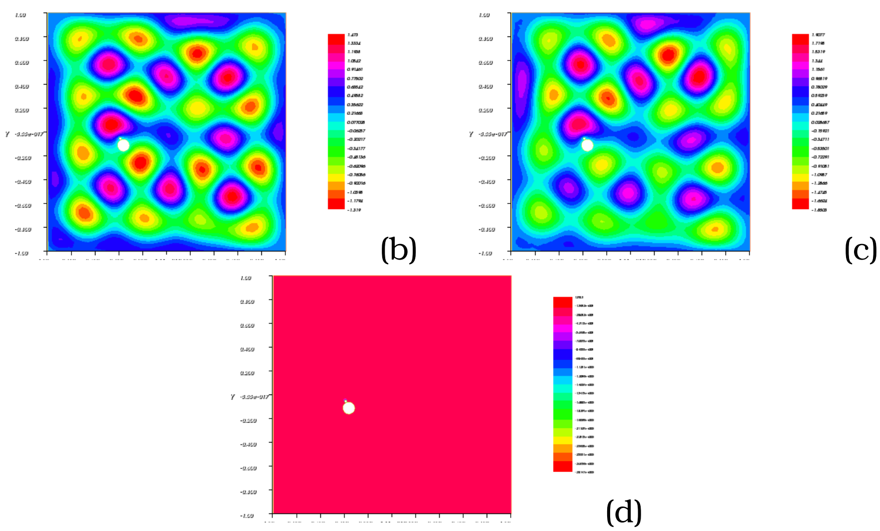

FIGURE 7. SECOND STEP : B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Second step : b)Direct State, c)Adjoint state d)Topological derivative

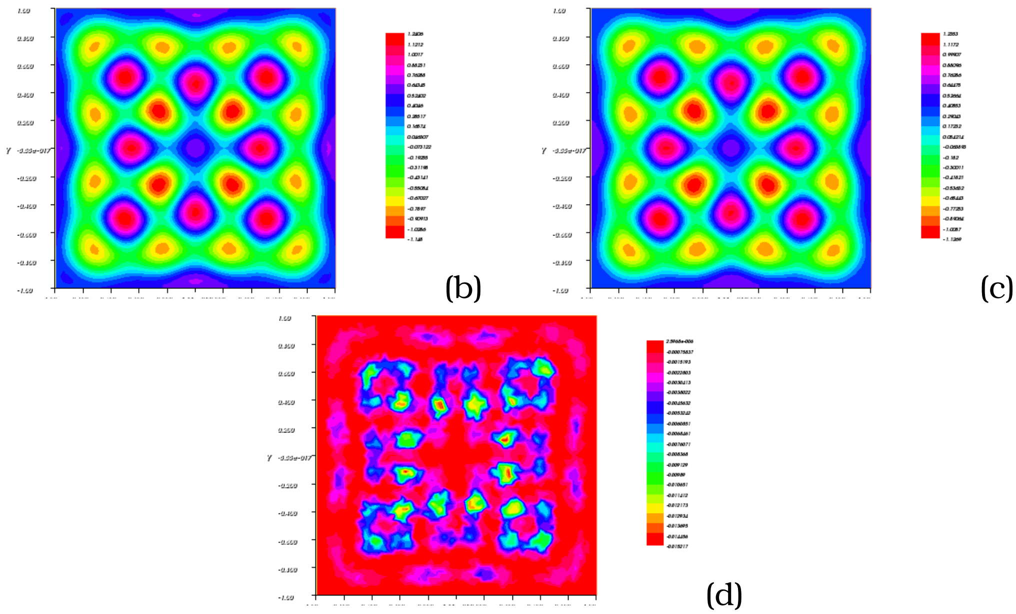

FIGURE 8. SECOND STEP : B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Second step : b)Direct State, c)Adjoint state d)Topological derivative

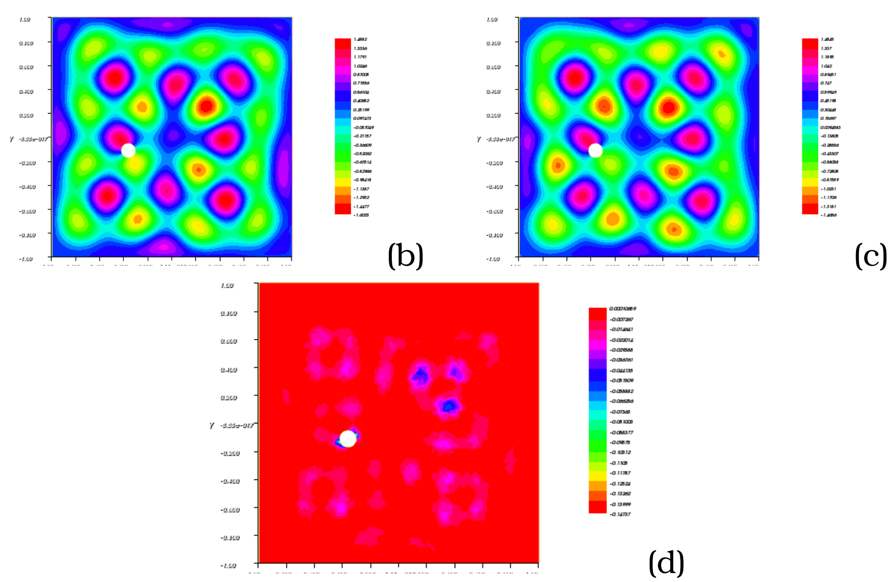

FIGURE 9. SECOND STEP : B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Second step : b)Direct State, c)Adjoint state d)Topological derivative

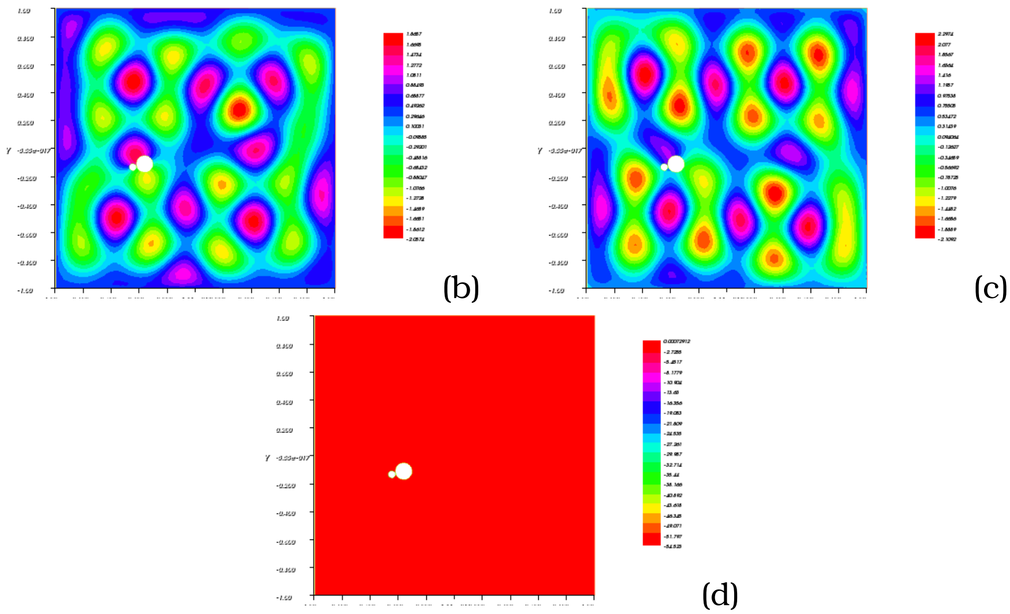

FIGURE 10. SECOND STEP : B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . Second step : b)Direct State, c)Adjoint state d)Topological derivative

The optimal design is given when ther is no possibilities to put a hole.

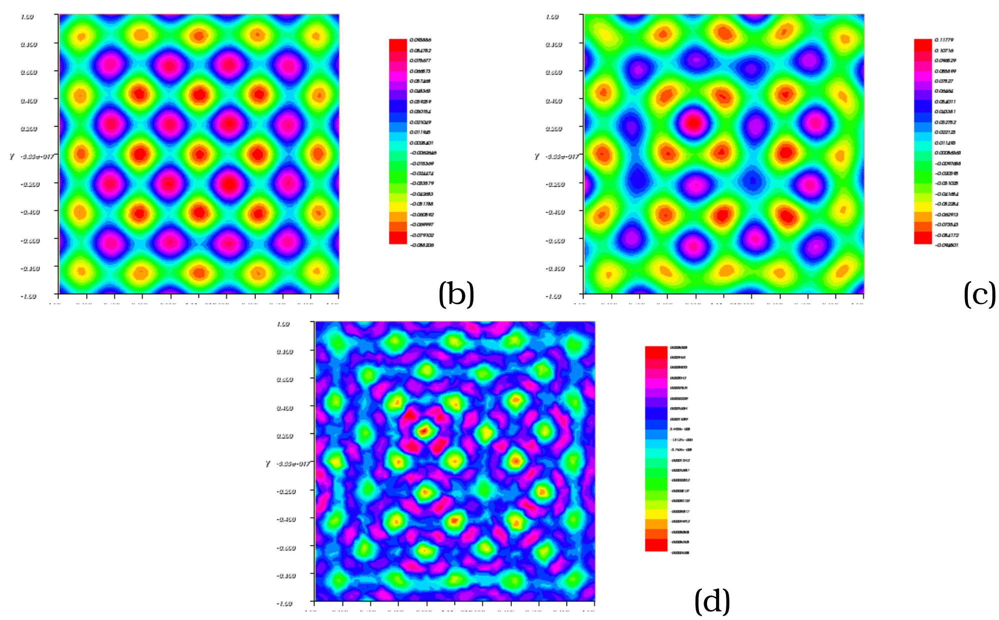

FIGURE 11. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . First step: b)Direct State, c)Adjoint state d)Topological derivative

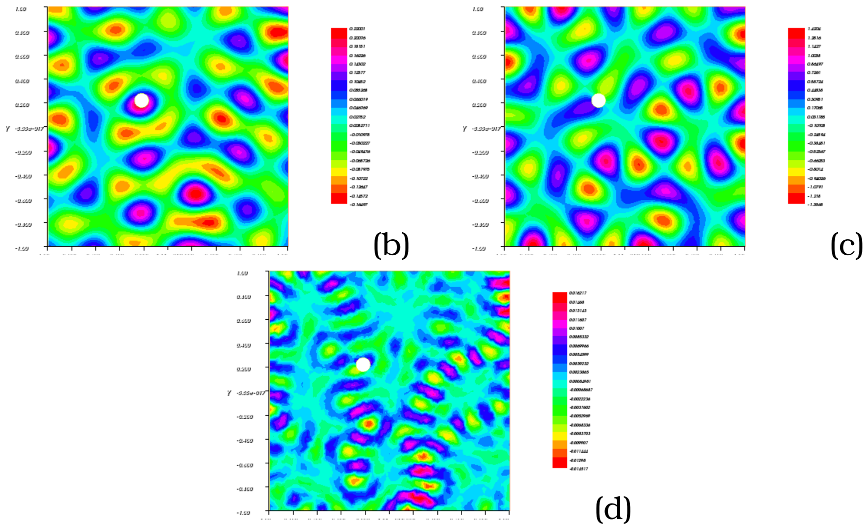

FIGURE 12. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . First step: b)Direct State, c)Adjoint state d)Topological derivative

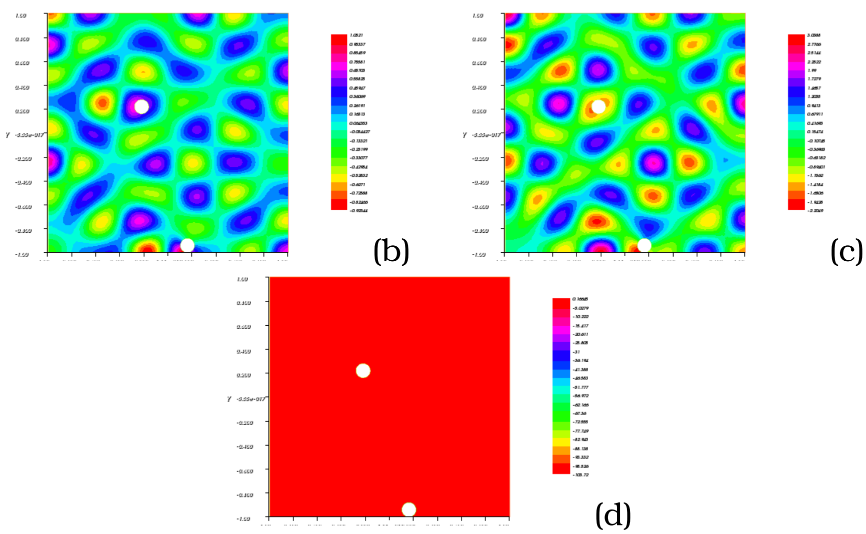

FIGURE 13. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . First step: b)Direct State, c)Adjoint state d)Topological derivative

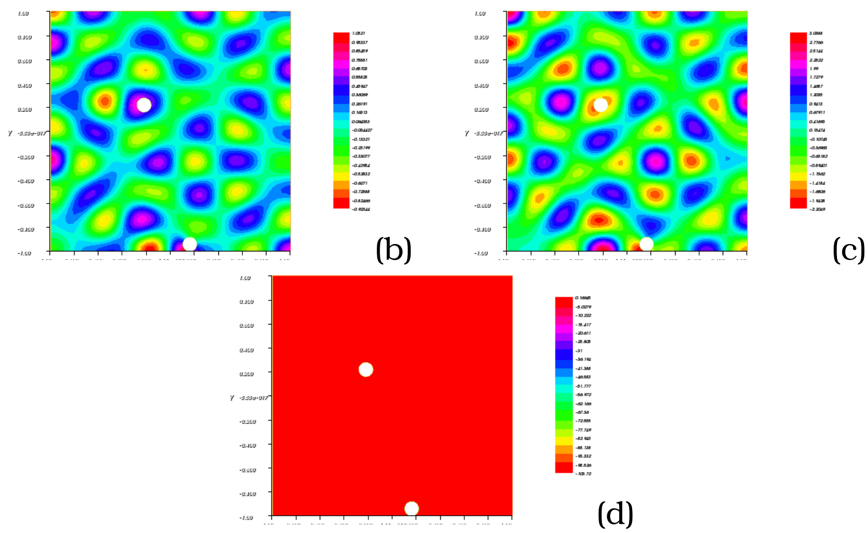

FIGURE 14. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE . First step: b)Direct State, c)Adjoint state d)Topological derivative

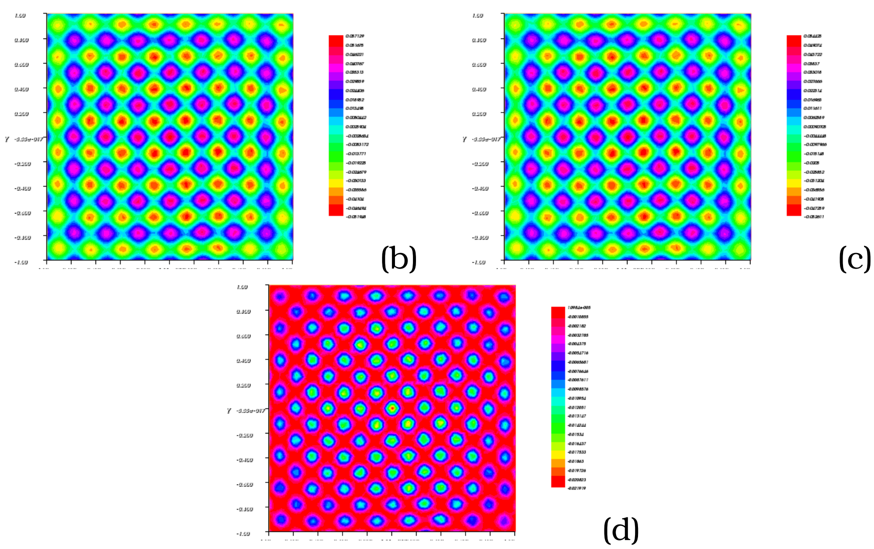

FIGURE 15.FIRST STEP: DOMAIN B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: Domain b)Direct State, c)Adjoint state d)Topological derivative

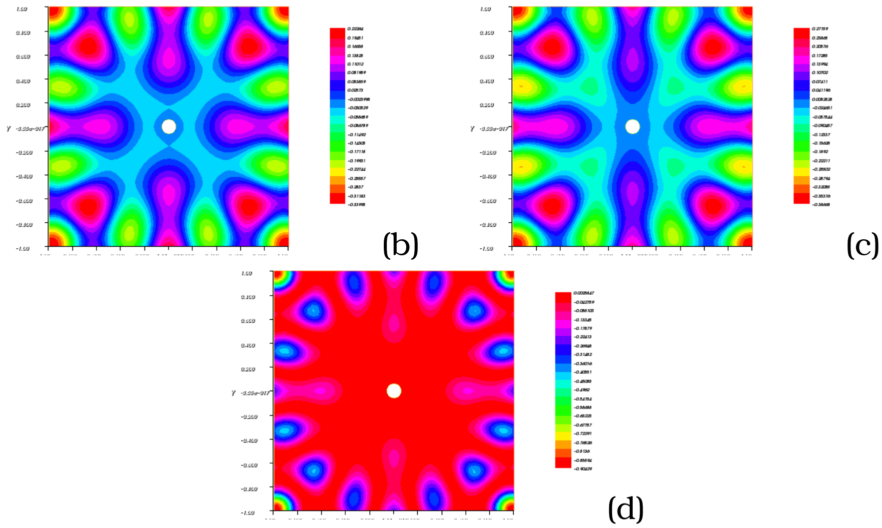

FIGURE 16. FIRST STEP: DOMAIN B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: Domain b)Direct State, c)Adjoint state d)Topological derivative

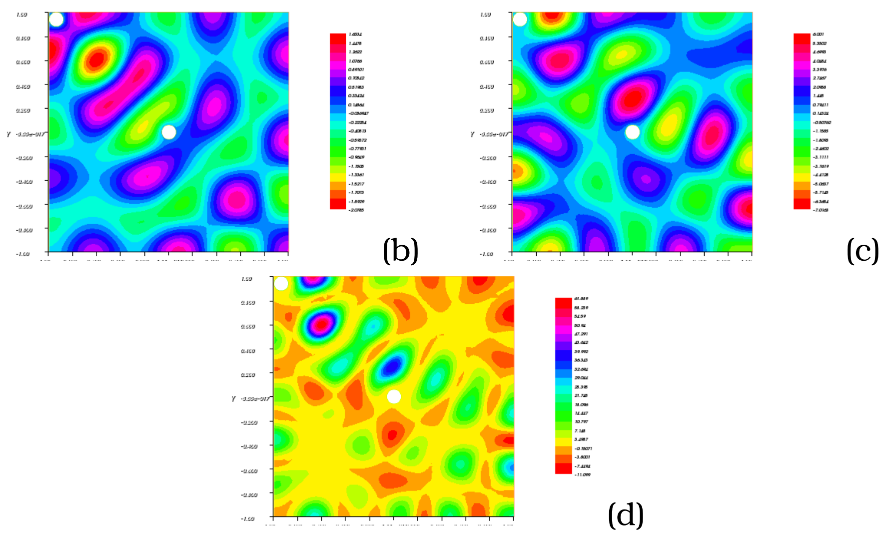

FIGURE 17. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

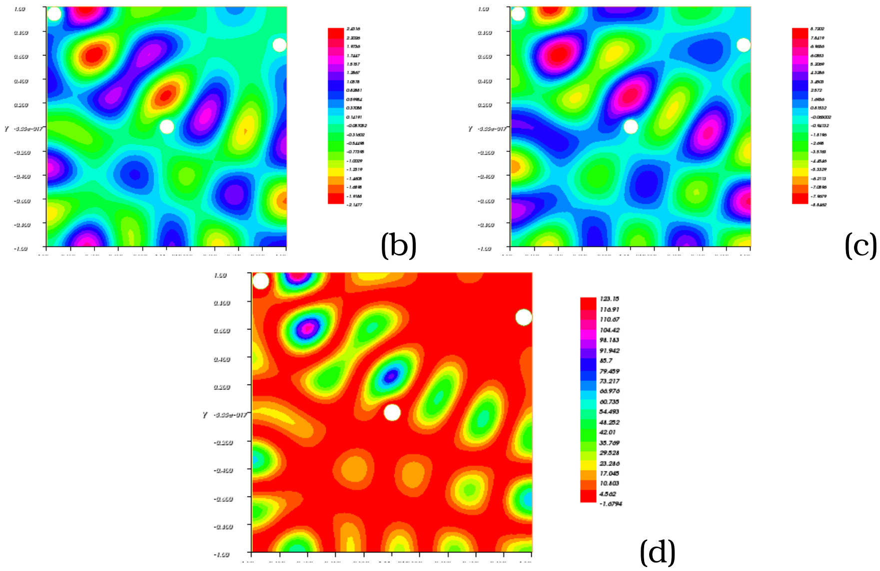

FIGURE 18. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

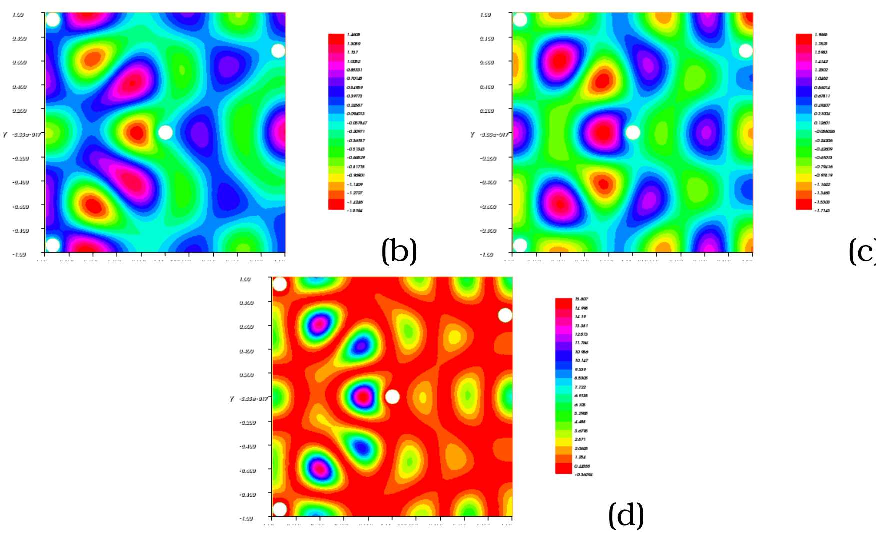

FIGURE 19. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

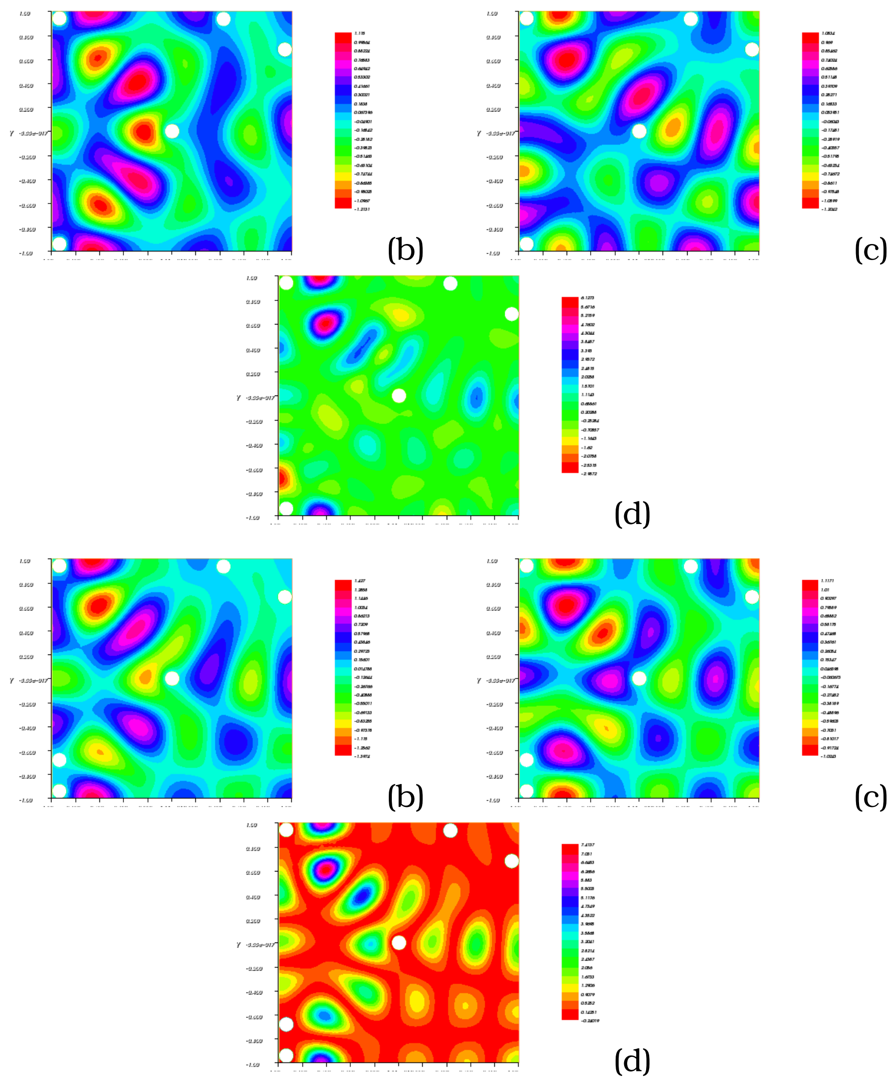

FIGURE 20. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

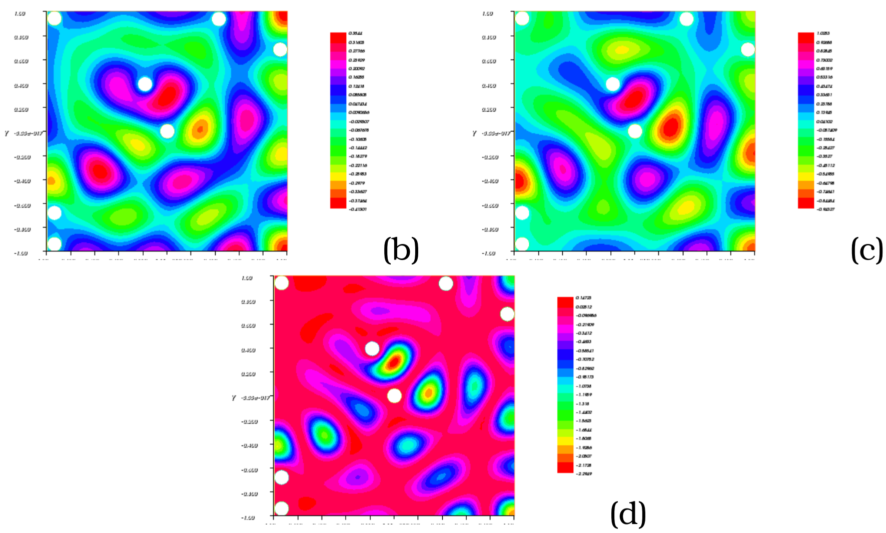

FIGURE 21. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

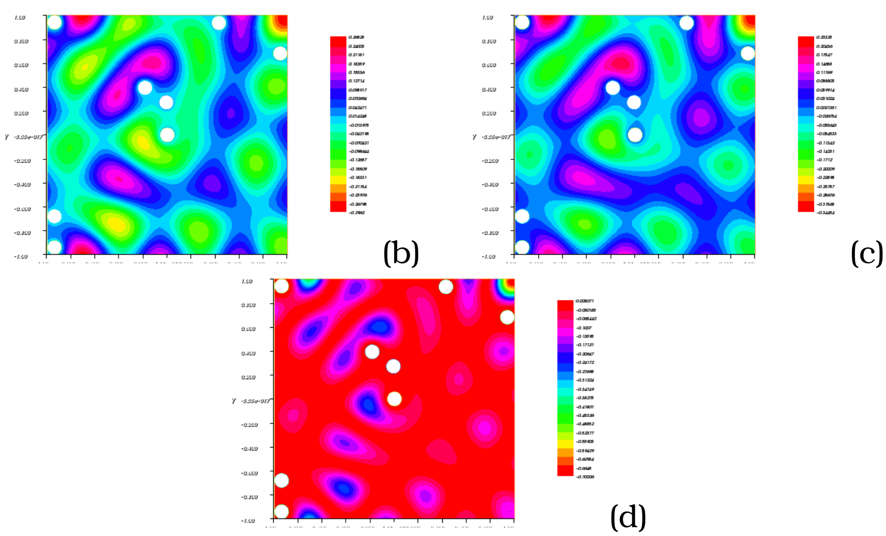

FIGURE 22. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

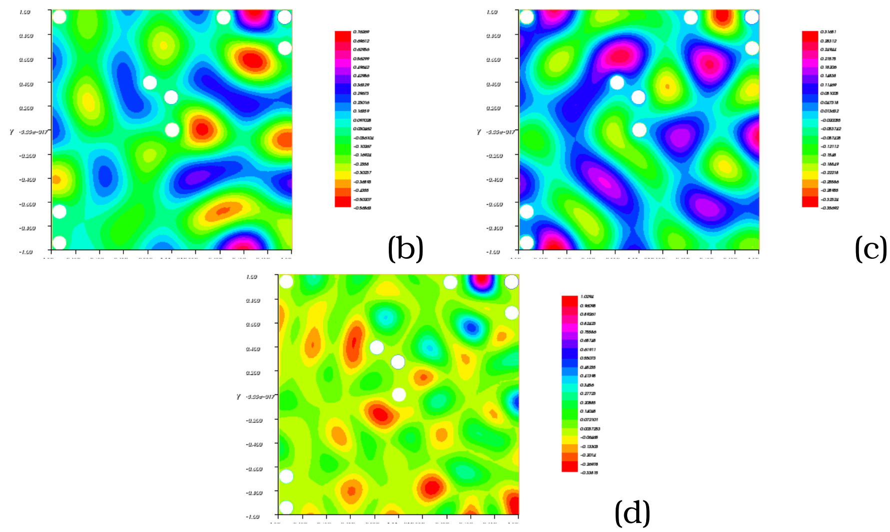

FIGURE 23. FIRST STEP: B)DIRECT STATE, C) ADJOINT STATE D) TOPOLOGICAL DERIVATIVE K=10A. First step: b)Direct State, c) Adjoint state d) Topological derivative k=10a

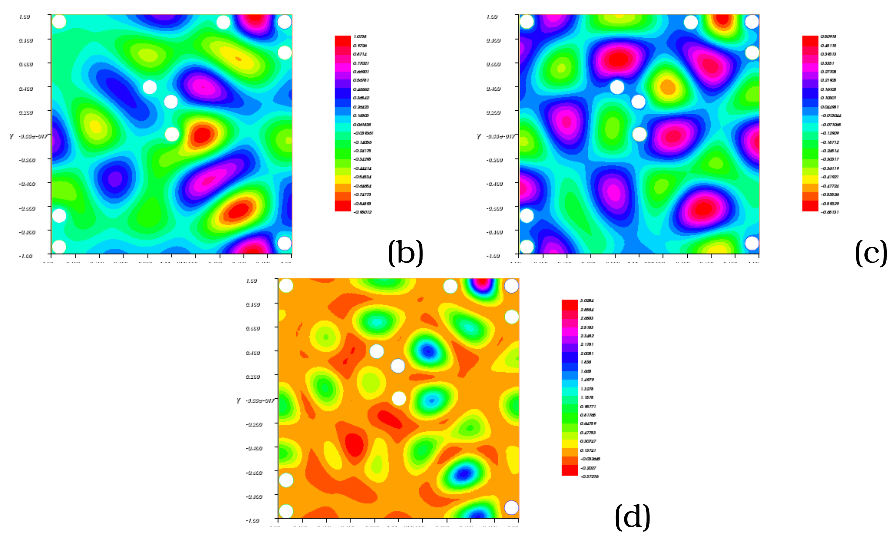

FIGURE 24. FIRST STEP: B)DIRECT STATE, C)ADJOINT STATE D)TOPOLOGICAL DERIVATIVE. First step: b)Direct State, c)Adjoint state d)Topological derivative

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