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CHAPTER 7

Bidimensional Asymptotic Normality of the Moving Kernel Poverty Index Estimate, *by Y. Ciss and A. Diakhaby*

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Abstract. In this paper we study the kernel estimator for the bidimensional extension of Foster, Greer and Thorbecke class of measures The asymptotic normality of the estimator is established. We next indicate how the proposed estimator can generate sequential confidence intervals by a moving kernel process. ◊

Keywords. Poverty line; Poverty aversion; Moving kernel; Bi-dimensional extension of Foster; Greer and Thorbecke; Uniform convergence; Asymptotic normality .

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Full Abstract. In this paper we study the kernel estimator for the bidimensional extension of Foster, Greer and Thorbecke class of measures by Duclos *et al.* (2006a) for the purpose of a dominance approach to multidimensional poverty. The measure they used in their dominance exercise is essentially a generalization, from one to two dimensions, of the FGT index separate poverty aversion parameters for each dimension. The asymptotic normality of the estimator is established. We next indicate how the proposed estimator can generate sequential confidence intervals by a moving kernel process. Our results are extensions of those of Dia (2009) and of Ciss *et al.* (2016) in one dimension.

Résumé. *Dans ce papier, nous proposons un estimateur pour la version bidimensionnelle de l'indice de pauvreté de Foster, Greer et Thorbecke, introduit par Duclos et al. (2006a) pour l'étude de pauvreté dans un cadre multidimensionnel grâce à la dominance stochastique. La mesure qu'ils utilisent dans cet exercice est en fait une extension bidimensionnelle de l'indice FGT avec deux paramètres pour l'aversion de la pauvreté, un dans chaque direction. La normalité asymptotique de l'estimateur à Noyau est d'abord établie. Nous montrons ensuite que l'estimateur proposé génére des intervalles de confiance séquentiels construits à l'aide de noyaux mobiles. Nos résultats constituent aussi une extension au cas bidimensionnel de ceux de Dia (2009) et de Ciss et al. (2016).*

1. Introduction and definition of the estimator

Duclos *et al.* (2006a) considered a multidimensional extension of the FGT class of measures, to address robustness analysis of the choice of poverty indices and poverty lines. They used the dominance approach for poverty comparisons, as initially developed in Atkinson (1987), in Foster *et al.* (1988a), Foster *et al.* (1988b) and in Foster *et al.* (1988c). A major advantage of this approach is its ability in generating poverty orderings that are robust with respect to the determination of poverty lines. Then the sensivity of most of poverty measures to the poverty line makes this approach more important. Besides, it also ensures robustness with respect to the choice of a multidimensional poverty index over broad classes of them, as well as robustness over the manner in which multidimensional indicators interact between them, when describing overall individual well-being. Duclos *et al.* (2006b) also used the bivariate stochastic dominance techniques to investigate the incidence of poverty, measured in terms of household expenditures per capita and child height-for-age indicators.

Such important traits of this measure motivated us to have an asymptotic theory based on estimators constructed on random samples that provide accurate approximations for small sizes. The results presented below are extensions of those of Dia (2009), for $\alpha = 0$ and $\alpha \geq 1$ and of Ciss *et al.* (2016), for $\alpha \in]0,1[$ in one dimension. These later have proved that this kernel estimator introduced par Dia can be used for each value of index $\alpha \in [0, \infty[$, which is important since the FGT measures concerns tests for poverty ordering or, equivalently, stochastic dominance and also optimal policy design (or program) for reducing poverty where the case $\alpha < 1$ is specific and useful as pointed out in Foster *et al.* (2010). By the way the results exposed in these papers (Dia (2009) and Ciss *et al.* (2016)) will be particular cases of results of the current paper.

Let x and y be two indicators of individual well-being among, for example, income, expenditures, caloric consumption, life expectancy, height, body weight, extent of personal safety and freedom, etc. Throughout this paper (X, Y) stands for the value of (x, y) for a randomly selected individual of the population. Then (X, Y) is a random couple of nonnegative real numbers defined of a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose cumulative distribution function (cdf) is denoted by $F(\cdot,\cdot)$ and we suppose that it admits a pdf $f(\cdot,\cdot)$.

The bi-dimensional extension of the FGT Foster *et al.* (1984) class of poverty measures by Duclos *et al.* (2006a) is denoted by $P(z_1, z_2, \alpha_1, \alpha_2)$ and is defined as follows, for $(\alpha_1 \geq 0, \alpha_2 \geq 0)$

$$(1.1) \qquad = \begin{cases} \int_0^{z_1}\int_0^{z_2} \left(\frac{z_1-x}{z_1}\right)^{\alpha_1}\left(\frac{z_2-y}{z_2}\right)^{\alpha_2} f(x,y)\,dxdy & \text{if } z_1 > 0 \text{ and } z_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

where z_1 (resp. z_2) represents the poverty line for the dimension x (resp. y). This index is useful for ordinal robust comparisons of poverty, even when the measurements are made across the intersection of the two dimensions considered.

Now let us consider, for $n \geq 1$, a random sample $(X_1, Y_1), ..., (X_n, Y_n)$ from (X, Y), defined of the probability space defined above. The empirical plug-in estimator of (1.1) is given by

$$\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-\frac{X_i}{z_1}\right)_+^{\alpha_1}\left(1-\frac{Y_j}{z_2}\right)_+^{\alpha_2} \quad \text{where} \quad x_+ = \max(0, x).$$

From there, we use the Parzen (1962) kernel estimator of the density $f(x, y)$

$$\hat{f}(x,y) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_1h_2}K\Big(\frac{x-X_i}{h_1}\Big)K\Big(\frac{x-Y_i}{h_2}\Big) \tag{1.2}$$

In a previous work Ciss *et al.* (2014), combining these two last facts, and based on Riemann sum, we have proposed the following kernel estimator of the DSY index (1.1) $P_n(z_1, z_2, \alpha_1, \alpha_2)$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{[\frac{z_1}{h_1}]}\sum_{j=1}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}K\Big(\frac{X_k-ih_1}{h_1}\Big)K\Big(\frac{Y_k-ih_2}{h_2}\Big) \tag{1.3}$$

where $[\frac{\cdot}{h_i}]$ stands for the integer part of $\frac{\cdot}{h_i}, i = 1, 2,\ \alpha_i \geq 0, h_i = h_i(n), i = 1, 2$ are positive nonrandom sequences of real numbers tending to zero as n tends to infinity, and finally K, a Riemann integrable kernel which satisfies the following hypotheses:

$$(\mathbf{H_1}) \sup_{-\infty<x<+\infty}|K(x)| < +\infty, (\mathbf{H_2}) \int_{\mathbb{R}} K(x)\,dx = 1, (\mathbf{H_3}) \lim_{(x)\to(\pm\infty)}|x||K(x)| = 0.$$

The rest of the paper is organized as follows. In section 2, we will state full details of the results. In section 3, as an illustrative example for the product of Gauss's kernels, we will determine the confidence intervals generated by a **moving kernel** process. The complete proofs are then given in section 4.

2. Asymptotic Normality

We will need a number of hypotheses and conditions for our theorems. We need to derive the following one $\mathbb{K}$, from the function K, defined on $\mathbb{R}^2$ by $\mathbb{K}(x, y) = K(x)K(y)$. This latter inherits from K these two properties :

$$\sup_{-\infty<x,y<+\infty}|\mathbb{K}(x,y)| < +\infty$$

and

$$\int\int_{\mathbb{R}^2}\mathbb{K}(x,y)\,dxdy = 1, \qquad \lim_{\|(x,y)\|\to(\pm\infty)}\|(x,y)\|\,|\mathbb{K}(x,y)| = 0.$$

Now additional hypotheses on $\mathbb{K}$ or K are the following:

$(\mathbf{H_4})$ K is of bounded variation function $V_{-\infty}^{u}K$ on $\mathbb{R}$ and we denote by $V(\mathbb{R})$ be its total variation.
$(\mathbf{H_5})$ $\int_{\mathbb{R}^2}|uv|\,|\mathbb{K}(u,v)|<+\infty.$
$(\mathbf{H_6})$ There exists a nonincreasing function λ such as $\lambda(\frac{u}{h_1},\frac{v}{h_2})=O(h_1h_2)$ on any bounded rectangle and for two couple of real numbers $x=(x_1,x_2)$ and $y=(y_1,y_2)$

$$|\mathbb{K}(x)-\mathbb{K}(y)|\leq\lambda\,\|x-y\|\quad\text{and}\quad\lambda(u,v)\longrightarrow0,(u,v)\to(0,0),u\geq0,v\geq0,$$

where $\|\cdot\|$ stands for the Euclidean norm.
$(\mathbf{H_7})$ $\frac{|xy|}{|h_1|^{1+\varepsilon}|h_2|^{1+\varepsilon}}|\mathbb{K}(\frac{x}{h_1},\frac{y}{h_2})|\to0,\quad0<\varepsilon<\frac{1}{2},$ as $\frac{|xy|}{h_1h_2}\to+\infty.$

Next, these conditions depend of the *pdf* $f(x,y)$:
$\mathbf{C_1}$: $f(x,y)$ is uniformly continuous.
$\mathbf{C_2}$: $f(x,y)$ admits almost everywhere a derivative $f^{'}(x,y)\in L_1(\mathbb{R}\times\mathbb{R})$.
$\mathbf{C_3}$: $f(x,y)$ satisfies a $C-$Lipschitz condition of order $\gamma,\ 0<\gamma\leq1$.

Finally, we consider a family of kernels $\mathbb{K}_\nu,\ \nu\in\Gamma\subset\mathbb{R}_+^*,\ \mathbb{R}_+^*$ being the set of strictly positive real numbers, about which we made respectively the same hypotheses.

$\mathbf{H_1}-\mathbf{H_7}$. Denote by P_n^ν the estimator of $P(z_1,z_2,\alpha_1,\alpha_2)$ when we replace $\mathbb{K}$ by $\mathbb{K}_\nu$ in P_n.

We suppose $\int_{\mathbb{R}^2}\mathbb{K}_\nu^2(y_1,y_2)\,dy_1dy_2<1$ for all $\nu\in\Gamma$ and $\sup_{\nu\in\Gamma}\int_{\mathbb{R}^2}\mathbb{K}_\nu^2(y_1,y_2)\,dy_1dy_2=1$. Let $N(0,1)$ be the standardized bivariate normal distribution function. We are now able to describe our main results.

THEOREM 17. Assume the hypotheses $\mathbf{C_3},\mathbf{H_6},\mathbf{H_7}$ hold and $\int_{\mathbb{R}^2}\|(y_1,y_2)\|^\gamma\,|\mathbb{K}_\nu^2(y_1,y_2)|\,dy_1dy_2<+\infty$ for all $\nu\in\Gamma$.
If $n\,\|(h_1,h_2)\|^{2\gamma}\to0$ as $n\to+\infty$,
then

$$\frac{P_n^\nu(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2)}{\sqrt{\mathbb{V}ar(P_n^\nu(z_1,z_2,\alpha_1,\alpha_2))}}\to N(0,1)$$

in distribution as $n\to+\infty$, provided
$(\int_{\mathbb{R}^2}\mathbb{K}_\nu^2(y_1,y_2)\,dy_1dy_2)P(z_1,z_2,2\alpha_1,2\alpha_2)-(P(z_1,z_2,\alpha_1,\alpha_2))^2>0.$

THEOREM 18. Assume the hypotheses $\mathbf{C_2}, \mathbf{H_5}, \mathbf{H_7}$ hold.
If $n\,\|(h_1,h_2)\|^{2\gamma} \to 0$ as $n \to +\infty$,
then

$$\frac{P_n^{\nu}(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2)}{\sqrt{\mathbb{V}ar(P_n^{\nu}(z_1,z_2,\alpha_1,\alpha_2))}} \to N(0,1)$$

in distribution as $n \to +\infty$, provided

$$(\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1,y_2)\,dy_1dy_2)P(z_1,z_2,2\alpha_1,2\alpha_2) - (P(z_1,z_2,\alpha_1,\alpha_2))^2 > 0.$$

We establish these two theorems by proving the two following lemmas which are respectively a generalization in three dimension of the function $\mathbb{K}$ of **Lemma 3** Ciss *et al.* (2014) and **Theorem 5** Ciss *et al.* (2014).

LEMMA 10. Let $0 \le \theta_i^j \le 1; i=1,2,3, \quad j=1,2$. Then for all $x=(x_1,x_2)$, $y=(y_1,y_2), \quad t=(t_1,t_2)$ pairwise different we have

$$\lim_{n\to+\infty} \sup_{(\theta_1^1,\theta_2^1,\theta_3^1)\in[0,1]\times[0,1]\times[0,1]} \sup_{(\theta_1^2,\theta_2^2,\theta_3^2)\in[0,1]\times[0,1]\times[0,1]} \Big((h_1h_2)^{-3}$$
$$\times \int_{-\infty}^{+\infty} |\mathbb{K}(\frac{u_1-x_1+\theta_1^1h_1}{h_1}, \frac{u_2-x_2+\theta_1^2h_2}{h_2})\mathbb{K}(\frac{u_1-y_1+\theta_2^1h_1}{h_1}, \frac{u_2-y_2+\theta_2^2h_2}{h_2})$$
$$\times \mathbb{K}(\frac{u_1-t_1+\theta_3^1h_1}{h_1}, \frac{u_2-t_2+\theta_3^2h_2}{h_2})|f(u_1,u_2)\,du_1du_2\Big) = 0.$$

LEMMA 11. Assume the hypotheses $\mathbf{C_1}$ or $\mathbf{C_2}$ holds.

Then under the hypotheses $\mathbf{H_6}$ and $\mathbf{H_7}$ we have for all $b=(b_1,b_2)>0$,

$$\lim_{n\to+\infty} \sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]} \sum_{0\le i_1\ne j_1\ne l_1\le[\frac{z_1}{h_1}]} \sum_{0\le i_2\ne j_2\ne l_2\le[\frac{z_2}{h_2}]} \Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}$$
$$\times \Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{l_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{l_2h_2}{z_2}\Big)^{\alpha_2}$$
$$\times \int_{\mathbb{R}^2} K\Big(\frac{u_1-i_1h_1}{h_1}\Big)K\Big(\frac{u_1-j_1h_1}{h_1}K\Big(\frac{u_1-l_1h_1}{h_1}\Big)K\Big(\frac{u_2-i_2h_2}{h_2}\Big)$$
$$\times K\Big(\frac{u_2-j_2h_2}{h_2}\Big)K\Big(\frac{u_2-l_2h_2}{h_2}f(u_1,u_2)\,du_1du_2 = 0.$$

Remark 1. To construct a confidence interval, we proceed as follow: For $0 < \beta = \beta_1 \times \beta_2 < 1$, let $q_{1-\frac{\beta}{2}} = q_{1-\frac{\beta_1}{2}} \times q_{1-\frac{\beta_2}{2}}$ be the $\beta-$quantile of the standardized normal bivariate distribution. Since $\mathbb{V}ar(P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2)) \geq 0$ for all (z_1, z_2) and by the theorems 3 and 4 Ciss *et al.* (2014)

$$\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2)) = \left(\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2\right) P(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P(z_1, z_2, \alpha_1, \alpha_2))^2, \tag{2.1}$$

we have $\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2)) = 0$ for all (z_1, z_2) such that

$$\left(\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2\right) P(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P(z_1, z_2, \alpha_1, \alpha_2))^2 \leq 0.$$

It follows that the asymptotic efficiency $e^{\mathbb{K}_\nu}(z_1, z_2, \alpha_1, \alpha_2)$ verifies

$$\begin{aligned} 0 \leq e^{\mathbb{K}_\nu}(z_1, z_2, \alpha_1, \alpha_2) &= \lim_{n\to+\infty} \frac{n\mathbb{V}ar(P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2))}{n\mathbb{V}ar(\widehat{P}_n(z_1, z_2, \alpha_1, \alpha_2))} \\ &\leq \int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2 < 1 \end{aligned}$$

for $\mathbb{K}$ product of two conventional kernels Parzen (1962) p.1068.

We define $100(1-\beta)\%$ the confidence interval CI_ν for $P(z_1, z_2, \alpha_1, \alpha_2)$ in the following form

$$\begin{aligned} CI_\nu &= P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2) \pm q_{1-\frac{\beta_1}{2}} \times q_{1-\frac{\beta_2}{2}} \\ &\times \{(\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2) P_n^{\nu}(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2))^2\}^{\frac{1}{2}} / \sqrt{n} \end{aligned} \tag{2.2}$$

as long as $(\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2) P_n^{\nu}(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P_n^{\nu}(z_1, z_2, \alpha_1, \alpha_2))^2 > 0$. Denote this inequality by (**C**). If it is not verified, we increase the size of the sample from n to $n+1$. If for all n the inequality (**C**) is not satisfied, we vary ν to increase $\int_{\mathbb{R}^2} \mathbb{K}_{\nu}^2(y_1, y_2)\, dy_1 dy_2$. There exists then $\nu \in \Gamma$ and a integer n_0 from which the interval CI_ν is defined. Indeed : let $\nu_k \in \Gamma$ be a sequence such that $\int_{\mathbb{R}^2} \mathbb{K}_{\nu_k}^2(y_1, y_2)\, dy_1 dy_2$ converges to 1. Then, since $P(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P(z_1, z_2, \alpha_1, \alpha_2))^2 > 0$ according to the empirical estimator, we have

$$(\int_{\mathbb{R}^2} \mathbb{K}^2_{\nu_k}(y_1, y_2)\, dy_1 dy_2) P(z_1, z_2, 2\alpha_1, 2\alpha_2) - (P(z_1, z_2, \alpha_1, \alpha_2))^2 > 0$$

for k large enough greater than or equal to k_0. The inequality $(\mathbf{C})$ is then verified from an integer n_0 and for $\nu = \nu_{k_0}$, under the conditions of the theorems 3, 4 Ciss *et al.* (2014) and the convergence in mean square of $P^{\nu}_n(z_1, z_2, \alpha_1, \alpha_2)$ to $P(z_1, z_2, \alpha_1, \alpha_2)$ theorems 3, 4 Ciss *et al.* (2014).

We conclude that, confidence the length of the interval CI_ν associated with our estimator is asymptotically lower than that of the empirical estimator, the coefficient being $\{\int_{\mathbb{R}^2} \mathbb{K}^2_\nu(y_1, y_2)\, dy_1 dy_2\}^{\frac{1}{2}} < 1$.

Remark 2. Consider the inequality $(\mathbf{C})$. The quantity $P_\nu = \int_{\mathbb{R}^2} \mathbb{K}^2_\nu(y_1, y_2)\, dy_1 dy_2$ may be considered as a weight placed in $P(z_1, z_2, 2\alpha_1, 2\alpha_2)$. Greater weight is attached to higher poverty line (It is even heavier than the ratio or proportion $Q_n = \dfrac{P^\nu_n(z_1, z_2, \alpha_1, \alpha_2)}{P^\nu_n(z_1, z_2, 2\alpha_1, 2\alpha_2)}$ is high. So, in order to perform a normality test or to construct a confidence interval of $P(z_1, z_2, \alpha_1, \alpha_2)$, we must calculate $Q_n(z_1, z_2, \nu, \alpha_1, \alpha_2)$ for a kernel $\mathbb{K}_{\nu_0}$. If $P_{\nu_0} > Q_n(z_1, z_2, \alpha_1, \alpha_2)$, we determinate CI_{ν_0} by the equality (2.2), else a greater weight kernel $\mathbb{K}_\nu$ is chosen such that the inequality is verified.

Remark 3. Consider the following family of kernels $\mathbb{K}_\nu(x_1, x_2) = \dfrac{\mathbb{K}(\nu x_1, \nu x_2)}{\int_{\mathbb{R}^2} \mathbb{K}(\nu x_1, \nu x_2)}$, $\nu > 0$. We verify that they satisfy the hypotheses $\mathbf{H_1} - \mathbf{H_7}$. Moreover

$$\int_{\mathbb{R}^2} \mathbb{K}^2_\nu(y_1, y_2)\, dy_1 dy_2 = \nu \int_{\mathbb{R}^2} \mathbb{K}^2(y_1, y_2)\, dy_1 dy_2.$$

Therefore Γ can be the interval $\left]0, \dfrac{1}{\int_{\mathbb{R}^2} \mathbb{K}^2(y_1, y_2)\, dy_1 dy_2}\right[$.

3. Application

As an illustrative example for the bivariate Gauss's kernel, let ν equal to $\frac{1}{\sigma^2}$, then $\mathbb{K}_\nu(x_1, x_2) = \dfrac{1}{2\pi\sigma^2} e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$. We have $\int_{\mathbb{R}^2} \mathbb{K}_\nu(y_1, y_2) dy_1 dy_2 = \dfrac{1}{4\pi\sigma^2}$. Let p_k,
$k = 1, 2, ...,$ be any increasing sequence of real numbers, $0 < p_k < 1$, and converging to 1. (For example $p_k = \frac{k}{10}, k = 1, ..., 9$ if we are interested in the deciles or $p_k = \frac{k}{4}$,
$k = 1, 2, 3$ for the quartiles). By setting $\frac{1}{4\pi\sigma^2} = p_k$ for $k = 1, 2, ...,$ we obtain successive kernels corresponding to $\sigma = \frac{1}{2\sqrt{p_k\pi}}$. Therefore, the test or the confidence interval will concern all z_p quantile of the distribution $F(z_1, z_2) = P(z_1, z_2, 0, 0)$ such that $p < p_k, k = 1, 2, ...$

This just described process can be thought of as a test or confidence intervals generated by a **moving kernel** process.**

4. Details of the Proofs

Proof of Lemma 10. We assume $\mathbf{C_1}$ holds. Let $\delta = (\delta_1, \delta_2) > 0$. Note that the notations θ_1^i and θ_2^i $i = 1, 2$ represent indices and not powers.
Define

$$\begin{aligned}
I_n(x_1, x_2, y_1, y_2) &= (h_1h_2)^{-3} \int\int_{-\infty}^{+\infty} |\mathbb{K}(\frac{u_1 - x_1 + \theta_1^1 h_1}{h_1}, \frac{u_2 - x_2 + \theta_1^2 h_2}{h_2}) \\
&\times \mathbb{K}(\frac{u_1 - y_1 + \theta_2^1 h_1}{h_1}, \frac{u_2 - y_2 + \theta_2^2 h_2}{h_2}) \\
&\times \mathbb{K}(\frac{u_1 - t_1 + \theta_3^1 h_1}{h_1}, \frac{u_2 - t_2 + \theta_3^2 h_2}{h_2})| f(u_1, u_2)\, du_1 du_2 \\
&= \int\int_{-\infty}^{+\infty} ((h_1h_2)^{-1}\mathbb{K}(\frac{v_1}{h_1}, \frac{v_2}{h_2}))|((h_1h_2)^{-1} K(\frac{v_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1 h_1}{h_1}) \\
&\times K(\frac{v_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2}))((h_1h_2)^{-1} K(\frac{v_1 + x_1 - \theta_1^1 - t_1 + \theta_3^1 h_1}{h_1}) \\
&\times K(\frac{v_2 + x_2 - \theta_1^2 - t_2 + \theta_3^2 h_2}{h_2}))| f(x_1 + v_1 - \theta_1^1 h_1, x_2 + v_2 - \theta_1^2 h_2)\, du_1 du_2 \\
&= \int_{|v_1| \leq \delta_1} \int_{|v_2| \leq \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| \leq \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| \leq \delta_2} .
\end{aligned}$$

Since f is continuous, it is so bounded on $I_1 \times I_2 = [x_1 - \delta_1, x_1 + \delta_1] \times [x_2 - \delta_2, x_2 + \delta_2]$. We assume n large enough such that $(x_1 + v_1 \pm \theta_1^1 h_1, x_2 + v_2 \pm \theta_1^2 h_2) \in I_1 \times I_2$. Therefore

(4.1)

$$\begin{aligned}
&\int_{|v_1|\leq\delta_1}\int_{|v_1|\leq\delta_2}\\
&\leq \sup_{|v_1|\leq\delta_1}\sup_{|v_2|\leq\delta_2} f(x_1+v_1-\theta_1^1 h_1, x_2+v_2-\theta_1^2 h_2)\\
&\times \int_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}}\int_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}} |\mathbb{K}(u_1,u_2)|\\
&\times |\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - y_2+\theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}\\
&\times \mathbb{K}(\frac{x_1-\theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - t_2+\theta_3^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|\, du_1du_2.
\end{aligned}$$

$$\begin{aligned}
(4.1) = &\sup_{|v_1|\leq\delta_1}\sup_{|v_2|\leq\delta_2} f(x_1+v_1-\theta_1^1 h_1, x_2+v_2-\theta_1^2 h_2)\\
&\times \int\int_{-\infty}^{+\infty} \chi_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}}\chi_{-\frac{\delta_2}{h_2}\leq u_2\leq\frac{\delta_2}{h_2}}(u_2)|\mathbb{K}(u_1,u_2)|\\
&\times |\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - y_2+\theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}\\
&\times \mathbb{K}(\frac{x_1-\theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - t_2+\theta_3^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|\, du_1du_2.
\end{aligned}$$

For all (u_1, u_2)

$$\lim_{n\to+\infty}|\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - y_2+\theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}| = 0.$$

Write

$$\begin{aligned}
&|\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - y_2+\theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\
&\quad = |(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1)(\frac{x_2-\theta_1^2 h_2 - y_2+\theta_2^2 h_2}{h_2}+u_2)\\
&\quad \times K(\frac{x_1-\theta_1^1 - y_1 + \theta_2^1 h_1}{h_1}+u_1)K(\frac{x_2-\theta_1^2 - y_2+\theta_2^2 h_2}{h_2}+u_2)|\\
&\qquad \times |\frac{1}{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1 + h_1u_1} \times \frac{1}{x_2-\theta_1^2 h_2 - y_2 + \theta_2^2 h_2 + h_2u_2}|.
\end{aligned}$$

We have

$$|\frac{1}{x_i-\theta_1^i h_i - y_i + \theta_2^i h_i + h_i u_i}| = \frac{1}{|x_i-y_i||1-\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i|} \qquad i=1,2.$$

Since $|u_i| \leq \frac{\delta_i}{h_i}$ we may choose δ_i small enough such that for $n \geq n_0$ we have

$$|\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i| \leq \frac{3(h_i)_{n_0}+\delta_i}{|x_i-y_i|} = \eta_i < 1.$$

Therefore

$$|\frac{1}{x_i-\theta_1^i h_i - y_i + \theta_2^i h_i + h_i u_i}| \leq \frac{1}{|x_i-y_i|(1-\eta_i)}, \tag{4.2}$$

since $\mathbf{H_3}$ implies there exists a constant B such as

$$\begin{aligned}|(&\frac{x_1-\theta_1^1 h_i - y_1 + \theta_2^1 h_1}{h_1}+u_1)(\frac{x_2-\theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2}+u_2)\\ &\times K(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1)K(\frac{x_2-\theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2}+u_2)| \leq B.\end{aligned}$$

Then, we have

$$\begin{aligned}\chi_{-\frac{\delta_1}{h_1}\leq u_1 \leq \frac{\delta_1}{h_1}}(u_1)\chi_{-\frac{\delta_2}{h_2}\leq u_2 \leq \frac{\delta_2}{h_2}}(u_2)|K(&\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1)\\ &\times K(\frac{x_2-\theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\ &\leq \frac{2B}{|x_1-y_1|(1-\eta_1)|x_2-y_2|(1-\eta_2)}.\end{aligned}$$

Similarly, there exists a constant C such that

$$\begin{aligned}|(&\frac{x_1-\theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}+u_1)(\frac{x_2-\theta_1^2 h_2 - t_2 + \theta_3^2 h_2}{h_2}+u_2)\\ &\times K(\frac{x_1-\theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}+u_1)K(\frac{x_2-\theta_1^2 h_2 - t_2 + \theta_3^2 h_2}{h_2}+u_2)| \leq C\end{aligned}$$

and

$$\chi_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}}(u_1)\chi_{-\frac{\delta_2}{h_2}\leq u_2\leq\frac{\delta_2}{h_2}}(u_2)|K(\frac{x_1-\theta_1^1h_1-t_1+\theta_3^1h_1}{h_1}+u_1)| \times |K(\frac{x_2-\theta_1^2h_2-t_2+\theta_3^2h_2}{h_2}+u_2)(h_1h_2)^{-1}| \leq \frac{2C}{|x_1-t_1|(1-\eta_3)|x_2-t_2|(1-\eta_4)}.$$

Therefore, if δ_i is small enough and n sufficiently large we have, for $-\dfrac{\delta_i}{h_i}\leq u_i\leq\dfrac{\delta_i}{h_i}, i=1,2$
$|\mathbb{K}(u_1,u_2)|$ being integrable, by dominated convergence theorem, the integral in the right-hand side of 4.1 tends to zero as $n\rightarrow+\infty$, uniformly with respect to $((\theta_1^1,\theta_1^2),(\theta_2^1,\theta_2^2),(\theta_3^1,\theta_3^2))$. Hence we have

$$\int_{|v_1|\leq\delta_1}\int_{|v_1|\leq\delta_2}\rightarrow 0 \quad \text{as} \quad n\rightarrow+\infty \quad \text{uniformly with respect to}$$

$$((\theta_1^1,\theta_1^2),(\theta_2^1,\theta_2^2),(\theta_3^1,\theta_3^2)).$$

Let $\displaystyle\int_{|v_1|>\delta_1}\int_{|v_1|>\delta_2}$ write in the form:

$$\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} = \int_{|v_1-\theta_1^1h_1|>\delta_1}\int_{|v_2-\theta_1^2h_2|>\delta_2}|v_1v_2(h_1h_2)^{-1}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})((h_1h_2)^{-1} \times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}((h_1h_2)^{-1} \times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-t_1+\theta_3^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-t_2+\theta_3^2h_2}{h_2})) \times \frac{f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)}{v_1v_2}|\,dv_1dv_2.$$

We obtain

(4.3)

$$\begin{aligned}\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} &\leq \frac{2}{\delta_1}\times\frac{2}{\delta_2}\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}|\frac{v_1v_2}{h_1h_2}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})| \\ &\times \int_{|v_1|>\delta_1}\int_{|v_1|>\delta_2}((h_1h_2)^{-1} \\ &\times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}))((h_1h_2)^{-1} \\ &\times \mathbb{K}(\frac{v_1+x_1-\theta_1^1-t_1+\theta_3^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-t_2+\theta_3^2h_2}{h_2})) \\ &\times |f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)|\,dv_1dv_2.\end{aligned}$$

Making the change of variable:

$$v_i+x_i-\theta_1^ih_i=u_i,\quad i=1,2.$$

Then

$$\begin{aligned}&\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} \\ &\leq \frac{2}{\delta_1-\theta_1^1h_1}\times\frac{2}{\delta_2-\theta_1^2h_2}\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}|\frac{v_1v_2}{h_1h_2}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})| \\ &\times\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})((h_1h_2)^{-1} \\ &\times\mathbb{K}(\frac{v_1+x_1-\theta_1^1-t_1+\theta_3^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-t_2+\theta_3^2h_2}{h_2}))|f(u_1,u_2)\,du_1du_2 \\ &\leq\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})((h_1h_2)^{-1} \\ &\times\mathbb{K}(\frac{v_1+x_1-\theta_1^1-t_1+\theta_3^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-t_2+\theta_3^2h_2}{h_2}))|f(u_1,u_2)\,du_1du_2\end{aligned}$$

and this quantity in right-side tends to zero as $n\to+\infty$ uniformly with respect to $((\theta_2^1,\theta_2^2),(\theta_3^1,\theta_3^2))$ according to the **Lemma 3** Ciss *et al.* (2014) and the **Remark 1** Ciss *et al.* (2014) (when we replace $\mathbb{K}$ by $|\mathbb{K}|$) and $(\mathbf{H_3})$ (valid under assumption $\mathbf{C_1}$ or $\mathbf{C_2}$). Since

$$\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}\left|\frac{v_1v_2}{h_1h_2}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})\right|\to 0\quad\text{as}\quad n\to+\infty,$$

we have

$$|\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}| \to 0 \quad \text{as} \quad n \to +\infty$$

and in this case we have the uniform convergence .

Consider the case $\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot)(\cdot)$ according to Fubini theorem we obtain

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot)(\cdot) = \int_{|v_1|\leq\delta_1}(\cdot)\int_{|v_2|>\delta_2}(\cdot).$$

Using the two unidimensional cases (**Lemma 2.3**) Dia (2008) we obtain

$$\int_{|v_1|\leq\delta_1} \to 0 \quad \text{and} \int_{|v_2|>\delta_2} \to 0 \quad \text{as} \quad n \to +\infty$$

Therefore

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2} \to 0 \quad \text{as} \quad n \to +\infty.$$

Consider the case $\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2}(\cdot)(\cdot)$ similarly by the Fubini's theorem we obtain

$$\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2}(\cdot)(\cdot) = \int_{|v_1|>\delta_1}(\cdot)\int_{|v_2|\leq\delta_2}(\cdot).$$

Therefore

$$\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2} \to 0 \quad \text{as} \quad n \to +\infty.$$

The proof of the **Lemma 10** is complete.

Proof of Lemma 11. We Suppose condition C_1 verified.
Let $\Delta_1 = [0, b_1] \times [0, b_1] \times [0, b_1]; \Delta_2 = [0, b_2] \times [0, b_2] \times [0, b_2]$.
We can write for all $(z_1, z_2) \in [0, b_1] \times [0, b_2]$

$$\sum_{0 \leq i_1 \neq j_1 \neq l_1 \neq i_1 \leq [\frac{z_1}{h_1}]} \left(1 - \frac{i_1 h_1}{z_1}\right)^{\alpha_1} \left(1 - \frac{j_1 h_1}{z_1}\right)^{\alpha_1} \left(1 - \frac{l_1 h_1}{z_1}\right)^{\alpha_1}$$
$$\times \sum_{0 \leq i_2 \neq j_2 \neq l_2 \neq i_2 \leq [\frac{z_2}{h_2}]} \left(1 - \frac{i_2 h_2}{z_2}\right)^{\alpha_2} \left(1 - \frac{j_2 h_2}{z_2}\right)^{\alpha_2} \left(1 - \frac{l_2 h_2}{z_2}\right)^{\alpha_2}$$
$$\times \int\int_{\mathbb{R}^2} |\mathbb{K}(\frac{u_1 - i_1 h_1}{h_1}, \frac{u_1 - j_1 h_1}{h_1})\mathbb{K}(\frac{u_2 - i_2 h_2}{h_2}, \frac{u_2 - j_2 h_2}{h_2})$$
$$\times \mathbb{K}(\frac{u_1 - l_1 h_1}{h_1}, \frac{u_2 - l_2 h_2}{h_2})|f(u_1, u_2)\, du_1 du_2)$$
$$= \int\int\int_{\{(x_1, y_1, t_1) \in \Delta_1 : |x_1 - y_1||x_1 - t_1||t_1 - y_1| > 0\}} \Phi_n(x_1, y_1, t_1)\, dx_1 dy_1 dt_1$$
$$\times \int\int\int_{\{(x_2, y_2, t_2) \in \Delta_2 : |x_2 - y_2||x_2 - t_2||t_2 - y_2| > 0\}} \Phi_n(x_2, y_2, t_2)\, dx_2 dy_2 dt_2,$$

where

$$\Phi_n(x_1, y_1, t_1) = \frac{1}{h_1^3} \sum_{0 \leq i_1 \neq j_1 \neq l_1 \neq i_1 \leq [\frac{z_1}{h_1}]} \chi_{\Delta_{h_1, i_1}}(x_1) \chi_{\Delta_{h_1, j_1}}(y_1) \chi_{\Delta_{h_1, l_1}}(t_1)$$
$$\times \int_{\mathbb{R}} |K(\frac{u_1 - i_1 h_1}{h_1}) K(\frac{u_1 - j_1 h_1}{h_1}) K(\frac{u_1 - l_1 h_1}{h_1})| f_1(u_1)\, du_1$$

and we obtain $\Phi_n(x_2, y_2, t_2)$ by changing 1 by 2.
Note $\Phi_{n_{12}}(x_1, y_1, t_1, x_2, y_2, t_2) := \Phi_n(x_1, y_1, t_1)\Phi_n(x_2, y_2, t_2)$. Therefore

(4.4)

$$\Phi_{n_{12}}(x_1, y_1, t_1, x_2, y_2, t_2) = \frac{1}{h_1^3 h_2^3} \sum_{0 \leq i_1 \neq j_1 \neq l_1 \neq i_1 \leq [\frac{z_1}{h_1}]} \chi_{\Delta_{h_1, i_1}}(x_1) \chi_{\Delta_{h_1, j_1}}(y_1) \chi_{\Delta_{h_1, l_1}}(t_1)$$
$$\times \sum_{0 \leq i_2 \neq j_2 \neq l_2 \neq i_2 \leq [\frac{z_2}{h_2}]} \chi_{\Delta_{h_2, i_2}}(x_2) \chi_{\Delta_{h_2, j_2}}(y_2) \chi_{\Delta_{h_2, l_2}}(t_2)$$
$$\times \int\int_{\mathbb{R}^2} |\mathbb{K}(\frac{u_1 - i_1 h_1}{h_1}, \frac{u_1 - j_1 h_1}{h_1})\mathbb{K}(\frac{u_2 - i_2 h_2}{h_2}, \frac{u_2 - j_2 h_2}{h_2})$$
$$\times \mathbb{K}(\frac{u_1 - l_1 h_1}{h_1}, \frac{u_2 - l_2 h_2}{h_2})|f(u_1, u_2)\, du_1 du_2.$$

Let $(x_1, y_1, t_1) \in \Delta_{h_1,i_1} \times \Delta_{h_1,j_1} \times \Delta_{h_1,l_1}$ $i_1 \neq j_1 \neq l_1 \neq i_1$; $(x_2, y_2, t_2) \in \Delta_{h_2,i_2} \times \Delta_{h_2,j_2} \times \Delta_{h_2,t_2}$ $i_2 \neq j_2 \neq l_2 \neq i_2$ with the representations:

$$x_1 = h_1 i_1 + \theta_1^1 h_1, \quad y_1 = h_1 j_1 + \theta_2^1 h_1 \quad t_1 = h_1 l_1 + \theta_3^1 h_1 \quad 0 \leq \theta_i^1 < 1, \quad i = 1, 2, 3$$

and

$$x_2 = h_2 i_2 + \theta_1^2 h_2, \quad y_2 = h_2 j_2 + \theta_2^2 h_2 \quad t_2 = h_2 l_2 + \theta_3^2 h_2 \quad 0 \leq \theta_i^2 < 1, \quad i = 1, 2, 3.$$

4.4
becomes

$$\frac{1}{h_1^3 h_2^3} \int\int_{\mathbb{R}^2} |\mathbb{K}(\frac{u_1 - i_1 h_1}{h_1}, \frac{u_1 - j_1 h_1}{h_1}) \times \mathbb{K}(\frac{u_2 - i_2 h_2}{h_2}, \frac{u_2 - j_2 h_2}{h_2})\mathbb{K}(\frac{u_1 - l_1 h_1}{h_1}, \frac{u_2 - l_2 h_2}{h_2})|f(u_1, u_2)\, du_1 du_2.$$

Let $\delta_i = \min(\frac{|x_i - y_i|}{2}, \frac{|x_i - t_i|}{2}, \frac{|t_i - y_i|}{2})$ $i = 1, 2$. With the change of variable $v_i = u_i - x_i - \theta_1^i h_i$, $i = 1, 2$ we obtain

$$\begin{aligned}
&\frac{1}{h_1^3 h_2^3} \int\int_{\mathbb{R}^2} |\mathbb{K}(\frac{v_1}{h_1}, \frac{v_2}{h_2})\mathbb{K}(\frac{x_1 - \theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}, \frac{x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2}) \\
&\times \mathbb{K}(\frac{x_1 - \theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}, \frac{t_2 - \theta_1^2 h_2 - t_2 + \theta_3^2 h_2}{h_2})|f(u_1, u_2)\, du_1 du_2 \\
&= \int_{|v_1| \leq \delta_1} \int_{|v_2| \leq \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| \leq \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| \leq \delta_2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int\int\int_{\mathcal{D}_1} \Phi_n(x_1, y_1, t_1)\, dx_1 dy_1 dt_1 \int\int\int_{\mathcal{D}_2} \Phi_n(x_2, y_2, t_2)\, dx_2 dy_2 dt_2 \\
&\leq \int\int\int_{\mathcal{D}_1} \sum_{0 \leq i_1 \neq j_1 \neq l_1 \neq i_1 \leq [\frac{z_1}{h_1}]} \chi_{\Delta_{h_1,i_1} \times \Delta_{h_1,j_1} \times \Delta_{h_1,l_1}}(x_1, y_1, t_1) \\
&\times \int\int\int_{\mathcal{D}_2} \sum_{0 \leq i_2 \neq j_2 \neq l_2 \neq i_2 \leq [\frac{z_2}{h_2}]} \chi_{\Delta_{h_2,i_2} \times \Delta_{h_2,j_2} \times \Delta_{h_2,l_2}}(x_2, y_2, t_2) \\
&\times (\int_{|v_1| \leq \delta_1} \int_{|v_2| \leq \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| \leq \delta_1} \int_{|v_2| > \delta_2} + \int_{|v_1| > \delta_1} \int_{|v_2| \leq \delta_2}).
\end{aligned}$$

The proof is conducted as follow : First consider

$$\int\int\int_{\mathcal{D}_1} \sum_{0\leq i_1\neq j_1\neq l_1\neq i_1\leq [\frac{z_1}{h_1}]} \chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}\times\Delta_{h_1,l_1}}(x_1,y_1,t_1)$$
$$\times\int\int\int_{\mathcal{D}_1} \sum_{0\leq i_2\neq j_2\neq l_2\neq i_2\leq [\frac{z_2}{h_2}]} \chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}\times\Delta_{h_2,2_2}}(x_2,y_2,t_2)$$
$$\times(\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2})$$

where

$$\mathcal{D}_1=\{(x_1,y_1,t_1)\in\Delta_1 : |x_1-y_1||x_1-t_1||t_1-y_1|>0\}$$

and

$$\mathcal{D}_2=\{(x_2,y_2,t_2)\in\Delta_2 : |x_2-y_2||x_2-t_2||t_2-y_2|>0\}.$$

Let $A=\sup_{(x,y)\in[0,z_1]\times[0,z_2]} f(x,y)$. The notations being as in the proof of **Lemma 10** and we suppose $h_i\leq\frac{b_i}{4}$ with $\delta_i=\frac{z_i}{2}$, $i=1,2$. We have, following inequality (4.1)

(4.5)
$$\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2}$$
$$\leq A\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_1}{h_1}\leq u_1\leq\frac{\delta_1}{h_1}}\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_2}{h_2}\leq u_2\leq\frac{\delta_2}{h_2}}|\mathbb{K}(u_1,u_2)|$$
$$\times|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|$$
$$\times|\mathbb{K}(\frac{x_1-\theta_1^1h_1-t_1+\theta_3^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-t_2+\theta_3^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\,du_1du_2.$$

For all (u_1,u_2)

$$\lim_{n\to+\infty}|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|=0.$$

Let us write

$$
\begin{aligned}
&|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\\
&=|\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)\\
&-\mathbb{K}(\frac{2b_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{b_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)\\
&+\mathbb{K}(\frac{2b_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{b_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|(h_1h_2)^{-1}\\
&\leq(\lambda(\frac{2b_1}{h_1},\frac{2b_2}{h_2})\\
&+|\mathbb{K}(\frac{2b_1+x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{b_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|)(h_1h_2)^{-1}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&|\mathbb{K}(\frac{2b_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{b_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|(h_1h_2)^{-1}\\
&=|\frac{2b_1+x_1-y_1+h_1u_1}{h_1}||\frac{2b_2+x_2-y_2+h_2u_2}{h_2}|\\
&\times|\mathbb{K}(\frac{2b_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{b_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}+u_2)|\\
&\times\frac{1}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}.
\end{aligned}
$$

We have

$$
\begin{aligned}
TT &= |\frac{2b_1 + x_1 - y_1 + h_1u_1}{h_1}||\frac{2b_2 + x_2 - y_2 + h_2u_2}{h_2}| \\
&= |\frac{2b_1 + x_1 - y_1 + h_1u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1} + \frac{\theta_1^1 h_1 - \theta_2^1 h_1}{h_1}| \\
&\times |\frac{2b_2 + x_2 - y_2 + h_2u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2} + \frac{\theta_2^2 h_2 - \theta_2^2 h_2}{h_2}| \\
&\leq |\frac{2b_1 + x_1 - y_1 + h_1u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}| + |\frac{\theta_1^1 h_1 - \theta_2^1 h_1}{h_1}| \\
&\times |\frac{2b_2 + x_2 - y_2 + h_2u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| + |\frac{\theta_1^2 h_2 - \theta_2^2 h_2}{h_2})| \\
&\leq |\frac{2b_1 + x_1 - y_1 + h_1u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}| \\
&\times |\frac{2b_2 + x_2 - y_2 + h_2u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| \\
&+ |\frac{2b_1 + x_1 - y_1 + h_1u_1 - \theta_1^1 h_1 + \theta_2^1 h_1}{h_1}| \\
&+ |\frac{2b_2 + x_2 - y_2 + h_2u_2 - \theta_1^2 h_2 + \theta_2^2 h_2}{h_2}| + 1.
\end{aligned}
$$

Let $B = \sup_{y \in \mathbb{R}} |y||K(y)|$ and $C = \sup_{y \in \mathbb{R}} |K(y)|$, then we have

$$
\begin{aligned}
&|\frac{2b_1 + x_1 - y_1 + h_1u_1}{h_1}||\frac{2b_2 + x_2 - y_2 + h_2u_2}{h_2}||K(\frac{2z_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1 h_1}{h_1} + u_1) \\
&\qquad \times K(\frac{z_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2} + u_2)| \\
&\qquad\qquad \leq B^2 + 2BC + C.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&|K(\frac{2z_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1 h_1}{h_1} + u_1)K(\frac{z_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2 h_2}{h_2} + u_2)|(h - 1h_2)^{-1} \\
&\qquad \leq \frac{B^2 + 2BC + C}{|2b_1 + x_1 - y_1 + h_1u_1||2b_2 + x_2 - y_2 + h_2u_2|}.
\end{aligned}
$$

Therefore

$$|\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|$$
$$\leq (\lambda(\frac{2b_1}{h_1},\frac{2b_2}{h_2}) + \frac{B^2+2BC+C}{|2b_1+x_1-y_1+h_1u_1||2b_2+x_2-y_2+h_2u_2|}.$$

Similarly

$$|\mathbb{K}(\frac{x_1-\theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2 - t_2 + \theta_3^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}|$$
$$\leq (\lambda(\frac{2b_1}{h_1},\frac{2b_2}{h_2}) + \frac{B^2+2BC+C}{|2b_1+x_1-t_1+h_1u_1||2b_2+x_2-t_2+h_2u_2|}.$$

We may conclude that for $h_i \quad i=1,2$ small enough

$$\begin{aligned}\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2} &\leq \frac{A}{|2b_1+x_1-y_1+h_1u_1||2b_2+x_2-y_2+h_2u_2|}\\ &\times \int\int_{\mathbb{R}^2}|\mathbb{K}(u_1,u_2)|(B^2+2BC+C)\,du_1du_2\\ &< \frac{AD}{|2b_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\ &\leq \frac{AD}{|2b_1+x_1-y_1+h_1u_1||2b_2+x_2-y_2+h_2u_2|}\\ &\leq \frac{AD}{(2b_1+x_1-y_1+h_1u_1)(2z_2+x_2-y_2+h_2u_2)},\end{aligned}$$

D being the finite of $\int\int_{\mathbb{R}^2}|\mathbb{K}(u_1,u_2)|(B^2+2BC+4C)\,du_1du_2$.

Then, we have

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2} \leq \frac{AD}{(2b_1+x_1-y_1+h_1u_1)(2b_2+x_2-y_2+h_2u_2)}+O(h_1h_2).$$

Since $-\delta_i \leq h_iu_i \leq \delta_i$ we have
$\frac{b_i}{4} \leq 2b_i+x_i-y_i+h_iu_i$, and $\frac{b_i}{4} \leq 2b_i+x_i-t_i+h_iu_i \quad i=1,2$.
Hence

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|\leq\delta_2} \leq \frac{16AD}{b_1^2b_2^2}+O(h_1h_2).$$

This inequality is true for all
$(x_i,y_i,t_i)\in\{(x_i,y_i,t_i)\in\Delta_i : |x_i-y_i||x_i-t_i||t_i-y_i|>0\} \quad i=1,2.$

For all (u_1, u_2) we have from the proof of the **Lemma 10**

$$\sup_{(\theta_1^1,\theta_2^1,\theta_3^1)\in[0,1]^3}\ \sup_{(\theta_1^2,\theta_2^2,\theta_3^2)\in[0,1]^3}\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_1}{h_1}\le u_1\le\frac{\delta_1}{h_1}}(u_1)\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_2}{h_2}\le u_2\le\frac{\delta_2}{h_2}}(u_2)|$$
$$\times\,\mathbb{K}(u_1,u_2)||\mathbb{K}(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|$$
$$\times\,|\mathbb{K}(\frac{x_1-\theta_1^1h_1-t_1+\theta_3^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-t_2+\theta_3^2h_2}{h_2}+u_2)(h_1h_2)^{-1}|\,du_1du_2$$

tends to zero as $n\to+\infty$ except on the complement in $\Delta=\Delta_1\times\Delta_2$ of
$\{((x_1,y_1,t_1),(x_2,y_2,t_2))\in\Delta_1\times\Delta_2, x_i\neq y_i\neq t_i, i=1,2\}$,
which is $dxdydt$-measure null.
Therefore, by Lebesgue -dominated convergence theorem

$$\lim_{n\to+\infty}\int\int\int_{\Delta_1}\sum_{0\le i_1\neq j_1\neq l_1\neq i_1\le[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}\times\Delta_{h_1,l_1}}(x_1,y_1,t_1)$$
$$\times\int\int\int_{\Delta_2}\sum_{0\le i_2\neq j_2\neq l_2\neq i_2\le[\frac{z_2}{h_2}]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}\times\Delta_{h_2,l_2}}(x_2,y_2,t_2)$$
$$\times\,(\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2})\,dx_1dy_1dt_1dx_2dy_2dt_2$$
$$=\int\int\int_{\Delta_1}\int\int\int_{\Delta_2}\lim_{n\to+\infty}(\int_{|v_1|\le\delta_1}\int_{|v_2|\le\delta_2})\,dx_1dy_1ddt_1dx_2y_2dt_2=0. \tag{4.6}$$

Then consider

$$\int_{\mathcal{D}_1}\sum_{0\le i_1\neq j_1\neq l_1\neq i_1\le[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}\times\Delta_{h_1,l_1}}(x_1,y_1,t_1)$$
$$\times\int_{\mathcal{D}_2}\sum_{0\le i_2\neq j_2\neq l_2\neq i_2\le[\frac{z_2}{h_2}]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}\times\Delta_{h_2,2_2}}(x_2,y_2,t_2)$$
$$\times\,(\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}), \tag{4.7}$$

where

$$\mathcal{D}_1=\{(x_1,y_1,t_1)\in\Delta_1:|x_1-y_1||x_1-t_1||t_1-y_1|>0\}$$

and

$$\mathcal{D}_2 = \{(x_2, y_2, t_2) \in \Delta_2 : |x_2 - y_2||x_2 - t_2||t_2 - y_2| > 0\}.$$

We use expression (3) in the second part of the proof of **Lemma 10** by analogous reasoning we obtain

(4.8)
$$\begin{aligned}
&\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} \\
&\quad \leq \frac{1}{\delta_1}\frac{1}{\delta_2}\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}|\frac{v_1}{h_1}\frac{v_2}{h_2}||\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})| \times \int\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}|(h_1h_2)^{-1} \\
&\quad \times \mathbb{K}(\frac{v_1 + x_1 - \theta_1^1 h_1 - y_1 + \theta_2^1 h_1}{h_1}, \frac{v_2 + x_2 - \theta_1^2 h_2 - y_2 + \theta_2^2 h_2}{h_2}) \\
&\quad \times (h_1h_2)^{-1}\mathbb{K}(\frac{v_1 + x_1 - \theta_1^1 h_1 - t_1 + \theta_3^1 h_1}{h_1}, \frac{v_2 + x_2 - \theta_1^2 h_2 - t_2 + \theta_3^2 h_2}{h_2} \\
&\quad \times |f(x_1 + v_1 - \theta_1^1 h_1, x_2 + v_2 - \theta_1^2 h_2)\, du_1 du_2.
\end{aligned}$$

Making the change of variable $u_i = x_i + v_i - \theta_1^i h_i, i = 1, 2$, the integral of the right-hand side of (4.9) does not exceed

$$\begin{aligned}
&\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{u_1 - y_1 - \theta_2^1 h_1}{h_1}, \frac{u_2 - y_2 + \theta_2^2 h_2}{h_2}) \\
&\qquad\qquad \times (h_1h_2)^{-1}\mathbb{K}(\frac{u_1 - t_1 + \theta_3^1 h_1}{h_1}, \frac{u_2 - t_2 + \theta_3^2 h_2}{h_2}|f(u_1, u_2)\, du_1 du_2.
\end{aligned}$$

Let $\delta_i \geq h_i^{\varepsilon}, 0 < \varepsilon < \dfrac{1}{2}$, we have

(4.9)
$$\begin{aligned}
&\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2} \\
&\quad \leq \sup_{|v_1|>h_1^{\varepsilon}}\sup_{|v_2|>h_2^{\varepsilon}}\frac{|v_1|}{h_1^{1+\varepsilon}}\frac{|v_2|}{h_2^{1+\varepsilon}}|\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})| \sup_{(\theta_2^1,\theta_3^1)\in[0,1]^2}\sup_{(\theta_2^2,\theta_3^2)\in[0,1]^2}\int_{\mathbb{R}^2}|(h_1h_2)^{-1} \\
&\quad \times \mathbb{K}(\frac{u_1 - y_1 - \theta_2^1 h_1}{h_1}, \frac{u_2 - y_2 + \theta_2^2 h_2}{h_2})(h_1h_2)^{-1} \\
&\quad \times \mathbb{K}(\frac{u_1 - t_1 + \theta_3^1 h_1}{h_1}, \frac{u_2 - t_2 + \theta_3^2 h_2}{h_2}|f(u_1, u_2)\, du_1 du_2.
\end{aligned}$$

When writing

$$\int\int_{\mathbb{R}^2} |(h_1h_2)^{-1}\mathbb{K}(\frac{u_1-y_1-\theta_2^1h_1}{h_1}, \frac{u_2-y_2+\theta_2^2h_2}{h_2})$$
$$\times (h_1h_2)^{-1}\mathbb{K}(\frac{u_1-t_1+\theta_3^1h_1}{h_1}, \frac{u_2-t_2+\theta_3^2h_2}{h_2}|f(u_1,u_2)\,du_1du_2$$
$$\leq \int_{|v_1|\leq\bar{\delta}_1}\int_{|v_2|\leq\bar{\delta}_2} + \int_{|v_1|>\bar{\delta}_1}\int_{|v_2|>\bar{\delta}_2} + \int_{|v_1|\leq\bar{\delta}_1}\int_{|v_2|>\bar{\delta}_2} + \int_{|v_1|>\bar{\delta}_1}\int_{|v_2|\leq\bar{\delta}_2},$$

with the change of variable $v_i = u_i - t_i + \theta_3^i h_i$ and $\bar{\delta}_i = \frac{|t_i-y_i|}{2}, i=1,2$ we proved by the proof of **Theorem 5** Ciss *et al.* (2014) that , for $\bar{\delta}_i > h_i^{\varepsilon}$

$$|\int_{|v_1|\leq\bar{\delta}_1}\int_{|v_2|\leq\bar{\delta}_2|} \leq \frac{16AD}{\sqrt{b_1b_2}} + O(h_1h_2), \qquad y_i \neq t_i$$

and

$$|\int_{|v_1|>\bar{\delta}_1}\int_{|v_2|>\bar{\delta}_2}| \leq \sup_{|v_1|>h_1^{\varepsilon}}\sup_{|v_2|>h_2^{\varepsilon}} \frac{|v_1|}{h_1^{1+\varepsilon}}\frac{|v_2|}{h_2^{1+\varepsilon}}|\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}$$
$$\times \mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|f(u_1+y_1-\theta_2^1h_1, u_2+y_2-\theta_2^2h_2)\,du_1du_2.$$

Since under the hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$

$$\int\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathbb{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})|f(u_1+y_1-\theta_2^1h_1, u_2+y_2-\theta_2^2h_2)\,du_1du_2$$

is bounded, we deduce that the right-hand side of 4.9 is bounded except over the two-dimensional rectangle $\bar{\delta}_i = 0; \quad i=1,2$. Hence

$$\int_{|v_1|>\delta_1}\int_{|v_2|>\delta_2}$$

is bounded except on $\{((x_1,y_1,t_1),(x_2,y_2,t_2)) \in \Delta_1\times\Delta_2 : \delta_i = 0\}$ which is $(dx_1dy_1dt_1)(dx_2dy_2dt_2)$-measure nul.
$\Delta_i, i=1,2$ being bounded, hypothesis $\mathbf{H_7}$ implies that the integral in (4.7) tends to zero as $n\to+\infty$.
Consider $\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot)$ by Fubini's theorem, we obtain

$$\int_{|v_1|\leq\delta_1}\int_{|v_2|>\delta_2}(\cdot) = \int_{|v_1|\leq\delta_1}(\cdot)\int_{|v_2|>\delta_2}(\cdot) \to 0*0=0, \quad n\to+\infty$$

according the **Theorem 2.3** in the unidimensional case Dia (2009).

Similarly, we have

$$\int_{|v_1|>\delta_1}\int_{|v_2|\leq\delta_2}(\cdot)=\int_{|v_1|>\delta_1}(\cdot)\int_{|v_2|\leq\delta_2}(\cdot)\to 0*0=0,\quad n\to+\infty.$$

Consequently

$$\lim_{n\to+\infty}\int\int_{\Delta_1}\int\int_{\Delta_2}\int\int_{\mathbb{R}^2}\to 0,\quad n\to+\infty$$

since Δ_1 and Δ_2 are bounded. The proof of the lemma is complete.

Proof of Theorem 17. For of simplicity in the notations, $\mathbb{K}$ stands for $\mathbb{K}_\nu$ and P_n for P_n^ν.

It is sufficient to prove:

On the one hand
1) $\sqrt{n}(\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2)-P(z_1,z_2,\alpha_1,\alpha_2))\to n$ as $n\to+\infty$

and on the other hand

$$2)\ \frac{P_n(z_1,z_2,\alpha_1,\alpha_2)-\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))}{\sqrt{\mathbb{V}(P_n(z_1,z_2,\alpha_1,\alpha_2))}}\to N(0,1)\quad\text{as}\quad n\to+\infty.$$

Let (x_0,y_0) the infinum of the support of f.
Let's firstly observe that

$$\lim_{(z_1,z_2)\to(0,0)}\frac{F(z_1,z_2)-F(z_1,0)-F(0,z_2)+F(0,0)}{z_1z_2}=f(0,0)\quad\text{for}\quad(x_0,y_0)=(0,0)$$

Therefore $\dfrac{F(z_1,z_2)}{z_1z_2}$ is bounded.

Let $\bar{\Delta}_{h_1,i}=\Delta_{h_1,i}\cap[0,z_1];\quad \bar{\Delta}_{h_2,j}=\Delta_{h_2,j}\cap[0,z_2]$ and χ_B the indicator function of the two-dimensional rectangle $B=B_1\times B_2$. Let $(z_1,z_2)\in[0,b_1]\times[0,b_2]$. We have

$$\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2))=\sum_{i=1}^{[\frac{z_1}{h_1}]}\sum_{j=1}^{[\frac{z_2}{h_2}]}\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}\int_{\mathbb{R}^2}\mathbb{K}(u_1,u_2)$$
$$\times f(u_1h_1-ih_1,u_1h_2-jh_2)\,du_1du_2;$$

which can be written in the following form:

$$\begin{aligned}(4.10)\quad & \int_0^{z_1}\int_0^{z_2}\sum_{i=1}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=1}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\\ &\times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1-ih_1,u_2h_2-jh_2)\,du_1du_2dxdy\\ &+(h_1([\frac{z_1}{h_1}]+1)-z_1)(h_2([\frac{z_2}{h_2}]+1)-z_2)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\\ &\times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2})\,du_1du_2.\end{aligned}$$

We have

(4.11)

$$\begin{aligned}&\sup_{(z_1,z_2)\in\mathbb{R}^2}|(h_1([\frac{z_1}{h_1}]+1)-z_1)(h_2([\frac{z_2}{h_2}]+1)-z_2)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\\ &\times\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2})\,du_1du_2|\\ &\leq h_1h_2\sup_{(x,y)\in\mathbb{R}^2}f(x,y)\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2.\end{aligned}$$

Since we have $|h_i([\frac{z_i}{h_i}]+1)-z_i|\leq h_i\quad i=1,2.$

Because of $\mathbf{H_2}$ we can write
(4.12)

$$P(z_1,z_2,\alpha_1,\alpha_2)=\int_0^{z_1}\int_0^{z_2}\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\mathbb{K}(u_1,u_2)\,du_1du_2f(x,y)\,dxdy.$$

Let $(x,y) \in \bar{\Delta}_{h_1,i} \times \bar{\Delta}_{h_2,j}$. By considering the terms (4.10) and (4.12) we get (4.13)

$$\begin{aligned}
&|\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}\int\int_{-\infty}^{+\infty}\mathbb{K}(u_1,u_2)f(u_1h_1-ih_1,u_1h_2-jh_2)\,du_1du_2\\
&\quad -\left(1-\frac{x}{z_1}\right)^{\alpha_1}\left(1-\frac{y}{z_2}\right)^{\alpha_2}\mathbb{K}(u_1,u_2)\,du_1du_2 f(x,y)|\\
&\quad = |\int\int_{-\infty}^{+\infty}\Big[\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}f(u_1h_1-ih_1,u_2h_2-jh_2)\\
&\quad -\left(1-\frac{x}{z_1}\right)^{\alpha_1}\left(1-\frac{y}{z_2}\right)^{\alpha_2}f(x,y)\Big]\mathbb{K}(u_1,u_2)\,du_1du_2|\\
&\quad \leq \int\int_{-\infty}^{+\infty}|\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}\\
&\quad -\left(1-\frac{x}{z_1}\right)^{\alpha_1}\left(1-\frac{y}{z_2}\right)^{\alpha_2}|f(x,y)|\mathbb{K}(u_1,u_2)|\,du_1du_2\\
&\quad +\int\int_{-\infty}^{+\infty}|\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}|\\
&\quad \times|f(u_1h_1-ih_1,u_1h_2-jh_2)-f(x,y)||\mathbb{K}(u_1,u_2)|\,du_1du_2
\end{aligned}$$

Let $(x,y) \in \Delta_{h_1,i} \times \Delta_{h_2,j}$ by the first order Taylor formula applied to the function we have

$$g(x,y) = \left(1-\frac{x}{z_1}\right)^{\alpha_1}\left(1-\frac{y}{z_2}\right)^{\alpha_2},$$

for $c_1 \in]h_1 i, x[$ and $c_2 \in]h_2 j, y[$

$$\begin{aligned}
&\left|\left(1-\frac{ih_1}{z_1}\right)^{\alpha_1}\left(1-\frac{jh_2}{z_2}\right)^{\alpha_2}-\left(1-\frac{x}{z_1}\right)^{\alpha_1}\left(1-\frac{y}{z_2}\right)^{\alpha_2}\right|\\
&\qquad = \left|\left(1-\frac{c_1}{z_1}\right)^{\alpha_1-1}\frac{\alpha_1}{z_1}\left(1-\frac{c_2}{z_2}\right)^{\alpha_2}\Big(ih_1-x\Big)\right.\\
&\qquad \left.+\left(1-\frac{c_1}{z_1}\right)^{\alpha_1}\frac{\alpha_2}{z_2}\left(1-\frac{c_2}{z_2}\right)^{\alpha_2-1}\Big(jh_2-y\Big)\right|\\
&\qquad \leq 2\left(\frac{\alpha_1h_1}{z_1}+\frac{\alpha_2h_2}{z_2}\right) = 2\left(\frac{z_2\alpha_1h_1+z_1\alpha_2h_2}{z_1z_2}\right).
\end{aligned}$$

Therefore, denoting by $I_1^{i,j}(x,y)$ the first integral of the right hand-side of (4.13) and

$$I_1(x,y) = \sum_{i=1}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=1}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)I_1^{i,j}(x,y),$$

we have

(4.14)

$$\int_0^{z_1}\int_0^{z_2} I_1(x,y)\,dxdy$$
$$\le \frac{2(\alpha_1 z_2+\alpha_2 z_1)\left\|\Big(h_1,h_2\Big)\right\|}{z_1 z_2}\int_0^{z_1}\int_0^{z_2}\Big(\int\int_{-\infty}^{+\infty} f(x,y)|\mathbb{K}(u_1,u_2)|\,du_1du_2\Big)\,dxdy$$
$$= 2(\alpha_1 z_2+\alpha_2 z_1)\left\|\Big(h_1,h_2\Big)\right\|\Big(\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2\Big)\frac{F(z_1,z_2)}{z_1 z_2}.$$

Denoting by $I_2^{i,j}(x,y)$ the second integral of (4.13) and

$$I_2(x,y)=\sum_{i=1}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=1}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)I_2^{i,j}(x,y),$$

we have

$$I_2^{i,j}(x,y)\le\int\int_{-\infty}^{+\infty}|f(u_1h_1-ih_1,u_2h_2-jh_2)-f(ih_1,jh_2)||\mathbb{K}(u_1,u_2)|\,du_1du_2$$
$$+\int\int_{-\infty}^{+\infty}|f(ih_1,jh_2)-f(x,y)||\mathbb{K}(u_1,u_2)|\,du_1du_2.$$

Therefore

(4.15)

$$\int_0^{z_1}\int_0^{z_2} I_2(x,y)\,dxdy$$
$$\le\sum_{i=1}^{[\frac{z_1}{h_1}]}\sum_{j=1}^{[\frac{z_2}{h_2}]}h_1h_2\int\int_{-\infty}^{+\infty}$$
$$\times|f(u_1h_1+x-\theta_1^1h_1,u_2h_2+y-\theta_2^2h_2)-f(x,y)||\mathbb{K}(u_1,u_2)|\,du_1du_2$$
$$+\varepsilon\int\int_{-\infty}^{+\infty}|\mathbb{K}(u_1,u_2)|\,du_1du_2dxdy$$
$$\le C\,\|(h_1,h_2)\|^{\gamma}\,z_1z_2\int_0^{z_1}\int_0^{z_2}\Big(\int_{\mathbb{R}^2}(\|(u_1,u_2)\|^{\gamma}+1)|\mathbb{K}(u_1,u_2)|\,du_1du_2dxdy$$

Because of $\mathbf{C_3}$ we have, on the one hand

$$\begin{aligned}
|f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)| &\leq C \left\|(h_1(u_1-\theta_1), h_2(u_2-\theta_2))\right\|^{\gamma} \\
&\leq C \left\|(h_1u_1, h_2u_2) - (h_1\theta_1, h_2\theta_2)\right\|^{\gamma} \\
&\leq C \left\|(h_1u_1, h_1u_1) - (h_1, h_2)\right\|^{\gamma} \\
&\leq C \left\|(h_1u_1, h_2u_2)\right\| + \left\|(h_1, h_2)\right\|)^{\gamma} \\
&= C \left\|(h_1, h_2)\right\|^{\gamma} (\left\|(u_1, u_2)\right\| + 1)^{\gamma}
\end{aligned}$$

since

$$\left\|(h_1u_1, h_2u_2)\right\| \leq \left\|(h_1, h_2)\right\| \left\|(u_1, u_2)\right\|$$

and on the other hand, the inequality (4.11) is bounded by

$$\left\|(h_1, h_2)\right\|^2 \sup_{(x,y)\in\mathbb{R}^2} f(x,y) \int\int_{-\infty}^{+\infty} |\mathbb{K}(u_1,u_2)|\, du_1du_2$$

since $h_1^2 + h_2^2 \geq 2h_1h_2$.

Moreover

$$\int_0^{z_1}\int_0^{z_2} I_1(x,y)\, dxdy \leq 2\left\|(h_1, h_2)\right\| (\alpha_1z_2 + \alpha_2z_1)(\int_{\mathbb{R}^2} |\mathbb{K}(u_1,u_2)|\, du_1du_2)\frac{F(z_1,z_2)}{z_1z_2}.$$

Since $h_i \leq \left\|(h_1, h_2)\right\|, i = 1, 2$. Consequently, we get

$$\begin{aligned}
&\sqrt{n}(\mathbb{E}(P_n(z_1,z_2,\alpha_1,\alpha_2) - P(z_1,z_2,\alpha_1,\alpha_2)) \leq \left\|(h_1, h_2)\right\|^{\gamma}\sqrt{n} \\
&\times \Big\{Cz_1z_2\int_0^{z_1}\int_0^{z_2}(\int_{\mathbb{R}^2}(\left\|(u_1,u_2)\right\|^{\gamma}+1)|\mathbb{K}(u_1,u_2)|\, du_1du_2)dxdy \\
&+ \left\|(h_1, h_2)\right\|^{2-\gamma} \sup_{(x,y)\in\mathbb{R}^2} f(x,y)\int\int_{-\infty}^{+\infty} |\mathbb{K}(u_1,u_2)|\, du_1du_2 \\
&+ 2\left\|(h_1, h_2)\right\|^{1-\gamma} (\alpha_1z_2+\alpha_2z_1)(\int\int_{-\infty}^{+\infty} |\mathbb{K}(u_1,u_2)|\, du_1du_2)\frac{F(z_1,z_2)}{z_1z_2}\Big\}.
\end{aligned}$$

The integrals in the braces exist. Hence the first part is proved. For the second part, define

$$U_i = \frac{1}{n} \sum_{0 \leq l_1 \leq [\frac{z_1}{h_1}]} (1 - \frac{l_1 h_1}{z_1})^{\alpha_1} K(\frac{X_i - l_1 h_1}{h_1}) \sum_{0 \leq l_2 \leq [\frac{z_2}{h_2}]} (1 - \frac{l_2 h_2}{z_2})^{\alpha_2} K(\frac{Y_i - l_2 h_2}{h_2}),$$

$i = 1...n, \quad \mu_i = \mathbb{E}(U_i)$ and $\beta_i = \mathbb{E}(|U_i - \mu_i|^3)$.
Let $B_n = (\sum_{i=1}^{n} \beta_i)^{\frac{1}{3}}$. We shall obtain the second statememt 2) if, by Liapounov's theorem, we prove

$$\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z_1, z_2, \alpha_1, \alpha_2))}} \to 0 \quad \text{as}\, n \to +\infty.$$

Now in the following of the proof, consider $n^3\mathbb{E}(U_i^3)$. If we develop it, we have

$$
\begin{aligned}
&n^3\mathbb{E}(U_i^3)\\
&=\mathbb{E}\Big[\Big(\sum_{0\leq l_1\leq[\frac{z_1}{h_1}]}(1-\frac{l_1h_1}{z_1})^{\alpha_1}K(\frac{X_i-l_1h_1}{h_1})\Big)^3\Big(\sum_{0\leq l_2\leq[\frac{z_2}{h_2}]}(1-\frac{l_2h_2}{z_2})^{\alpha_2}K(\frac{Y_i-l_2h_2}{h_2})\Big)^3\Big]\\
&=\mathbb{E}\Big[\Big\{\sum_{0\leq l_1\leq[\frac{z_1}{h_1}]}(1-\frac{l_1h_1}{z_1})^{3\alpha_1}K^3(\frac{X_i-l_1h_1}{h_1})\\
&+\sum_{0\leq l_1\neq i_1\leq[\frac{z_1}{h_1}]}(1-\frac{l_1h_1}{z_1})^{2\alpha_1}K^2(\frac{X_i-l_1h_1}{h_1})(1-\frac{i_1h_1}{z_1})^{\alpha_1}K(\frac{X_i-i_1h_1}{h_1})\\
&+\sum_{0\leq l_1\neq i_1\neq j_1\neq l_1\leq[\frac{z_1}{h_1}]}(1-\frac{l_1h_1}{z_1})^{\alpha_1}K(\frac{X_i-l_1h_1}{h_1})(1-\frac{i_1h_1}{z_1})^{\alpha_1}\\
&\times K(\frac{X_i-i_1h_1}{h_1})(1-\frac{j_1h_1}{z_1})^{\alpha_1}K(\frac{X_i-j_1h_1}{h_1})\Big\}\\
&\times\Big\{\sum_{0\leq l_2\leq[\frac{z_2}{h_2}]}(1-\frac{l_2h_2}{z_2})^{3\alpha_2}K^3(\frac{X_j-l_2h_2}{h_2})\\
&+\sum_{0\leq l_2\neq i_2\leq[\frac{z_2}{h_2}]}(1-\frac{l_2h_2}{z_2})^{2\alpha_2}K^2(\frac{Y_i-l_2h_2}{h_2})(1-\frac{i_2h_2}{z_2})^{\alpha_2}K(\frac{Y_i-i_2h_2}{h_2})\\
&+\sum_{0\leq l_2\neq i_2\neq j_2\neq l_2\leq[\frac{z_2}{h_2}]}(1-\frac{l_2h_2}{z_2})^{\alpha_2}K(\frac{Y_i-l_2h_2}{h_2})(1-\frac{i_2h_2}{z_2})^{\alpha_2}\\
&\times K(\frac{Y_i-i_2h_2}{h_2})(1-\frac{j_2h_2}{z_2})^{\alpha_2}K(\frac{Y_i-j_2h_2}{h_2})\Big\}\Big]\\
&=\mathbb{E}\Big[m_1m_4+m_1m_5+m_1m_6+m_2m_4+m_2m_5+m_2m_6+m_3m_4+m_3m_5+m_3m_6\Big].
\end{aligned}
$$

with

$$
\begin{aligned}
m_1 &:= \sum_{0\leq l_1\leq [\frac{z_1}{h_1}]} (1-\frac{l_1h_1}{z_1})^{3\alpha_1} K^3(\frac{X_i-l_1h_1}{h_1}) \\
m_2 &:= \sum_{0\leq l_1\neq i_1\leq [\frac{z_1}{h_1}]} (1-\frac{l_1h_1}{z_1})^{2\alpha_1} K^2(\frac{X_i-l_1h_1}{h_1})(1-\frac{i_1h_1}{z_1})^{\alpha_1} K(\frac{X_i-i_1h_1}{h_1}) \\
m_3 &:= \sum_{0\leq l_1\neq i_1\neq j_1\neq l_1\leq [\frac{z_1}{h_1}]} (1-\frac{l_1h_1}{z_1})^{\alpha_1} K(\frac{X_i-l_1h_1}{h_1})(1-\frac{i_1h_1}{z_1})^{\alpha_1} \\
&\times \quad K(\frac{X_i-i_1h_1}{h_1})(1-\frac{j_1h_1}{z_1})^{\alpha_1} K(\frac{X_i-j_1h_1}{h_1}) \\
m_4 &:= \sum_{0\leq l_2\leq [\frac{z_2}{h_2}]} (1-\frac{l_2h_2}{z_2})^{3\alpha_2} K^3(\frac{X_j-l_2h_2}{h_2}) \\
m_5 &:= \sum_{0\leq l_2\neq i_2\leq [\frac{z_2}{h_2}]} (1-\frac{l_2h_2}{z_2})^{2\alpha_2} K^2(\frac{Y_i-l_2h_2}{h_2})(1-\frac{i_2h_2}{z_2})^{\alpha_2} K(\frac{Y_i-i_2h_2}{h_2}) \\
m_6 &:= \sum_{0\leq l_2\neq i_2\neq j_2\neq l_2\leq [\frac{z_2}{h_2}]} (1-\frac{l_2h_2}{z_2})^{\alpha_2} K(\frac{Y_i-l_2h_2}{h_2})(1-\frac{i_2h_2}{z_2})^{\alpha_2} \\
&\times \quad K(\frac{Y_i-i_2h_2}{h_2})(1-\frac{j_2h_2}{z_2})^{\alpha_2} K(\frac{Y_i-j_2h_2}{h_2}).
\end{aligned}
$$

We have

$$\mathbb{E}(m_1m_4) \leq \sum_{0\leq l_1\leq[\frac{z_1}{h_1}]}\ \sum_{0\leq l_2\leq[\frac{z_2}{h_2}]} \int_{\mathbb{R}^2} |K^3(\frac{u_1-l_1h_1}{h_1})||K^3(\frac{u_2-l_2h_2}{h_2})|f(u_1,u_2)\,du_1du_2$$
$$\to \Big(\int_{\mathbb{R}^2}|\mathbb{K}^3(y_1,y_2)|\,dy_1dy_2\Big)F(z_1,z_2) \quad \text{as} \quad n\to+\infty,$$

because of the **Corollary 1** Ciss *et al.* (2014) (with $(\alpha_1,\alpha_2)=(0,0)$, $\mathbb{K}^3$ replace by $|\mathbb{K}^3|$).

By Fubini, we have

$$\begin{aligned}
\mathbb{E}(m_1m_5) &\leq \Big\{\int_{\mathbb{R}}|K^3(\frac{u_1-l_1h_1}{h_1})|f_1(u_1)\,du_1\Big\}\\
&\quad\times\Big\{\sum_{0\leq l_2\neq i_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K^2(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})|f_2(u_2)\,du_2\Big\}\\
&\leq \Big\{\int_{\mathbb{R}}|K^3(\frac{u_1-l_1h_1}{h_1})|f_1(u_1)\,du_1\Big\}\\
&\quad\times\Big\{\sup_{y_2\in\mathbb{R}}|K(y_2)|\sum_{0\leq l_2\neq i_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})|f_2(u_2)\,du_2\Big\}.
\end{aligned}$$

Yet

$$\int_{\mathbb{R}}|K^3(\frac{u_1-l_1h_1}{h_1})|f_1(u_1)\,du_1 \to \Big(\int_{\mathbb{R}}|K^3(y_1)|\,dy_1\Big)F_1(z_1) \quad \text{as} \quad n\to+\infty$$

and because of **Theorem 2.5** Dia (2008) in the unidimensional we have

$$\sup_{y_2\in\mathbb{R}}|K(y_2)|\sum_{0\leq l_2\neq i_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})|f_2(u_2)\,du_2\to 0$$

as $n\to+\infty$.

Still, by Fubini, we have

$$\begin{aligned}
\mathbb{E}(m_1m_6) \quad &\leq \quad \Big\{\sum_{0\leq l_1\neq i_1\leq[\frac{z_1}{h_1}]}\int_{\mathbb{R}}|K^3(\frac{u_1-l_1h_1}{h_1})|f_1(u_1)\,du_1\Big\}\\
&\quad\times\quad\Big\{\sum_{0\leq l_2\neq i_2\neq j_2\neq l_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})K(\frac{u_2-j_2h_2}{h_2})|f_2(u_2)\Big\}
\end{aligned}$$

and by **Theorem 2.3** Dia (2009), we have

$$\sum_{0\leq l_2\neq i_2\neq j_2\neq l_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})K(\frac{u_2-j_2h_2}{h_2})|f_2(u_2)\to 0$$

as $n\to+\infty$.

Similarly we have

$$\mathbb{E}(m_2m_4)\leq\Big\{\sup_{y_1\in\mathbb{R}}|K(y_1)|\sum_{0\leq l_1\neq i_1\leq[\frac{z_1}{h_1}]}\int_{\mathbb{R}}|K(\frac{u_1-l_1h_1}{h_1})K(\frac{u_1-i_1h_1}{h_1})|f_1(u_1)\,du_1\Big\}$$
$$\times\Big\{\int_{\mathbb{R}}|K^3(\frac{u_2-l_2h_2}{h_2})|f_2(u_2)\,du_2\Big\},$$

$$\mathbb{E}(m_2m_5)\leq\sup_{(y_1,y_2)\in\mathbb{R}^2}|\mathbb{K}(y_1,y_2)|\sum_{0\leq l_1\neq i_1\leq[\frac{z_1}{h_1}]}\int_{\mathbb{R}}|K(\frac{u_1-l_1h_1}{h_1})K(\frac{u_1-i_1h_1}{h_1})|$$
$$\times\sum_{0\leq l_2\neq i_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})|f(u_1,u_2)\,du_1du_2,$$

which tends to zero as $n\to+\infty$ according to the **Theorem 5** Ciss *et al.* (2014) in the two-dimensional case,

$$\mathbb{E}(m_2m_6)\leq\Big\{\sup_{y_1\in\mathbb{R}}|K(y_1)|\sum_{0\leq l_1\neq i_1\leq[\frac{z_1}{h_1}]}\int_{\mathbb{R}}|K(\frac{u_1-l_1h_1}{h_1})K(\frac{u_1-i_1h_1}{h_1})|f_1(u_1)\Big\}$$
$$\times\Big\{\sum_{0\leq l_2\neq i_2\neq j_2\neq l_2\leq[\frac{z_2}{h_2}]}\int_{\mathbb{R}}|K(\frac{u_2-l_2h_2}{h_2})K(\frac{u_2-i_2h_2}{h_2})K(\frac{u_2-j_2h_2}{h_2})|f_2(u_2)\Big\},$$

which tends to zero as $n\to+\infty$ according to **Theorem 2.5** Dia (2008) and **Theorem 2.3** Dia (2009) unidimensional case,

$$\mathbb{E}(m_3m_4)\leq\Big\{\sum_{0\leq l_1\neq i_1\neq j_1\neq l_1\leq[\frac{z_1}{h_1}]}\int_{\mathbb{R}}|K(\frac{u_1-l_1h_1}{h_1})K(\frac{u_1-i_1h_1}{h_1})K(\frac{u_1-j_1h_1}{h_1})|f_1(u_1)\Big\}$$
$$\times\Big\{\int_{\mathbb{R}}|K^3(\frac{u_2-l_2h_2}{h_2})|f_2(u_2)\,du_2\Big\},$$

which tends to zero as $n \to +\infty$ by **Theorem 2.3** Dia (2009),

$$\mathbb{E}(m_3m_5) \leq \Big\{ \sum_{0 \leq l_1 \neq i_1 \neq j_1 \neq l_1 \leq [\frac{z_1}{h_1}]} \int_{\mathbb{R}} |K(\frac{u_1 - l_1h_1}{h_1})K(\frac{u_1 - i_1h_1}{h_1})K(\frac{u_1 - j_1h_1}{h_1})|f_1(u_1)\Big\}$$

$$\times \Big\{ \sum_{0 \leq l_2 \neq i_2 \leq [\frac{z_2}{h_2}]} \int_{\mathbb{R}} |K(\frac{u_2 - l_2h_2}{h_2})K(\frac{u_2 - i_2h_2}{h_2})|f(u_2)\, du_2\Big\},$$

$$\mathbb{E}(m_3m_5) \leq \Big\{ \sum_{0 \leq l_1 \neq i_1 \neq j_1 \neq l_1 \leq [\frac{z_1}{h_1}]} \int_{\mathbb{R}} |K(\frac{u_1 - l_1h_1}{h_1})K(\frac{u_1 - i_1h_1}{h_1})K(\frac{u_1 - j_1h_1}{h_1})|f_1(u_1)\Big\}$$

$$\times \Big\{ \sup_{y_2 \in \mathbb{R}} |K(y_2)| \sum_{0 \leq l_2 \neq i_2 \leq [\frac{z_2}{h_2}]} \int_{\mathbb{R}} |K(\frac{u_2 - l_2h_2}{h_2})K(\frac{u_2 - i_2h_2}{h_2})|f_2(u_2)\, du_2\Big\},$$

which tends to zero as $n \to +\infty$ according to **Theorem 2.5** Dia (2008) and **Theorem 2.3** Dia (2009) unidimensional case,

$$\mathbb{E}(m_3m_6) \leq \Big\{ \sum_{0 \leq l_1 \neq i_1 \neq j_1 \neq l_1 \leq [\frac{z_1}{h_1}]} \int_{\mathbb{R}} |K(\frac{u_1 - l_1h_1}{h_1})K(\frac{u_1 - i_1h_1}{h_1})K(\frac{u_1 - j_1h_1}{h_1})|\Big\}$$

$$\times \Big\{ \sum_{0 \leq l_2 \neq i_2 \neq j_2 \neq l_2 \leq [\frac{z_2}{h_2}]} \int_{\mathbb{R}} |K(\frac{u_2 - l_2h_2}{h_2})K(\frac{u_2 - i_2h_2}{h_2})K(\frac{u_2 - j_2h_2}{h_2})|\Big\}$$

$$\times f(u_1, u_2)\, du_1du_2,$$

which tends to zero as $n \to +\infty$ according to **Lemma 2** bidimensional case Ciss *et al.* (2014). We have

$$\mathbb{E}(|U_i - \mu_i^3) \leq \mathbb{E}\big\{(|U_i| + |\mu_i|)^3\big\} = \mathbb{E}\big\{|U_i|^3 + |\mu_i|^3 + 3|U_i|^2|\mu_i| + 3|U_i||\mu_i|^2\big\}.$$

Therefore

$$\mathbb{E}(|U_i - \mu_i^3) \leq \mathbb{E}(|U_i|^3) + |\mu_i|^3 + 3\mathbb{E}(|U_i|^2)|\mu_i| + 3\mathbb{E}(|U_i|)|\mu_i|^2.$$

Yet

$$\mathbb{E}(|U_i) \leq \sum_{0 \leq i \leq [\frac{z_1}{h_1}]} \sum_{0 \leq j \leq [\frac{z_2}{h_2}]} \int_{\mathbb{R}^2} |K(\frac{u_1 - ih_1}{h_1})K(\frac{u_2 - jh_2}{h_2})|f(u_1, u_2)\, du_1du_2$$

which tends to $\Big(\int_{\mathbb{R}^2} |\mathbb{K}(y_1, y_2)|\, dy_1dy_2\Big)F(z_1, z_2)$ as $n \to +\infty$ with $\alpha_i = 0, \quad i = 1, 2$.

We also have

$$\begin{aligned}
\mathbb{E}(U_i^2) \leq & \sum_{0\leq i\leq[\frac{z_1}{h_1}]}\ \sum_{0\leq j\leq[\frac{z_2}{h_2}]} \int_{\mathbb{R}^2} |K^2(\frac{u_1-ih_1}{h_1})K^2(\frac{u_2-jh_2}{h_2})| f(u_1,u_2)\, du_1du_2 \\
& + \sum_{0\leq i_1\neq j_1\leq[\frac{z_1}{h_1}]}\ \sum_{0\leq i_2\neq j_2\leq[\frac{z_2}{h_2}]} \int_{\mathbb{R}^2} |K(\frac{u_1-i_1h_1}{h_1}) \\
& \times K(\frac{u_1-j_1h_1}{h_1})K(\frac{u_2-i_2h_2}{h_2})K(\frac{u_2-j_2h_2}{h_2})| f(u_1,u_2)\, du_1du_2 \\
& + \Big\{ \sum_{0\leq i_1\leq[\frac{z_1}{h_1}]} \int_{\mathbb{R}} |K^2(\frac{u_1-i_1h_1}{h_1})| f(u_1)\, du_1 \Big\} \\
& \times \Big\{ \sum_{0\leq i_2\neq j_2\leq[\frac{z_2}{h_2}]} \int_{\mathbb{R}} |K(\frac{u_2-i_2h_2}{h_2})K(\frac{u_2-j_2h_2}{h_2})| f_2(u_2)\, du_2 \Big\} \\
& + \Big\{ \sum_{0\leq i_1\neq j_1\leq[\frac{z_1}{h_1}]} \int_{\mathbb{R}} |K(\frac{u_1-i_1h_1}{h_1})K(\frac{u_1-j_1h_1}{h_1})| f_1(u_1)\, du_1 \Big\} \\
& \times \Big\{ \sum_{0\leq i_2\leq[\frac{z_2}{h_2}]} \int_{\mathbb{R}} |K^2(\frac{u_2-i_2h_2}{h_2})| f_2(u_2)\, du_2 \Big\}
\end{aligned}$$

The first term of the right hand-side of this inequality tends to $\left(\int_{\mathbb{R}^2} |\mathbb{K}^2(y_1,y_2)|\, dy_1dy_2\right) F(z_1,z_2)$ as $n \to +\infty$ with $\alpha_i = 0, i = 1,2.$
The second term of the right hand-side of this inequality tends to zero as $n \to +\infty$ according to **Therem 5** bidimensional case Ciss *et al.* (2014).

The two latter terms of the right hand-side of this inequality tends to zero as $n \to +\infty$ according to Fubini's theorem, **Theorem 2.5** and **Corollary 2.1** Dia (2008) unidimensional case.

Therefore the limits of these terms exist. Hence the condition of Lyapounov holds.

We have

$$n^3\mathbb{E}(|U_i-\mu_i|))^3<+\infty \quad \text{and} \quad B_n=\frac{(\sum_{i=1}^{n} n^3\mathbb{E}(|U_i-\mu_i|^3))^{1/3}}{n}.$$

Let cst be the constant that dominates $n^3\mathbb{E}(|U_i-\mu_i|))^3$ therefore

$$\begin{aligned}\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))}} &= \frac{(\sum_{i=1}^{n} n^3\mathbb{E}(|U_i-\mu_i|^3))^{1/3}}{n\sqrt{\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))}}\\ &= \frac{n^{1/3}cst}{n\sqrt{\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))}}\end{aligned}$$

we have

$$\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))=O(\frac{1}{n})>0.$$

Hence

$$\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z_1,z_2,\alpha_1,\alpha_2))}}=O(\frac{n^{1/3}}{nO(\frac{1}{n})})\cong(\frac{n^{1/3}}{n^{\frac{1}{2}}})$$

which tends to zero as $n\to+\infty$. This completes the proof.

Proof of Theorem 18. We have

$$\sqrt{n}\Big(\mathbb{E}(P_n^{\nu}(z_1,z_2,\alpha_1,\alpha_2))-P(z_1,z_2,\alpha_1,\alpha_2)\Big)\to 0 \quad \text{as} \quad n\to+\infty$$

by **Lemma 2** Ciss *et al.* (2014) bidimensional case. The second part remains unchanged. The proof is complete.

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