

HYPERBOLIC–CONCAVE FUNCTIONS AND HARDY–LITTLEWOOD MAXIMAL FUNCTIONS

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A class of generalized convex functions, the hyperbolic–concave functions, is defined, and used to characterize the collection of Hardy–Littlewood maximal functions. These maximal functions and the probability measures associated with these maximal functions, the maximal probability measures, are used in representations and inequalities within martingale theory. A related collection of minimal probability measures is also characterized, through a class of hyperbolic–concave envelopes.

1. Introduction

In this paper, a new class of functions, the collection of hyperbolic–concave functions, is introduced to give natural characterizations of the collections of Hardy–Littlewood maximal probability measures (p.m.’s) and a related collection of minimal p.m.’s. These collections of probability measures play an important part in martingale theory and other areas of probability theory.

The Hardy–Littlewood maximal p.m.’s can be described as follows. Let μ be any p.m. on \mathbb{R} with distribution function $F_\mu = F$ and left continuous inverse F_μ^{-1} , satisfying $\int_0^\infty x d\mu(x) < \infty$. The Hardy–Littlewood maximal function associated with μ is the function $H^{-1} = H_\mu^{-1}$ defined by

$$H^{-1}(u) := (1 - u)^{-1} \int_u^1 F^{-1}(t) dt.$$

As a random variable on $[0,1]$, with Borel sets and Lebesgue measure, H^{-1} has an associated p.m. μ^* , called the Hardy–Littlewood maximal p.m. associated with μ . These maximal p.m.’s appear in many areas of probability theory (Blackwell and Dubins (1963), Dubins and Gilat (1978), Hardy and Littlewood (1930), Kertz and Rösler (1990)).

In martingale theory the maximal p.m.’s appear in the following characterizations, see Blackwell and Dubins (1963), Dubins and Gilat (1978),

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Kertz and Rösler (1990): for any p.m. μ on \mathbb{R} with $\int |x|d\mu(x) < \infty$,

$$(1.1) \quad \mu^* = \sup_{\prec_s} \{ \nu : \text{there is a martingale } (X_t)_{0 \leq t \leq 1} \text{ satisfying} \\ X_1 \stackrel{\mathcal{D}}{=} \mu \text{ and } \sup_{0 \leq t \leq 1} X_t \stackrel{\mathcal{D}}{=} \nu \}$$

and

$$\{ \nu : \text{there is a martingale } (X_t)_{0 \leq t \leq 1} \text{ satisfying } X_1 \stackrel{\mathcal{D}}{=} \mu \\ \text{and } \sup_{0 \leq t \leq 1} X_t \stackrel{\mathcal{D}}{=} \nu \} = \{ \nu \text{ is a p.m. on } \mathbb{R} : \mu \prec_s \nu \prec_s \mu^* \}.$$

Here \prec_s denotes the stochastic order on p.m.'s, and $Y \stackrel{\mathcal{D}}{=} \mu$ denotes that Y has associated p.m. μ . Two collections of p.m.'s important for the characterizations in (1.1) are

$$(1.2) \quad \mathcal{P}^* := \{ \nu \text{ is a p.m. on } \mathbb{R} : \nu \prec_s \mu^* \\ \text{for some p.m. } \mu \text{ with } \int_0^\infty x d\mu(x) < \infty \}$$

and

$$\mathcal{P}_0^* := \{ \nu \text{ is a p.m. on } \mathbb{R} : \nu = \mu^* \text{ for some p.m. } \mu \\ \text{with } \int_0^\infty x d\mu(x) < \infty \}$$

In Kertz and Rösler (1991a), it was shown that \mathcal{P}^* , the set of p.m.'s dominated by maximal p.m.'s in the stochastic order, equals the set of p.m.'s ν on \mathbb{R} satisfying $\limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0$ (see also Kertz and Rösler (1991b)). The collection \mathcal{P}_0^* is the set of Hardy-Littlewood maximal p.m.'s.

In the main section of this paper, Section 4, connections between maximal p.m.'s and hyperbolic-concave functions are given. In Theorem 4.2, it is shown that \mathcal{P}_0^* is isomorphic to the set of hyperbolic-concave functions $\mu^*[\cdot, \infty)$ associated with p.m.'s in \mathcal{P}^* . Theorem 4.3 shows that maximal p.m.'s can be expressed in terms of their 'hyperbolic derivatives,' as defined in (2.7).

A related collection of 'minimal' p.m.'s is described as follows. For each p.m. $\nu \in \mathcal{P}^*$ (i.e., each p.m. ν on \mathbb{R} with $\limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0$), the minimal p.m. ν_Δ associated with ν is the p.m. on \mathbb{R} satisfying

$$\nu_\Delta := \inf_{\prec_c} \{ \mu \text{ is a p.m. on } \mathbb{R} : \int_0^\infty x d\mu(x) < \infty \text{ and } \nu \prec_s \mu^* \}.$$

Here \prec_c denotes the convex order on right-tail-integrable p.m.'s. The existence of minimal p.m.'s ν_Δ was proved in Theorem 2.4 of Kertz and

Rösler (1991a). Also the importance of these minimal p.m.'s in martingale theory was made explicit there through the following characterizations: for any p.m. ν on \mathbb{R} satisfying $\limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0$ with finite $x_0 := \inf\{z : \nu(-\infty, z] > 0\}$,

$$\nu_\Delta = \inf_{\prec_c} \{ \mu : \text{there is a martingale } (X_t)_{0 \leq t \leq 1} \text{ satisfying} \\ X_1 \stackrel{\mathcal{D}}{=} \mu \text{ and } \sup_{0 \leq t \leq 1} X_t \stackrel{\mathcal{D}}{=} \nu \}$$

and

$$\{ \mu : \text{there is a martingale } (X_t)_{0 \leq t \leq 1} \text{ satisfying } X_1 \stackrel{\mathcal{D}}{=} \mu \text{ and } \sup_{0 \leq t \leq 1} X_t \stackrel{\mathcal{D}}{=} \nu \} \\ = \{ \mu \text{ is a p.m. on } \mathbb{R} : \int x d\mu(x) = x_0 \text{ and } \nu_\Delta \prec_c \mu \prec_s \nu \}.$$

In Theorem 4.4 it is shown that the minimal p.m.'s are characterized by the hyperbolic-concave envelopes of p.m.'s in \mathcal{P}^* .

The concept of hyperbolic-concave functions is defined in Section 2. Through a natural connection to convex functions, given by the map $x \rightarrow 1/x$, properties of convex functions carry over to give desirable properties of hyperbolic-concave functions. Some of these properties are listed in Sections 2 and 3. In particular, hyperbolic-concave envelopes are defined, and their properties are identified, in Section 3. The results in Sections 2 and 3 are applied to give characterizations and identifications in the central Section 4.

2. Hyperbolic-Concave Functions

In this Section, the concept of hyperbolic-concave functions is defined (Definition 2.2); and properties of these functions are given (Lemmas 2.5 and 2.7 and Proposition 2.6). Proofs are facilitated through a key equivalence between hyperbolic-concave functions and convex functions, given in Theorem 2.3. Standard definitions and properties associated with convex functions are used throughout this paper; for reference see Roberts and Varberg (1970) and Rockafellar (1970). Within this paper, intervals in the real numbers \mathbb{R} may or may not contain their endpoints, are nonempty, but may be a singleton.

For any real numbers a and b with $a < b$, $k(\cdot, a, b)$ denotes the hyperbolic function from \mathbb{R} into $(0, \infty]$ given by

$$k(x; a, b) = (b - a)/(x - a) \text{ if } x > a, \text{ and } = +\infty \text{ if } x \leq a.$$

Let \mathcal{H}_0 denote the collection of such functions. The following two properties of these hyperbolic functions are easily verified.

LEMMA 2.1 (i) Let (x_1, y_1) and (x_2, y_2) be any pairs of real numbers satisfying $x_1 < x_2$ and $y_1 > y_2 > 0$. Then there exists exactly one pair of real numbers a and b with $a < b$ for which the function $k(\cdot; a, b)$ satisfies $k(x_1; a, b) = y_1$ and $k(x_2; a, b) = y_2$. The numbers a and b are given by $a = (x_1y_1 - x_2y_2)/(y_1 - y_2)$ and $b = (x_1y_1 - x_2y_2 + y_1y_2(x_2 - x_1))/(y_1 - y_2)$.

(ii) Let $k_1(\cdot) = k(\cdot; a_1, b_1)$ and $k_2(\cdot) = k(\cdot; a_2, b_2)$ be two different functions in \mathcal{H}_0 . Then there is some number \bar{x} for which either (a) $k_1(x) < k_2(x)$ if $a_1 < x < \bar{x}$, and $k_2(x) < k_1(x)$ if $\bar{x} < x < \infty$; or (b) $k_2(x) < k_1(x)$ if $a_2 < x < \bar{x}$, and $k_1(x) < k_2(x)$ if $\bar{x} < x < \infty$.

Function g from \mathbb{R} into $[0, \infty]$ is said to be *nondegenerate* if g takes on at least one value in $(0, \infty)$. Let \mathcal{G} denote the collection of nondegenerate, nonincreasing functions from \mathbb{R} into $[0, \infty]$. For each function g in \mathcal{G} , let $w_0 := \sup\{x : g(x) = \lim_{y \rightarrow -\infty} g(y)\}$ if this set $\neq \emptyset$, and $= -\infty$ otherwise; and $x_0 := \inf\{x : g(x) = 0\}$ if this set $\neq \emptyset$, and $= +\infty$ otherwise. Observe that $w_0 \leq x_0$ if g is not identically constant; if $-\infty < w_0$, then $g(x) = \lim_{y \rightarrow -\infty} g(y)$ for all $x \in (-\infty, w_0)$; and if $x_0 < \infty$, then $g(x) = 0$ for all $x \in (x_0, \infty)$. For each g in \mathcal{G} , the function $1/g$ is defined on the interval $I_g := \text{Dom}(1/g) = \{x : 0 < g(x) < \infty\}$. Observe that I_g is nonempty. Also, let w_1 be the extended real number

$$(2.1) \quad w_1 := \sup\{x : g(x) = \text{limit of } g(y) \text{ as } y \downarrow \inf(I_g) \text{ over } y \text{ in } I_g\}$$

if this set is nonempty, and $= w_0$ otherwise.

To aid in understanding these definitions, consider the following function: $g(x) = +\infty$ if $x \in (-\infty, 0]$, $= 1$ if $x \in (0, 2]$, $= -(1/2)(x - 3)$ if $x \in (2, 3]$, and $= 0$ if $x \in (3, \infty)$. For this function $w_0 = 0$, $w_1 = 2$, and $x_0 = 3$.

DEFINITION 2.2 Let f be any function in \mathcal{G} . We say that f is a *hyperbolic-concave function* if for any two pairs $(x, f(x))$ and $(y, f(y))$ with $x < y$ and $0 < f(y) < f(x) < \infty$ and any associated function $k(\cdot) = k(\cdot; a, b)$ from \mathcal{H}_0 satisfying $k(x) = f(x)$ and $k(y) = f(y)$, it follows that $k(z) \leq f(z)$ for all z in $[x, y]$.

Let \mathcal{H} denote the collection of hyperbolic-concave functions. The approach taken here in defining the hyperbolic-concave functions is analogous to the approach taken to define \mathcal{F} -convex functions (see Roberts and Varberg (1970, Section 84)). The following result is very useful in the analysis of hyperbolic-concave functions.

THEOREM 2.3 Let $f \in \mathcal{G}$. Then f is a hyperbolic-concave function if and only if $1/f$ is a convex function.

PROOF This equivalence follows in a straightforward way, upon observing the following. Let x_1 and x_2 be in \mathbb{R} with $x_1 < x_2$ and $\infty > f(x_1) > f(x_2) >$

0, and let $k(x) = k(x; a, b)$ be in \mathcal{H}_0 with $k(x_i) = f(x_i)$ for $i = 1, 2$. Then for all z in $[x_1, x_2]$,

$$k(z) = \left(\left(\frac{x_2 - z}{x_2 - x_1} \right) \frac{1}{f(x_1)} + \left(\frac{z - x_1}{x_2 - x_1} \right) \frac{1}{f(x_2)} \right)^{-1}$$

(from substituting into $k(z; a, b)$ the expressions for a and b given in Lemma 2.1(i)). \square

To obtain another useful characterization of hyperbolic-concave functions as a corollary of Theorem 2.3, we introduce the following definition. We say that function g in \mathcal{G} has a *hyperbola of support* at x in I_g if there is a function k in \mathcal{H}_0 for which $k(x) = g(x)$ and $g \leq k$.

COROLLARY 2.4 *Let $f \in \mathcal{G}$. Then f is a hyperbolic-concave function if and only if, for each x in (w_1, x_0) , there is a hyperbola of support for f at x , and f is right continuous at w_1 , if $[w_1, w_1 + \epsilon) \subseteq I_f$ for some $\epsilon > 0$.*

PROOF To prove this equivalence, observe only that

- f has hyperbola of support at x , for all x in (w_1, x_0)
- \Leftrightarrow for each $x \in (w_1, x_0)$ there is a function $\ell(y) = ((y - a)/(b - a))_+$, for some $a < b$, with $\ell(x) = 1/f(x)$ and $\ell \leq 1/f$ on I_f
- \Leftrightarrow for each $x \in (w_1, x_0)$, $1/f$ has a line of support at x . \square

Using Theorem 2.3, one sees that the following are hyperbolic-concave functions: functions in \mathcal{H}_0 ; the functions $f(x) = x^{-\alpha}$ if $x > 0$, and $= \infty$ if $x \leq 0$, for $\alpha > 1$; the functions e^{-x} and $(1 + e^x)^{-1}$; the constant functions $f(x) \equiv a$, for $a > 0$; the function $f(x) = +\infty$ if $x < c$, $= a \in (0, \infty)$ if $x = c$, and $= 0$ if $x > c$; and the function $f(x) = 1$ if $x \leq 0$, $= 1 - x$ if $0 \leq x \leq 1$, and $= 0$ if $x \geq 1$. For comparison, some functions in $\mathcal{G} \setminus \mathcal{H}$ are the following: $g(x) = x^{-\alpha}$ if $x > 0$, and $= \infty$ if $x \leq 0$, for $0 < \alpha < 1$; the function $g(x) = (\log x)^{-1}$ if $x > 1$, and $= \infty$ if $x \leq 1$; and the function g given immediately after Expression (2.1). The following closure properties also follow from Theorem 2.3:

- (2.2) (i) if $f \in \mathcal{H}$ and $0 < a < \infty$, then $\alpha f \in \mathcal{H}$;
 (ii) for any index set Γ , if $f_\gamma \in \mathcal{H}$ for each $\gamma \in \Gamma$ and if $f(x) := \inf_{\gamma \in \Gamma} f_\gamma(x)$ is nondegenerate, then $f \in \mathcal{H}$.

The following lemma contains properties of hyperbolic-concave functions.

LEMMA 2.5 *Let $f \in \mathcal{H}$.*

- (i) The function f is continuous on the interior of I_f . If w_0 is the left endpoint of I_f , $w_0 \in I_f$, and $-\infty < w_0 < x_0 \leq \infty$, then f is right continuous at w_0 .
- (ii) For the number w_1 of (2.1), $f(x)$ is identically constant if $x < w_1$, $x \in I_f$, and $f(x)$ is strictly decreasing if $w_1 < x$, $x \in I_f$.
- (iii) For $w_1 \leq x < y \leq x_0$, $(yf(y) - xf(x))/(f(y) - f(x))$ is nondecreasing in x and y ; for $w_1 \leq x < x' < y < y' \leq x_0$,

$$\frac{yf(y) - xf(x)}{f(y) - f(x)} \leq \frac{y'f(y') - x'f(x')}{f(y') - f(x')}.$$

PROOF The conclusions are straightforward from Theorem 2.3 and properties of convex functions. \square

For each nonconstant function g in \mathcal{G} , define sets $\mathcal{S}_- = \mathcal{S}_-(g)$ and $\mathcal{S}_+ = \mathcal{S}_+(g)$ by

$$\begin{aligned} (2.3) \quad \mathcal{S}_- &:= \{x : x > w_0 \text{ and } g(x) < g(y) \text{ for all } y < x\} \\ \mathcal{S}_+ &:= \{x : g(y) < g(x) < \infty \text{ for all } y > x\} \end{aligned}$$

and define functions Λ^-g and Λ^+g on \mathcal{S}_- and \mathcal{S}_+ respectively by

$$\begin{aligned} (2.4) \quad \Lambda^-g(x) &:= \sup_{w \in (w_0, x)} (wg(w) - xg(x))/(g(w) - g(x)), \text{ and} \\ \Lambda^+g(x) &:= \inf_{y \in (x, \infty)} (yg(y) - xg(x))/(g(y) - g(x)). \end{aligned}$$

Observe that sets \mathcal{S}_- and \mathcal{S}_+ are contained in $[w_1, x_0]$. From Lemma 2.5(iii), it follows that if $f \in \mathcal{H}$, then $(w_1, x_0) \subseteq \mathcal{S}_- \cap \mathcal{S}_+$. Let $D^-g(x)$ and $D^+g(x)$ denote respectively the left-hand derivative and the right-hand derivative of function g at x , and let $Dg(x)$ denote the derivative of g at x .

PROPOSITION 2.6 *Let f be any nonconstant function in \mathcal{H} .*

- (i) Functions Λ^-f and Λ^+f have representations

$$\begin{aligned} (2.5) \quad \Lambda^-f(x) &= \lim_{y \rightarrow x^-} (yf(y) - xf(x))/(f(y) - f(x)) \text{ and} \\ \Lambda^+f(x) &= \lim_{y \rightarrow x^+} (yf(y) - xf(x))/(f(y) - f(x)). \end{aligned}$$

- (ii) $\Lambda^-f(x) \leq \Lambda^+f(x)$.

- (iii) Functions Λ^-f and Λ^+f are finite-valued and nondecreasing.

- (iv) Λ^+f is right continuous on $\mathcal{S}_+ \setminus \{x_0\}$. Λ^-f is left continuous on $\mathcal{S}_- \setminus \{x_0\}$. If f is left continuous at x_0 , then so is Λ^-f .

(v) $\Lambda^- f$ and $\Lambda^+ f$ have representations

$$(2.6) \quad \Lambda^- f(x) = x - \{f(x)D^-(1/f)(x)\}^{-1} = x + \{f(x)/D^- f(x)\};$$

and

$$\Lambda^+ f(x) = x - \{f(x)D^+(1/f)(x)\}^{-1} = x + \{f(x)/D^+ f(x)\}$$

if $x \in \mathcal{S}_+ \setminus \{x_0\}$, and $\Lambda^+ f(x_0) = x_0$ if $x_0 \in \mathcal{S}_+$.

(vi) $\Lambda^+ f(x) \leq \Lambda^- f(y)$ if $x < y$; for $x \in (w_1, x_0)$, $\Lambda^- f(x+) = \Lambda^+ f(x)$ and $\Lambda^- f(x) = \Lambda^+ f(x-)$.

PROOF Conclusions (i), (ii) and (iii) follow directly from Lemma 2.5(iii). For conclusion (iv), use also Lemma 2.5(i); and for conclusion (v), use (2.5). For conclusion (vi), again use Lemma 2.5(iii), and use part (iv). \square

For any nonconstant function f in \mathcal{H} , define the *hyperbolic derivative* Λf for x in $\mathcal{S}_- \cap \mathcal{S}_+$ by

$$(2.7) \quad \Lambda f(x) = \lim_{y \rightarrow x} (yf(y) - xf(x))/(f(y) - f(x)), \text{ if this limit exists.}$$

LEMMA 2.7 *Let f be any nonconstant function in \mathcal{H} .*

(i) *Within $\mathcal{S}_- \cap \mathcal{S}_+$, the set of discontinuity points of $\Lambda^+ f$ and of $\Lambda^- f$ coincide, and equals the set of discontinuity points within $\mathcal{S}_- \cap \mathcal{S}_+$ of $D^+ f$ and of $D^- f$. This set, denoted by \mathcal{D} , is countable.*

(ii) *Within $\mathcal{S}_- \cap \mathcal{S}_+$,*

$$x \in (\mathcal{S}_- \cap \mathcal{S}_+) \setminus \mathcal{D}$$

$\Leftrightarrow \Lambda^- f(x) = \Lambda^+ f(x), \Lambda f(x)$ exists and equals this common value

\Leftrightarrow there is a unique hyperbola of support for f at x .

In this case,

$$\Lambda f(x) = x + \{f(x)/Df(x)\} = x - \{f(x)D(1/f)(x)\}^{-1};$$

and the unique hyperbola of support at x is given by

$$k(y; a, b) = f(x)(x - \Lambda f(x))/(y - \Lambda f(x)) \text{ if } y > \Lambda f(x), \text{ and } = +\infty \text{ otherwise}$$

(iii) *For all x, y with $w_1 < x < y < x_0$,*

$$(2.8) \quad f(y)/f(x) = \exp \int_x^y \{\Lambda^+ f(t) - t\}^{-1} dt,$$

and f can be identified from its hyperbolic derivative.

PROOF To obtain (2.8), use Theorem 2.3, Lemma 2.6(v) and standard results on convex functions found, e.g., in Freedman (1971, pp. 359–363). To verify the other conclusions, use Theorem 2.3 and Corollary 2.4, the representations in (2.6), and standard results for convex functions. \square

3. Hyperbolic-Concave Envelopes

In this Section, hyperbolic-concave envelopes are defined (Definition 3.1), and properties of hyperbolic-concave envelopes are given. These hyperbolic-concave envelopes have a direct connection to convex envelopes of functions, given in (3.5). For this comparison, we recall the definition of convex envelopes; and for a class of functions of interest here, we recall some properties of these convex envelopes (for references to these results, see Rockafellar (1970, pp. 36, 51, 103, 157) and Roberts and Varberg (1970, p. 21). For any function f from an interval I to \mathbb{R} which majorizes at least one affine function on I , the *convex envelope of f* , written $\text{env } f$, is the function defined on I by

$$(3.1) \quad (\text{env } f)(x) = \sup\{A(x) : A \text{ is an affine function, } A \leq f\}.$$

The basic class of functions of interest in this paper is the class \mathcal{G}^1 , the subset of \mathcal{G} given by

$$(3.2) \quad \mathcal{G}^1 := \{g(\cdot) : g \text{ is a left-continuous, nonincreasing function from } \mathbb{R} \text{ into } [0, 1] \text{ with } \lim_{x \rightarrow -\infty} g(x) = 1 \text{ and } \limsup_{x \rightarrow \infty} xg(x) = 0\}.$$

Observe that if $g \in \mathcal{G}^1$, then g is nondegenerate and $\lim_{x \rightarrow \infty} g(x) = 0$; and if w_0 and x_0 are the numbers associated with g in Section 2, then $w_0 = \sup\{x : g(x) = 1\}$ if this set $\neq \emptyset$, and $= -\infty$ otherwise. For any function g in \mathcal{G}^1 , the function $\text{env}(1/g)$, defined on $I_g := \text{Dom}(1/g)$ through (3.1), is a closed function. Thus it follows from the defining properties of \mathcal{G}^1 and properties of convex envelopes that for $g \in \mathcal{G}^1$,

- (3.3) (i) $\text{env}(1/g)$ is the greatest convex function which is majorized by $1/g$ on I_g ;
- (ii) $\text{env}(1/g)$ is continuous on I_g ;
- (iii) $(\text{env}(1/g))(x) = \inf\{\mu : (x, \mu) \text{ is in the convex hull of the epigraph of } 1/g\} = \inf\{\lambda(1/g)(x_1) + (1 - \lambda)(1/g)(x_2) : \lambda x_1 + (1 - \lambda)x_2 = 1 \text{ for some } 0 \leq \lambda \leq 1, x_1 \leq x \leq x_2\}$; and
- (iv) if $1/g$ is convex in I_g , then $1/g = \text{env}(1/g)$.

Let $\mathcal{H}^1 := \mathcal{G}^1 \cap \mathcal{H}$, the collection of hyperbolic-concave functions in \mathcal{G}^1 . Numbers w_0, w_1 and x_0 associated with each function f in \mathcal{H}^1 satisfy $-\infty \leq w_0 = w_1 \leq x_0 \leq +\infty$; it is the case that $w_0 = x_0$ if and only if $f(x) = 1$ if $x \leq c$, and $= 0$ if $x > c$, for some $c \in \mathbb{R}$. If $f \in \mathcal{H}^1$ and $x_0 \in \mathbb{R}$, then $I_f = \text{Dom}(1/f) = (-\infty, x_0)$ if $f(x_0) = 0$, and $= (-\infty, x_0]$ if $f(x_0) > 0$; $\mathcal{S}_- = (w_0, x_0]$, and if also $w_0 \in \mathbb{R}$, then $\mathcal{S}_+ = [w_0, x_0)$ if $f(x_0) = 0$, and $= [w_0, x_0]$ if $f(x_0) > 0$.

DEFINITION 3.1 For each g in \mathcal{G}^1 , the *hyperbolic-concave envelope* of g is the function \widehat{g} from \mathbb{R} into $[0,1]$ defined by

$$(3.4) \quad \widehat{g}(x) := \inf\{k(x) : k \in \mathcal{H}_0, g \leq k\}.$$

(Observe that the set in (3.4) is nonempty.) The collection of hyperbolic-concave envelopes of functions in \mathcal{G}^1 is denoted by $\widehat{\mathcal{G}}^1$. We show that $\widehat{\mathcal{G}}^1 = \mathcal{H}_1$ in Theorem 3.4. Connections between hyperbolic-concave envelopes and convex envelopes, together with some other properties of hyperbolic-concave envelopes are given in the following.

LEMMA 3.2 Let $g \in \mathcal{G}^1$ and let $I_g = \text{Dom}(1/g)$. The hyperbolic-concave envelope \widehat{g} has the following properties:

(i) \widehat{g} takes values in $[0, 1]$; \widehat{g} is nonincreasing; $\lim_{x \rightarrow -\infty} \widehat{g}(x) = 1$ and $\limsup_{x \rightarrow \infty} x\widehat{g}(x) = 0$; $w_0 := w_0(g) = w_0(\widehat{g})$, $x_0 := x_0(g) = x_0(\widehat{g})$, and $I_g = \text{Dom}(1/g) = \text{Dom}(1/\widehat{g}) = I_{\widehat{g}}$ with $\widehat{g}(x) = 1$ for all x in $(-\infty, w_0]$ and $\widehat{g}(x) = 0$ for all x in (x_0, ∞) ; and \widehat{g} is continuous on $\mathbb{R} \setminus \{x_0\}$ and is left continuous at x_0 ;

(ii) $g \leq \widehat{g}$; if $g(x) = 1$ for $x \leq x_0$, and $= 0$ for $x > x_0$, for some $x_0 \in \mathbb{R}$, then $g = \widehat{g}$;

(iii) on I_g ,

$$(3.5) \quad \text{env}(1/g) = 1/\widehat{g};$$

and

(iv) For each $x \in I_g$, one and only one of the following hold:

(a) $g(x) = \widehat{g}(x)$ and $1/g(x) = \text{env}(1/g)(x)$; and for some $a < b$, $k(\cdot) = k(\cdot; a, b)$ is in \mathcal{H}_0 with $g \leq k$ and $g(x) = k(x)$, and $\ell(y) = ((y - a)/(b - a))_+$ satisfies $\ell \leq 1/g$ and $\ell(x) = 1/g(x)$; or

(b) $g(x) < \widehat{g}(x)$ and $1/g(x) > \text{env}(1/g)(x)$; and for some $a < b$, and x_1, x_2, x_3, x_4 with $x_1 \leq x_2 < x < x_3 \leq x_4$, $k(\cdot) = k(\cdot; a, b)$ in \mathcal{H}_0 satisfies $\widehat{g} \leq k$, $g(x_i) = \widehat{g}(x_i) = k(x_i)$ for $i = 1, \dots, 4$, $\widehat{g}(y) = k(y)$ for $y \in [x_1, x_4]$, $\widehat{g}(y) < k(y)$ for $y \in (I_g \setminus [x_1, x_4]) \cup (x_0, \infty)$, and $g(y) < k(y)$ for $y \in (x_2, x_3)$; and $\ell(y) = ((y - a)/(b - a))_+$ satisfies $\ell \leq \text{env}(1/g)$, $1/g(x_i) = \text{env}(1/g)(x_i) = \ell(x_i)$ for $i = 1, \dots, 4$, $\text{env}(1/g)(y) = \ell(y)$ for $y \in [x_1, x_4]$, $\text{env}(1/g)(y) > \ell(y)$ for $y \in (I_g \setminus [x_1, x_4]) \cup (x_0, \infty)$, and $1/g(y) > \ell(y)$ for $y \in (x_2, x_3)$.

PROOF From the definition of \widehat{g} , it is immediate that $g \leq \widehat{g}$ and \widehat{g} is nonincreasing; thus, also $0 \leq \widehat{g}$. The limit property $\limsup_{x \rightarrow \infty} x\widehat{g}(x) = 0$ follows easily from the analogous property of g .

We show $\widehat{g} \leq 1$. For any $0 < \epsilon < 1$, there is an $N > 1$ sufficiently large such that, for all $x \geq N$, $g(x) \leq \epsilon x^{-1} = k(x; 0, \epsilon)$. Let $\bar{x} < N$, and let δ be chosen sufficiently small so that $0 < \delta < 1$ and $\bar{x} + (1 + 2\delta)\delta^{-1} > N$. Let $k(\cdot) = k(\cdot; a, b)$ be the function in \mathcal{H}_0 passing through $(x_1, y_1) = (\bar{x} - \delta^{-1}, 1 + 2\delta)$ and $(x_2, y_2) = (\bar{x}, 1 + \delta)$. Then $g(x) \leq k(x)$ for all x (since if $x < N < \bar{x} + (1 + 2\delta)\delta^{-1}$, $g(x) \leq 1 \leq k(x)$; and if $x > N$, $g(x) \leq \epsilon x^{-1} < k(x)$ by Lemma 2.1(ii)); and thus $\widehat{g}(x) \leq k(x)$ for all x . Also, for $x \geq \bar{x}$, $\widehat{g}(x) \leq k(\bar{x}) = 1 + \delta$; since this can be done for each $0 < \delta < 1$ small, it follows that $\widehat{g} \leq 1$. It is immediate that $\widehat{g}(x) = 1$ for all x in $(-\infty, w_0]$, where $w_0 := w_0(g)$. From the definition of \widehat{g} and appropriate choice of k 's in \mathcal{H}_0 majorizing g , it similarly follows that $\widehat{g}(x) = 0$ for all x in (x_0, ∞) , where $x_0 := x_0(g)$; and that if $g(x) = 1$ for $x \leq x_0$, and $= 0$ for $x > x_0$, for some $x_0 \in \mathbb{R}$, then $g = \widehat{g}$.

Next, to obtain (3.5) observe that for all $x \in I_g := \text{Dom}(1/g)$,

$$\begin{aligned}
 (3.6) \quad \text{env}(1/g)(x) &= \sup\{A(x) : A \text{ is an affine function, } A \leq 1/g\} \\
 &= \sup\{A(x) : A(x) = ((x - a)/(b - a))_+ \\
 &\quad \text{for some } a < b, A \leq 1/g\} \\
 &= \{\inf\{k(x) : k \in \mathcal{H}_0, g \leq k\}\}^{-1} \\
 &= 1/\widehat{g}(x).
 \end{aligned}$$

As immediate consequences of (3.3), (3.5), and the properties of g , one obtains that $w_0(g) = w_0(\widehat{g})$, $x_0(g) = x_0(\widehat{g})$, and $\text{Dom}(1/g) = \text{Dom}(1/\widehat{g})$; and that \widehat{g} is continuous on $\mathbb{R} \setminus \{x_0\}$ and left continuous at x_0 . By exploiting the correspondence between functions $k(\cdot; a, b)$ in \mathcal{H}_0 with $g \leq k$ and functions $\ell(y) = ((y - a)/(b - a))_+$ with $\ell \leq 1/g$, as in (3.6), and by using the non-negativity and left continuity of g and the property $\limsup_{x \rightarrow \infty} xg(x) = 0$, one obtains that Lemma 3.2(iv) holds. \square

PROPOSITION 3.3 *Let $g \in \mathcal{G}^1$. The hyperbolic-concave envelope \widehat{g} is in \mathcal{H}^1 and is the smallest hyperbolic-concave function which majorizes g . The function \widehat{g} has representation*

$$\begin{aligned}
 (3.7) \quad \widehat{g}(x) &= \sup\{k(x) : k \text{ is in } \mathcal{H}_0 \text{ and passes through } (x_1, g(x_1)) \\
 &\quad \text{and } (x_2, g(x_2)) \text{ for some } x_1 < x < x_2\}.
 \end{aligned}$$

If g is a hyperbolic-concave function, then $g = \widehat{g}$.

PROOF Use Lemma 3.2(i) to obtain that \widehat{g} is in \mathcal{G}^1 . The results that \widehat{g} is hyperbolic-concave, and is the smallest hyperbolic-concave function which

majorizes g , follow from $g \leq \widehat{g}$ and (3.5), (3.3)(i), and Theorem 2.3. The remaining conclusions follow from (3.5) and (3.3)(iii),(iv). \square

THEOREM 3.4 *The collection of hyperbolic–concave functions in \mathcal{G}^1 and the collection of hyperbolic–concave envelopes of functions in \mathcal{G}^1 are equal; that is, $\widehat{\mathcal{G}}^1 = \mathcal{H}^1$.*

PROOF If $\widehat{g} \in \widehat{\mathcal{G}}^1$ for some $g \in \mathcal{G}^1$, then $\widehat{g} \in \mathcal{H}^1$ from Proposition 3.3. If $h \in \mathcal{H}^1$, then $\widehat{h} = h$, from Proposition 3.3, and thus $h \in \widehat{\mathcal{G}}^1$. \square

4. Characterizations of the Sets \mathcal{P}^* , \mathcal{P}_0^* , and the Set of Minimal p.m.’s

From the Introduction, recall the definitions of the collections of p.m.’s \mathcal{P}^* , the set of p.m.’s dominated by maximal p.m.’s (in the \prec_s order); and \mathcal{P}_0^* , the set of Hardy–Littlewood maximal p.m.’s. Also recall, from Section 3, the collections of functions \mathcal{G}^1 of (3.2); and \mathcal{H}^1 , the set of hyperbolic–concave functions in \mathcal{G}^1 , which equals the set $\widehat{\mathcal{G}}_1$ of hyperbolic–concave envelopes of functions in \mathcal{G}^1 by Theorem 3.4. In this Section, the sets \mathcal{P}^* and \mathcal{P}_0^* are shown to be isomorphic to the sets \mathcal{G}^1 and \mathcal{H}^1 . Moreover, an explicit identification between minimal p.m.’s ν_Δ , associated with p.m.’s ν in \mathcal{P}^* , and hyperbolic–concave envelopes \widehat{f} , associated with functions f in \mathcal{G}^1 , is given in Theorem 4.4.

LEMMA 4.1 *There is an isomorphism between \mathcal{G}^1 and \mathcal{P}^* identified by $g(x) = \nu[x, \infty)$ for $g \in \mathcal{G}^1, \nu \in \mathcal{P}^*$.*

PROOF As stated in the Introduction, Proposition 2.1 of Kertz and Rösler (1991a) gives that $\mathcal{P}^* = \{\nu \text{ is a p.m. on } \mathbb{R} : \limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0\}$. From this representation and the usual identification of p.m.’s on \mathbb{R} and distribution functions (see e.g., Section 10 of Loève (1963)), the conclusion is immediate. \square

THEOREM 4.2 *There is an isomorphism between \mathcal{H}^1 , the set of hyperbolic–concave functions in \mathcal{G}^1 , and \mathcal{P}_0^* , the set of maximal p.m.’s on \mathbb{R} , identified by $f(x) = \mu^*[x, \infty)$ for $f \in \mathcal{H}^1, \mu^* \in \mathcal{P}_0^*$.*

PROOF In Lemma 2.6 of Kertz and Rösler (1991a), it was shown that for any p.m. ν on \mathbb{R} , $\nu \in \mathcal{P}_0^*$ if and only if the following holds

- (4.1) (i) $\limsup_{w \nearrow 1} (1 - w)F_\nu^{-1}(w) = 0$, and
- (ii) $(1 - w)F_\nu^{-1}(w)$ is a concave function.

Now, assume $\nu = \mu^* \in \mathcal{P}_0^*$; so, also F_ν^{-1} satisfies (4.1). First, observe that if $x_0 = x_0(\nu) < \infty$, then $\limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0$; and if $x_0 = +\infty$, then

$$\begin{aligned} \limsup_{x \rightarrow \infty} x\nu[x, \infty) &= \limsup_{x \rightarrow \infty} F_\nu^{-1}(F_\nu(x))(1 - F_\nu(x)) \\ &= \limsup_{w \nearrow 1} (1 - w)F_\nu^{-1}(w) = 0. \end{aligned}$$

Second, observe that

$$(4.2) \quad (1 - w)F_\nu^{-1}(w) \text{ is concave iff } 1/(1 - F_\nu(x)) \text{ is convex,}$$

for example, from a calculation based on the definitions of convexity and concavity for functions; and so $f(x) := \nu[x, \infty)$ is a hyperbolic-concave function, from Theorem 2.3. Thus f is a hyperbolic-concave function in \mathcal{G}^1 , from Lemma 4.1, i.e., $f \in \mathcal{H}^1$.

On the other hand, assume $f \in \mathcal{H}^1$, i.e., f is a hyperbolic-concave function in \mathcal{G}^1 . Let $\nu[x, \infty) := f(x)$; then ν is a p.m. in \mathcal{P}^* , from Lemma 4.1. From Lemma 2.5, f is continuous on $(-\infty, x_0)$ and strictly decreasing on $[w_0, x_0]$. It follows that $\limsup_{w \nearrow 1} (1 - w)F_\nu^{-1}(w) = 0$ if $w_1 < 1$; and if $w_1 = 1$,

$$\begin{aligned} \limsup_{w \nearrow 1} (1 - w)F_\nu^{-1}(w) &= \limsup_{x \rightarrow \infty} (1 - F_\nu(x))F_\nu^{-1}(F_\nu(x)) \\ &= \limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0. \end{aligned}$$

From the hyperbolic-concavity of $\nu[x, \infty)$, Theorem 2.3, and (4.2), it follows that $(1 - w)F_\nu^{-1}(w)$ is concave. Thus F_ν^{-1} satisfies (4.1), and $\nu \in \mathcal{P}_0^*$. \square

The following theorem shows that any Hardy-Littlewood maximal p.m. can be identified through its hyperbolic derivative.

THEOREM 4.3 *Let μ be any p.m. on \mathbb{R} with $\int_0^\infty x d\mu(x) < \infty$, with associated Hardy-Littlewood maximal p.m. μ^* ; and let f denote the function in \mathcal{H}^1 defined by $f(x) = \mu^*[x, \infty)$ for all $x \in \mathbb{R}$. Then μ^* can be identified through its hyperbolic derivative Λf by*

$$(4.3) \quad \mu^*[x, \infty) = \exp\left(\int_{w_0}^x (\Lambda f(t) - t)^{-1} dt\right) \text{ for all } x \in [w_0, x_0].$$

PROOF Let $w_0 = w_0(\mu)$ and $x_0 = x_0(\mu)$. The representation (4.3) follows immediately from Lemma 2.7, since the function f is in \mathcal{H}^1 . \square

THEOREM 4.4 (i) *For each $\nu \in \mathcal{P}^*$, the minimal p.m. ν_Δ satisfies $(\nu_\Delta)^*[x, \infty) = \hat{g}(x)$ for all $x \in \mathbb{R}$, where g is the function in \mathcal{G}^1 defined by $g(x) := \nu[x, \infty)$ for all $x \in \mathbb{R}$.*

(ii) For each $g \in \mathcal{G}^1$, the hyperbolic-concave envelope \hat{g} satisfies $\hat{g}(x) = (\nu_\Delta)^*[x, \infty)$ for all $x \in \mathbb{R}$, where ν is the p.m. in \mathcal{P}^* defined by $\nu[x, \infty) := g(x)$ for all $x \in \mathbb{R}$.

Thus, the following diagram commutes:

$$\begin{array}{ccc}
 \nu \text{ in } \mathcal{P}^* & \longrightarrow & g \text{ in } \mathcal{G}^1 \\
 \downarrow & & \downarrow \\
 \nu_\Delta & \longrightarrow & (\nu_\Delta)^*[\cdot, \infty) = \hat{g}(\cdot)
 \end{array}$$

PROOF Let $\nu \in \mathcal{P}^*$, so that ν is a p.m. on \mathbb{R} satisfying $\limsup_{x \rightarrow \infty} x\nu[x, \infty) = 0$, and consider the associated minimal p.m. ν_Δ , as defined in the Introduction, and p.m. $(\nu_\Delta)^*$; thus, ν_Δ is the unique p.m. on \mathbb{R} satisfying

- (4.4) (i) $\int_0^\infty x d\nu_\Delta(x) < \infty$;
- (ii) $\nu \prec_s (\nu_\Delta)^*$; and
- (iii) if $\bar{\mu}$ is any p.m. on \mathbb{R} with $\int_0^\infty x d\bar{\mu}(x) < \infty$ and $\nu \prec_s \bar{\mu}^*$, then $(\nu_\Delta)^* \prec_s \bar{\mu}^*$.

(see Theorem 2.4 of Kertz and Rösler (1991a) for verification that such a p.m. ν_Δ exists). Define $g(x) := \nu[x, \infty)$ for all $x \in \mathbb{R}$; from Lemma 4.1, this function g is in \mathcal{G}^1 . From Proposition 3.3, we know that \hat{g} is the unique function on \mathbb{R} satisfying

- (4.5) (i) \hat{g} is a hyperbolic-concave function in \mathcal{G}^1 ;
- (ii) $g(x) \leq \hat{g}(x)$ for all $x \in \mathbb{R}$; and
- (iii) if h is any hyperbolic-concave function in \mathcal{G}^1 satisfying $g(x) \leq h(x)$ for all $x \in \mathbb{R}$, then $\hat{g}(x) \leq h(x)$ for all $x \in \mathbb{R}$.

Now, define $\bar{g}(x) := (\nu_\Delta)^*[x, \infty)$ for $x \in \mathbb{R}$. We claim that $\bar{g} = \hat{g}$. From Theorem 4.2, \bar{g} is a hyperbolic-concave function in \mathcal{G}^1 ; and from (4.4)(ii), it follows that $g(x) \leq \bar{g}(x)$ for all $x \in \mathbb{R}$. To verify (4.5)(iii), we let h be any hyperbolic-concave function in \mathcal{G}^1 satisfying $g(x) \leq h(x)$ for all $x \in \mathbb{R}$. From Theorem 4.2, there exists a p.m. $\bar{\mu}$ on \mathbb{R} with $\int_0^\infty x d\bar{\mu}(x) < \infty$ for which $h(x) = (\bar{\mu})^*[x, \infty)$ for all $x \in \mathbb{R}$; and we have that $\nu[x, \infty) = g(x) \leq h(x) = (\bar{\mu})^*[x, \infty)$ for all $x \in \mathbb{R}$. It follows from (4.4)(iii) that $\bar{g}(x) = (\nu_\Delta)^*[x, \infty) \leq (\bar{\mu})^*[x, \infty) = h(x)$ for all $x \in \mathbb{R}$. Thus, $\hat{g}(x) = (\nu_\Delta)^*[x, \infty)$ for all $x \in \mathbb{R}$.

For part (ii), let g be a function in \mathcal{G}^1 ; \hat{g} denotes the hyperbolic-concave envelope of g . From Lemma 4.1, $\nu[x, \infty) := g(x)$ defines a p.m. ν in \mathcal{P}^* ; and from Theorem 4.2, there is a p.m. ρ on \mathbb{R} with $\int_0^\infty x d\rho(x) < \infty$ and $\rho^*[x, \infty) = \hat{g}(x)$ for all $x \in \mathbb{R}$. We claim that $\rho = \nu_\Delta$; and thus $\hat{g}(x) = (\nu_\Delta)^*[x, \infty)$ for all $x \in \mathbb{R}$. Now, $\nu[x, \infty) = g(x) \leq \hat{g}(x) = \rho^*[x, \infty)$ for all

$x \in \mathbb{R}$ from (4.5)(ii). Also, if $\bar{\mu}$ is any p.m. on \mathbb{R} with $\int_0^\infty x d\bar{\mu}(x) < \infty$ and $\nu \prec_s \bar{\mu}^*$, then $h(x) := \bar{\mu}^*[x, \infty)$ is in \mathcal{H}^1 and satisfies $g(x) \leq h(x)$ for all $x \in \mathbb{R}$. It follows from (4.5)(iii) that $\rho^*[x, \infty) = \widehat{g}(x) \leq h(x) = (\bar{\mu})^*[x, \infty)$ for all $x \in \mathbb{R}$. Thus, ρ satisfies (4.4) and we have that $\rho = \nu_\Delta$. \square

We remark that the second part of the proof of Theorem 4.4 gives another proof of the existence of the minimal p.m. ν_Δ associated with a p.m. $\nu \in \mathcal{P}^*$.

To illustrate these ideas, we include the following example. For $n \geq 1$, let $\nu = \sum_{i=0}^n p_i \epsilon_{y_i}$ where $y_0 < \dots < y_n$ and $0 < p_i < 1$ for $i = 0, \dots, n$ with $p_0 + \dots + p_n = 1$, and ϵ_z = point mass at z . Then $\nu[x, \infty) = 0$ if $y_n < x < \infty$; $= \sum_{i=k+1}^n p_i$ if $y_k < x \leq y_{k+1}$ for $k = 0, \dots, n - 1$; and $= 1$ if $-\infty < x \leq y_0$. From the definition of ν^* , one obtains that

$$\begin{aligned} \nu^*[x, \infty) &= \left(\sum_{i=k+1}^n p_i (y_i - y_k) \right) / (x - y_k) \\ &\quad \text{if } \sum_{i=k}^n p_i y_i / \sum_{i=k}^n p_i < x \leq \sum_{i=k+1}^n p_i y_i / \sum_{i=k+1}^n p_i \\ &\quad \text{for } k = 0, \dots, n - 1 \\ &= 0 \text{ if } y_n < x < \infty, \text{ and } = 1 \text{ if } -\infty < x \leq \sum_{i=0}^n p_i y_i; \end{aligned}$$

and $f(x) = \nu^*[x, \infty)$ is a hyperbolic-concave function in \mathcal{G}^1 . One can obtain for example from Kertz and Rösler (1991a) that $\nu_\Delta = \sum_{i=0}^k \pi_i \epsilon_{\Lambda(x_i)}$ and that

$$\begin{aligned} (\nu_\Delta)^*[x, \infty) &= \left(\sum_{m=\ell+1}^k \pi_m \right) (x_{\ell+1} - \Lambda(x_\ell)) / (x - \Lambda(x_\ell)) \\ &\quad \text{if } x_\ell \leq x \leq x_{\ell+1} \text{ for } \ell = 0, \dots, k - 1 \\ &= 0 \text{ if } x_k < x < \infty, \text{ and } = 1 \text{ if } -\infty < x \leq x_0, \end{aligned}$$

where

$$\Lambda(x) = \inf_{y>x} \{ (y\nu[y, \infty) - x\nu[x, \infty)) / (\nu[y, \infty) - \nu[x, \infty)) \};$$

x_0, \dots, x_k are chosen as follows: $x_0 = y_0$, and having chosen $x_0 = y_0 < x_1 = y_{i_1} < \dots < x_j = y_{i_j}$, the next number $x_{j+1} = y_{i_{j+1}}$ is the maximal number $y_\ell > x_j$ for which

$$\Lambda(x_j) = (y_\ell \nu[y_\ell, \infty) - x_j \nu[x_j, \infty)) / (\nu[y_\ell, \infty) - \nu[x_j, \infty)),$$

and the last number $x_k = y_n$; and $\pi_0 = \nu(-\infty, x_1) = \sum_{i=0}^{i_1-1} p_i$, $\pi_j = \nu[x_j, x_{j+1}) = \sum_{i=i_j}^{i_{j+1}-1} p_i$ for $j = 1, \dots, k - 1$, and $\pi_k = \nu[x_k, \infty) = p_n$.

The function $h(x) := (\nu_\Delta)^*[x, \infty)$ is the hyperbolic-concave envelope of $g(x) := \nu[x, \infty)$.

In particular, let $\nu = \frac{1}{3}\epsilon_0 + \frac{1}{3}\epsilon_1 + \frac{1}{3}\epsilon_2$. Then $g(x) = \nu[x, \infty) = 0$ if $2 < x < \infty$, $= 1/3$ if $1 < x \leq 2$, $= 2/3$ if $0 < x \leq 1$, and $= 1$ if $x \leq 0$; $h(x) = \nu^*[x, \infty) = 0$ if $2 < x < \infty$, $= (3(x-1))^{-1}$ if $3/2 \leq x \leq 2$, and $= x^{-1}$ if $1 \leq x \leq 3/2$, and $= 1$ if $x \leq 1$. Also, $\nu_\Delta = \frac{1}{3}\epsilon_{-2} + \frac{1}{3}\epsilon_0 + \frac{1}{3}\epsilon_2$ and $\hat{g}(x) = (\nu_\Delta)^*[x, \infty) = 0$ if $2 < x < \infty$, $= (2/3)x^{-1}$ if $1 \leq x \leq 2$, $= 2(x+2)^{-1}$ if $0 \leq x \leq 1$, and $= 1$ if $x \leq 0$.

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