DEPENDENCE IN MULTIVARIATE EXTREME VALUES

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We review the limiting behavior of extreme values of sequences of random vectors in R^d by considering mainly the dependence properties of its nondegenerate limit laws. We treat separately the i.i.d. case, the stationary case, the independent non-identically distributed case, and the general nonstationary case. As dependence concepts we discuss total dependence, association, positive lower orthant dependence, and independence.

1. Introduction. Consider a sequence $\{\mathbf{X}_i, i \geq 1\}$ of d-dimensional random vectors with (multivariate) distribution F_i. In this paper, we discuss the behavior of the maximum $\mathbf{M}_n = (M_{n1}, M_{n2}, \ldots, M_{nd})'$ where M_{nj} denotes the maximum up to time n of the j-th components of $\mathbf{X}_i$:

$$M_{nj} = \max(X_{1j}, \ldots, X_{nj}), \;\; j \leq d.$$

Our main interest is the dependence structure of the limiting distribution of properly normalized $\mathbf{M}_n$. More precisely, we deal with the convergence of

$$\begin{aligned} &P\{(\mathbf{M}_n - \mathbf{b}_n)/\mathbf{a}_n \leq \mathbf{z}\} = P\{\mathbf{X}_i \leq \mathbf{a}_n\mathbf{z} + \mathbf{b}_n, \; i \leq n\} \\ &\xrightarrow{w} G(\mathbf{z}) = P\{\mathbf{Z} \leq \mathbf{z}\} \;\text{ as }\; n \to \infty, \end{aligned} \tag{1}$$

and the dependence properties of G. Note that all algebraic operations are componentwise and that the normalization constants satisfy $\mathbf{a}_n > 0$. The univariate case has been treated by many authors; c.f. the textbooks by Leadbetter et al. (1983), Galambos (1987), and Resnick (1987). One additional aspect of extreme value theory in the multivariate case is the dependence properties of the limit law G in (1). This important and interesting question is the primary focus of our review. We investigate the extreme value distribution G for independence, total dependence, association, and positive lower orthant dependence (see Section 2 for definitions).

In Section 2 we quickly review the case of i.i.d. sequences, whose study was initiated by Geffroy (1958/59), Tiago de Oliveira (1958), and Sibuya (1960) for

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$d = 2$. In the remaining sections we consider more general cases, first beginning with the case of stationary sequences. Most of the classical results remain valid in this situation provided a certain mixing condition is satisfied.

Next we treat the case of independent but non-identically distributed random vectors. This case can only be reasonably treated by introducing a certain uniform asymptotic negligibility condition. Without this restriction, every (multivariate) distribution G can occur as a limit in (1). This condition, however, can be interpreted as a natural extension of the conditions used implicitly in the classical i.i.d. case. Finally, we deal with the general case of non-independent, non-stationary sequences.

2. The i.i.d. Case. For the i.i.d. case ($F_i \equiv F$), equation (1) becomes

$$F^n(\mathbf{a}_n\mathbf{z} + \mathbf{b}_n) \xrightarrow{w} G(\mathbf{z}) \tag{2}$$

or

$$n(1 - F(\mathbf{a}_n\mathbf{z} + \mathbf{b}_n)) \to -\log G(\mathbf{z}).$$

This situation is rather completely discussed in the literature (cf. Galambos (1987) and Resnick (1987) for references). Therefore we only mention the results which are relevant to the following discussion of non i.i.d. random vectors. The limit G is called an extreme value distribution and is characterized by the max stability property, i.e., for every $s > 0$ there exist c_s and d_s such that $G^s(\cdot) = G(c_s \cdot + d_s)$. Note that the univariate marginals G_j of an extreme value distribution G are obviously univariate extreme value distributions. Generally a multivariate distribution on $[0,1]^d$, a so-called dependence function, is used to discuss dependence properties of G (de Haan and Resnick (1977), Deheuvels (1984)). For the purpose of our discussion, however, the following results are more informative and useful in applications. Because of max stability the extreme value distribution G is max i.d. (max infinitely divisible), i.e. G^s is a multivariate distribution for every $s > 0$, (cf. Balkema and Resnick (1977)). Hence G is associated: $\mathrm{Cov}(\phi(\mathbf{Z}), \psi(\mathbf{Z})) \geq 0$ for any (componentwise) nondecreasing functions ϕ and ψ where G is the distribution of $\mathbf{Z}$ (cf. Esary et al. (1967), Resnick (1987)). The result that extreme value distributions are associated, is due to Marshall and Olkin (1983).

THEOREM 2.1. *Assume that (2) holds for the sequence of random vectors* $\{\mathbf{X}_i, i \geq 1\}$ *in* R^d, *with normalization* $\mathbf{a}_n$ *and* $\mathbf{b}_n$. *Then* G *is associated since it is max stable and max i.d.*

Obviously, it also implies the weaker PLOD (positive lower orthant dependence) property: $G(\mathbf{z}) \geq \Pi_{j=1}^d G_j(z_j)$. If $G(\mathbf{z}) = \Pi_{j=1}^d G_j(z_j)$, we say that $\mathbf{Z}$ has independent components where $\mathbf{Z}$ has distribution G. As an upper bound for any multivariate distribution, we have the inequality $G(\mathbf{z}) \leq G_j(z_j)$ for any $j \leq d$. If this statement holds as an equality, more precisely, if $G(\mathbf{z}) = \min(G_j(z_j), j \leq d)$ for all $\mathbf{z}$, we say that $\mathbf{Z}$ has totally dependent components. Also assuming $G_j \equiv G_1$, this means that $P\{Z_1 = Z_2 = \ldots = Z_d\} = 1$. It defines the strongest possible dependence.

The two questions of independence and of total dependence were treated in the previously mentioned papers by Geffroy (1958/59), Tiago de Oliveira (1958), and Sibuya (1960) for the bivariate case. Since the limit G is max i.d., bivariate independence implies joint multivariate independence (cf. Newman and Wright (1981)). The result on independence was recently improved by Takahashi (1987, 1988). He gave necessary and sufficient conditions such that an extreme value distribution is characterized by its marginal distributions. Combining these facts we obtain the following statement.

THEOREM 2.2. *Let $\{\mathbf{X}_i, i \geq 1\}$ be a sequence of random vectors in R^d. Assume that (2) holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$. Then $\mathbf{Z}$ has independent components iff for every $1 \leq j < j' \leq d$*

$$\lim_{n\to\infty} nP\{X_{1j} > a_{nj}z_j + b_{nj},\ X_{1j'} > a_{nj'}z_{j'} + b_{nj'}\} = 0 \tag{3}$$

for some z_j, $z_{j'}$ such that $G_{j,j'}(z_j, z_{j'}) \in (0,1)$, where $G_{j,j'}$ is a bivariate marginal of G.

A similar statement holds for the total dependence in place of independence. The bivariate case was treated by Sibuya (1960). Takahashi's characterization (1988) also improves upon this statement.

THEOREM 2.3. *Let $\{\mathbf{X}_i, i \geq 1\}$ be a sequence of random vectors in R^d. Assume that (2) holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$, and that $G_j \equiv G_1$, $j \leq d$. Then $\mathbf{Z}$ has totally dependent components iff for every $1 \leq j \neq j' \leq d$*

$$\lim_{n\to\infty} n(P\{X_{1j} > u_{nj},\ X_{1j'} > u_{nj'}\} - P\{X_{1j} > u_{nj}\}) = 0 \tag{4}$$

for some z, such that $G_1(z) \in (0,1)$, where u_{nj} is defined by $u_{nj} = a_{nj}z + b_{nj}$.

Note that (4) is equivalent to

$$\lim_{n\to\infty} nP\{X_{1j} > a_{nj}z + b_{nj},\ X_{1j'} > a_{nj'}z + b_{nj'}\} = -\log G_1(z)$$

in the i.i.d. case. Equation (4) will be used in the following more general case.

3. The Stationary Case. In this section we consider stationary sequences of random vectors, with $F_i \equiv F$. In this case the extreme value theory is mainly discussed for Gaussian sequences (Lindgren (1974), Amram (1985), Hüsler and Schüpbach (1988)) or for more general sequences in R ($d = 1$) which satisfy a mild mixing condition (cf. Leadbetter et al. (1983)).

In the multivariate situation, if the conditions are such that

$$P\{\mathbf{M}_n \leq \mathbf{a}_n\mathbf{z} + \mathbf{b}_n\} - F^n(\mathbf{a}_n\mathbf{z} + \mathbf{b}_n) \to 0 \text{ as } n \to \infty, \tag{5}$$

then all the results of Section 2 remain valid in this more general situation. However, even under less restrictive conditions, when (5) does not hold, we can still discuss the dependence properties of the limit G, provided it exists.

As mentioned, the univariate problem for stationary sequences is discussed in detail (cf. Leadbetter et al. (1983), Galambos (1987)). The general multivariate stationary problem, however, was only considered in a few papers. An attempt was made by Villasenor (1976), for the case of bivariate exchangeable sequences. Hsing (1987) and Hüsler (1987) independently extended the univariate results, related to Leadbetter's mixing conditions, to the multivariate case. Sbihi (1987) also discusses the multivariate stationary case. In addition, Hüsler (1987) focused more on the dependence properties. These results are reviewed below.

We introduce the following mixing conditions. Given $\mathbf{z}$, we set $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$ and $B_n(I) = \{\mathbf{X}_i \le \mathbf{u}_n, i \in I\}$, where $I \subset \{1, \ldots, n\}$. The set I will also usually depend on n.

Condition $\mathbf{D}_d = \mathbf{D}_d(\{\mathbf{u}_n, n \ge 1\})$ holds for a given $\mathbf{z}$ with normalization $\mathbf{a}_n(> 0)$ and $\mathbf{b}_n$, if there exists an array $\{\alpha_{nm}, n \ge 1, m \le n\}$ such that

i) $|P(B_n(I \cup J)) - P(B_n(I))P(B_n(J))| \le \alpha_{nm}$

for every pair of subsets I and J of $\{1, \ldots, n\}$ which are m-separated (i.e. $\min_{i \in J}(i) - \max_{i \in I}(i) \ge m$ or $\min_{i \in I}(i) - \max_{i \in J}(i) \ge m$) and

ii) $\lim_{n\to\infty} \alpha_{n,m_n^*} = 0$ for some sequence $\{m_n^*, n \ge 1\}$ with $m_n^* \to \infty$ and $m_n^*(1 - F(\mathbf{u}_n)) \to 0$ as $n \to \infty$.

This condition restricts the so-called long-range dependence since it implies that extreme values are asymptotically independent when they occur largely separated in time. Note that this condition is weaker than the usual mixing condition. The following local dependence condition $\mathbf{D}_d'$ excludes the clustering of extreme values in a small time interval.

Condition $\mathbf{D}_d' = \mathbf{D}_d'(\{\mathbf{u}_n, n \ge 1\})$ holds for a given $\mathbf{z}$ with a normalization $\mathbf{a}_n(> 0)$ and $\mathbf{b}_n$, if

$$\lim_{r\to\infty} \limsup_{n\to\infty} n \sum_{1 < i \le n/r} P\{\mathbf{X}_1 \not\le \mathbf{u}_n, \mathbf{X}_i \not\le \mathbf{u}_n\} = 0.$$

These two conditions imply that (5) holds. Hence

THEOREM 3.1. *Let $\{\mathbf{X}_i, i \ge 1\}$ be a stationary sequence of random vectors in R^d. Assume that $\mathbf{D}_d$ and $\mathbf{D}_d'$ hold for every $\mathbf{z}$ with $G(\mathbf{z}) > 0$ and $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$, $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$ the normalization. Then (1) is equivalent to (2). Hence*

i) (Association) G is associated, since G is max stable and max i.d.

ii) (Independence) If, in addition, condition (3) holds, then $\mathbf{Z}$ has independent components, and conversely.

iii) *(Total Dependence) If, in addition, (4) holds, then* $\mathbf{Z}$ *has totally dependent components, and conversely.*

Some of the above statements hold even under weaker conditions.

THEOREM 3.2. *Assume that for some stationary sequence* $\{\mathbf{X}_i, i \geq 1\}$ *in* R^d *the limit* G *in (1) exists and that the condition* $\mathbf{D}_d$ *holds for every* $\mathbf{z}$ *with* $G(\mathbf{z}) > 0$ *and* $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$, $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$ *the normalization. Then* G *is associated since it is max stable. Hence* G *is also PLOD and satisfies the inequality*

$$G(\mathbf{z}) \geq \max(G^*(\mathbf{z}),\ \Pi_{j=1}^d G_j(z_j)),$$

where $G^*(\mathbf{z}) = \lim_n F^n(\mathbf{a}_n\mathbf{z} + \mathbf{b}_n)$.

This statement follows since extreme values which occur in m_n-separated intervals are asymptotically independent by $\mathbf{D}_d$. This implies the max stability in the stationary case.

The case of independence also occurs if the random vectors $\mathbf{X}_i$ are negative dependent in some sense, since the above result shows that under Condition $\mathbf{D}_d$ the limit law G is associated and hence positive dependent. We assume a rather weak form of negative dependence, namely PNQD (pair-wise negative quadrant dependence): F is PNQD if every bivariate marginal $F_{jj'}$ is NQD (negative quadrant dependent), i.e., for every $1 \leq j < j' \leq d$

$$F_{jj'}(x, y) \leq F_j(x)F_{j'}(y) \text{ for all } x, y.$$

THEOREM 3.3. *Let* $\{\mathbf{X}_i, i \geq 1\}$ *be a stationary sequence of random vectors in* R^d *such that* F *is PNQD. Then*

i) *Condition (3) holds if* $n(1 - F(\mathbf{u}_n)) = O(1)$ *as* $n \to \infty$.

ii) *Assume also that (1),* $\mathbf{D}_d$ *and* $\mathbf{D}'_d$ *hold for every* $\mathbf{z}$ *with* $G(\mathbf{z}) > 0$ *and* $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$, $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$ *the normalization. Then the limit* $\mathbf{Z}$ *has independent components.*

Independent asymptotic components can also occur in another situation where $G_j \neq G_j^*$. This means that extreme values may occur locally in clusters. G_j is still an extreme value distribution if we assume Condition $\mathbf{D}_d$; more precisely, there exists an extremal index θ_j such that $G_j = (G_j^*)^{\theta_j}$ (cf. Leadbetter (1983)). By Theorem 3.2, G is also an extreme value distribution. By the result of Takahashi, if $G(\mathbf{z}) = \Pi_{j \leq d} G_j(z_j)$ for some $\mathbf{z}$ with $G_j(z_j) \in (0, 1)$ for all $j \leq d$, then $\mathbf{Z}$ has independent components. Therefore the joint behavior of the components has to be restricted in a suitable way to verify the condition of Takahashi. The following result is a slightly extended version of Theorem 3.4 of Hüsler (1987) and follows by similar arguments.

Condition $\mathbf{D}_d''$ holds for $\mathbf{z}$ with normalization $\mathbf{a}_n(>0)$ and $\mathbf{b}_n$ if

$$\lim_{r\to\infty}\limsup_{n\to\infty} r \sum_{1\le i,h\le n/r}\sum_{1\le j<l\le d} P\{X_{ij}>u_{nj},\ X_{hl}>u_{nl}\}=0.$$

THEOREM 3.4. *Let $\{\mathbf{X}_i, i\ge 1\}$ be a stationary sequence of random vectors in R^d, such that (1) holds. Assume that Condition $\mathbf{D}_d$ holds for every $\mathbf{z}$ with $G(\mathbf{z})>0$ and $\mathbf{u}_n=\mathbf{a}_n\mathbf{z}+\mathbf{b}_n$, $\mathbf{a}_n(>0)$, $\mathbf{b}_n$ the normalization. If $\mathbf{D}_d''$ holds for some $\mathbf{z}$ with $G_j(z_j)\in(0,1)$, then the limit $\mathbf{Z}$ has independent components.*

Finally we discuss the case of total dependence, without assuming Condition $\mathbf{D}_d$ and $\mathbf{D}_d'$. This result is not stated in Hüsler (1987), but it is an immediate consequence of the more general statement in Section 5, Theorem 5.4.

THEOREM 3.5. *Let $\{\mathbf{X}_i, i\ge 1\}$ be a stationary sequence of random vectors in R^d. Assume that $P\{M_{nj}\le a_{nj}z+b_{nj}\}\xrightarrow{w} G_1(z)$ holds for every $j\le d$ with a normalization $a_{nj}(>0)$, b_{nj}. If (4) holds for all z with $G_1(z)\in(0,1)$, then the limit $\mathbf{Z}$ exists and is totally dependent.*

In general, condition (4) is not necessary as is shown in Section 5 by an example. Note also, that if $\mathbf{Z}$ exists with G_1 being an extreme value distribution, it is sufficient that (4) holds only for some z with $G_1(z)\in(0,1)$ by Takahashi's result.

We also mention that Condition $\mathbf{D}_d$, $\mathbf{D}_d'$ and $\mathbf{D}_d''$ can be verified for a Gaussian sequence which satisfies a Berman type condition, i.e.,

$$r_{jj'}(n)\log n\to 0 \text{ as } n\to\infty.$$

Here $r_{jj'}(n)$ is the correlation of X_{1j} and $X_{nj'}$. This verification uses the technique developed in Berman (1964) (cf. Leadbetter et al. (1983)).

In Theorem 3.2–3.4 we did not assume Condition $\mathbf{D}_d'$. These results, however, heavily depend upon Condition $\mathbf{D}_d$. Without Condition $\mathbf{D}_d$ it would not be possible to give such a unified treatment of the behavior of extreme values. Note, for instance, that even negative dependent distributions G could occur as limits in (1) by taking a sequence of random vectors $\mathbf{X}_i\equiv\mathbf{X}_1$ for all $i\ge 1$ with $\mathbf{X}_1$ distributed as G.

Many of the statements can also be formulated for triangular arrays of random vectors. We only mention that, in general, a larger class of limit laws occurs for $\mathbf{M}_n$ (cf. Hüsler and Reiss (1989) for the Gaussian case).

4. The Nonstationary Independent Case. In the independent but nonstationary case, we need to consider the convergence of

$$\Pi_{i\le n}F_i(\mathbf{a}_n\mathbf{z}+\mathbf{b}_n)\xrightarrow{w} G(\mathbf{z}) \text{ as } n\to\infty. \tag{6}$$

The following results are contained in Hüsler (1988a). As mentioned in the introduction, we need to impose some restrictions in this case. We assume the following condition $\mathbf{A}_d$; the first part is a uniform asymptotic negligibility (u.a.n.) condition. Without this restriction, any G can occur as a limit in (6).

Condition $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n > 0$ and $\mathbf{b}_n$, if

$$F^*_{\max,n} = \max_{i \le n}\{1 - F_i(\mathbf{a}_n \mathbf{z} + \mathbf{b}_n)\} \to 0 \text{ as } n \to \infty$$

for all $\mathbf{z}$ and if

$$\sum_{i \le n}(1 - F_i(\mathbf{a}_n \mathbf{z} + \mathbf{b}_n)) \xrightarrow{w} w(\mathbf{z}) \textit{ as } n \to \infty.$$

Assume that $w(\mathbf{z}) < \infty$ for some $\mathbf{z} \in R^d$.

Note that $\mathbf{A}_d$ implies the existence of G. Conversely, if the u.a.n. condition holds, then the existence of G in (6) implies the second part of Condition $\mathbf{A}_d$ with $w(\mathbf{z}) = -\log G(\mathbf{z})$. Note also that in the stationary case, $F^*_{\max,n} = 1 - F(\mathbf{u}_n)$. Thus $n(1 - F(\mathbf{u}_n)) = O(1)$ implies the u.a.n. condition, i.e., the u.a.n. condition is implicitly assumed in the stationary case. In particular, we proved the following result.

THEOREM 4.1. *Let $\{\mathbf{X}_i, i \ge 1\}$ be an independent sequence of random vectors in R^d. Assume that $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$. Then the limit law G in (6) is max i.d., hence associated and PLOD.*

This follows by a result of Balkema and Resnick (1977). By assuming a slightly extended version of Condition $\mathbf{A}_d$, the limits G can be totally characterized. This extended version also implies that the limits of partial maxima $\mathbf{M}_{[nt]}, 0 < t \le 1$ have a max i.d. limit distribution. Since $\mathbf{M}_n = \mathbf{M}_{[nt]} \vee \mathbf{M}_{(nt,n]}$ with $\mathbf{M}_{(m,n]} = \max_{m<i\le n}\{\mathbf{X}_i\}$, a general decomposition of the limit G can be obtained (see Hüsler (1988a)) in an analogous way as for the sup self-decomposable distributions G (see Gerritse (1986)). These arise from the assumption $\mathbf{a}_n \equiv \mathbf{1}$ in (6).

The limit law G in (6) also has a positive dependence structure. If every $\mathbf{X}_i$ has a negative dependence structure, we again expect a limit with independent components as in the former sections. The following condition (7) is weaker and is, in general, the equivalent statement for independence. It follows as in the i.i.d. case and by Theorem 4.1.

THEOREM 4.2. *Let $\{\mathbf{X}_i, i \ge 1\}$ be an independent sequence of random vectors in R^d. Assume that $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$. Then*

$$(7) \qquad S_n^{(2)} = \sum_{i=1}^{n} \sum_{1 \le j < j' \le d} P\{X_{ij} > u_{nj},\, X_{ij'} > u_{nj'}\} \to 0 \textit{ as } n \to \infty$$

for all $\mathbf{z}$ such that $G(\mathbf{z}) > 0$, with $\mathbf{u}_n = \mathbf{a}_n \mathbf{z} + \mathbf{b}_n$, is equivalent to

$$(8) \qquad G(\mathbf{z}) = \Pi_{j=1}^{d} G_j(z_j),$$

where G_j is the j-th marginal of G.

COROLLARY 4.3. *Let $\{\mathbf{X}_i, i \geq 1\}$ be an independent sequence of random vectors in R^d. Assume that $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$. If for every $i \geq 1$, F_i is PNQD, then the limit $\mathbf{Z}$ in (6) has independent components.*

These two results on the independence can be slightly improved by only assuming the u.a.n. condition and the existence of all the univariate marginal limits G_j, instead of Condition $\mathbf{A}_d$. This weaker assumption will be used in the last section.

The total dependence case can be treated as before but with an obvious change. Here we assume $\mathbf{A}_d$ which implies the equivalence of (9) and (10).

THEOREM 4.4. *Let $\{\mathbf{X}_i, i \geq 1\}$ be an independent sequence of random vectors in R^d. Assume that $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$ and that the existing G satisfies $G_j \equiv G_1$, for every $j \leq d$. Then*

$$G(\mathbf{z}) = G_1(\min_j(z_j)) \tag{9}$$

is equivalent to

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} P\{X_{ij} > u_{nj},\ X_{ij'} > u_{nj'}\} - \sum_{i=1}^{n} P\{X_{ij} > u_{nj}\}\right) = 0 \tag{10}$$

for all $1 \leq j \neq j' \leq d$ and every z with $G_1(z) > 0$ and $u_{nj} = a_{nj}z + b_{nj}$.

In this case, it is generally necessary to assume (7) and (10) for all $\mathbf{z}$ since the results of Takahashi, which were proved for max stable distributions, do not hold for general max i.d. distributions (Hüsler (1989)). Note that because of independence of the random vectors, the second sum in (10) converges to $-\log G_1(z)$. In this case the extreme values M_{nj}, $j \leq d$, occur jointly at the same time point, asymptotically.

In the following situation, a rather restricted but interesting case occurs where we obtain an associated limit law without assuming Condition $\mathbf{D}_d$. This follows by simple properties of association. The PLOD property follows similarly.

THEOREM 4.5. *Let $\{\mathbf{X}_i, i \geq 1\}$ be an independent sequence of random vectors in R^d. Assume that $\mathbf{A}_d$ holds with normalization $\mathbf{a}_n$ and $\mathbf{b}_n$. If every F_i is associated (PLOD), then the distributions of $\mathbf{M}_n$ and of $\mathbf{Z}$ are associated (PLOD), respectively.*

5. The General Nonstationary Case. The extension of the results in Section 4 to this more general situation is carried out along the same lines as the extension of the classical i.i.d. case to the stationary situation. If we find conditions such that for every $\mathbf{z}$,

$$P\{\mathbf{X}_i \leq \mathbf{a}_n\mathbf{z} + \mathbf{b}_n\} - \Pi_{i \leq n} F_i(\mathbf{a}_n\mathbf{z} + \mathbf{b}_n) \to 0 \text{ as } n \to \infty, \tag{11}$$

all the results of Section 4 can be reformulated in this general case. But again, we are interested in finding weaker conditions such that the four dependence properties hold for a possible limit G in (1). We obviously assume the u.a.n. condition $\mathbf{A}_d$ in this section again. The following results are discussed and proved in Hüsler (1988b).

The following mixing condition $\mathbf{D}_d$ is an extension of the mixing condition in the stationary case. We use the same notation as before since both mixing conditions are equivalent in the stationary case.

Condition $\mathbf{D}_d = \mathbf{D}_d(\{\mathbf{u}_n, n \geq 1\})$. We assume that there exists an array $\{\alpha_{nm}, n \geq 1, m \leq n\}$ such that

i) $|P(B_n(I \cup J)) - P(B_n(I))P(B_n(J))| \leq \alpha_{nm}$ for every pair of subsets I and J of $\{1, \ldots, n\}$ which are m-separated and

ii) $\lim_{n\to\infty} \alpha_{n,m_n^*} = 0$ for some sequence $\{m_n^*, n \geq 1\}$ with $m_n^* \to \infty$ and $m_n^* F_{\max,n}^* \to 0$ as $n \to \infty$.

Note that in the stationary case, $m_n^* F_{\max,n}^* = m_n^*(1 - F(\mathbf{u}_n)) = O(m_n^*/n)$. To prove (11) we use the following extension of the local mixing condition in the stationary case, which we again denote by $\mathbf{D}_d'$. For any $I \subset \{1, \ldots, n\}$ and $\delta > 0$, define $d_n'(I, \delta)$ by

$$d_n'(I,\delta) = \min_{I^* \subset I} \sum_{i<h\in I^*} P\{\mathbf{X}_i \not\leq \mathbf{u}_n, \mathbf{X}_h \not\leq \mathbf{u}_n\}$$

where $\Sigma_{i\in I\setminus I^*} P\{\mathbf{X}_i \not\leq \mathbf{u}_n\} < \delta$. Let $F_n^*(I) = \Sigma_{i\in I} P\{\mathbf{X}_i \not\leq \mathbf{u}_n\}$ and $F_n^* = F_n^*(\{1, \ldots, n\})$. Note that we define $d_n'(I, \delta)$ as the sum on a suitable subset I^* of I. This idea is very useful in the Gaussian case, where some random vectors may have a heavy weight in the sum $\Sigma_{i<h\in I} P\{\mathbf{X}_i \not\leq \mathbf{u}_n, \mathbf{X}_h \not\leq \mathbf{u}_n\}$, but not in the sum $\Sigma_{i\in I} P\{\mathbf{X}_i \not\leq \mathbf{u}_n\}$ (cf. Hüsler (1983) in the univariate case and Hüsler and Schüpbach (1988) in the multivariate case). Obviously, this idea is also useful in the general non Gaussian case.

Condition $\mathbf{D}_d' = \mathbf{D}_d'(\{\mathbf{u}_n, n \geq 1\})$. We assume that there exist an array $\{\alpha_{nr}', n \geq 1, r \geq 1\}$ and a sequence $\{g_r, r \geq 1\}$ such that $\lim_{r\to\infty} r g_r = 0$, $\lim_{r\to\infty} \limsup_{n\to\infty} r\alpha_{nr}' = 0$ and for every $r \geq 1$ and for all $n \geq n_0(r) : d_n'(I, g_r) \leq \alpha_{nr}'$ for all $I \subset \{1, \ldots, n\}$ such that $F_n^*(I) \leq F_n^*/r$.

Both conditions $\mathbf{D}_d$ and $\mathbf{D}_d'$ together imply that $\mathbf{M}_n$ behaves asymptotically as if $\{\mathbf{X}_i, i \geq 1\}$ would be an independent sequence.

THEOREM 5.1. *Let $\{\mathbf{X}_i, i \geq 1\}$ be a general sequence of random vectors in R^d. Assume that $\mathbf{A}_d$, $\mathbf{D}_d$ and $\mathbf{D}_d'$ hold for every $\mathbf{z}$ with $G(\mathbf{z}) > 0$ (or $w(\mathbf{z}) < \infty$) and with $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$ and normalization $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$. Then (1) is equivalent to (6). Hence*

i) (Association) G is associated, since G is max stable and max i.d.

ii) (Independence) If, in addition, condition (7) holds, then $\mathbf{Z}$ has independent components and conversely.

iii) (Total Dependence) If, in addition, (10) holds, then $\mathbf{Z}$ *has totally dependent components and conversely.*

We can still prove that G has a positive dependence structure without assuming the local mixing condition, as in the stationary case.

THEOREM 5.2. *Let* $\{\mathbf{X}_i, i \geq 1\}$ *be a general nonstationary sequence of random vectors in* R^d. *Assume that the conditions* $\mathbf{A}_d$ *and* $\mathbf{D}_d$ *hold for every* $\mathbf{z}$ *such that* $G(\mathbf{z}) > 0$, *with* $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$ *and normalization* $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$. *If*

$$P\{\mathbf{M}_n \leq \mathbf{a}_n\mathbf{z} + \mathbf{b}_n\} \xrightarrow{w} G(\mathbf{z})$$

as $n \to \infty$, *then* G *is max i.d., hence associated.*

The next statement considers the asymptotic independence of the components for the extreme values. If the local mixing condition $\mathbf{D}'_d$ is not assumed, there exists the possibility that the extreme values will cluster in small time intervals. However, the components of $\mathbf{Z}$ can still be independent, if we assume a condition $\mathbf{D}''_d$, similar to the stationary case. To illustrate the possibility of clustering, consider, e.g., the simple case of independent bivariate random vectors with $X_{i1} \equiv X_{i2} - \gamma_i, \gamma_i$ real. Such a sequence satisfies the conditions $\mathbf{D}_d$ and $\mathbf{D}'_d$. Clustering occurs at the same time point, jointly in the two components. An asymptotic result for such an example would still follow from Theorem 5.1. Another interesting case of clustering arises if we consider, in every component, the joint clustering of extreme values in small time intervals. For instance, let Y_{ij} be independent random variables and define $X_{ij} = Y_{[i/\gamma_j]+1,j}$ with γ_j integer. Obviously $\mathbf{M}_n$ has independent components, but $\mathbf{D}'_d$ does not hold if $\gamma_j > 1$. A limit $\mathbf{Z}$ with independent components Z_j is still possible for such a clustering. Hence we define the following restriction.

Condition $\mathbf{D}''_d$ is defined in the same way as Condition $\mathbf{D}'_d$, where the expression $d'_n(I, \delta)$ is replaced by

$$d''_n(I, \delta) = \min_{I^* \subset I} \sum_{i,h \in I^*} \sum_{j < \ell} P\{X_{ij} > u_{nj},\ X_{hl} > u_{nl}\}.$$

Condition $\mathbf{D}''_d = \mathbf{D}''_d(\{\mathbf{u}_n, n \geq 1\})$. We assume that there exist an array $\{\alpha''_{nr}, n \geq 1, r \geq 1\}$ and a sequence $\{g''_r, r \geq 1\}$ such that $\lim_{r\to\infty} r g''_r = 0$, $\lim_{r\to\infty} \limsup_{n\to\infty} r\alpha''_{nr} = 0$ and for every $r \geq 1$ and for all $n \geq n_0(r) : d''_n(I, g_r) \leq \alpha''_{nr}$ for all $I \subset \{1, \ldots, n\}$ such that $F^*_n(I) \leq F^*_n/r$.

Then, analogous to the results in the stationary case, the independence of the components of $\mathbf{Z}$ occurs in the following situation described by Theorem 5.3. Note, however, that the limit G is, in general, not a max stable distribution. Hence we cannot make use of the results of Takahashi to improve upon the general statement of the theorem. For the last two statements we use only the u.a.n. condition $F^*_{\max,n} \to 0$ and the existence of the univariate marginal limits G_j, i.e.,

$$P\{M_{nj} \le a_{nj}z + b_{nj}\} \xrightarrow{w} G_j(z) \text{ as } n \to \infty \tag{12}$$

for every $j \le d$ with normalization a_{nj} and b_{nj}. The convergence in (12) is discussed in Hüsler (1983, 1986), where sufficient conditions such as the univariate versions $\mathbf{D}_1$ and $\mathbf{D}_1'$ are formulated. The results in Section 3 do not follow as a special case of Theorem 5.3 since we are assuming slightly different conditions.

THEOREM 5.3. *Let $\{\mathbf{X}_i, i \ge 1\}$ be a general nonstationary sequence of random vectors in R^d. Assume that the u.a.n. condition, (12) and $\mathbf{D}_d$ hold for every $\mathbf{z}$ such that $G_j(z_j) > 0$, for all $j \le d$ with $\mathbf{u}_n = \mathbf{a}_n\mathbf{z} + \mathbf{b}_n$ and normalization $\mathbf{a}_n(> 0)$, $\mathbf{b}_n$. Then $G(\mathbf{z}) = \Pi_{j=1}^d G_j(z_j)$ if either, for every $\mathbf{z}$ with $\mathbf{a}_n$ and $\mathbf{b}_n$,*

i) $\mathbf{D}_d'$ holds and every F_i is PNQD, or

ii) $\mathbf{D}_d''$ holds.

Finally we consider the total dependence case. The total dependence result follows rather easily without assuming condition $\mathbf{D}_d$.

THEOREM 5.4. *Let $\{\mathbf{X}_i, i \ge 1\}$ be a sequence of random vectors in R^d. Assume that (12) and (10) hold for every $u_{nj} = a_{nj}z + b_{nj}$ with z such that $G_1(z) > 0$ and $G_j \equiv G_1$ for all $j \le d$. Then the limit distribution G in (1) exists with*

$$G(\mathbf{z}) = G_1(\min_j(z_j)).$$

Consequently G is totally dependent.

This statement holds for extreme values of random sequences which exhibit a behavior similar to that of independent sequences. More precisely, this means that the extreme values M_{nj}, $j \le d$, mainly occur jointly at the same time point. It does not include random sequences such as, e.g., $\mathbf{X}_i = (Y_{i+1}, Y_i)$ where $\{Y_i\}$ is an i.i.d. sequence of random variables satisfying (12).

Note also that a general nonstationary Gaussian sequence satisfies Condition $\mathbf{D}_d$, $\mathbf{D}_d'$ and $\mathbf{D}_d''$ for any normalization $\mathbf{u}_n$ satisfying Condition $\mathbf{A}_d$ if a Berman type condition holds (Hüsler and Schüpbach (1988)).

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