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COPULAS AND MARKOV OPERATORS

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For any pair of random variables X and Y with common domain, there is a function $C : [0,1]^2 \to [0,1]$ such that

$$F_{XY}(x,y) = C(F_X(x), F_Y(y)).$$

The function C is called a copula; it is a continuous, monotonic function satisfying boundary conditions $C(x,0) = C(0,y) = 0$, $C(x,1) = x$ and $C(1,y) = y$. Let $C_{,1} = \partial C/\partial x$ and $C_{,2} = \partial C/\partial y$. Then

$$\begin{aligned} E(I_{X\leq x} \,|\, Y) &= C_{,2}(F_X(x), F_Y(Y)) \quad \text{a.s.} \\ E(I_{Y\leq y} \,|\, X) &= C_{,1}(F_X(X), F_Y(y)) \quad \text{a.s.} \end{aligned}$$

Some consequences of these relations are explored. In particular, they allow the conditional independence condition of a Markov process to be expressed in terms of the copulas of pairs of random variables in the process. The resulting conditional independence condition gives a natural product operation on the set of copulas:

$$A * B(x,y) = \int_0^1 A_{,2}(x,s) B_{,1}(s,y)\, ds.$$

The set of copulas under this product is isomorphic to the set of Markov operators T on $L^\infty[0,1]$ under composition, via the correspondence

$$\begin{aligned} [T_C f](x) &= \frac{d}{dx} \int_0^1 C_{,2}(x,t) f(t)\, dt \\ C_T(x,y) &= \int_0^x [TI_{[0,y]}](s)\, ds. \end{aligned}$$

This correspondence is discussed.

1. Introduction. A copula is a function $C : [0,1]^2 \to [0,1]$ satisfying

Boundary conditions: For all x, $y \in [0,1]$,

$$C(x,0) = C(0,y) = 0$$

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$$C(x,1) = x$$
$$C(1,y) = y.$$

Monotonicity condition: Whenever $x_1 \leq x_2$ and $y_1 \leq y_2$

$$C(x_1, y_1) - C(x_2, y_1) - C(x_1, y_2) + C(x_2, y_2) \geq 0.$$

Copulas have several nice properties. Among the properties we shall use here are:

(a) A copula C is a Lipschitz continuous function satisfying

$$|C(x_1, y_1) - C(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|.$$

(b) If C is a copula, then $x \to C(x,y)$ and $y \to C(x,y)$ are nondecreasing functions.

(c) First partial derivatives of copulas exist almost everywhere, and $x \to C_{,2}(x,y)$ and $y \to C_{,1}(x,y)$ are almost certainly nondecreasing.

(d) The set $\mathcal{C}$ of copulas is a compact subset of $L^\infty([0,1]^2)$.

Here and elsewhere we use the notation

$$C_{,1}(x,y) = \partial C(x,y)/\partial x$$
$$C_{,2}(x,y) = \partial C(x,y)/\partial y.$$

Note that the Lipschitz property of a copula implies that the sections $x \to C(x,y)$ and $y \to C(x,y)$ are absolutely continuous functions, so that C can be recovered from either of its first partial derivatives by integration, using the boundary condition of the left boundary (if it is $C_{,1}$ which is integrated) or the lower boundary (if it is $C_{,2}$ which is integrated). While the differentiability properties of copulas are well known, they are often not exploited; the results reported here exploit the differentiability properties. For a discussion of properties of copulas, see Darsow et al. (1992).

Copulas are of interest because of the following theorem:

THEOREM 1.1 (Sklar (1959), Sklar (1973)). *For any real valued random variables X and Y with joint distribution F_{XY} there is a copula C such that*

$$F_{XY}(x,y) = C(F_X(x), F_Y(y))$$

where F_X and F_Y denote the distribution functions of X and Y, respectively. If X and Y are continuous, the copula C is unique. If not, the values of C are uniquely determined at points $(F_X(x), F_Y(y))$ where x is in the range of X and y is in the range of Y, and the values of C at other points can be assigned using bilinear interpolation.

The proof for the general n–dimensional case is outlined in Appendix A of Sklar (1995), in this volume.

When we speak of *the* copula of a pair of random variables X and Y, we will mean the copula whose existence is guaranteed by the theorem, using bilinear interpolation, if one or both of the random variables are not continuous (i.e., have discontinuous cumulative distribution functions). When both are continuous, the copula is uniquely determined without the need of an interpolation convention. We remark that the bilinear interpolation convention is used in the proofs of both Theorem 1.2 and Theorem 1.3, below, to handle cases when random variables are not continuous, see Darsow et al. (1992).

Theorem 1.1 states that copulas are related to joint distribution functions. The derivatives of copulas are directly related to conditional expectations:

THEOREM 1.2. *If X and Y are real valued random variables with distribution functions F_X and F_Y and joint distribution function F_{XY}, and if C is the copula of X and Y, then*

$$\begin{aligned} E(I_{X\le x}\,|\,Y) &= C_{,2}(F_X(x), F_Y(Y)) \quad \text{a.s.} \\ E(I_{Y\le y}\,|\,X) &= C_{,1}(F_X(X), F_Y(y)) \quad \text{a.s.} \end{aligned}$$

Here and elsewhere I_G denotes the characteristic function of a set G.

PROOF. A proof is given in Darsow et al. (1992). A heuristic proof, which is rigorous when F_X and F_Y are continuous and strictly increasing, is as follows:

$$P(X \le x\,|\,y \le Y \le y + \Delta y) = \frac{C(F_X(x), F_Y(y+\Delta y)) - C(F_X(x), F_Y(y))}{F_Y(y+\Delta y) - F_Y(y)}.$$

Take the limit as $\Delta y \to 0$ to obtain the first conclusion of the theorem. ∎

Now if random variables X, Y and Z are continuous random variables, and X and Z are conditionally independent given Y, then by definition, for all real numbers x and z,

$$E([I_{X\le x}][I_{Z\le z}]\,|\,Y) = E(I_{X\le x}\,|\,Y)E(I_{Z\le z}\,|\,Y) \qquad \text{a.s.}$$

Integrate this expression over the common domain of X, Y and Z. The integral of the left hand side can be replaced by the joint distribution function of X and Z; substitute the copulas of X and Z, X and Y and Y and Z, using

Theorems 1.1 and 1.2, to obtain

$$\begin{aligned}
&C_{XZ}(F_X(x), F_Z(z)) \\
&\quad = \int_\Omega C_{XY,2}(F_X(x), F_Y(Y(\omega)))C_{YZ,1}(F_Y(Y(\omega)), F_Z(z))\, dP(\omega) \\
&\quad = \int_0^1 C_{XY,2}(F_X(x), \eta)C_{YZ,1}(\eta, F_Z(z))\, d\eta.
\end{aligned}$$

The condition that X and Z are conditionally independent given Y implies, therefore, the following relation among the copulas:

$$C_{XZ}(x, z) = \int_0^1 C_{XY,2}(x, y)C_{YZ,1}(y, z)\, dy.$$

This leads to a way to state the conditional independence condition of a real valued Markov process in terms of the copulas of random variables of the process. An explicit expression for the finite dimensional distributions of a Markov process in terms of the copulas pairs of random variables, is given in Darsow et al. (1992). The content of the Chapman–Kolmogorov equations can be stated in terms of the copulas of the process in a particularly simple way:

THEOREM 1.3. *Let $t \to X_t$, $t \in T$, denote a stochastic process. Let C_{st} denote the copula of X_s and X_t, $s < t$. The following statements are equivalent:*

(1) *The transition probabilities $P(s, x, t, E) = P(X_t \in E \mid X_s = x)$ of the process satisfy the Chapman–Kolmogorov equations*

$$P(s, x, t, E) = \int_{-\infty}^{\infty} P(u, \xi, t, E)P(s, x, u, d\xi)$$

for all Borel sets E, for all $s < t$ in T, for all $u \in (s, t) \cap T$ and for almost all $x \in R$.

(2) *For all s, u, $t \in T$ satisfying $s < u < t$, and for all $(x, y) \in [0, 1]^2$*

$$C_{st}(x, y) = \int_0^1 C_{su,2}(x, t)C_{ut,1}(t, y)\, dt.$$

A proof is given in Darsow et al. (1992). We remark that the theorem is not restricted to continuous random variables and that the bilinear interpolation convention mentioned above is used in the proof of the theorem in case not all of the random variables X_t are continuous. Theorem 1.3 says roughly that copulas capture the dependence structure of real valued Markov processes in a manner equivalent to that of the Chapman–Kolmogorov equations. Note, however, that they do so without any information about the marginal distributions of the process.

Theorem 1.3 motivates the following definition of a product on the set $\mathcal{C}$ of copulas: For copulas A and B define

$$A * B(x,y) = \int_0^1 A_{,2}(x,t)B_{,1}(t,y)\,dt. \tag{1.1}$$

The product defined in (1.1) has the following properties:

(a) $A * B$ is a copula.

(b) $*$ is an associative binary operation.

(c) $*$ is continuous in each place with respect to the uniform topology.

These properties are established in Darsow et al. (1992). Using the $*$ product notation, we can restate Theorem 1.3 in compact form:

THEOREM 1.3. (Restated) *A real stochastic process X_t satisfies the Chapman–Kolmogorov equations if and only if, for $s < u < t$,*

$$C_{st} = C_{su} * C_{ut} \tag{1.2}$$

where C_{st} denotes the copula of X_s and X_t.

To emphasize the content of Theorem 1.3, we indicate how a discrete time Markov process can be constructed by specifying marginal distributions and copulas:

Let $T = N =$ natural numbers. The construction is as follows:

(a) Assign copulas $C_{n;n+1}$ in any manner.

(b) For $k > 1$ set

$$C_{n;n+k} = C_{n;n+1} * C_{n+1;n+2} * \ldots * C_{n+k-1;n+k}.$$

(c) Assign (continuous) marginal distributions F_n in any manner.

(d) Require that the n–dimensional distributions for $n > 2$ satisfy the conditional independence condition for a Markov process.

(e) Apply Kolmogorov's fundamental theorem to obtain a stochastic process X_t with the specified finite dimensional distributions.

Observe that the copulas assigned in (a) and (b) are the copulas of a Markov process, regardless of what distributions are assigned in step (c) of the construction.

The integral operators suggested by the Chapman–Kolmogorov equations are Markov operators; the content of the Chapman–Kolmogorov equations can be stated as a condition involving the composition of Markov operators. This implies that there is a natural relation between Markov operators and copulas. The goal of this paper is to make that relationship explicit and then to investigate the relationship. In Section 2 of the paper, we define Markov operators and establish an isomorphism between copulas under the $*$ product and Markov operators on $L^\infty[0,1]$ under composition. In Section 3, we investigate the relationship. In particular, we show how to obtain proofs of some known properties of Markov operators via the isomorphism theorem and some known properties of copulas, and we translate some statements about copulas to Markov operators using the isomorphism. Section 4 contains discussion and conclusions.

2. An Isomorphism between Markov Operators and Copulas. Let $\{\Omega;\sigma;\mu\}$ be a measure space. We assume Ω is a compact set in R^n. A linear operator $T : L^\infty(\Omega) \to L^\infty(\Omega)$ is a Markov operator if

(a) T is positive, that is $f \geq 0$ implies $Tf \geq 0$;

(b) The constant function $f = 1$ is a fixed point of T; and

(c) $\int_\Omega Tf\,d\mu = \int_\Omega f\,d\mu$ for all $f \in L^\infty$. (2.1)

It follows easily from the defining properties that

$$\|T\| = \sup_{\|f\|_\infty = 1} \|Tf\|_\infty = 1.$$

It is also not difficult to show that T extends to a bounded linear operator on $L^p(\Omega)$ for $p \in [1,\infty)$ and that the $L^1(\Omega)$ operator norm of T is 1.

We prove the isomorphism theorem by way of some lemmas; we will state the theorem after proving the lemmas.

LEMMA 2.1. *For a copula C, define T_C via*

$$[T_C f](x) = \frac{d}{dx}\int_0^1 C_{,2}(x,t)f(t)\,dt. \tag{2.2}$$

Then T_C is a Markov operator on $L^\infty([0,1])$.

PROOF. We show first that if $f \in L^\infty([0,1])$ then

$$x \to \int_0^1 C_{,2}(x,t)f(t)\,dt = g(x)$$

is Lipschitz continuous with Lipschitz constant $\|f\|_{0,\infty}$. For if $x_1 < x_2$ then

$$\begin{aligned}
|g(x_2) - g(x_1)| &= |\int (C_{,2}(x_2,t) - C_{,2}(x_1,t))f(t)\,dt| \\
&\leq \|f\|_{0,\infty} \int |C_{,2}(x_2,t) - C_{,2}(x_1,t)|\,dt \\
&= \|f\|_{0,\infty} \int (C_{,2}(x_2,t) - C_{,2}(x_1,t))\,dt \\
&= \|f\|_{0,\infty}(C(x_2,1) - C(x_1,1)) \\
&= \|f\|_{0,\infty}(x_2 - x_1).
\end{aligned}$$

The statement in the third line above makes use of the fact that $x \to C_{,2}(x,t)$ is almost surely increasing, the statement in the fourth line makes use of the fact that C is absolutely continuous in each place, and the last statement makes use of the boundary conditions satisfied by C.

We show next that the derivative in (2.2) exists. Since $x \to C_{,2}(x,t)$ is a.s. increasing, the functions

$$x \to \int_0^1 C_{,2}(x,t)(|f(t)| - f(t))\,dt \qquad \text{and}$$

$$x \to \int_0^1 C_{,2}(x,t)(|f(t)| + f(t))\,dt$$

are both a.s. increasing and thus have derivatives pointwise a.e. Since

$$x \to \int_0^1 C_{,2}(x,t)f(t)\,dt$$

is a linear combination of the foregoing functions, it follows that the derivative in (2.2) exists pointwise a.e.

The result of the foregoing paragraph shows that the derivative is bounded above by $\|f\|_{0,\infty}$, so $T_C f \in L^\infty([0,1])$. Positivity of T_C follows from similar considerations.

That $\int Tf\,d\mu = \int f\,d\mu$ and that $T_C 1 = 1$ are direct calculations. We indicate the former calculation:

$$\begin{aligned}
\int_0^1 T_C f\,dx &= \int_0^1 \frac{d}{dx}\int_0^1 C_{,2}(x,t)f(t)\,dt\,dx \\
&= \int_0^1 (C_{,2}(1,t) - C_{,2}(0,t))f(t)\,dt \\
&= \int_0^1 f(t)\,dt.
\end{aligned}$$

The statement on the second line uses the fact, proved above, that

$$x \to \int_0^1 C_{,2}(x,t)f(t)\,dt$$

is Lipschitz continuous, hence absolutely continuous, and the statement in the third line uses the fact that since $C(1,t) = t$ and $C(0,t) = 0$, necessarily $C_{,2}(1,t) = 1$ and $C_{,2}(0,t) = 0$. ∎

Conversely, if T is a Markov operator on $L^\infty([0,1])$, we may define a function C_T via

$$C_T(x,y) = \int_0^x [TI_{[0,y]}](s)\,ds. \tag{2.3}$$

LEMMA 2.2. *C_T as so defined is a copula, and if we write $T_C = \Phi(C)$ for the function which maps C into T_C via (2.2), and $C_T = \Psi(T)$ for the map defined by (2.3), then $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity operators.*

PROOF. To establish that C_T is a copula, observe first that if $x_1 < x_2$ and $y_1 < y_2$ then

$$C_T(x_1,y_1) - C_T(x_2,y_1) - C_T(x_1,y_2) + C_T(x_2,y_2) = \int_{x_1}^{x_2} [TI_{(y_1,y_2]}](s)\,ds \geq 0$$

using the positivity of T. Hence, C_T is monotonic. As to boundary conditions, observe that

$$\begin{aligned} C_T(x,1) &= \int_0^x [TI_{[0,1]}](s)\,ds = \int_0^x ds = x \\ C_T(1,y) &= \int_0^1 [TI_{[0,y]}](s)\,ds = \int_0^1 I_{[0,y]}(s)\,ds = y. \end{aligned}$$

The first statement uses the fact that the constant function $f = 1$ is a fixed point of T and the second uses the fact that $\int Tf\,d\mu = \int f\,d\mu$.

We turn to the last assertion of the Lemma. Observe that

$$\begin{aligned} [\Psi \circ \Phi(C)](x,y) &= C_{\Phi(C)}(x,y) \\ &= \int_0^x [T_C I_{[0,y]}](s)\,ds \\ &= \int_0^x \frac{d}{ds}\int_0^1 C_{,2}(s,t) I_{[0,y]}(t)\,dt\,ds \\ &= \int_0^1 C_{,2}(x,t) I_{[0,y]}(t)\,dt \\ &= \int_0^y C_{,2}(x,t)\,dt \\ &= C(x,y). \end{aligned}$$

Observe next that, applying $\Phi \circ \Psi(T)$ to the characteristic function $I_{[0,y]}$, we obtain

$$\begin{aligned}
[\Phi \circ \Psi(T) I_{[0,y]}](x) &= [T_{\Psi(T)} I_{[0,y]}](x) \\
&= \frac{d}{dx} \int_0^1 \frac{\partial}{\partial t} \Big(\int_0^x [T I_{[0,t]}](s)\, ds \Big) I_{[0,y]}(t)\, dt \\
&= \frac{d}{dx} \int_0^y \frac{\partial}{\partial t} \Big(\int_0^x [T I_{[0,t]}](s)\, ds \Big)\, dt \\
&= \frac{d}{dx} \int_0^x [T I_{[0,y]}](s)\, ds \\
&= [T I_{[0,y]}](x).
\end{aligned}$$

It follows that $\Phi \circ \Psi(T)$ agrees with T on the set of functions which are linear combinations of characteristic functions of intervals $(y_1, y_2] \subset [0,1]$; since this set is dense in L^1 and both $\Phi \circ \Psi(T)$ and T are Markov operators, hence bounded, on L^1, they necessarily agree everywhere on L^1. Since $L^p \subset L^1$ for $p \in (1, \infty]$, we have the desired result. ∎

LEMMA 2.3. *Let B and C be copulas. Then $T_B \circ T_C = T_{B*C}$.*

PROOF. Let $f \in C^\infty$. Then

$$\begin{aligned}
[T_B \circ T_C f](x) &= \frac{d}{dx} \int_0^1 B_{,2}(x,t) [T_C f](t)\, dt \\
&= \frac{d}{dx} \int_0^1 B_{,2}(x,t) \frac{d}{dt} \int_0^1 C_{,2}(t,s) f(s)\, ds\, dt \\
&= \frac{d}{dx} \int_0^1 B_{,2}(x,t) \frac{d}{dt} \Big(t f(1) - \int_0^1 C(t,s) f'(s)\, ds \Big)\, dt \\
&= \frac{d}{dx} \Big(x f(1) - \int_0^1 \int_0^1 B_{,2}(x,t) C_{,1}(t,s) f'(s)\, dt\, ds \Big) \qquad (2.4) \\
&= \frac{d}{dx} \Big(x f(1) - \int_0^1 B * C(x,s) f'(s)\, ds \Big) \\
&= \frac{d}{dx} \Big(x f(1) - x f(1) + \int_0^1 (B * C)_{,2}(x,s) f(s)\, ds \Big) \\
&= [T_{B*C} f](x).
\end{aligned}$$

Since (2.4) holds for a dense set in L^1, it holds for all $f \in L^1$ and thus, since $L^\infty \subset L^1$, for all $f \in L^\infty$. ∎

It is obvious that the map $C \to T_C$ also preserves convex combinations. It is also true that transposes are mapped into adjoints: $C^\top \to (T_C)^\dagger$.

LEMMA 2.4. *Extend T_C to $L^p([0,1])$, $p \in (1, \infty)$. Then $T_{C^\top} = (T_C)^\dagger$.*

PROOF. For all $f \in L^p$ and for all test functions $g \in C_0^\infty([0,1])$ (the set of infinitely differentiable functions vanishing at 0 and 1), we have

$$\begin{aligned}\int_0^1 g(x)[T_C f](x)\,dx &= \int_0^1 g(x)\frac{d}{dx}\int_0^1 C_{,2}(x,t)f(t)\,dt\,dx\\ &= -\int_0^1 g'(x)\int_0^1 C_{,2}(x,t)f(t)\,dt\,dx\\ &= -\int_0^1 f(t)\Big(\frac{d}{dt}\int_0^1 g'(x)C(x,t)\,dx\Big)\,dt\\ &= \int_0^1 f(t)\Big(\frac{d}{dt}\int_0^1 g(x)C_{,1}(x,t)\,dx\Big)\,dt\\ &= \int_0^1 [T_{C^\top} g](t)f(t)\,dt.\end{aligned}$$

Since the test functions are dense in the dual of L^p, we have the desired result. ∎

The foregoing lemmas yield the following result:

THEOREM 2.1. (Isomorphism Theorem). *The correspondence $T \to C_T$ of (2.3) is an isomorphism of the set of Markov operators on $L^\infty([0,1])$, under composition, and the set $\mathcal{C}$ of copulas, under the $*$ product. That is, if we set $\Phi(C) = T_C$, then Φ is one–to–one and onto and*

(a) $\Phi(C_1 * C_2) = \Phi(C_1) \circ \Phi(C_2)$,

(b) $\Phi(\lambda C_1 + (1-\lambda)C_2) = \lambda\Phi(C_1) + (1-\lambda)\Phi(C_2)$, and

(c) $\Phi(C^\top) = \Phi(C)^\dagger$.

We remark that Ryff (1963), utilizing a prior characterization of L_1 operators established by Kantorovich and Vulich (1937), showed that every Markov operator T on $L^\infty([0,1])$ has the representation

$$[Tf](x) = \frac{d}{dx}\int_0^1 K(x,y)f(y)\,dy$$

where the kernel K is measurable and satisfies the following six conditions:

(a) $K(0,y) = 0$, $0 \le y \le 1$.

(b) $\operatorname{Ess\,sup} V[K(\cdot,y)] = C < \infty$, where by $V[K(\cdot,y)]$ is meant the total variation of K for fixed y.

(c) $x \to \int_0^1 K(x,y)f(y)\,dy$ is absolutely continuous for every $f \in L^1$.

(d) $x = \int_0^1 K(x,y)\,dy$ for all $x \in [0,1]$.

(e) $x_1 \le x_2$ implies $K(x_1,\cdot) \le K(x_2,\cdot)$.

(f) $K(1,y)=1,\ 0\le y\le 1$.

It is easy to verify that if C is a copula then $C_{,2}$ possesses these six properties, and conversely that if K possesses the six properties, then C defined by

$$C(x,y)=\int_0^y K(x,t)\,dt \tag{2.5}$$

is a copula (the existence of the integral in (2.5) follows from property (d), monotonicity follows from (e), and the boundary conditions follow from (a), (d) and (f)). Thus, Ryff's characterization plus a few calculations give most of the results above. We note that Ryff did not obtain (but easily could have obtained) the law of composition of Markov operators implied by his characterization; this law does not have a nice form, unless it is formulated in terms of copulas. We note also that the interpretation of Ryff's result seems to demand Theorem 1.2 concerning copulas and conditional expectations, and hence seems to demand reformulation in terms of copulas.

3. Applications of the Isomorphism Theorem. We address first some topological issues.

THEOREM 3.1. *Let C_n denote a sequence of copulas, and write T_n for $\Phi(C_n)$, where Φ is the isomorphism of the preceding theorem. $T_n\to T$ in the weak operator topology of L^p, for $p\in[1,\infty)$, if and only if $C_n\to C_T$ uniformly.*

We prove this result by way of two lemmas.

LEMMA 3.1. *Let C_n and C be copulas, and let $p\in(1,\infty]$. The following statements are equivalent:*

(a) $\|C-C_n\|_{0,\infty}\to 0$.

(b) For all $f\in L^p([0,1]^2)$,

$$\int_0^1\int_0^1 f(x,y)C_{n,2}(x,y)\,dx\,dy\to\int_0^1\int_0^1 f(x,y)C_{,2}(x,y)\,dx\,dy.$$

(c) For all $f\in L^p([0,1]^2)$,

$$\int_0^1\int_0^1 f(x,y)C_{n,1}(x,y)\,dx\,dy\to\int_0^1\int_0^1 f(x,y)C_{,1}(x,y)\,dx\,dy.$$

PROOF. We will show that statement (a) implies statement (b) and that statement (b) implies statement (a). The arguments that statement (a) implies statement (c) and that statement (c) implies statement (a) are similar.

Suppose statement (a) holds. We will show that every subsequence of $C_{n,2}$ possesses a subsubsequence converging weakly to $C_{,2}$. Let $C_{n_k,2}$ be a subsequence; relabel: $C_{k,2}$. Since the unit ball in L^q, $q \in (1,\infty)$, is weakly sequentially compact, there is a subsubsequence $C_{k_j,2}$ (relabel: $C_{j,2}$) and a function $g \in L^q$ such that

$$\int_0^1 \int_0^1 f(x,y) C_{j,2}(x,y)\,dx\,dy \to \int_0^1 \int_0^1 f(x,y) g(x,y)\,dx\,dy \tag{3.1}$$

for all $f \in L^p$, where $1/p + 1/q = 1$; in particular, (3.1) holds for all test functions $f \in C_0^\infty$. But for any such test function, we have

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y) C_{j,2}(x,y)\,dx\,dy &= -\int_0^1 \int_0^1 f_{,2}(x,y) C_j(x,y)\,dx\,dy \\ &\to -\int_0^1 \int_0^1 f_{,2}(x,y) C(x,y)\,dx\,dy.\end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we conclude that necessarily $g = C_{,2}$. This implies the desired result.

Conversely, suppose statement (b) holds. We will show that every subsequence of C_n possesses a subsubsequence converging uniformly to C. Let C_{n_k} be a subsequence; relabel: C_k. Since the set of all copulas $\mathcal{C}$ is compact in the uniform topology, there is a subsubsequence C_{k_j} (relabel: C_j) and a copula $B \in \mathcal{C}$ such that $\|B - C_j\|_{0,\infty} \to 0$. But then for all test functions f, we have

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y) C_{j,2}(x,y)\,dx\,dy &= -\int_0^1 \int_0^1 f_{,2}(x,y) C_j(x,y)\,dx\,dy \\ &\to -\int_0^1 \int_0^1 f_{,2}(x,y) B(x,y)\,dx\,dy.\end{aligned}$$

Compare this with statement (b) and conclude that $B_{,2} = C_{,2}$ a.s., so that necessarily $B = C$. ∎

LEMMA 3.2. Let $T_n = T_{C_n}$ and $T = T_C$. $T_n \to T$ in the weak operator topology of L^p, for $p \in (1,\infty)$, if and only if for all $f \in L^p$,

$$\int_0^1 \int_0^1 f(x,y) C_{n,2}(x,y)\,dx\,dy \to \int_0^1 \int_0^1 f(x,y) C_{,2}(x,y)\,dx\,dy. \tag{3.3}$$

PROOF. Let $\phi,\ \psi \in C^\infty([0,1])$. Then

$$\begin{aligned}\int_0^1\int_0^1 (C_{,2}(x,y) - C_{n,2}(x,y))\phi(x)\psi(y)\,dx\,dy \\ = -\int_0^1 \Phi(x)[(T-T_n)\psi](x)\,dx\end{aligned} \tag{3.4}$$

where Φ is an antiderivative of ϕ. If $T_n \to T$ in the weak operator topology, then the right hand side of (3.4) goes to zero for all ϕ and ψ, so that (3.3) holds whenever $f(x,y) = \phi(x)\psi(y)$. But linear combinations of functions of this form are dense in L^p, so (3.3) holds for all $f \in L^p$. On the other hand, if (3.3) holds, then the left hand side of (3.4) goes to zero for all ϕ and ψ. Since functions Φ and ψ are dense in L^p and L^q, respectively, where $1/p + 1/q = 1$, and $T - T_n$ is a bounded operator, it follows that $T_n \to T$ in the weak operator topology of L^p. ∎

It is known that Markov operators form a compact set under the weak operator topology, Brown (1965). This is a corollary of the foregoing theorem and the fact that the copulas are a compact set in the uniform topology.

Define three norms on span $\mathcal{C}$, the linear span of the set of copulas:

(a) Set $\mathcal{B} = \text{co}\,\{\mathcal{C} \cup (-\mathcal{C})\}$, where co denotes convex hull, and define a Minkowski functional via

$$\|A\|_M = \inf\{t > 0 \mid A \in t\mathcal{B}\}.$$

(b) Extend the isomorphism Φ to span $\mathcal{C}$ and define

$$\|A\|_{\text{Op}} = \|\Phi(A)\|.$$

(c) Let $A \in \text{span}\,\mathcal{C}$, let $\mu_A = \mu_A^+ - \mu_A^-$ be the Jordan decomposition of the finite signed measure induced by A. Set $A^+(x,y) = \mu_A^+([0,x]\times[0,y])$, and $A^-(x,y) = \mu_A^-([0,x]\times[0,y])$. Define

$$\begin{aligned}\|A\|_J = \max\{\|A^+_{,1}(\,\cdot\,,1)\|_\infty, \|A^+_{,2}(1,\,\cdot\,)\|_\infty\} \\ + \max\{\|A^-_{,1}(\,\cdot\,,1)\|_\infty, \|A^-_{,2}(1,\,\cdot\,)\|_\infty\}\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the $L^\infty([0,1])$ norm.

THEOREM 3.2. $\|A\|_M = \|A\|_J$ for all $A \in \text{span}\,\mathcal{C}$.

A proof of this result can be found in Darsow and Olsen (1995). We conjecture that $\|\cdot\|_M$ and $\|\cdot\|_{\text{Op}}$ are equivalent norms on span $\mathcal{C}$. It is easy to see that $\|\cdot\|_M$ dominates $\|\cdot\|_{\text{Op}}$. It would be of interest to establish equivalence, since the linear span of the copulas is a (real) Banach algebra under $\|\cdot\|_M$, see Darsow and Olsen (1995).

We turn now to some other results for Markov operators which are obtained easily via the isomorphism theorem by translating known results for copulas to the Markov operator setting.

We will say $f : [0,1] \to [0,1]$ is measure preserving if for any Borel set E, $f^{-1}(E)$ is a Lebesgue measurable set, and $\lambda(f^{-1}(E)) = \lambda(E)$ where λ denotes Lebesgue measure. Set

$$\mathcal{F} = \{f : [0,1] \to [0,1] \mid f \text{ is measure preserving }\},$$

and for $f,\ g \in \mathcal{F}$, define a function $C_{f;g} : [0,1]^2 \to [0,1]$ via

$$C_{f;g}(x,y) = \lambda(f^{-1}([0,x]) \cap g^{-1}([0,y])\,). \tag{3.5}$$

THEOREM 3.3. (Representation Theorem). *The function $C_{f;g}$ of (3.5) is a copula. Furthermore, for any copula C there exist functions $f,\ g \in \mathcal{F}$ such that*

$$C = C_{f;g}.$$

For a proof, see Darsow and Olsen (1993) or Vitale (1995).

Set $M(x,y) = \min\{x,y\}$, then M is a copula, and is a unit for the $*$ product. The corresponding Markov operator T_M is the identity map on $L^\infty([0,1])$. We say a copula A is left (right) invertible if there is a copula B such that $B * A = M$ $(A * B = M)$, and we say an operator T is left (right) invertible if there is an operator S such that $S \circ T$ $(T \circ S)$ is the identity. We shall use the isomorphism theorem to translate known facts about left and right invertible copulas to the Markov operator setting.

First, it is known that if a copula is left (right) invertible, its left (right) inverse is its transpose, Darsow et al. (1992). We then have the following result, via the isomorphism theorem:

THEOREM 3.4. *If a Markov operator on $L^\infty([0,1])$ is invertible, its extension to $L^2([0,1])$ is necessarily unitary.*

Second, there exist copulas which are invertible on one side but not both, Darsow et al. (1992), so there exist Markov operators on $L^\infty([0,1])$ which are invertible on one side but not both.

Third, using the representation theorem stated above and related results, we can prove a curious representation theorem for Markov operators on

$L^\infty([0,1])$. Let e denote the identity map on $[0,1]$. It is easy to see that for any $f,\ g \in \mathcal{F}$

$$C_{f;g} = C_{f;e} * C_{e;g}. \tag{3.6}$$

It is also easy to see that for any $f \in \mathcal{F}$, $C_{f;f} = M$, Darsow and Olsen (1993). It follows that for any $f \in \mathcal{F}$, $C_{f;e}$ is right invertible and $C_{e;f}$ is left invertible. Thus, using the representation theorem and (3.6), every copula is the product of a right invertible copula and a left invertible copula. It then follows via the isomorphism theorem that:

THEOREM 3.5. *Every Markov operator T on $L^\infty([0,1])$ can be factored as a right invertible Markov operator composed with a left invertible Markov operator.*

4. Discussion and Conclusions. We have used the fact that the conditions $C_{st} = C_{su} * C_{ut}$ are equivalent to the Chapman–Kolmogorov equations, together with the fact that the composition of Markov operators follows the Chapman–Kolmogorov equations, to formulate and prove an isomorphism between the set $\mathcal{C}$ of copulas under the $*$ product and the set of Markov operators on $L^\infty([0,1])$ under composition. We have also shown how the resulting characterization of Markov operators fits with work done by J. V. Ryff some thirty years ago, and we have explored some of the consequences of the isomorphism.

The relation of copulas under the $*$ product and Markov operators under composition is like the relation of matrices under matrix multiplication and linear operators on finite dimensional vector spaces under composition.

The work described here exploits well known but, to our knowledge, seldom exploited, differentiability properties of copulas. It has been our experience that confusing issues in stochastic processes can often be reduced to clear issues in real analysis by reformulation in terms of copulas and further that, because of the nice properties of copulas and their first partial derivatives, the resulting issues can often be addressed and answered by simple classical arguments.

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