Investigating the Structure of Truncated Lévy-stable Laws

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Abstract

Truncations of stable laws have been proposed in the econophysics literature for modelling financial returns, often with imprecise definitions. This paper sharpens definitions of exponential truncations and attempts to expose underlying structure. Analytical comparisons are made with alternative models, leading to a tentative conclusion that the generalized hyperbolic family is more attractive for empirical work.

Keywords: Truncation; Lévy-stable laws

1 Introduction

Extensive empirical research shows that (log-)return data obtained from frequently sampled financial time series is not well fitted by a normal (Gaussian) law. Rather, the 'true' population law is more peaked around its median and it has fatter tails. Many analytically specified laws have been proposed and found to give a good fit to selected data sets. For example McDonald [22], Rydberg [31], and Voit [35, §5.3,5.4] are recent reviews representing the finance, statistics, and physics disciplines, respectively. In particular, Mandelbrot [23, E14,15] champions validity of non-normal stable laws. In fact, many return series exhibit tail behaviour which is intermediate to normal and non-normal stable behaviour. As a result, various more complicated models built from stable laws are found to mimic the stylized features of real data; see [31, 3].

The 'econophysics' school of modellers support use of so-called truncated Lévy (*i.e.*, stable) laws. See [7] for a general discussion of their use in finance, and [25] for pricing options. Let $g(x;\alpha)$ denote the density function of a stable law having index $\alpha \in (0,2)$ and symmetric about the origin, and let X be a random variable having this law. If $1 < \alpha < 2$ then $E(X) = 0$ and $var(X) = \infty$, but if $0 < \alpha \leq 1$ then neither the mean nor the variance can be defined. Econophysicists hold this to be unsatisfactory on the reasonable grounds that returns cannot be arbitrarily large in magnitude, and hence admissible models should possess finite moments of all orders. In general terms, the solution they propose is to use weighted densities

$$f(x;\alpha,w) = w(x)g(x;\alpha), \tag{1.1}$$

where $w(x) \geq 0$ is a weighting function satisfying $w(x) \to 0$ as $|x| \to \infty$, $\int f(x;\alpha,w)dx = 1$, and $\int |x|^r f(x;\alpha,w)dx < \infty$ for all $r > 0$.

This idea was introduced by Mantegna and Stanley [24] in the specific case $w(x) = c1_{[-l,l]}(x)$ where $0 < l < \infty$ is a truncation level and c is a normalization constant. Let S_n denote the sum of n independent copies of a random variable having this truncated stable law, let $f_n(x)$ denote its density function, and $v_l = E(S_1^2)$. The local version of the central limit theorem asserts that

$$\lim_{n\to\infty} \sqrt{n} f_n(0) = \frac{1}{\sqrt{2\pi v_l}}. \tag{1.2}$$

Mantegna and Stanley [24] provide simulations showing that the asymptotic regime (1.2) is approached more and more slowly as l increases. In addition, they found evidence of a quite long-lived pre-asymptotic regime during which $f_n(0)$ decays in proportion to $n^{-1/\alpha}$, the asymptotic behaviour for the parent stable law.

This modification of stable laws is analytically awkward and hence Koponen [19] recommends versions where $w(x) > 0$ for all x but decaying exponentially fast as $|x| \to \infty$. In fact, the precise nature of his definition is not clear. He asserts (for the symmetric case) that the truncated density function is

$$f(x) = c|x|^{-1-\alpha} e^{-\gamma|x|}, \tag{1.3}$$

where $\gamma > 0$ is an additional parameter and c is the normalizing constant. But clearly (1.3) does not define a density function since $\int_{[-\varepsilon,\varepsilon]} f(x)dx = \infty$ for any $\varepsilon > 0$. Koponen [19] mentions a lengthy calculation of a characteristic function (CF) whose symmetric version is

$$\sigma(t) = E\left(e^{itX}\right) = \exp\left[-A(t^2+\gamma^2)^{\alpha/2} \frac{\cos(\alpha \arctan(|t|/\gamma))}{\cos(\frac{1}{2}\alpha\pi)}\right]. \tag{1.4}$$

Paul and Baschnagel [29, p. 123] specify (1.3) holding in an asymptotic sense as $|x| \to \infty$, thus removing the singularity problem. They give a detailed derivation of (1.4) (see their Appendix D) where it is evident that the right-hand side of (1.3) is taken as the density of the Lévy measure of an infinitely divisible law. Consequently the nature of the law whose CF is (1.4) is unresolved.

Our aim here is to illumine this obscurity. We will distinguish three operations: (i) truncation as envisaged by Koponen [19], that is, exponential down-weighting a parent density function; exponential tilting, which involves multiplying a parent density by a decreasing exponential function (thus inflating the left-hand tail); and (iii) pruning an infinitely divisible (infdiv) law by truncating its Lévy measure. Pruning is implicit in Paul and Baschnagel's calculation. Briefly, it seems that truncation does not support a useful theory, whereas shrinking and tilting are almost equivalent, and they support a richer theory.

A key reason for considering truncated/pruned stable laws is to give a parametric family which exhibits a wider spectrum of tail behaviours than the stable laws, normal and non-normal. Thus in §2 we review relevant results about infinite divisibility and convolution equivalent laws. In particular, Theorem 2.1 gives conditions ensuring the right-hand tail of the law is asymptotically proportional to the right-hand tail of its Lévy measure. This is a simple extension of known results for one-sided laws, and its proof is given in Pakes [27]. In §3 we define an exponential truncation operation on two-sided laws, observing that even though tail behaviour is in principal accessible, other structural properties such as determination of its moments present substantial analytical difficulties.

Section 4 reviews a tilting operation, familiar in other contexts, and relates it to the pruning of spectrally positive infinitely divisible laws. Particular application is made to extreme stable laws, and some limit distributions are obtained which illuminate the simulation results in Mantegna and Stanley [24]. In §5 we define the pruning of two-sided stable laws as the difference of independent tilted extreme stable random variables and Theorem 5.1 gives its characteristic function. Representation as either a tilted or truncated law is examined. We explore the representation of differences of tilted laws in terms of a truncated law, showing in particular that these pruned stable laws cannot be represented as a truncation of any two-sided stable law. In §6 we observe that pruned stable laws are generalized gamma convolutions, (GGC's) and hence their symmetric versions are normal-variance mixtures. This form of mixing is significant in financial modelling as a representation of stochastic volatility. Unfortunately, the mixing law appears to be quite complicated. Simpler representations as GGC's are found. These representations suggest comparisons with other laws which can be obtained from normal-variance mixing and tilting, and we look briefly at two special families, one being the generalized hyperbolic laws. Process and series representations of tilted and pruned stable laws are examined in §7. Here we give a self-contained and elementary account of random series representations of a broad class of infinitely divisible laws, and demonstrate that although there are many representations of tilted and pruned stable laws, finding one with simple explicit generating elements is problematic. Some final comments are given in §8, where we recommend the generalized hyperbolic family as being far better suited for empirical work than truncated or pruned stable laws.

2 Infinitely divisible laws

Our context will be the infinitely divisible (infdiv) laws, and there are several equivalent ways of defining this notion. We will agree that the random variable X with distribution function $F(x)$ has an infdiv law if its characteristic function (CF) $\phi(t) = E[e^{itX}]$ has the form $\phi(t) = \exp(-\psi(t))$ where the *characteristic exponent* is

$$\psi(t) = -Ait + \tfrac{1}{2}Vt^2 + \int_{|x|<1} \left[1 - e^{itx} + itx\right] \nu(dx) + \int_{|x|\geq 1} \left[1 - e^{itx}\right] \nu(dx), \qquad (2.1)$$

A is a real constant, $V \geq 0$, and ν is a measure on $(-\infty, \infty)$ satisfying $\nu\{0\} = 0$ and $\int (x^2 \wedge 1)\nu(dx) < \infty$, and called the Lévy measure. (We use the notation $\nu\mathcal{E}$ to denote the measure assigned by ν to the set $\mathcal{E}$.) Thus $\mathcal{L}(X)$, the law of X, is comprised of three independent components, the constant A, a $\mathcal{N}(0,V)$ normal component, and a superposition of compound Poisson laws (Sato (1999) for example). Infdiv laws comprise the totality of laws which satisfy the partition property: For any integer $n \geq 1$, X can be written as a sum $\sum_{j=1}^{n} \varepsilon_{jn}$ of independent and identically distributed random variables (and clearly their law has the CF $(\phi(t))^{1/n})$). The centering term itx in the first integral of (2.1) is often expressed in different ways, but this only affects the value of A. Infdiv laws are always unbounded in at least one direction and hence the Mantegna-Stanley [24] truncation always results in a law which is not infdiv.

We identify the important special case of spectrally positive infdiv laws (SPID laws), defined by the constraint $\nu(-\infty,0) = 0$. In this case the Laplace-Stieltjes transform (LST) $E(e^{-\theta X}) := L(\theta) = \exp(-\kappa(\theta))$ is finite ($\theta \geq 0$) where the *cumulant function* has the representation

$$\kappa(\theta) = A\theta - \tfrac{1}{2}V\theta^2 + \int_0^{1-} \left[1 - e^{-\theta x} - \theta x\right] \nu(dx) + \int_1^{\infty} \left[1 - e^{-\theta x}\right] \nu(dx). \tag{2.2}$$

To minimise algebraic details, we will always assume that $V = 0$. It is clear that the general infdiv law can be decomposed as $X = A + X_1 - X_2$, where X_1 and X_2 are independent SPID random variables with zero constant terms. In this sense SPID laws are fundamental.

There are two types of SPID law. Type 2 is defined by the condition $\int_0^1 x\nu(dx) = \infty$, in which case $\operatorname{supp}(\mathcal{L}(X)) = \mathbb{R}$; X can assume any positive or negative value. But since $L(\theta) < \infty$ if $\theta > 0$, the left-hand tail $P(X < -x)$ ($x > 0$) decreases to zero faster than any exponentially decreasing function. Indeed, Ohkubo [26, p. 78] shows that $P(X < -x) = O(\exp(-x\log x))$ for large x. Thus $\mathcal{L}(X)$ is 'almost' one-sided.

Type 1 is defined by the condition $\int_0^1 x\nu(dx) < \infty$ in which case the cumulant function has the slightly simpler representation

$$\kappa(\theta) = B\theta + \int_0^{\infty} \left[1 - e^{-\theta x}\right] \nu(dx), \tag{2.3}$$

where $B = A - \int_0^{1-} x\nu(dx)$, and $\mathcal{L}(X)$ is one-sided with support $[B,\infty)$. This includes the fundamental compound Poisson case where $\rho = \nu[0,\infty) < \infty$. In this case $\rho^{-1}\nu[0,x]$ is a distribution function and we can write

$$X = B + \sum_{j=1}^{N} \eta_j,$$

where N has a Poisson(ρ) law and the η_j are independent with distribution function $\rho^{-1}\nu[0,x]$ and independent of N. We shall see below that the asymptotic behaviour of $P(X - A > x)$ is determined by the rate at which $\nu[x,\infty)$ tends to zero as $x \to \infty$; in other words, it is determined by the compound Poisson component of the infdiv law. It typically is the case that an infdiv or SPID law is such that it is not possible to explicitly exhibit its distribution function, whereas its Lévy measure often has a simple form. (This is certainly true for stable laws which we later consider.) Thus, it is important to somehow relate the upper tail $P(X > x)$ to properties of $\nu(dx)$. Sato [33] discusses these matters for cases where there exists a constant $\gamma > 0$ such that the transform $\hat{\nu}(\theta) = \int_1^\infty e^{-tx}\nu(dx)$ converges in the open interval $(-\gamma, \infty)$ and diverges otherwise. Here we are concerned with cases where $\gamma \geq 0$ and $\hat{\nu}(-\gamma) < \infty$. The following concepts embrace this situation.

We begin the following definition, slightly extending Definition 1 in Cline [10]. Denote the survivor function of a distribution function $G(x)$ by $\overline{G}(x) := 1 - G(x)$.

Definition 2.1. A distribution function $G(\cdot)$ has an exponential tail with rate $\gamma \geq 0$, written $G(\cdot) \in \mathcal{L}_\gamma$, if

$$\lim_{x\to\infty} \frac{\overline{G}(x-y)}{\overline{G}(x)} = e^{\gamma y} \qquad (-\infty < y < \infty).$$

For each $y' > 0$ the limit holds uniformly for $y \leq y'$ if $\gamma > 0$, and uniformly in $[-y', y']$ if $\gamma = 0$. Speaking of an exponential tail with rate $\gamma = 0$ is somewhat contradictory, and we observe that, for our purposes, $\mathcal{L}_0$ is a very substantial class of long-tailed distribution functions in that $\lim_{x\to\infty} e^{\varepsilon x}\overline{G}(x) = \infty$ for each $\varepsilon > 0$.

We will see that stable and pruned stable laws belong to the following general class of laws. Denote the convolution of distribution functions G and H by $G * H$, and convolution powers by, for example, G^{*2}.

Definition 2.2. If $G \in \mathcal{L}_\gamma$ for some $\gamma \geq 0$, say that it is convolution equivalent, written $G \in \mathcal{S}_\gamma$, if

$$\lim_{x\to\infty} \frac{\overline{G^{*2}}(x)}{\overline{G}(x)} = 2M, \tag{2.4}$$

where $M < \infty$.

Bingham *et al.* [5] make some remarks about convolution equivalence on the line, and the unpublished report [37] develops some properties of two-sided convolution equivalence. Apart from these references, general theory for convolution equivalent distribution functions assumes that $G(0-) = 0$, that is, that the corresponding random variables are non-negative. In this case Cline [11, p. 355] shows that $M = M_G(\gamma) := \int e^{\gamma x} dG(x)$, the moment generating function of G. (Unrestricted integrals are taken over the real line.) Pakes [27] extend this to the two-sided case. For positive laws, $G(0-) = 0$, the boundary case $\gamma = 0$ usually is defined by (2.4) alone with the additional condition $M = 1$, giving the so-called subexponential class $\mathcal{S}$, which is a proper subset of $\mathcal{L}_0$. The subexponential class was introduced by Chistyakov [8] for estimating the long-term mean size of certain population processes, and it contains virtually any long-tailed law occuring in financial modelling and other applications. We define $\mathcal{S}$ to comprise the laws satisfying (2.4) with $M = 1$. An important example is $\overline{G} \in \mathcal{R}_{-a}$, the class of regularly varying (at infinity) functions with index $-a$. Another is $\overline{G}(x) = const.\exp(-cx^a)$ $(x > 0)$ where $c > 0$ and $0 < a < 1$. In these cases $G \in \mathcal{S}$.

The definition (2.4) is equivalent to $\lim_{x\to\infty} \overline{G^{*n}}(x)/\overline{G}(x) = nM^{n-1}$ for some (and hence all) integers $n \geq 1$. Thus if $\gamma = 0$, the definition has the probabilistic meaning

$$\lim_{x\to\infty} \frac{P(Y_1 + \cdots + Y_n > x)}{P(\max(Y_1, \ldots, Y_n) > x)} = 1,$$

where the Y_j are independent with distribution function $G(\cdot)$. If $\gamma > 0$, then $\overline{G}(x) = e^{-\gamma x}\tau(x)$, where $\int_1^\infty \tau(x)dx < \infty$. Bingham *et al.* [5] thus use the term 'close to exponential' for members of $\cup_{\gamma>0}\mathcal{S}_\gamma$. Observe that exponential and gamma laws with scale parameter γ belong to $\mathcal{L}_\gamma$ but not to $\mathcal{S}_\gamma$. Cline [10] gives several criteria for membership of $\mathcal{S}_\gamma$ $(\gamma \geq 0)$. The following lemma, embracing the laws we consider here, is a direct consequence of his Corollary 2.

Lemma 2.3. Suppose that

$$\overline{G}(x) = x^{-\delta}L(x)e^{-\gamma x - cx^\omega},$$

where $\gamma, c \geq 0$, $\omega < 1$, $L(\cdot)$ is normalized slowly varying, and if $c = 0$ then either $\delta > 1$ or $\delta = 1$ and $\int_1^\infty (L(x)/x)dx < \infty$. Then $G \in \mathcal{S}_\gamma$.

Proof. Observe that if $c = 0$ then $M_G(\gamma) < \infty$ iff the conditions on δ hold. Write $\overline{G}(x) = \exp[-\xi(x)]$ and observe that

$$\xi'(x) = \gamma + c\omega x^{\omega-1} + \delta/x - \varepsilon(x)/x$$

where $\varepsilon(x) \to 0$ $(x \to \infty)$ is the index function of $L(\cdot)$; see [5, pp. 12-15]. The function $\xi_1(x) = \gamma x + cx^\omega + \delta \log x$ is concave and $x|\xi'(x) - \xi_1'(x)| \to 0$, thus fulfilling Cline's conditions. □

The following theorem relates the asymptotic behaviour of $\overline{F}(x)$ for an infdiv law and the distribution function

$$J(x) = \lambda^{-1}\nu(x,\infty)1_{[1,\infty)}(x),$$

where $\lambda = \nu(1,\infty)$, of its positive jump components exceeding unity. Its proof with other details and references are given in Pakes [27].

Theorem 2.1. Suppose that $\gamma \geq 0$ and F is an infdiv distribution function. The following assertions are equivalent:

(i) $J \in \mathcal{S}_\gamma$;
(ii) $J \in \mathcal{L}_\gamma$ and

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\nu(x,\infty)} = M_F(\gamma); \tag{2.5}$$

(iii) $F \in \mathcal{S}_\gamma$.

As an application of his results for two-sided convolution equivalence, Willekens [37] proved the equivalence of (i) and (ii) and that each implies (iii). In addition to the Tauberian converse, Pakes [27] shows Theorem 2.1 is essentially a consequence of the one-sided case. We emphasize that for our purposes the most significant part of Theorem 2.1 is the assertion that (i) implies (ii) and (iii). Observe also that reflection about the origin shows that if $\tilde{J}(x) = \nu[-x,-1)/\nu(-\infty,-1) \in \mathcal{S}_\gamma$ then

$$\lim_{x\to\infty} \frac{P(X<-x)}{\nu(-\infty,-x)} = M_F(-\gamma).$$

Consider the important case of the general stable law with index $\alpha \in (0,2)$, denoted by stable(α,b,p). This is defined by the absolutely continuous Lévy measure $\nu(dx) = n(x)dx$ where

$$n(x) = \begin{cases} \frac{\alpha b p}{\Gamma(2-\alpha)} x^{-1-\alpha} & \text{if } x > 0, \\ \frac{\alpha b q}{\Gamma(2-\alpha)} |x|^{-1-\alpha} & \text{if } x < 0, \end{cases}$$

where $b > 0$ and $0 \le p = 1-q \le 1$. Then the CF is given by

$$\psi(t) = -Ait + \begin{cases} c|t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan(\tfrac{1}{2}\pi\alpha)\right) & \text{if } \alpha \neq 1, \\ c|t| \left(1 + i\beta \operatorname{sgn}(t) \tfrac{2}{\pi} \log(|t|\right) & \text{if } \alpha = 1, \end{cases}$$

where

$$c = \begin{cases} \frac{b}{\alpha-1} \cos(\tfrac{1}{2}\pi\alpha) & \text{if } \alpha \neq 1, \\ \tfrac{1}{2} b\pi & \text{if } \alpha = 1, \end{cases}$$

and $\beta = p - q$. Note that many textbook renditions err in the sign attached to β; see [15]. The most error-proof derivation of this result is synthesizing it from the cumulant function of the spectrally positive version, $p = 1$. In this case

$$\kappa(\theta) = A\theta - \begin{cases} \frac{b}{\alpha-1}\theta^\alpha & \text{if } \alpha \neq 1, \\ b\theta\log\theta & \text{if } \alpha = 1, \end{cases}$$

obtained by integrating the relation $\kappa''(\theta) = \int_0^\infty e^{-\theta x} x^2 n(x) dx = -\alpha b \theta^{-2+\alpha}$. Some manipulation with complex algebra leads to the above CF.

Stable laws are subexponential ($\gamma = 0$), and hence Theorem 2.1 gives the asymptotic estimate

$$P(X > x) \sim \nu(x, \infty) = \frac{bp}{\Gamma(2-\alpha)} x^{-\alpha} \qquad (x \to \infty). \tag{2.6}$$

For later reference observe that the parameter b functions as a scale constant, and that it affects only the constant multiplier in (2.6). In particular, all one-dimensional laws of the embedding process are tail equivalent in the sense that

$$\lim_{x\to\infty} \frac{P(\Lambda(\tau_1) > x)}{P(\Lambda(\tau_2) > x} = \frac{\tau_1}{\tau_2}.$$

3 Truncating the density

Suppose a random variable X has a density $f(x) > 0$ for all real x, and with $\gamma > 0$ let X_γ denote the random variable having the symmetrically truncated density

$$f_\gamma(x) = Ke^{-\gamma|x|} f(x),$$

where K is a normalizing constant. In principle, this specifies an explicit density function with easily determined tail behaviour. Thus if $f(\cdot)$ is a stable density, it follows from (2.6) that $P(X_\gamma > x) \sim const.x^{-1-\alpha}e^{-\gamma x}$. However, it seems difficult to gain further structural information, such as moments or the CF, or to determine if $\mathcal{L}(X_\gamma)$ is infdiv, or to give a probabilistic characterization of this construction. In addition, it is hard to see how truncation could be put into a process framework. Specifically, if $X = \Lambda(1)$ where $(\Lambda(s))$ is a Lévy process then is there a process $(\Lambda_\gamma(s))$ such that $\Lambda_\gamma(1)$ has density $f_\gamma(x)$ and properties which are relevant to the modelling context?

Recalling that $\phi(t)$ is the CF of $f(\cdot)$ and observing that the Fourier transform of the kernel $e^{-\gamma|x|}$ equals $\gamma/(\gamma^2 + t^2)$, we can write the CF of the truncated density as the convolution (or Poisson integral)

$$\phi_\gamma(t) = \gamma \int \frac{\phi(u)}{\gamma^2 + (t-u)^2} du.$$

This however seems to offer little insight into the nature of $\mathcal{L}(X_\gamma)$, even for quite specific cases such as symmetric stable laws.

One exception is the Cauchy density $f(x) = c/\pi(c^2 + x^2)$. In this case, reference to a table of integrals [14] shows that the truncated law has the LST

$$E\left(e^{-\theta X_\gamma}\right) = \frac{I(\gamma + \theta) + I(\gamma - \theta)}{2I(\gamma)} \qquad (|\theta| < \gamma)$$

where

$$I(\theta) = \int_0^\infty e^{-\theta x} \frac{c}{c^2 + x^2} dx = \mathrm{ci}(c\theta)\sin(c\theta) - \mathrm{si}(c\theta)\cos(c\theta)$$

and

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos v}{v} dv \qquad \& \qquad \mathrm{si}(x) - \int_x^\infty \frac{\sin v}{v} dv$$

are cosine and sine integrals, respectively. In addition, $E(X_\gamma) = 0$ (by symmetry) and

$$var(X_\gamma) = \frac{c}{\gamma I(\gamma)} - c^2 \sim \frac{2c}{\pi\gamma} \qquad (\gamma \downarrow 0).$$

We conclude that truncation is not a fruitful concept.

4 Tilting and pruning

In this section, we recall an asymmetric exponential weighting operation which has been much studied in other contexts. So we let X be a random variable with arbitrary distribution function $F(\cdot)$ satisfying $L(\theta) = \int e^{-\theta x} dF(x) < \infty$ for all $\theta \geq 0$. Fix a constant $\gamma > 0$ and define the law of a random variable X_γ, the (exponential) γ-tilt of $\mathcal{L}(X)$, which has the distribution function

$$F(x;\gamma) = \frac{\int_{-\infty}^x e^{-\gamma x} dF(x)}{L(\gamma)} \qquad (x \in \mathbb{R}). \tag{4.1}$$

The LST of X_γ is

$$L(\theta;\gamma) = \frac{L(\theta+\gamma)}{L(\gamma)}.$$

The family of laws obtained by varying γ through the largest open interval such that $L(\gamma) < \infty$ is called the natural exponential family (NEF) generated by $\mathcal{L}(X)$. See Seshadri [34] for these matters, but note that we adopt an opposite sign convention for the exponent parameter. Tilting is used for obtaining asymptotic expansions for sums of random variables, and in the theory of random walk. See Feller [13] for the latter application, where he uses the term 'associated distribution'.

If X has a SPID law, as defined by (2.2), we can define an operation of *pruning* whereby ν is replaced by the truncation

$$\nu_\gamma(dx) = \tau e^{-\gamma x} \nu(dx), \tag{4.2}$$

where $\tau > 0$ is a constant. Some manipulation shows that the cumulant function κ is transformed into

$$\kappa(\theta;\gamma,\tau) = \tau[\kappa(\theta+\gamma) - \kappa(\gamma) + B_\gamma \theta],$$

where

$$B_\gamma = \int_0^{1-} \left(1 - e^{-\gamma x}\right) x \nu(dx).$$

A random variable $\hat{X}_{\gamma,\tau}$ with this pruned law has the LST is

$$L(\theta;\gamma,\tau) := E\left(e^{-\theta \hat{X}_{\gamma,\tau}}\right) = \left[\frac{L(\theta+\gamma)}{L(\gamma)}\right]^\tau e^{-\tau B_\gamma \theta}. \tag{4.3}$$

This shows that the exponential tilt of $\mathcal{L}(X)$ is equivalent to weighting ν according to (4.2) with $\tau = 1$, together with a shift B_γ to the left of the pruned law. For the reverse direction, let $(\Lambda(\tau) : \tau \geq 0)$ denote the spectrally positive Lévy process with cumulant function $\kappa(\theta)$. Then the transformation (4.2) maps $\mathcal{L}(X)$ to $\mathcal{L}(\Lambda_\gamma(\tau) + \tau B_\gamma)$, the exponential tilt of the process at time τ with a translation τB_γ to the right. In particular, if $\tau = 1$ then, in obvious notation,

$$\hat{X}_\gamma \stackrel{L}{=} X_\gamma + B_\gamma.$$

In the Type 1 case, we can apply the tilting operation to (2.3) to obtain the same form with B unchanged (*i.e.*, $B_\gamma = 0$) and Lévy measure (4.2) with $\tau = 1$. In any event, we see that the non-symmetric tilting operation thins the right-hand tail of $\mathcal{L}(X)$ in the manner recommended by [19, 24]. In contrast, the left-hand tail, if it is non-trivial, is inflated by exponential tilting but it still decreases faster than exponential. In addition, if $\nu(\cdot)$ has a density $n(\cdot)$, that is $\nu(0,x] = \int_0^x n(y)dy$, then $F(\cdot)$ has a density function $f(x)$, and $\mathcal{L}(X_\gamma)$ has a density function and an absolutely continuous Lévy measure given respectively by

$$f(x;\gamma) = e^{-\gamma x} f(x)/L(\gamma) \qquad \& \qquad \nu_\gamma(dx) = e^{-\gamma x} n(x)dx.$$

Finally, any relation connecting the tail behaviours of $F(x)$ and $\nu(dx)$ translates to a parallel relation between $F(x;\gamma)$ and $\nu_\gamma(dx)$. In general, it is analytically more convenient to work with tilting rather than pruning.

We shall now consider the effect of these transformations on the spectrally positive stable law stable$(\alpha, b, 1)$. First, ignoring the change of location, note that the effect of the parameter τ in (4.2) is simply to multiply the parameter b. Consequently, for our present considerations, we lose no generality in setting $\tau = 1$, and we do this until further notice. Tilting shrinks the Lévy measure to $\nu_\gamma(dx) = [\alpha b/\Gamma(2-\alpha)]e^{-\gamma x}x^{-1-\alpha}dx$, yielding the cumulant function

$$\kappa(\theta;\gamma) = \kappa(\theta+\gamma) - \kappa(\gamma) = A\theta - \begin{cases} \frac{b}{\alpha-1}\left((\theta+\gamma)^\alpha - \gamma^\alpha\right) & \text{if } \alpha \neq 1, \\ b\left[(\theta+\gamma)\log(\theta+\gamma) - \gamma\log\gamma\right] & \text{if } \alpha = 1. \end{cases} \tag{4.4}$$

We denote this tilted law by t-stable$(\alpha, b; \gamma)$, where the notation is understood to imply that the asymmetry parameter $\beta = 1$. The case $\alpha < 1$ defines the so-called Hougaard laws [17], used to model lifetime distributions in a heterogeneous population. Seshadri [34] mentions some earlier formulations. The special case $\alpha = \frac{1}{2}$ gives the inverse-Gaussian law whose density function in the case $A = 0$ is

$$f(x;\gamma) = \frac{b}{\sqrt{\pi x^3}} \exp\left[2b\sqrt{\gamma} - \left(\frac{b^2}{x} + \gamma x\right)\right], \qquad (x > 0).$$

The gamma laws occur as the limit of the t-stable$(\alpha, b/\alpha; \gamma)$ as $\alpha \to 0$. In no other case is it possible to express $f(x;\gamma)$ in terms of elementary functions. This is a consequence of the corresponding intractability of stable densities. However Hoffmann-Jorgensen

[16] gives expressions for stable densities in terms of an incomplete hypergeometric function, valid in all cases except $\alpha = 1$ and $\beta \neq 0$. Williams [38] gives an elegant demonstration that the stable density for the case $\alpha = 1/3$ and $\beta = 1$ has a simple form in terms of a Bessel function. See Zolotarev [39, pp. 155–158] for some representations in terms of Whittaker functions.

The mean and variance of X_γ are given respectively by

$$\mu_\gamma = A - \begin{cases} \frac{\alpha b}{\alpha - 1}\gamma^{\alpha-1}, \\ b(1+\log\gamma), \end{cases} \qquad \& \quad \sigma_\gamma^2 = \begin{cases} \alpha b \gamma^{\alpha-2} & \text{if } \alpha \neq 1, \\ \frac{b}{\gamma} & \text{if } \alpha = 1. \end{cases}$$

These quantities are finite, and indeed all moments are finite. Observing that

$$x^{1+\alpha}\int_x^\infty y^{-1-\alpha}e^{-\gamma y}dy = e^{-\gamma x}\int_0^\infty \left(\frac{x}{x+y}\right)^{1+\alpha} e^{-\gamma y}dy \sim \gamma^{-1}e^{-\gamma x} \qquad (x \to \infty),$$

we see that Theorem 2.1, or the tilting construction itself, implies that

$$P(X_\gamma > x) \sim \frac{\alpha b}{\gamma L(\gamma)\Gamma(2-\alpha)} e^{-\gamma x}x^{-1-\alpha} \qquad (x \to \infty). \tag{4.5}$$

Note that in the case of heavy tilting, $\gamma \gg 1$, $\mu_\gamma \approx 0$ if $\alpha < 1$ and $\mu_\gamma \approx -\infty$ if $\alpha \geq 1$, and $\sigma_\gamma \approx 0$ in both cases. It is easy to show that $(X_\gamma - \mu_\gamma)/\sigma_\gamma$ is approximately standard normal when γ is large, a regime which is unlikely to be relevant for financial applications.

As mentioned in §1, Mantegna and Stanley [24], on the basis of simulations, identify two limit regimes as n increases for the sum S_n of n identical copies of random variables having their version of the truncated Lévy law. In addition, they assert a value of n, denoted by $n_\times$, which is claimed to characterize the transition from stable limit behaviour to ultimate normal limit behaviour. The basis for this is an assertion that the density function of S_n, evaluated at the origin, has a stable form when n is small and a normal form when n is large. (The precise nature of these forms result from local limit theorems.) The critical value $n_\times$ is obtained by equating the two density values.

The following considerations identify three limit regimes for the tilted spectrally positive stable law. The cumulant function of S_n is $n\kappa(\theta;\gamma)$ and since, from (4.4), the factor n merely inflates the paramter b we can, and shall, set $n = 1$ and let $b \to \infty$. We will see that the limit behaviour of $\mathcal{L}(X_\gamma)$ is determined by a critical parameter $\xi = \alpha b \gamma^\alpha$. There is some tension in the literature on modelling financial returns between whether they exhibit algebraically decreasing (heavy) tails or whether there also is a truncation factor $e^{-\gamma x}$ present (called semi-heavy tails by some). If this factor is present, then it may be that $\gamma \ll 1$: see §8 for further remarks. In such a case we can envisage that even with b large, there are three possible regimes, $\xi \approx 0$, $\xi = O(1)$, and $\xi \gg 1$, the last being attained in the limit $b \to \infty$. The following theorem deals with the first two possibilities. The proof is omitted since it involves only a straightforward manipulation of $\kappa(\theta b^{-1/\alpha};\gamma)$.

Theorem 4.1. Let $b \to \infty$ and $\xi \to \alpha\zeta$ where $0 \le \zeta < \infty$. If $\alpha \ne 1$ then

$$V_b(\alpha) := b^{-1/\alpha}(X_\gamma - A) \xrightarrow{L} \text{t-stable}(\alpha, 1; \zeta^{1/\alpha}).$$

If $\alpha = 1$ then

$$V_b(1) := b^{-1}(X_\gamma - A - \log b) \xrightarrow{L} \text{t-stable}(1, 1; \zeta).$$

The proof shows in the case $\alpha = 1$ that the limit actually is an identity in law if $\zeta \equiv \xi/\alpha$. (We define $\zeta \log \zeta := 0$ if $\zeta = 0$.) Theorem 4.1 asserts that if b is large but γ is so small that $\xi \ll 1$ then $\mathcal{L}(V_b)$ is close to stable. As b grows further so that ξ is moderate, then $\mathcal{L}(V_b)$ retains the tilted stable form.

As b becomes larger still the critical parameter ξ becomes large. A straightforward application of the binomial theorem to the cumulant function of the normed variable $Z_b := (X_\gamma - \mu_\gamma)/\sigma_\gamma$ yields the expansion

$$\log E\left(e^{-\theta Z_b}\right) = \tfrac{1}{2}\theta^2 + \sum_{j=3}^{\infty} a_j(-\theta)^j \xi^{-\frac{1}{2}j+1}, \tag{4.6}$$

where

$$a_j = \frac{\alpha\Gamma(j-\alpha)}{j!\Gamma(2-\alpha)}.$$

This expansion is valid for $0 < \alpha < 2$. The following result characterizing the third (limiting) regime follows immediately.

Theorem 4.2. If $\xi \to \infty$ as $b \to \infty$ then $Z_b \xrightarrow{L} \mathcal{N}(0,1)$, the standard normal law.

Observe that the norming in Theorem 4.1 when $\zeta > 0$ is equivalent to that used for Theorem 4.2 since $\sigma_\gamma \sim \sqrt{\alpha\zeta}(b/\zeta)^{1/\alpha}$ and, if $\alpha \ne 1$, $(\mu_\gamma - A)/\sigma_\gamma \to (\alpha-1)^{-1}\sqrt{\alpha\zeta}$. It is not at all clear how one might quantify the transition from one regime to the next.

The expansion (4.6) makes it clear that the normal limit is approached only after $\sqrt{\xi}$ becomes large. Indeed, this expansion can be inverted using the Fourier methodology described by Feller [13, Chapter XVI]. If $g_b(x)$ denotes the density function of Z_b and $\varphi(x)$ is the standard normal density function, then

$$g_b(x) - \varphi(x) = \varphi(x)\sum_{j=3}^{r} \xi^{-\frac{1}{2}j+1}P_j(x) = o(\xi^{-\frac{1}{2}r+1}),$$

where $P_j(x)$ is a polynomial of degree j which is independent of ξ. The case $r \le 5$ gives the approximation

$$g_b(x) - \varphi(x) = \tfrac{1}{2}\varphi(x)\sum_{j=3}^{r} a_j \xi^{-\frac{1}{2}j+1}H_j(x) + O(\xi^{-\frac{1}{2}(r-1)}),$$

where

$$H_j(x) = (-1)^j e^{\frac{1}{2}x^2}\frac{d^j}{dx^j}e^{-\frac{1}{2}x^2}$$

is a (version of a) Hermite polynomial. Thus $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$ and $H_5(x) = x^5 - 10x^3 + 15x$.

5 Pruning two-sided stable laws

The construction implicit in the calculation of Paul and Baschnagel [29] can be expressed in general terms as follows. Let $X(j)$ $(j = 1,2)$ be independent random variables having a SPID law with Lévy measure ν_j, constant term A_j, and LST $L_j(\theta)$. Next, let $\hat{X}_\gamma(j)$ denote a random variable with the pruned law obtained from (4.2), that is, with Lévy measure $\nu_{j,\gamma}(dx) = \tau_j e^{-\gamma x}\nu_j(dx)$. Thus the scaling constant, but not the shrinkage parameter γ, may depend on j. Alteration of details below allows relaxation of this restriction. Then $X = \hat{X}_\gamma(1) - \hat{X}_\gamma(2)$ has a two-sided infdiv law, and from (4.3) its mgf is

$$M_F(\theta) = \left(\frac{L_1(\gamma-\theta)}{L_1(\gamma)}\right)^{\tau_1}\left(\frac{L_2(\gamma+\theta)}{L_2(\gamma)}\right)^{\tau_2} e^{\theta(\tau_1 B_{1,\gamma} - \tau_2 B_{2,\gamma})},$$

which is finite in an interval containing $[-\gamma,\gamma]$. Equivalently, we can define $X = X_\gamma(1) - X_\gamma(2)$, in which case the exponential factor is absent.

Assume X is centered so that $\tau_1(A_1 + B_{1,\gamma}) - \tau_2(A_2 + B_{2,\gamma}) = 0$. If

$$J_1(x) = [\nu_{1,\gamma}(x,\infty)/\nu_{1,\gamma}(1,\infty)]1_{[1,\infty)}(x) \in \mathcal{S}_\gamma$$

then Theorem 2.1 implies that

$$\lim_{x\to\infty}\frac{P(X>x)}{\nu_{1,\gamma}(x,\infty)} = (L_1(\gamma))^{-\tau_1}\left(\frac{L_2(2\gamma)}{L_2(\gamma)}\right)^{\tau_2}, \tag{5.1}$$

with a similar relation for $P(X < -x)$.

In the sequel, we will work with the tilted rather than the pruned version. So we let $X(j) - A_j \sim$stable$(\alpha, b_j, 1)$ $(j = 1,2)$ where $b_j > 0$, A_j is real, $\gamma > 0$. The CF of $X = X_\gamma(1) - X_\gamma(2)$ can easily be computed from (4.4) by substituting $\theta = -it$ and converting to the polar representation $\gamma - it = \sqrt{\gamma^2+t^2}e^{-i\omega \operatorname{sgn}(t)}$ where $\omega = \arctan(|t|/\gamma)$. Algebraic manipulation yields the following result, agreeing with Paul and Baschnagel [29], and with Koponen [19], apart from scaling of some of the parameters.

Theorem 5.1. The CF of the pruned law $\mathcal{L}(X)$, where $X = X_\gamma(1) - X_\gamma(2)$, is $\exp(-\psi(t))$ where

$$\psi(t) - Ait = \begin{cases} -\frac{b}{\alpha-1}\left\{(\gamma^2+t^2)^{\alpha/2}[\cos(\alpha\omega) - i\beta \operatorname{sgn}(t)\sin(\alpha\omega)] - \gamma^\alpha\right\} & \text{if } \alpha \neq 1 \\ -b\left\{(\gamma^2+t^2)^{\frac{1}{2}}[\frac{1}{2}\log(\gamma^2+t^2)\cos\omega - \omega\sin\omega - i\beta\operatorname{sgn}(t)(\frac{1}{2}\log(\gamma^2+t^2)\sin\omega + \omega\cos\omega)] - \gamma\log\gamma\right\} & \text{if } \alpha = 1, \end{cases}$$

where

$$A = A_1 - A_2, \qquad b = \tau_1 b_1 + \tau_2 b_2, \quad \& \quad \beta = \frac{\tau_1 b_1 - \tau_2 b_2}{\tau_1 b_1 + \tau_2 b_2}.$$

We shall, without any real loss in generality, take $\tau_1 = \tau_2 = 1$, and let p-stable$(\alpha, b, \beta; \gamma)$ denote the resulting law.

Moments of $\mathcal{L}(X)$ and its asymptotic behaviour for large b can be inferred from the results in the previous section. For later reference we record tail estimates, assuming the location parameter $A = 0$:

$$\bar{F}(x) \sim \frac{B_+}{\gamma} x^{-1-\alpha} e^{-\gamma x} \quad \& \quad F(-x) \sim \frac{B_-}{\gamma} x^{-1-\alpha} e^{-\gamma x} \quad (x \to \infty) \tag{5.2}$$

where

$$B_+ = \frac{\alpha b p}{\Gamma(2-\alpha)} \frac{L_2(2\gamma)}{L_1(\gamma) L_2(\gamma)} \qquad \& \qquad B_- = \frac{\alpha b q}{\Gamma(2-\alpha)} \frac{L_1(2\gamma)}{L_1(\gamma) L_2(\gamma)}.$$

The following considerations will make it clear that $\mathcal{L}(X)$ cannot be realized as a two-sided exponential truncation (1.1) of a stable law.

We can gain some appreciation of the structure of $\mathcal{L}(X)$ as follows. Suppose for now that $X(j)$ $(j = 1,2)$ are independent with density functions $f_j(x)$ positive at least in $(0,\infty)$ and that $L_j(\theta) = \int e^{-\theta x} f_j(x)dx < \infty$ if $0 \le \theta \le 2\gamma$. Then the density of $X = X_\gamma(1) - X_\gamma(2)$ is

$$f(x) = \frac{e^{-\gamma x}}{L_1(\gamma) L_2(\gamma)} \int f_1(x+y) f_2(y) e^{-2\gamma y} dy. \tag{5.3}$$

Observe that

$$\int e^{-\theta x} \left(\int f_1(x+y) f_2(y) e^{-2\gamma y} dy \right) dx = L_1(\theta) \int f_2(y) e^{-y(2\gamma - \theta)} dy = L_1(\theta) L_2(2\gamma - \theta),$$

which is finite if $0 \le \theta \le 2\gamma$. Hence,

$$g(x) := \frac{\int f_1(x+y) f_2(y) e^{-2\gamma y} dy}{L_2(2\gamma)} \qquad (-\infty < x < \infty) \tag{5.4}$$

is a density function positive on the real line and with (bilateral) LST

$$L_G(\theta) = L_1(\theta) L_2(2\gamma - \theta)/L_2(2\gamma).$$

This is the LST of a random variable $U := X(1) - X_{2\gamma}(2)$, which clearly is infdiv if the $X(j)$ are SPID. In particular, if $X(j)$ has a stable$(\alpha, b_j, 1)$ law then U has the Lévy density

$$n_G(x) = \begin{cases} \frac{\alpha b_1}{\Gamma(2-\alpha)} x^{-1-\alpha} & \text{if } x > 0, \\ \frac{\alpha b_2}{\Gamma(2-\alpha)} |x|^{-1-\alpha} e^{-2|x|} & \text{if } x < 0, \end{cases}$$

showing that $\mathcal{L}(U)$ is asymmetric with tail probabilities $P(U > x) = O(x^{-\alpha})$ and $P(U < -x) = O(x^{-1-\alpha} e^{-2\gamma x})$.

Returning to the general case, we now can interpret (5.3) as specifying the exponential tilt $\mathcal{L}(X) = \mathcal{L}(U_\gamma)$, noting that $\mathcal{L}(U)$ depends on γ. This interpretation can be extended to show that $\mathcal{L}(X)$ can indeed be represented as an exponential truncation of a two-sided law. Denoting the integral in (5.3) by $I(x)$, observe that

$$K_1 := \int_0^\infty I(x)dx = \int \bar{F}_1(y)f_2(y)e^{-2\gamma y}dy = L_2(2\gamma)P(X(1) > X_{2\gamma}(2)).$$

Similarly

$$K_2 := \int_{-\infty}^0 e^{-2\gamma x}I(x)dx = L_1(2\gamma)P(X(2) > X_{2\gamma}(1)).$$

Thus $h_+(x;\gamma) := (I(x)/K_1)1_{(0,\infty)}(x)$ is the conditional density of $X(1) - X_{2\gamma}(2)$, given this difference is positive. Similarly $h_-(x;\gamma) := (e^{-2\gamma x}I(x)/K_2)1_{(-\infty,0)}(x)$ is the conditional density of $X_{2\gamma}(1) - X(2)$, given this difference is negative. Now define the family of two-sided densities $h_r(x;\gamma) = rh_+(x;\gamma) + (1-r)h_-(x;\gamma)$, where $0 < r < 1$. For each such r let $m_1(r) = (K_1/rL_1(\gamma)L_2(\gamma))$ and $m_2(r) = (K_2/(1-r)L_1(\gamma)L_2(\gamma))$. Then we have the truncation (*c.f.* (1.1))

$$f(x) = \begin{cases} m_1(r)e^{-\gamma x}h_r(x;\gamma) & \text{if } x > 0, \\ m_2(r)e^{\gamma x}h_r(x;\gamma) & \text{if } x < 0, \end{cases}$$

thus representing $f(x)$ somewhat in the manner (apparently) envisaged by Koponen [19]. We emphasize that the law we are truncating here depends on the parameter γ, and it is obvious that this construction gives the only possible truncation representation. Observe however that in the stable case (5.2) implies that $f(x) \sim B_+x^{-1-\alpha}e^{-\gamma x}$ $(x \to \infty)$ and hence that $h_r(x;\gamma) \sim const.|x|^{-1-\alpha}$ as $|x| \to \infty$, with a different constant for each tail. Hence the truncated density has tails which decay at the same algebraic rate as a stable$(\alpha,\cdot,\beta)$ with $|\beta| < 1$. The nature of these laws is an open question, for example, it is not clear whether or not they are infdiv.

6 Normal-variance mixtures

Following Bondesson [6], let $\mathcal{T}_e$ denote the class of extended generalized gamma convolutions, that is, the closure of laws obtained as finite linear combinations of independent gamma distributed random variables. The subclass $\mathcal{T}$ of generalized gamma convolutions (GGC's) is generated from the linear combinations having positive coefficients. Members of $\mathcal{T}_e$ are infdiv and absolutely continuous. Moreover the symmetric members of $\mathcal{T}_e$ are normal-variance mixtures. More specifically, if $\mathcal{L}(X) \in \mathcal{T}_e$ and it is symmetric, then $X \stackrel{L}{=} Z\sqrt{Y}$ where $\mathcal{L}(Z) = \mathcal{N}(0,2)$, $\mathcal{L}(Y) \in \mathcal{T}$ and Y and Z are independent. The parametrization for $\mathcal{L}(Z)$ is chosen so that

$$M_F(\theta) = M_G(\theta^2), \tag{6.1}$$

where $G(\cdot)$ is the distribution function of Y, and we assume the mgf's are finite. Normal-variance mixing is important in financial modelling as a way of modelling stochastic volatility. More precisely, if $(Y(\tau):\tau\geq 0)$ is the Lévy process with $Y(1)\stackrel{L}{=}Y$ and $(B(\tau):\tau\geq 0)$ is a Brownian motion process with $B(1)\stackrel{L}{=}Z$ then the *subordinated* process $\Lambda(\tau):=B(Y(\tau))$ is an embedding Lévy process, $\Lambda(1)\stackrel{L}{=}X$. Many marginal laws used for financial modelling can be represented in this way. See [18] for a catalogue and references. The following considerations show that pruning is accomodated by this framework.

The Laplace transform relation

$$x^{-1-\alpha}e^{-\gamma x}=\frac{1}{(\Gamma(1+\alpha))}\int_0^\infty[(v-\gamma)^+]^\alpha e^{-xv}dv$$

implies that t-stable$(\alpha,b;\gamma)\in\mathcal{T}$ and p-stable$(\alpha,b,\beta;\gamma)\in\mathcal{T}_e$; see Bondesson [6, pp. 30,107]. In particular, p-stable$(\alpha,b,0;\gamma)$ is a normal-variance mixture, a fact which is not evident from inspection of its CF in Theorem 5.1 or its cumulant function

$$\kappa_s(\theta;\gamma)=-\frac{b}{2(\alpha-1)}\left[(\gamma+\theta)^\alpha+(\gamma-\theta)^\alpha-2\gamma^\alpha\right],\tag{6.2}$$

finite if $|\theta|<\gamma$. The following results will show that the mixing law arises from tilting another positive law, and that this more basic law has a complicated form.

Suppose that $\mathcal{L}(X)$ =p-stable$(\alpha,b,0;\gamma)$ and denote its Lévy density by $n(x)$. Hence, $n(x)=(\alpha b/2\Gamma(2-\alpha))|x|^{-1-\alpha}e^{-\gamma|x|}$ (all real x). As above, $X\stackrel{L}{=}Z\sqrt{Y}$, and $m(x)$ will denote the Lévy density of the mixing law $\mathcal{L}(Y)$. The general relation (6.1) can be expressed for infdiv laws as

$$\log M_F(\theta)=\int_0^\infty\left(e^{\theta^2y}-1\right)m(y)dy.\tag{6.3}$$

We use the following general result relating the Lévy densities of a normal-variance mixture.

Lemma 6.1. Suppose $F(x)$ is an absolutely continuous and symmetric infdiv law and that it is a normal-variance mixture as specified by (6.1). Then the corresponding Lévy densities are related by

$$n(x)=\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-x^2/4y}m(y)\frac{dy}{\sqrt{y}},$$

or

$$4\sqrt{\pi}n(\sqrt{s})=\int_0^\infty e^{-sv}m(1/4v)v^{-3/2}dv.\tag{6.4}$$

Proof. The right-hand side of (6.3) has the representation $\int_0^\infty E\left[e^{\theta Z\sqrt{y}}-1\right]m(y)dy$. Equating this expression to $\kappa(-\theta)$, as obtained from (2.3) with $A=V=0$, and differentiating twice with respect to θ yields the identity

$$\int e^{\theta x}x^2 n(x)dx = E\left[Z^2\int_0^\infty e^{\theta Z\sqrt{y}}ym(y)dy\right] = E\left[Z^2\int_0^\infty\left(\int e^{\theta x}\delta(x-Z\sqrt{y})dx\right)ym(y)dy\right],$$

where $\delta(\cdot)$ is the Dirac delta-function. Interchanging the integrals on the right-hand side and inverting the bilateral Laplace transforms yields

$$x^2 n(x) = E\left[Z^2\int_0^\infty \delta(x-Z\sqrt{y})ym(y)dy\right].$$

Evaluating the expectation and cancelling the x^2 factor common to both sides leads to (6.4). □

Lemma 6.2. If $\hat{\eta}(\theta)$ is the Laplace transform of the function $\eta(x)$, then $\hat{\eta}(\sqrt{s})=\hat{h}(s)$ where

$$h(v)=\frac{1}{2\sqrt{\pi v^3}}\int_0^\infty ue^{-u^2/4v}\eta(u)du.$$

Proof. Simply observe that $\hat{h}(s)=\int_0^\infty e^{-u\sqrt{s}}\eta(u)du$ and that the exponential factor in the integrand is the Laplace transform of the stable$(\frac{1}{2},\frac{1}{2}u,1)$ law. □

Theorem 6.1. The Lévy density of the mixing law $\mathcal{L}(Y)$ for the p-stable$(\alpha,b,0;\gamma)$ is

$$m(x)=\frac{2\alpha b}{\Gamma(\alpha)\Gamma(1-\alpha)x}\int_0^\infty u^{\alpha-1}e^{-x(u+\gamma)^2}du=\frac{2^{1-\alpha}\alpha b}{\Gamma(1-\alpha)}\cdot x^{-\frac{1}{2}\alpha-1}e^{-\gamma^2 x}\Psi(\tfrac{1}{2}\alpha,\tfrac{1}{2};\gamma^2 x), \tag{6.5}$$

where Ψ is the Kummer (confluent hypergeometric) function of the second kind.

Proof. The left-hand side of (6.4) has the form $\hat{\eta}(\sqrt{s})$ where

$$\eta(u)=\frac{4\sqrt{\pi}\alpha b}{\Gamma(2-\alpha)}\cdot\frac{[(u-\gamma)^+]^\alpha}{\Gamma(1+\alpha)},$$

and hence Lemma 6.2 leads to the evaluation

$$h(v)=\frac{2bv^{-3/2}}{\Gamma(\alpha)\Gamma(2-\alpha)}I(1/v) \qquad \& \qquad I(x)=\int_0^\infty (u+\gamma)u^\alpha e^{-x(u+\gamma)^2}du.$$

But $\hat{h}(s)$ must equal the right-hand side of (6.4), that is, $h(v)=v^{-3/2}m(1/4v)$. Integrating $I(x)$ by parts leads to the integral representation in (6.5). The final form comes from a substitution in the identity $\int_0^\infty u^{\alpha-1}e^{-u^2-2uz}du = \Gamma(\alpha)H_{-\alpha}(z)$, where the Hermite function $H_{-\alpha}(z) = 2^{-\alpha}\Psi(\frac{1}{2}\alpha, \frac{1}{2}; z^2)$ [20, pp. 285,290]. □

Expanding the exponent in (6.5) yields $m(x) = e^{-\gamma^2 x}\ell(x)$ where

$$\ell(x) = \frac{2\alpha K}{x}\int_0^\infty u^{\alpha-1}e^{-x(u^2+2\gamma u)}du \qquad \& \qquad K = \frac{b}{\Gamma(\alpha)\Gamma(2-\alpha)}. \tag{6.6}$$

Lemma 6.3. The mixing law is a tilted law, $Y \stackrel{L}{=} W_{\gamma^2}$ where $\mathcal{L}(W)$ is a positive infdiv law with Lévy density $\ell(x)$. Moreover, $\int_0^1 \ell(x)dx = \infty$, and

$$\ell(x) \sim \alpha K\Gamma(\tfrac{1}{2}\alpha)x^{-1-\alpha/2}\ (x\to 0) \qquad \& \qquad \ell(x) \sim \frac{\alpha b}{2^{\alpha-1}\gamma^\alpha\Gamma(2-\alpha)}x^{-1-\alpha}\ (x\to\infty).$$

Proof. We show first that $\ell(x)$ is a Lévy density. Clearly $\ell(x)\downarrow 0$ as $x\uparrow\infty$, and $\ell(0+) = \infty$. The substitution $v = u\sqrt{x}$ yields $\ell(x) = 2\alpha K x^{-1-\alpha/2}\int_0^\infty v^{\alpha-1}e^{-v^2-2\gamma v\sqrt{x}}dv$, and the integral converges to $\frac{1}{2}\Gamma(\frac{1}{2}\alpha)$ as $x\to 0$. Consequently $\int_0^1 x\ell(x)dx < \infty$. Next, observing that $v^{\alpha-1}e^{-v^2} \sim v^{\alpha-1}$ as $v\to 0$, a Tauberian theorem implies that the last integral is asymptotically equal to $\Gamma(\alpha)(2\gamma\sqrt{x})^{-\alpha}$ as $x\to\infty$, and the second asymptotic relation follows. In particular $\int_1^\infty \ell(x)dx < \infty$. □

It is clear from Lemma 6.3 that $\mathcal{L}(W)$ is not stable, although $P(W > x)$ is asymptotically proportional to the right-hand tail of a stable(α) law (provided it is not spectrally negative). We have not been able to relate $\mathcal{L}(W)$ to simpler known laws. The integral expression (6.6) yields the Laplace transform expression

$$\int_0^\infty x\ell(x)e^{-\theta x}dx = 2\alpha K\int_0^\infty \frac{u^{\alpha-1}}{u^2+2\gamma u+\theta}du = \frac{2\alpha K\pi}{\sin(\alpha\pi)}\cdot\frac{(z_-(\theta))^{\alpha-1}-(z_+(\theta))^{\alpha-1}}{z_+(\theta)-z_-(\theta)} \tag{6.7}$$

where $z_\pm = \gamma\pm\sqrt{\gamma^2-\theta}$, $|\theta| < \gamma^2$, and the second equality follows from Gradshteyn and Ryzhik [14, 3.223,#1], and it holds for $\alpha \neq 1$. The expression for $\alpha = 1$ is given by

$$2K[\log(z_+(\theta)/z_-(\theta))]/[z_+(\theta)-z_-(\theta)].$$

A further integration yields the explicit expression for the cumulant function of $\mathcal{L}(W)$,

$$\int_0^\infty (1-e^{-\theta x})m(x)dx = \frac{2\pi b}{\Gamma(\alpha)\gamma(2-\alpha)}\frac{(2\gamma)^\alpha-(z_+(\theta))^\alpha-(z_-(\theta))^\alpha}{\sin(\alpha\pi)}, \quad (\alpha\neq 1),$$

valid for $\theta\le\gamma^2$. This representation is too narrowly defined to give the cumulant function of $Y = W_{\gamma^2}$. An explicit expression for the cumulant function of Y can be given in terms of trigonometric functions, but it yields little insight.

We now explore the GGC properties of $\mathcal{L}(W)$ and $\mathcal{L}(Y)$, beginning by rendering Bondesson's [6, p. 29] definition as follows. We say that the positive law $\mathcal{L}(W)$ is a

GGC if

$$E(e^{-\theta W}) = \exp\left[-\Delta\theta - \int_0^\infty \log\left(1+\frac{\theta}{x}\right) d\mathcal{V}(x)\right], \tag{6.8}$$

where Δ is a constant and $\mathcal{V}(x) \geq 0$ is non-decreasing on $(0,\infty)$ and satisfies the conditions $\int_{0+}^1 |\log x| d\mathcal{V}(x) < \infty$ and $\int_1^\infty x^{-1} d\mathcal{V}(x) < \infty$. We have the following stochastic integral representation,

$$W \stackrel{L}{=} \int_0^\infty \tau^{-1} d_\tau \mathcal{G}(\mathcal{V}(\tau)) \tag{6.9}$$

where $(\mathcal{G}(\tau) : \tau \geq 0)$ is a (standard) gamma process with cumulant function $\log(1+\theta)$, and $\mathcal{V}(\tau)$ functions as a deterministic time transformation. This representation arises from the easily demonstrated result for the stochastic integral $I = \int_0^T k(\tau) d\Lambda(\mathcal{V}(\tau))$, where $k(\tau)$ and $T \leq \infty$ are deterministic, and the L'evy process Λ has characteristic exponent $\psi(t)$: The CF of I is $\exp[-\int_0^T \psi(tk(\tau)) d\mathcal{V}(\tau)]$.

The following result shows that $\mathcal{L}(W)$ is a GGC.

Lemma 6.4. The cumulant function $\kappa_W(\theta)$ of $\mathcal{L}(W)$ has the canonical form (6.8) with $\Delta = 0$ and

$$\mathcal{V}(x) = K\left[\sqrt{\gamma^2+x} - \gamma\right]^\alpha. \tag{6.10}$$

Proof. Observing that the left-hand side of (6.7) is $\kappa_W'(\theta)$, integration of the second term yields

$$\kappa_W(\theta) = 2\alpha K \int_0^\infty u^{\alpha-1} \log\left(1+\frac{\theta}{u^2+2\gamma u}\right) du.$$

The change of variable $y = u^2 + 2\gamma u$ reduces this to the GGC canonical form (6.8) as asserted. The density $v(x)$ of the measure (6.10) satisfies $v(x) \sim K(x/2\alpha)^{\alpha-1}$ as $x \to 0+$, showing that $\int_0^1 |\log x| v(x) dx < \infty$, and $v(x) \sim \alpha K x^{\frac{1}{2}\alpha-1}$ as $x \to \infty$, showing $\int_1^\infty [v(x)/x] dx < \infty$. Thus all conditions for GGC membership are satisfied. □

A little manipulation with (6.8) shows that the a-tilt of a GGC is again a GGC with canonical measure $\mathcal{V}(x-a)$. Applying this to (6.10) shows that $Y = W_{\gamma^2}$ is a GGC with canonical measure $\mathcal{V}_\gamma(x) = K([\sqrt{x}-\gamma]^+)^\alpha$. The stochastic integral representations (6.9) of these laws could form the basis of data simulation.

The above decompositions leading to Lemma 6.3 shows the existence of a positive law $\mathcal{L}(W)$ which can be tilted, then used to mix a normal variance, thus yielding the symmetric pruned stable law $X = Z\sqrt{W_{\gamma^2}}$. If desired, asymmetry can be introduced by a further tilting operation as follows. Let $-\gamma < \zeta < \gamma$, and define the law $\mathcal{L}(X_\zeta)$ whose density is proportional to $e^{-\zeta x} f(x)$. From (6.2), we see that its cumulant function is

$$-\frac{b}{2(\alpha-1)}\left[(\gamma+\zeta+\theta)^\alpha + (\gamma-\zeta-\theta)^\alpha - 2\gamma^\alpha\right],$$

showing that this law is realized as $X_\zeta \stackrel{L}{=} V_{\gamma+\zeta}(1) - V_{\gamma-\zeta}(2)$ where the $V(j)$ are independent stable$(\alpha, b, 1)$ variates.

This chain of construction can be applied to any initial positive law. For example, a structurally simpler way of truncating the tails of stable laws could begin with W having a positive stable$(a, b, 1)$ law where $0 < a < 1$. Then W_{γ^2} has LST

$$L(\theta; \zeta) = \exp\left(-\frac{b}{1-a}[(\gamma^2 + \theta)^a - \gamma^{2a}]\right).$$

As before, let Z be independent of W_{γ^2} with a normal $\mathcal{N}(0,2)$ law and $X = Z\sqrt{W_{\gamma^2}}$. The CF of X is $\phi(t) = \exp(-\psi(t))$ where

$$\psi(t) = \frac{b}{1-a}\left[(\gamma^2 + t^2)^{\alpha/2} - \gamma^\alpha\right], \tag{6.11}$$

where $\alpha = 2a$. The normal inverse Gaussian family corresponds to $\alpha = 1$ [1], and the normal-variance gamma laws arise as the limiting case $\alpha \to 0$ after replacing b with $2b/\alpha$. Note the similarity of (6.11) and the case $\alpha \neq 1$ and $\beta = 0$ of Theorem 5.1; there is no $\cos(\alpha\omega)$ term or dependence on the sign of $\alpha - 1$ in (6.11). The following result lists properties relevant to our theme of this tilted-stable mixture law. Let $K_\lambda(\cdot)$ denote the modified Bessel function of the third kind.

Theorem 6.2. The law defined by (6.11) is infdiv with a symmetric Lévy density

$$n(x) = \frac{ab}{\Gamma(2-a)\sqrt{\pi}}\left(\frac{2\gamma}{|x|}\right)^{a+\frac{1}{2}} K_{a+\frac{1}{2}}(\gamma|x|). \tag{6.12}$$

As $x \to \infty$,

$$n(x) \sim \frac{ab}{\Gamma(2-a)}(2\gamma)^a x^{-1-a} e^{-\gamma x} \tag{6.13}$$

and

$$\bar{F}(x) \sim \frac{ab}{\gamma\Gamma(2-a)}(2\gamma)^a \exp\left(\frac{b\gamma^\alpha}{1-a}\right) x^{-1-a} e^{-\gamma x}. \tag{6.14}$$

The density function has the series representation

$$f(x) = -\frac{M}{|x|}\sum_{j=1}^{\infty} \frac{\Gamma(aj+1)\sin(a\pi j)}{j!}\left[-\rho\left(\frac{|x|}{2\gamma}\right)^a\right]^j K_{aj+\frac{1}{2}}(\gamma|x|), \tag{6.15}$$

where $\rho = b/(1-a)$ and $M = (2\pi)^{-3/2}\gamma^{-\frac{1}{2}}\exp(\rho\gamma^\alpha)$.

Proof. The Lévy density of W_{γ^2} is

$$m(x) = \frac{ab}{\Gamma(2-a)} x^{-1-a} e^{-\gamma^2 x} 1_{(0,\infty)}(x).$$

Lemma 6.1 yields a standard integral [14, p. 340,#9], giving (6.12). The asymptotic form (6.13) immediately follows [14, p. 963, #6], and then (6.14) from Theorem 2.1. For (6.15), let $g_{\gamma^2}(x)$ be the density of W_{γ^2} and observe that the specification of $f(x)$ as a normal-variance mixture entails

$$f(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-x^2/4y} g_{\gamma^2}(y) y^{-\frac{1}{2}} dy = \frac{\exp(\rho\alpha^a)}{2\sqrt{\pi}} \int_0^\infty e^{-vx^2/4-\gamma^2 v} g_0(1/v) v^{-3/2} dv$$

where $g_0(x)$ is the stable$(a,b,1)$ density. Inserting the power series representation of $g_0(1/v)$ and integrating term-by-term leads to (6.15). □

A key difference between the symmetric pruned stable law and this variance mixture is seen in the differing algebraic factors in the expansions (5.2) and (6.14), $x^{-1-\alpha}$ and $x^{-1-\alpha/2}$, respectively. The pruned stable law allows a little more scope for fitting tails than the tilted stable mixture. Another difference lies in their variances,

$$v_{PS} = \frac{2\alpha b}{2-\alpha} \gamma^{\alpha-2} \qquad \& \qquad v_{TSM} = \frac{2ab}{1-a} \gamma^{\alpha-2},$$

for the pruned stable law and the tilted-stable mixture, respectively. The first is obtained by differentiation of the characteristic exponent in Theorem 5.1 and the second from $v_{NM} = E(Z^2 W_{\gamma^2})$.

There is no reason to expect a closed expression for the sum (6.15), just as there is no closed expression for general stable densities. By contrast, the density for the pruned stable is completely intractable. The case $\alpha = 1$ for the tilted-stable mixture admits the explicit result

$$f(x) = \frac{2b\gamma\exp(2b\gamma)}{\pi} \cdot \frac{K_1(\gamma\sqrt{4b^2 x^2}}{\sqrt{4b^2+x^2}},$$

which we recognize as the symmetric normal inverse Gaussian law because the base law is the positive stable$(\frac{1}{2})$ law whose tilt is an inverse Gaussian. See [1, 2] for financial applications. An asymmetric extension of (6.11) is produced by a further ζ-tilting operation, that is, multiplying the Lévy density (6.13) by $e^{-\zeta x}$.

An algebraically even simpler starting point gives the very flexible *generalized hyperbolic* (GH) family. Define a measure μ on $(0,\infty)$ by $\mu(dx) = x^{\lambda-1} e^{-\delta^2/x} dx$, where $\delta \geq 0$ and $\lambda \in \mathbb{R}$. This measure is finite iff $\lambda < 0$ and $\delta > 0$, in which case normalization gives the density of the reciprocal gamma family. A normal-variance mixture using this family gives the Student t-laws. If $\delta > 0$ then the γ^2-tilt $e^{-\gamma^2 x}\mu(dx)$ after normalization gives the generalized inverse Gaussian family [34]. Using this family to form a normal-variance mixture gives the symmetric GH family, and ζ-tilting (with $|\zeta| < \gamma$) gives the full GH family, which we denote by GH$(\lambda, \delta; \gamma, \zeta)$. Note that we use a parametrization differing slightly to the usual accounts in order to more easily compare with the p-stable and t-stable mixture laws. Specifically, the mgf of the GH$(\lambda,\delta;\gamma,\zeta)$ law is

$$M_{GH}(\theta) = \left(\frac{\gamma^2-\zeta^2}{\gamma^2-(\theta+\zeta)^2}\right)^{\frac{1}{2}\lambda} \cdot \frac{K_\lambda(2\delta\sqrt{\gamma^2-(\theta+\zeta)^2})}{K_\lambda(2\delta\sqrt{\gamma^2-\zeta^2})}$$

and there is an explicit expression for the corresponding density, also in terms of a Bessel function. The tail behaviour of the GH law can be expressed via its density as $f(x) \sim Cx^{\lambda-1}e^{-(\gamma-\zeta)x}$ $(x \to \infty)$. Note that the exponent in the algebraic factor can take any real value. See [12] and references therein for accounts of the GH family.

A related construction of the GH family (*e.g.*, [3, p. 173]) is via $X \stackrel{L}{=} \zeta Y + Z\sqrt{Y}$, where Y has the generalized inverse Gaussian law. The random mean term accomplishes the second tilting operation, but with the modified parametrization GH$(\lambda,\delta;\sqrt{\gamma^2+\zeta^2},\zeta)$. Analagous outcomes occur if Y has the mixing law of Theorem 6.1, or the t-stable$(\alpha,b;\gamma^2)$ law.

Rydberg [31], and Barndorff-Nielsen and Shephard [3, 4] are recent surveys of the application to financial data of the GH and related models and the highly developed methodology developed for them.

7 Process and series representations

It hardly needs saying that t-stable and p-stable random variables can be embedded in a Lévy process. The random measure representation of this embedding process does not significantly simplify, except insofar as discussed by Eberlein [12, p.326] whose remarks apply whenever the process has a finite mean. Barndorff-Nielsen and Shephard [4] construct stationary models of Ornstein-Uhlenbeck (OU) type built on the fact that a law $\mathcal{L}(X)$ is self-decomposable iff $X = \int_0^\infty e^{-\tau}d\mathcal{B}(\tau)$ where the integrator is a Lévy process, called the background driving Lévy process (BDLP). The corresponding stationary OU process is $X(\tau) = e^{-\tau}X(0) + \int_0^t e^{-(\tau-u)}d\mathcal{B}(u)$, where $X(0) \stackrel{L}{=} X$. If n and ℓ denote the Lévy densities of X and $\mathcal{B}(1)$, respectively, then $\ell(x) = -(d/dx)(xn(x))$ [4, p. 302]. This construction fo~~rms the basis of their~~ coherent modelling methodology mentioned above.

Barndorff-Nielsen and Shephard [3, 4] choose the GH family for $\mathcal{L}(X)$. In principle their general approach is applicable to t-stable and p-stable laws. For example, if we fix $A > 0$ and let $\mathcal{L}(X)$ have the Lévy density

$$n_\gamma(x) = Ax^{-1-\alpha}e^{-\gamma x}1_{(0,\infty)}(x)$$

then

$$\ell(x) = \alpha n_\gamma(x) + \gamma A x^{-\alpha}e^{-\gamma x}1_{(0,\infty)}(x),$$

which clearly is a Lévy density. It follows that the t-stable law is self-decomposable and its BDLP is the sum of two independent Lévy processes. The first has Lévy density

$\alpha n_\gamma(x)$ corresponding to the time dilated embedding process $(\Lambda(\alpha\tau))$. If $\alpha < 1$ then the second component is a compound Poisson process having a gamma jump law which in obvious notation we denote by $\mathrm{Gam}(1-\alpha,\gamma)$. If $\alpha = 1$ then the second component is a gamma process, and if $1 < \alpha < 2$ then it is generated by a t-stable$(\alpha - 1, b; \gamma)$ law, where $b = (A/\alpha)\Gamma(2-\alpha)$. Similar representations hold for p-stable laws, even in the asymmetric case.

The tilted-stable mixture law of Theorem 6.2 being a normal-variance mixture with a GGC mixing law is a member of $\mathcal{T}_e$, and hence it too is self-decomposable. It follows that

$$\ell(x) = 2an(x) + \frac{2ab\gamma^2}{\Gamma(2-a)\sqrt{\pi}}\left(\frac{2\gamma}{|x|}\right)^{a-\frac{1}{2}} K_{a-\frac{1}{2}}(\gamma|x|).$$

The proof involves differentiating (6.12) and using the fact $xK'_\nu(x) + \nu K_\nu(x) = xK_{\nu-1}(x)$. So again the BDLP resolves into independent components with the first a time dilated version of the embedding process. The second component can be shown to be a compound Poisson process of normal-variance mixture type, $C_\tau = \sum_{j=1}^{N_\tau} Z_j\sqrt{V_j}$ where (N_τ) is a Poisson process with rate $2ab\gamma^2/(1-a)$, the Z_j are independent copies of $Z \sim \mathcal{N}(0,2)$, and the V_j are independent $\mathrm{Gam}(a,\gamma^2)$ variates. This decomposition generalizes Proposition 6.2 in [4] $(a = \frac{1}{2})$ and it represents the second component in a simpler and more explicit form than they achieve.

Motivated in part by the search for prior laws for Bayesian nonparametric inference, there is a body of work on random series representations of stable, and more generally, of infdiv variates. See [21, 30, 32, 36] for a fairly complete listing of the literature. Practicable ways of simulating stably distributed data appears to be a subsidiary motivation, but it is generally agreed now that the series converge too slowly to be useful for this purpose. As we now show, an elementary treatment results from imposing a regular variation condition on a Lévy measure. This condition holds for all modelling applications we know of.

Let (N_τ) be a unit rate Poisson process with event times $T_1 < T_2 < \cdots$. Given a Lévy measure μ, define $M(x) = \mu(x,\infty)$ and suppose there is a constant $\beta > 0$ and a function L slowly varying at infinity such that

$$M(x) = x^{-\beta}L(x^{-\beta}) \qquad (x > 0),$$

that is, M is regularly varying at the origin. The constraint $\int_0^1 x^2\mu(dx) < \infty$ implies that $\beta \le 2$. The function M has an asymptotic inverse

$$\rho(\nu) = \left(\nu L^{\#}(\nu)\right)^{-1/\beta} > 0, \qquad (x > 0) \tag{7.1}$$

where $L^{\#}$ is the slowly varying conjugate of L [5]. Finally, let $\{Y_n : n \ge 1\}$ denote independent copies of Y which has CF σ and first moment ξ_1, when it is defined.

Theorem 7.1. (i) Suppose $I_\rho := \int_1^\infty \rho(v)dv < \infty$. Then the series

$$X = \sum_{n=1}^{\infty} Y_n \rho(T_n) \tag{7.2}$$

converges absolutely almost surely iff

$$\beta < 1 \qquad \& \qquad E|Y|^\beta L(|Y|^\beta) < \infty$$

or

$$\beta = 1 \qquad \& \qquad E|Y|\ell(|Y|) < \infty,$$

where $\ell(x) = \int_x^\infty y^{-1}L(y)dy < \infty$. If either condition holds then the CF of X is

$$\phi(t) = \exp\left[-\int_0^\infty (1-\sigma(tx))\mu(dx)\right]. \tag{7.3}$$

(ii) Suppose $1 \le \beta < 2$, $I_\rho = \infty$, ζ_1 is finite, and L is normalized slowly varying. Then the series

$$\bar{X} = \sum_{n=1}^{\infty} (Y_n\rho(T_n) - \zeta_1\rho(n)) \tag{7.4}$$

converges unconditionally almost surely if $E|Y|^p < \infty$ for some $p > \beta$. If $\zeta_1 = 0$ then $\bar{X}$ has the CF (7.3). If $\zeta_1 \neq 0$ then

$$\dot{X} = \sum_{n=1}^{\infty} \left(Y_n\rho(T_n) - \zeta_1 \int_n^{n+1} \rho(v)dv\right) - B, \tag{7.5}$$

where $B = \zeta_1 \int_1^{M(1)} \rho(v)dv$, converges unconditionally almost surely under the above moment condition, and its CF is

$$\tilde{\phi}(t) = \exp\left[-\int_1^\infty (1-\sigma(tx))\mu(dx) - \int_0^1 (1-\sigma(tx) + i\zeta_1 tx)\mu(dx)\right].$$

Proof. (i) The law of large numbers and (7.1) imply that $\rho(T_n) \sim \rho(n)$ and the absolute convergence assertions are an immediate consequence of general convergence criteria for random Dirichlet series: See Corollary 2.2(b,c) in Pakes [28], observing that $\beta \le 1$ is necessary for $I_\rho < \infty$.

The form (7.5) of the CF is derived essentially as in [21]. If $X_n = \sum_{j=1}^{N_n} Y_j\rho(T_j)$ then

$$\begin{aligned} E\left(e^{itX_n}\right) &= E\left[E\left(e^{itX_n}\right)|N_n\right] = E\left[\left(n^{-1}\int_0^n \sigma(t\rho(v)dv\right)^{N_n}\right], \\ &= \exp\left[-\int_0^n (1-\sigma(t\rho(v)))dv\right] = \exp\left[-\int_{\rho(n)}^\infty (1-\sigma(tx))\mu(dx)\right] \end{aligned} \tag{7.6}$$

and (7.3) follows since $X_n \overset{a.s.}{\to} X$.
(ii) Our assumptions imply that $I_\rho = \infty$ whence $\sum_{n\geq 1} Y_n\rho(T_n)$ is almost surely divergent (Pakes [28, Corollary 2(a,c)]). Express the summands in (7.4) as $Y_n[\rho(T_n) - \rho(n)] + [Y_n - \zeta_1]\rho(n) \equiv U_{1n} + U_{2n}$ and let $\bar{\kappa} = \sup\{\kappa \geq 1 : E|Y|^\kappa < \infty\}$. Now $\sum_{n\geq 1} U_{2n}$ is a random Dirichlet series if it is regarded as a function of β^{-1}, and Theorem 3.2 (b) of Pakes [28] asserts that its abscissa of unconditional convergence is $\max(\frac{1}{2}, \bar{\kappa}^{-1})$. Choose $p < 2$ and note that since $\beta^{-1} > \frac{1}{2}$ we have $\beta^{-1} > p^{-1} \geq \bar{m}^{-1}$, and hence this series is almost surely unconditionally convergent under our moment hypothesis.

Observe that the normalization assumption on L implies that $L^\#$ is normalized slowly varying, and hence that $|L^\#(T_n)/L^\#(n) - 1| = o(|T_n - n|/n)$. This estimate and the mean value theorem imply that

$$|\rho(T_n) - \rho(n)| \sim \frac{\rho(n)}{\beta n}|T_n - n|.$$

We infer from the Marcinkiewicz-Zygmund strong law [9, p.122] that a.s. $|T_n - n| = o(n^{1/p})$ and hence that $|\rho(T_n) - \rho(n)| = o(n^{-1-(\beta^{-1}-p^{-1})}L^\#(n))$. It follows that $\sum_{n\geq 1} U_{1n}$ is almost surely absolutely convergent. If $\zeta_1 = 0$ then (7.6) still holds and hence the integral has a finite limit as $n \to \infty$.

If $\zeta_1 \neq 0$ write the nth partial sum of (7.5) is

$$\mathcal{P}(n) = \sum_{j=1}^{n} (Y_j\rho(T_j) - \zeta_1\rho(j)) + \sum_{j=1}^{n} \left(\rho(j) - \int_j^{j+1} \rho(v)dv\right) - B.$$

The terms in the second sum are non-negative and bounded above by $\rho(j) - \rho(j+1)$, and hence that sum converges as $n \to \infty$. This establishes the unconditional convergence of the series (7.5). Observe now that

$$X_n'' := X_n - \zeta_1 \int_{N_n}^{n} \rho(v)dv = \mathcal{P}(N_n) + \zeta_1 H(n) - B,$$

where

$$H(n) = \int_n^{1+N_n} \rho(v)dv = O(\rho(n)|N_n - n|) = \rho(n)o(\sqrt{n}\log n) \to 0,$$

and the final estimate is a consequence of a strong law of Kolmogorov. (In Feller's rendering [13, p. 239], for example, take his independent summands X_k to have the same law and $b_k = \sqrt{k}\log k$. Of course, the above estimate follows from the more abstruse law of the iterated logarithm.) It follows that X_n'' has a limit law coinciding with $\mathcal{L}(\tilde{X})$.

But since $\int_{N(1)}^{n} \rho(v)dv = \int_{\rho(n)}^{1} x\mu(dx)$, it follows from (7.6) that

$$E\left(e^{itX_n''}\right) = \exp\left[-\int_1^\infty (1 - \sigma(tx))\mu(dx) - \int_{\rho(n)}^1 (1 - \sigma(tx) + i\zeta_1 tx)\mu(dx)\right],$$

and this converges to $\tilde{\phi}(t)$ because the series (7.5) converges. □

Two boundary cases, which are not covered by Theorem 7.1, are stated without proof in the next result. Its proof is similar to that above, but using Corollary 2.3 and Theorem 3.2 in Pakes [28].

Theorem 7.2. If M is slowly varying at the origin then the series (7.2) converges absolutely almost surely iff $EM(|Y|^{-1}) < \infty$. If $\beta = 2$ and $\zeta_1 = 0$ then (7.2) converges unconditionally almost surely iff $\tilde{\ell}(x) = \int_x^\infty M(y^{-2})dy/y < \infty$ $(x > 0)$ and $E|Y^2|\tilde{\ell}(|Y|) < \infty$. If either convergence criterion is satisfied then (7.3) holds.

Observe that if $M(0+) < \infty$ then $\rho(v) \equiv 0$ if $v > M(0+)$ and the series (7.2) has finitely many non-zero terms, that is, X has a compound Poisson law. In the case $\beta = 2$ note that $\tilde{\ell}$ is slowly varying. Finally, we mention that representations for the embedding process, with its time parameter restricted to $[0,1]$, are obtained by replacing Y with $Y1_{(0,U]}(\tau)$ in the above series, where U has the uniform law on $(0,1]$.

It is clear that the laws of X and $\tilde{X}$ are infdiv since the Lévy measure $\tau\mu$ induces the function $\rho(v/\tau)$. In particular, if μ has density m and Y has density f, then X and $\tilde{X}$ have the Lévy density

$$n(x) = \int_0^\infty f(x/y)m(y)dy/y = \int_0^\infty f(y\mathrm{sgn}(x))m(|x|/y)dy/y. \tag{7.7}$$

Thus desired functional forms of n in principle can be tailored from convenient choices of m and f. For example, it is known that $m(x) = Ax^{-1-\alpha}$ yields spectrally positive (respectively, two-sided) stable(α)) laws for any one-sided (respectively, two-sided) $\mathcal{L}(Y)$, provided the convergence criteria are satisfied. The common choice is the point mass at unity (respectively, $P(Y = \pm 1) = \frac{1}{2}$). If $m(x) = Ax^{-1-\alpha}e^{-\gamma x}1_{(0,\infty)}(x)$ then these choices for $\mathcal{L}(Y)$ give $n(x) = m(x)$ in the first case (t-stable), and $n(x) = \frac{1}{2}m(|x|)$ in the second case (p-stable). Integration and changing variables yields

$$M(x) = ax^{-1-\alpha}E_{1+\alpha}(\gamma x)$$

where $E_{1+\alpha}$ is an exponential integral.

If f is the $\mathcal{N}(0,2)$ density, then (7.7) has the normal mixture form in Lemma 6.1 after replacing $m(x)$ there with $2xm(x^2)$. To realize the t-stable or symmetric p-stable laws, it follows from (6.5) that we must have

$$m(x) = A2^{1-\alpha}x^{-1-\alpha}e^{-\gamma^2x^2}\Psi(\tfrac{1}{2}\alpha, \tfrac{1}{2}; \gamma^2x^2).$$

Similarly, p-stable laws can be achieved by taking

$$f(x) = \frac{\gamma^\delta}{2\Gamma(\delta)} \cdot |x|^{\delta-1}e^{-\gamma|x|} \qquad \& \qquad m(x) = \frac{2A\Gamma(\delta)\gamma^\alpha}{\Gamma(\alpha+\delta)} \cdot x^{-1-\alpha}(1-x)^{\alpha+\delta-1}1_{(0,1)}(x).$$

Explicit determination of ρ in any of these cases is problematic.

8 Final remarks

We have illuminated confused definitions of exponentially truncated stable laws and elicited some properties of the pruned version. Symmetric pruned stable laws have been contrasted with normal variance mixtures using a tilted positive stable law or an inverse Gaussian law. Two characteristics stand out. The first is the absence of explicit expressions for the densities of the pruned stable or the tilted-stable mixture. The second is the restricted ranges which are permissable for the exponent values in the algebraic factors occuring in the tail estimates; $(1,3)$ for (5.2) and $(1,2)$ for (6.14).

Proponents of truncated/pruned stable laws argue their case in terms of the good fits obtained for selected data sets. It seems to us that a consistent application of this criterion should cause abandonment of these laws in favour of generalized hyperbolic laws. The GH family of laws is more useful because

- There are explicit expressions in terms of Bessel functions for their density functions, their Lévy densities, and for their moment generating functions for all parameter values;
- The family includes several commonly used sub-families;
- Members have the financially desirable property of being represented as normal variance mixtures;
- The family is more flexible for data fitting purposes, particularly by having greater scope in its tail behaviour.

The last point is nicely illustrated by Hurst and Platen [18] in their examination of five major world market index series. They fit eight types of symmetric law to these series, including the stable, Student-t, and the GH family. The Student-t family is declared the 'winner' in the sense of achieving a uniformly better fit according to a likelihood ratio criterion. The GH family fits equally well, but at the expense of an extra parameter. The Student t-law density function $f(x) \sim const.x^{-1-d}$ as $x \to \infty$, where $d > 0$ is the degrees-of-freedom parameter. In all cases its estimated value $\hat{d}$ lies outside the interval $(0,2)$, the permissable range of the stable index α. The estimates $\hat{\gamma}$ of the exponential decay factor for symmetric GH laws all appear to be very close to zero, though it should be noted that only estimates of $2\delta\gamma$ are actually reported. However, the fact that $\hat{\gamma} > 0$ implies that $-\hat{\lambda}$ should be smaller than $\hat{d}$, as indeed it is. In fact $-\hat{\lambda} > 0$, and it lies outside the interval $(0,2)$ for the Australian index series, but not for the other series. For all series, the best fitting laws have smaller tails than any non-normal stable law can attain. One anticipates that fitting a symmetric p-stable law will show little improvement over the stable, and not so good a fit as the symmetric GH. It would be worthwhile fitting the p-stable, if only to eliminate it from further consideration.

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