Addendum: Open Problems

Open problems abound. Here we merely give a representative sample. The problems fall into two categories—logic and analysis.

We begin with questions motivated by logic.

1. Our first problem is concerned with the following: What is the degree of difficulty of those analytic processes which have been proved to the computable? One topic of broad scope is to bring into analysis the whole complex of problems associated with $P = NP$ (cf. Cook [1971], Karp [1972], Friedman, Ko [1982], Friedman [1984], Ko [1983], Blum, Shub, Smale [to appear]). Thus we may ask which analytic processes are computable in polynomial time, polynomial space, exponential time, etc. In the same manner, we can ask about levels of difficulty within the Grzegorczyk hierarchy, or any other subrecursive hierarchy. Or we could fix our attention on the primitive recursive functions. There is no reason to believe that the answers to these questions will be automatic extensions of the general recursive case.
2. For processes proved to be noncomputable, we can also ask for fine structure—this time via the theory of degrees of unsolvability. Most of the noncomputability results in this book make use of an arbitrary recursively enumerable nonrecursive set. In fact, any recursively enumerable nonrecursive set—of any degree of unsolvability—will do. The question is: Can we replace results which merely assert that a certain process is noncomputable by a fine structure for that process, involving different degrees of unsolvability?
3. Our third problem is concerned with nonclassical reasoning. We recall that the reasoning in this book is classical—i.e. the reasoning used in everyday mathematical research. This contrasts with the intuitionist approach (e.g. of Brouwer), the constructivist approach (e.g. of Bishop), and the Russian school (e.g. Markov and Šanin). A natural question is: What are the analogs, within these various modes of reasoning, of the results in this book?

   In this connection, we cite the work of Feferman [1984], who originated the system T_0 for representing Bishop-style constructive mathematics. T_0 has both constructive and classical models. In particular, Feferman reformulated our First Main Theorem in T_0, and left as an open question the status of our Second Main Theorem.
4. Our fourth problem concerns higher order recursion theory. Let us set the stage.

   Higher order recursion theory, of course, deals with functionals of functions from $\mathbb{N}$ to $\mathbb{N}$, functionals of such functionals, etc. A functional approach to recursive analysis was given by Grzegorczyk [1955]. Here the real numbers are viewed as the set $\mathscr{R}$ of functions $\varphi: \mathbb{N} \to \mathbb{N}$. A function of a real variable is associated with a functional Φ mapping $\mathscr{R}$ into $\mathscr{R}$. The function of a real variable is "computable" if the associated functional Φ is computable—i.e. a general recursive functional.

Grzegorczyk proved that the functional approach is equivalent to the notion of Chapter-0-computability for continuous functions (Grzegorczyk [1957]). Chapter-0-computability, since it is tied more closely to standard analytic concepts (e.g. effective uniform continuity), appears to be more amenable to work in analysis.

The Chapter-0 definition leads in a natural way to certain generalizations—e.g. the definition of L^p-computability, and beyond that, the concept of a computability structure. The main results in this book are based on these generalizations. Higher order recursion theory leads to its own generalizations. The problem is: How do the concepts and results of higher order recursion theory relate to the concepts and results developed in this book?

We turn now to analysis. Here the set of problems is almost limitless, and we give only a few samples, which we find particularly appealing.

5. This problem combines ideas from topology, complex analysis, and recursion theory. Recursive topology provides a standard definition of "computable" open set in $\mathbb{R}^n$. However, it will emerge that this traditional definition may not be the right one.

   Let us consider this issue from the viewpoint of complex analysis. For this purpose, of course, we work within the 2-dimensional complex plane. We begin with a simply connected proper open subset of the plane. Here the crucial theorem is the Riemann Mapping Theorem, which asserts that every such region has a conformal mapping onto the open unit disk. This mapping is, up to trivial transformations, unique. Two obvious questions are: (1) If the mapping is computable, is the region computable? (2) If the region is computable, is the mapping computable? This in turn leads to the question of what we should mean by a computable region in the plane.

   The standard definition of a "computably open set" Ω goes as follows: Ω is *computably open* if it is the union of a sequence of disks $\{|z - a_i| < r_i\}$, where $\{r_i\}$ and $\{a_i\}$ are respectively computable sequences of real and complex numbers (Lacombe [1957b], Lacombe/Kreisel [1957]). This definition will not suffice for complex analysis. For, if we adopted it, the hope for any connection between computable region and computable function would be dashed. Here is a trivial counterexample. Consider the region $\Omega = \{z: |z| < \alpha\}$, where α is a noncomputable real which is the limit of a computable monotone sequence of rationals. Then the region Ω would be computably open in the sense defined above. But the natural conformal mapping onto the unit disk, namely z/α, is obviously not computable.

   The resolution of this question might provide an interesting interplay between plane set topology, complex analysis, and logic. The topological aspects could obviously be generalized to $\mathbb{R}^n$. For the analytic aspects, we might consider conformal mappings of multiply connected regions. Finally, one could generalize these problems to Riemann surfaces.

   The above discussion applies, of course, to recursive analysis with the usual classical reasoning. We mention that a discussion within the constructivist framework appears in Bishop [1967] and Bishop/Bridges [1985].

6. Many of the theorems in this book deal with the computability aspects of linear analysis. There are still many unsolved problems in this area. Here we mention two.

   Our first question is open ended. Throughout this book we have attempted to give general principles from which the effectivization or noneffectiveness of well-known classical theorems follows as corollaries. Obviously this program can be broadened in many ways. For example, it would be interesting to have a general principle which gave as a corollary the known facts concerning the Hahn-Banach Theorem. The facts are these. Metakides, Nerode, and Shore [1985] have proved a recursive version of the Hahn-Banach Theorem, in which they enlarge the norm of the functional by an arbitrary ε. They show that this enlargement is necessary. One particular question, which we would like to see emerge as an outgrowth of a more general principle, is the following. Characterize those Banach spaces for which we can obtain a recursive Hahn-Banach Theorem without an enlargement of the norm.

   Our second question is: Under what conditions are the eigenvalues of a bounded effectively determined operator T computable? For compact operators an affirmative answer is well-known. When T is self-adjoint or normal an affirmative answer is provided by our Second Main Theorem. On the other hand when T is neither self-adjoint nor compact, noncomputable eigenvalues can occur (Theorem 4.5). However, for many nonnormal operators, the eigenvalues are known to be computable—indeed they have been computed. The problem, then, is to find conditions, more general than normality, which cover important applications and imply that the eigenvalues are computable.

7. Finally, we give some open problems concerned with nonlinear analysis. So far as we know, the only major nonlinear problem which has been investigated from the viewpoint of recursion theory is the Cauchy-Peano existence theorem for ordinary differential equations (Aberth [1971], Pour-El/Richards [1979]). Nonlinear analysis is a vast area, and its connections with recursion theory, at the time of this writing, remain largely untouched.

   In many nonlinear problems, when they are dealt with classically, the technique of linearization plays an important role. This leads then to two questions. The first, absolutely untouched so far as we know, is the connection between the computability of the original nonlinear operator and the linear operator which results from it. The second concerns the computability of the eigenvalues of these linear operators. For self adjoint operators, this question has been answered by our Second Main Theorem. But for operators which are neither self-adjoint nor compact, the question remains open (cf. problem 6, above).

   Another problem is to extend the First Main Theorem to nonlinear operators. More precisely, we might ask to what extent, and under what side conditions, the First Main Theorem holds?

   A third area is the recursion theoretic study of particular nonlinear problems of classical importance. Examples are the Navier-Stokes equation, the KdV equation, and the complex of problems associated with Feigenbaum's constant.

   Obviously, this discussion provides but a small sample of the questions which can be asked in recursive nonlinear analysis.