Chapter XIII
E-Recursively Enumerable Degrees

Degree theory for subsets of E-closed structures differs markedly from the Σ_1 admissible case. On the surface the results are similar, but the modes of argument differ considerably. Post's problem once again has a positive solution, but this time without injuries and without repeated attempts to satisfy a given requirement. The presence of Moschovakis witnesses makes all the difference. Injuries do occur in the proof of Slaman's splitting theorem.

1. Regular Sets

Let $\mathscr{E}$ be transitive and $A, B \subseteq \mathscr{E}$. The relativization of E-recursiveness to B was introduced in Section 5.XI. In essence a new scheme,

$$\{c\}^B(x_1, \ldots, x_n) = B \cap x_i \quad (c = \langle 7, n, i \rangle)$$

is added to the original six. f is partial E-recursive relative to B if $f \simeq \{e\}^B$ for some $e < \omega$. D is E-recursively enumerable in p relative to B if

$$D = \{x \mid \{e\}^B(x, p)\downarrow\}$$

for some e. $\mathscr{E}$ is E-closed relative to B if

$$\{e\}^B(x)\downarrow \rightarrow \{e\}^B(x) \in \mathscr{E}$$

for all $e < \omega$ and $x \in \mathscr{E}$.

Assume $\mathscr{E}$ is E-closed. A is *E-reducible* to B on $\mathscr{E}$ (in symbols $A \leq_{\mathscr{E}} B$) if there exist $e < \omega$ and $p \in \mathscr{E}$ such that

(i) $\qquad \{e\}^B(x, p)\downarrow \quad$ for all $\quad x \in \mathscr{E}$,

(ii) $\qquad T_{\langle e, x, p; B \rangle} \in \mathscr{E} \quad$ for all $\quad x \in \mathscr{E}$, and

(iii) $\qquad A = \hat{x}[x \in \mathscr{E} \quad \& \quad \{e\}^B(x, p) = 0]$.

Note that $\{e\}^B(y)$ is not required to converge outside $\mathscr{E}$. (ii) says that the computation trees associated with the convergent computations of (i) belong to $\mathscr{E}$. The joint effect of (i) and (ii) is weaker than that of E-closure of $\mathscr{E}$ relative to B. (ii) is included for the sake of parity (cf. Part B): computations, arguments and values all have the same complexity as elements of $\mathscr{E}$. (ii) is costly; because of it, $\leq_{\mathscr{E}}$ need not be transitive.

If $\mathscr{E}$ is HF and $A, B \subseteq \omega$, then $A \leq_{\mathscr{E}} B$ iff A is Turing reducible to B.

If $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$, then A is said to be *subgeneric* if for all $e < \omega$ and $x \in \mathscr{E}$

$$\{e\}^A(x)\downarrow \rightarrow \{e\}^A(x) \in \mathscr{E}$$

In short $\mathscr{E}$ is E-closed relative to A, or shorter still, $\langle \mathscr{E}, A \rangle$ is E-closed.

If $\mathscr{E}$ is E-closed and $A \subseteq \mathscr{E}$, then A is said to be *E-recursively enumerable on $\mathscr{E}$* if there exist $e < \omega$ & $p \in \mathscr{E}$ such that

$$A = \hat{x}[x \in \mathscr{E} \ \ \& \ \ \{e\}(x, p)\downarrow].$$

If both A and $\mathscr{E} - A$ are E-recursively enumerable on $\mathscr{E}$, then A is *E-recursive on $\mathscr{E}$*.

Lemma 1.1. *Assume $L(\kappa)$ is E-closed. Suppose A, B, $C \subseteq L(\kappa)$ are E-recursively enumerable on $L(\kappa)$. If $A \leq_{L(\kappa)} B$ and $B \leq_{L(\kappa)} C$, then $A \leq_{L(\kappa)} C$.*

Proof. If C is subgeneric, then $A \leq_{L(\kappa)} C$ by composition.

Assume C is not subgeneric. Suppose $L(\kappa) = E(x)$ for some set x of ordinals. If $E(x)$ is Σ_1 admissible, then Proposition 1.2 (ii) implies $A \leq_{L(\kappa)} C$, since $\leq_\kappa$ (as defined in 3.2.VII) is transitive. If $E(x)$ is not Σ_1 admissible, then $E(x)$ admits Moschovakis witnesses by Theorem 5.8.X. If $L(\kappa) \neq E(x)$ for any set x of ordinals, then $L(\kappa) \neq E(x)$ for any $x \in L(\kappa)$. (Each $x \in L(\kappa)$ is coded by a relation (on ordinals) $\in L(\kappa)$.) Thus it is safe to assume $L(\kappa)$ admits Moschovakis witnesses.

Assume C is regular. Then any computation relative to C of height less than κ belongs to $L(\kappa)$. Since C is not subgeneric, there is a computation relative to C of height κ. Thus for some $e < \omega$ and $y \in L(\kappa) \cap 2^\kappa$:

$$(x)[x \in y \rightarrow T_{\langle e, x; C\rangle} \in L(\kappa)], \quad \text{and}$$
$$\kappa = \sup\{|\{e\}^C(x)| \mid x \in y\}.$$

Let $d < \omega$ and $p \in L(\kappa)$ be such that

$$A = \hat{z}\{z \in L(\kappa) \quad \& \quad \{d\}(z, p)\downarrow\}.$$

Then $z \notin A$ iff

(1) $\qquad (\mathrm{E}x)(\mathrm{E}w)[x \in y \quad \& \quad w \in L(|\{e\}^C(x)|) \quad \& \quad w \text{ witnesses } \{d\}(z, p)\uparrow].$

It follows from (1) and the regularity of C that $A \leq_{L(\kappa)} C$. The least x satisfying (1) is computable from C by effective transfinite recursion on $y \subseteq \kappa$.

Assume C is not regular. Suppose

$$C = \{x \mid \{f\}(x, q)\downarrow \quad \& \quad x \in L(\kappa)$$

for some $q \in L(\kappa)$. Choose $y \in L(\kappa)$ so that $y \cap C \notin L(\kappa)$. Then

$$\sup\{|\{f\}(x, q)| \mid x \in y \cap C\} = \kappa, \quad \text{and}$$

(2) $$z \notin A \rightarrow (\mathrm{Ex})(\mathrm{Ew})[x \in (y \cap C) \ \& \ w \in L(|\{f\}(x, q)|) \\ \& \quad w \text{ witnesses } \{d\}(z, p)\uparrow].$$

(2) implies $A \leq_{L(\kappa)} C$ as (1) did. □

1.2 Proposition. *Suppose x is a set of ordinals, $E(x)$ is Σ_1 admissible, and $A, B \subseteq E(x)$.*

(i) A is E-recursively enumerable on $E(x)$ iff A is $\Sigma_1^{E(x)}$.

(ii) $A \leq_{E(x)} B$ iff A is $E(x)$-recursive in B (in the sense of admissibility theory).

Proof. Same as that in Section 5.10.X. □

Suppose $L(\kappa)$ is E-closed and $A, B \subseteq L(\kappa)$ are E-recursively enumerable on $L(\kappa)$. A and B have the same *degree* (in symbols $A \equiv_{L(\kappa)} B$) if each is E-reducible to the other on $L(\kappa)$. By Lemma 1.1, $\equiv_{L(\kappa)}$ is an equivalence relation, and so the degrees are well defined. The degree of A (in symbols $\underset{\sim}{A}$) is

$$\{C \mid C \equiv_{L(\kappa)} A \ \& \ C \text{ is } E\text{-recursively enumerable on } L(\kappa)\}.$$

Proposition 1.2 solves Post's problem for $L(\kappa)$ when $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$ for some x a set of ordinals. Otherwise $L(\kappa)$ admits Moschovakis witnesses, and they carry the burden of proof in the solution of Post's problem.

A set $A \subseteq \mathscr{E}$, $\mathscr{E}$ E-closed, is said to be *regular* if $(A \cap x) \in \mathscr{E}$ for all $x \in \mathscr{E}$. Clearly subgenericity implies regularity. The converse is far from true (Exercise 1.4). If A is E-recursively enumerable on $\mathscr{E}$, then A is regular iff the "enumeration" of $A \cap x$ finishes inside $\mathscr{E}$. More precisely,

$$\{|\{e\}(y, p)| \mid y \in A \cap x\} \in \mathscr{E},$$

where $A = \{y \mid \{e\}(y, p)\downarrow\} \cap \mathscr{E}$ for some $p \in \mathscr{E}$. Consequently the next theorem is essential for dynamic arguments about $\mathscr{E}$ recursively enumerable degrees.

1.3 Theorem (Sacks 1985). *Assume $L(\kappa)$ is E-closed. Suppose $C \subseteq L(\kappa)$ is E-recursively enumerable on $L(\kappa)$. Then there exists a regular $B \subseteq L(\kappa)$, E-recursively enumerable on $L(\kappa)$, such that $B \equiv_{L(\kappa)} C$.*

Proof. Suppose $L(\kappa)$ is not of the form $E(x)$. Then every E-recursively enumerable-on-$L(\kappa)$ subset of $L(\kappa)$ is regular. Suppose A is E-recursively enumerable in p on $L(\kappa)$ and $x \in L(\kappa)$. Then $A \cap x$ is first order definable over $E(\{x, p\}) \in L(\kappa)$.

Now suppose $L(\kappa)$ is of the form $E(x)$. It is safe to assume x is a set of ordinals. If $E(x)$ is Σ_1 admissible, then apply Proposition 1.2 and Theorem 4.2.VII. Suppose $E(x)$ is not Σ_1 admissible and C is not regular. Then $A \leq_{L(\kappa)} C$ for every A E-recursively enumerable on $L(\kappa)$, as in the last part of the proof of Lemma 1.1. In other words, C is *complete*. It suffices to find a regular, complete B E-recursively enumerable on $L(\kappa)$. Recall that the completeness of C in the proof of 1.1 follows from the existence of a computation relative to C of height κ. B will be defined so that $\kappa \leq_E \rho_0, x; B$, where

$$(1) \qquad \rho_0 = \mu\rho(\mathrm{E}g)[g\colon \rho \to \kappa \quad \& \quad g \in \Sigma_1^{L(\kappa)} \quad \& \quad \sup \text{range } g = \kappa].$$

g_0 of Exercise 1.5 is a strictly increasing $\Delta_0^{L(\kappa)}$ map on ρ_0 and

$$\sup \text{range } g_0 = \kappa.$$

It follows that range g_0 is E-recursive on $L(\kappa)$. Intuitively κ is computable from ρ_0 and range g_0. That will in fact be the case after a slight modification of g_0 via f.

Since $L(\kappa) = E(x)$ there is, for each $\delta < \kappa$, an $\langle e, u\rangle \in \omega \times \sup x$ such that

$$(2) \qquad |\{e\}(u, x)| = \delta.$$

Let $f(\delta)$ be the "least" member of $\omega \times \sup x$ that satisfies (2). Thus $f(\delta) \leq_E \delta, x$, and $\delta \leq_E f(\delta), x$. A pairing device allows the range of f to be construed as a subset of x.

Assume g_0 has been altered so that it is continuous and $g_0(i) + \rho_0 < g_0(i+1)$ for all $i < \rho_0$. Define

$$B = \{g_0(i) + f(g_0(i+1)) \mid i < \rho_0\}.$$

B is E-recursively enumerable on $L(\kappa)$. Enumerate γ in B as follows. Compute i, the greatest j such that $g_0(j) < \gamma$. Let $v = \gamma - g_0(i)$. Compute $f^{-1}(v)$ from v, x as above. If $f^{-1}(v) = g_0(i+1)$, then enumerate γ. (1) implies B is regular.

Observe that $\{g_0(i) \mid i < \rho_0\}$ can be defined by an effective transfinite recursion relative to B.

$$g_0(\lambda) = \sup\{g_0(i) \mid i < \lambda\}.$$
$$\{g_0(i) + f(g_0(i+1))\} = B \cap (g_0(i), g_0(i+1)).$$
$$g_0(i+1) \leq_E f(g_0(i+1)), x.$$

But then κ is E-recursive in $\rho_0, x; B$ via a computation of height κ. Hence B is complete.

The main trick above is the use of f to provide enough power to pass from $g_0(i)$ to $g_0(i+1)$. This trick is due to Slaman, who greatly simplified the original proof of 1.3. $\square$

1.4–1.5 Exercises

1.4. Find an $A \subseteq E(\omega_\tau)$ such that A is regular but not subgeneric.

1.5. Suppose $L(\kappa) = E(x)$ for some set x of ordinals. Define ρ_0 as in (1) of the proof of Theorem 1.3. Show there exists a continuous $\Delta_0^{L(\kappa)}$ map g_0 such that $g_0(i) + \sup x < g_0(i+1)$ for all i and sup range $g_0 = \kappa$.

2. *Projecta and Cofinalities*

Some structural properties of $L(\kappa)$ are needed to establish suitable indexings of Friedberg–Muchnik requirements in the next two sections.

ρ, the greater, and η, the lesser, E-recursively enumerable projectum, are defined as follows.

$$\rho^\kappa = \mu\gamma_{\gamma \leq \kappa}(\mathrm{E}f)[f \text{ is a partial } E\text{-recursive-on-}L(\kappa) \text{ map of } \gamma \text{ onto } L(\kappa)].$$

$$\eta^\kappa = \mu\gamma_{\gamma \leq \kappa}(\mathrm{E}A)[A \in 2^\gamma - L(\kappa) \text{ and } A \text{ is } E\text{-recursively enumerable on } L(\kappa)].$$

As any student of Jensen's fine structure theory might expect, the next result says $\eta^\kappa = \rho^\kappa$. For simplicity it is assumed that $L(\kappa)$ is E-closed. Only small changes in the proof of Lemma 2.1 are needed when $L(\kappa)$ is rudimentarily closed rather than E-closed.

2.1. Lemma (Sacks 1985). $\eta^\kappa = \rho^\kappa$.

Proof. Let f be a partial E-recursive-on-$L(\kappa)$ map of ρ onto $L(\kappa)$.

If $\rho < \eta$, then $f^{-1}[\kappa] \in L(\kappa)$, and consequently $\kappa \in L(\kappa)$.

To see $\eta \geq \rho$, fix $\gamma < \rho$ and let $A \subseteq \gamma$ be E-recursively enumerable in $\rho \in L(\kappa)$ via Gödel number e with the intention of showing $A \in L(\kappa)$. It is safe to assume p is an ordinal, since each member of $L(\kappa)$ is E-recursive in some ordinal less than κ. Let g be a universal partial E-recursive function. The essential property of g is: for all x,

$$g[x] = \{\{e\}(z_0, \ldots, z_{n-1}) \mid \{e\}(z_0, \ldots, z_{n-1})\downarrow, e, n \langle \omega, z_i \in x\}.$$

Thus $E(x) = g[TC(\{x\})]$. Let

$$H = g[\gamma \cup \{p\}]. \tag{1}$$

H is said to be the *partial E-recursive hull* of $\gamma \cup \{p\}$. Observe that 2.8.X implies

$$z \in H \rightarrow O(z) \in H. \tag{1a}$$

$O(z)$ is the least δ such that $z \in L(\delta+1) - L(\delta)$.

(1) implies $H \prec_0 L(\kappa)$, that is, an Δ_0^{ZF} sentence with parameters true in H, is also true in $L(\kappa)$. The argument proceeds by induction on length of sentences. Suppose $\mathscr{F}(x)$ is a Δ_0^{ZF} formula with parameters in H such that

$$L(\kappa) \models (\mathrm{E}x)_{x \in b} \mathscr{F}(x), \tag{2}$$

where $b \in H$. The problem is to find some $x \in b \cap H$. that satisfies $\mathscr{F}(x)$. Each z put in H by g is accompanied by a wellordering of z. z is the value of some $\{e\}(z_0, \ldots, z_{n-1})$, where every z_i is an ordinal. Hence the convergent computation of $\{e\}(z_0, \ldots, z_{n-1})$ virtually includes a computation of a wellordering of z, as in Proposition 5.2.X. Let $w \in H$ be a wellordering of b, and x_0 the w-least element of b that satisfies $\mathscr{F}(x)$. Then x_0 is E-recursive in b, w and the parameters of $\mathscr{F}(x)$, and so belongs to H.

Since $H \prec_0 L(\kappa)$, H is extensional, hence isomorphic to a transitive set H_0 via the collapsing map t (cf. 2.6.VII). It follows from (1a) that $H_0 = L(\beta)$ for some $\beta \leq \kappa$ (cf. 2.5.VII). By (1)

$$L(\beta) = g[\gamma \cup \{t(p)\}]. \tag{3}$$

Since ρ is an $L(\kappa)$-cardinal, it follows from (3) that g, slightly altered, partially maps a bounded initial segment of ρ onto $L(\beta)$. Consequently $\beta < \rho$. Next

$$A = \{x \mid x < \gamma \quad \& \quad \{e\}(x, p)\downarrow\},$$

hence $A \in L(\beta + 1) \in L(\kappa)$, if t preserves convergence and divergence facts about elements of H. This last follows from $H_0 \prec_0 L(\kappa)$. Suppose $x < \gamma$. Then

$$\begin{aligned} &L(\beta) \models |\{e\}(x, t(p))| = \tau \\ \text{iff} \quad &L(\kappa) \models |\{e\}(x, p)| = t^{-1}(\tau). \qquad \square \end{aligned}$$

Let γ be an ordinal. A slight revision of the proof of Lemma 2.1 shows: if H is the partial E-recursive hull of a subset of γ, then the transitive collapse of H is isomorphic to some $L(\beta)$. The latter is an effective variation on the Jensen–Karp condensation lemma: if $H \prec_1 L(\kappa)$, then the transitive collapse of H is isomorphic to some $L(\beta)$. It is possible to develop a fine structure theory for L based on E-recursion. The idea is to replace Σ_1 by E-recursively enumerable systematically. It is not clear if any worthwhile applications result.

A relativization of Lemma 2.1 is needed for Slaman's density and splitting theorems in the next chapter. Lemma 2.2 relativizes 2.1 to B when $L(\kappa)$ is E-closed and B is regular and E-recursively enumerable on $L(\kappa)$. The following definitions assume nothing about κ or B.

ρ^B is the least $\gamma \leq \kappa$ such that for some $p \in L(\kappa; B)$, there exists a partial map

$$\lambda x \mid \{e\}^B(x, p)$$

of γ onto $L(\kappa; B)$ via computations in $L(\kappa; B)$.

η^B is the least $\gamma \leq \kappa$ such that there exists an $R \in 2^\gamma - L(\kappa; B)$ such that R is E-recursively enumerable on $L(\kappa; B)$ relative to B. This last means there are $e < \omega$ and $q \in L(\kappa; B)$ such that

$$R \cap \gamma = \{x \mid \{e\}^B(q, x)\downarrow \text{ via a computation in } L(\kappa; B)\}.$$

The proof of Lemma 2.2 makes strong use of the assumption that $L(\kappa)$ is E-closed. By contrast the proof of Lemma 2.1 succeeds when $L(\kappa)$ is rudimentarily closed.

2.2 Lemma (Slaman 1985). *Assume $L(\kappa)$ is E-closed. If $B \subseteq \kappa$ is regular and E-recursively enumerable on $L(\kappa)$, then $\eta^B = \rho^B$.*

Proof. Modeled on the proof of 2.1. The only difference is the presence of B, which causes no great conceptual difference save for the last part of the argument.

As in 2.1 $\eta^B \leq \rho^B$.

Fix $\gamma < \rho^B$ and let $A \subseteq \gamma$ be E-recursively enumerable in $p \in L(\kappa; B)$ relative to B via computations in $L(\kappa; B)$. Since B is regular, $L(\kappa; B) = L(\kappa)$. $\eta^B \geq \rho^B$ is proved by showing $A \in L(\kappa)$. Let g^B be a universal, partial E-recursive-in-B function such that for all x:

$$g^B[x] = \{\{e\}^B(z_0, \ldots, z_{n-1}) \mid \{e\}^B(z_0, \ldots, z_{n-1})\downarrow, e, n < \omega, z_i \in x\}.$$

Thus $E(x; B) = g^B[TC(\{x\})]$. Suppose B is E-recursively enumerable in q on $L(\kappa)$. Let

$$(1) \qquad H = \{z \mid z \in g^B[\gamma \cup \{p, q\}] \text{ via a computation in } L(\kappa)\}.$$

Define $O^B(z) = \mu\alpha[z \in L(\alpha+1; B) - L(\alpha; B)]$. As in the proof of 2.1,

$$(2) \qquad z \in H \to O^B(z) \in H.$$

It is safe to assume p and q are ordinals, since each element of $L(\kappa)$ is E-recursive in some ordinal below κ. B is a set of ordinals. Hence each z put in H by g^B is accompanied by a wellordering of z, as in Proposition 5.2.X. It follows, as in 2.1, that

$$(3) \qquad \langle H; H \cap B \rangle \prec_0 \langle L(\kappa; B) \rangle.$$

According to (3) each Δ_0^{ZF} sentence with parameters in H, and with $y \in B$ as an additional atomic formula, is true in H iff it is true in $L(\kappa)$. (3) implies H is extensional, hence isomorphic to a transitive set via a collapsing map t. By (2)

$$t: H \xrightarrow{\approx} L(\beta), \quad \text{and}$$

$$\langle H, H \cap B \rangle \approx \langle L(\beta); \bar{B} \rangle,$$

where $\beta \leq \kappa$ and $\bar{B} = t[H \cap B]$.

Suppose $\beta < \kappa$. A is definable over $\langle L(\beta); \bar{B}\rangle$. $\bar{B}$ is E-recursively enumerable in $t(q)$ on $L(\beta)$ thanks to (3) and the E-closedness of $L(\kappa)$. Consequently A is definable over $L(\beta)$ and so belongs to $L(\kappa)$.

Suppose $A \notin L(\kappa)$. By the previous paragraph $\beta = \kappa$. (1) implies g^B maps some $\gamma_0 < \rho^B$ onto H via computations in H. Hence $g^{\bar{B}}$ maps γ_0 onto $L(\kappa)$ via computations in $L(\kappa)$. The existence of $g^{\bar{B}}$ almost contradicts the definition of η^B; it fails to do so because it is computed relative to $\bar{B}$ rather than B. The defect is remedied by using A to approximate $g^{\bar{B}}$. Suppose

$$A = \{v \,|\, v < \gamma \quad \& \quad \{c\}^B(v, p)\downarrow\}$$

with the understanding that only computations in $L(\kappa)$ are acceptable. Define $|v| = |\{c\}^B(v, p)|$.

Consider the definition of $g^{\bar{B}}$: $\bar{B}$ is E-recursively enumerable in $t(q)$ on $L(\kappa)$; $g^{\bar{B}}$ is computed relative to $\bar{B}$ via computations in $L(\kappa)$. The definition of $g_v^{\bar{B}}$ is obtained by substituting $L(|v|)$ for $L(\kappa)$ in the definition of $g^{\bar{B}}$. If $t(p)$ or $t(q) \notin L(|v|)$, then $g_v^{\bar{B}}(u)$ is undefined. Set $f(u, v)$ equal to $g_v^{\bar{B}}(u)$. f is partial E-recursive in p, $t(p)$, $t(q)$ relative to A via computations in $L(\kappa)$.

It remains only to show f is a partial map of $\gamma_0 \times \gamma$ onto $L(\kappa)$. Fix $u < \gamma_0$ and suppose $g^{\bar{B}}(u)$ is defined by some computation d in $L(\kappa)$. d uses $\bar{B} \cap \sigma$ for some $\sigma < \kappa$. ($\bar{B}$ is regular because B is.) Since $L(\kappa)$ is E-closed, $\bar{B} \cap \sigma$ is enumerated via computations of height less than τ for some $\tau < \kappa$. Thus $g^{\bar{B}}(u) = g_v^{\bar{B}}(u)$ for all sufficiently large $|v|$. Note that

$$\sup\{|v| \;|v \in A\} = \kappa,$$

since $A \notin L(\kappa)$. $\square$

2.3 Proposition. *Let p be an ordinal and A a set of ordinals. If $\lambda x\,|\,\kappa_r^{p,x}$ does not attain a maximum on A, then*

$$\sup\{\kappa_r^{p,x} \,|\, x \in A\} \leq \sup\{\kappa_0^{p,y,z} \,|\, y, z \in A\}.$$

Proof. Suppose not. Then there are $c, d \in A$ such that

(1) $$\kappa_r^{p,c} > \kappa_r^{p,d} \geq \sup\{\kappa_0^{p,y,z} \,|\, y, z \in A\}.$$

By Lemma 5.5.X, $\kappa_r^{p,c,d} > \kappa_r^{p,d}$. The intended contradiction is: $\kappa_r^{p,c,d}$ is a $\langle p, d\rangle$-reflecting ordinal. Suppose some Σ_1 sentence $\mathscr{F}$ about $\langle p, d\rangle$ is true below $\kappa_r^{p,c,d}$. Then $\mathscr{F}$ is true below $\kappa_0^{p,c,d}$. But then (1) implies $\mathscr{F}$ is true below $\kappa_r^{p,d}$. $\square$

2.4 Proposition. *Assume $L(\kappa)$ is E-closed, and admits Moschovakis witnesses. If $A \subseteq \kappa$ is E-recursively enumerable on $L(\kappa)$ and incomplete, then A is subgeneric.*

Proof. The incompleteness of A implies A is regular as in the first half of the proof of Theorem 1.3. Hence $L(\kappa; A) = L(\kappa)$. If $x \in L(\kappa)$ and $\{e\}^A(x)\downarrow$, then $T_{\langle e, x; A\rangle} \in L(\kappa)$ by recursion on $|\{e\}^A(x)|$. Suppose e is $2^m \cdot 3^n$. Then by recursion $T_{\langle m, x; A\rangle}$ and $T_{\langle n, y; A\rangle}$ (for all $y \in \{m\}^A(x)$) belong to $L(\kappa)$. If $|\{m\}^A(x)| < \beta$ and

$$(1) \qquad \sup^+ \{|\{n\}^A(y)| \mid y \in \{m\}^A(x)\} \leq \beta < \kappa,$$

then $T_{\langle e, x; A\rangle}$ is first order definable over $L(\beta, x; A)$, and so belongs to $L(\kappa)$ by the regularity of A.

Suppose there is no β that satisfies (1). Then for all $d < \omega$ and all $z \in L(\kappa)$,

$$(2) \qquad \{d\}(z)\uparrow \leftrightarrow (\mathrm{Ey})[y \in \{m\}^A(x) \quad \& \quad \text{some } w \text{ first order definable over } L(|\{n\}^A(y)|) \text{ witnesses } \{d\}(z)\uparrow].$$

The right side of (2) can be verified by a computation from A in $L(\kappa)$, because A is regular. But then A is complete. (cf. proof of Lemma 1.1.) $\square$

The next theorem is needed for priority constructions in which requirements are indexed by ordinals less than ρ^κ. It will imply a block of requirements bounded below ρ^κ can be satisfied by some stage prior to the end of the construction.

2.5 Theorem (Sacks 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. If $p \in \kappa$ and $\gamma < \rho^\kappa$, then*

$$\sup\{\kappa_r^{p,\delta} \mid \delta < \gamma\} < \kappa.$$

Proof. Since ρ is either κ or an $L(\kappa)$-cardinal, it can be assumed that γ is closed under pairing in the sense that

$$(1) \qquad x, y < \gamma \rightarrow (\mathrm{Ew})[w < \gamma \quad \& \quad x, y \leq_E w].$$

To verify (1) let z be the standard 1-1 onto map from pairs of ordinals to ordinals. z is defined by recursion; at stage σ, $z(\sigma, \gamma)$ is defined for all $\gamma \leq \sigma$. Both z and z^{-1} are partial E-recursive. The w of (1) is $z(x, y)$.

(1) permits

$$(2) \qquad \{\langle e, \delta\rangle \mid \{e\}(p, \delta)\downarrow \quad \& \quad e < \omega \quad \& \quad \delta < \gamma\}$$

to be treated as a subset of γ. It follows from Lemma 2.1 that $(2) \in L(\kappa)$. Consequently

$$(3) \qquad \sup\{\kappa_0^{p,\delta} \mid \delta < \gamma\} < \kappa.$$

It follows from Theorem 5.8.X that $\kappa_r^{p,\delta} < \kappa$ for all $\delta < \kappa$. Suppose $\lambda w \mid \kappa_r^{p,w}$ does not attain a maximum on $\{\delta \mid \delta < \gamma\}$. Then Proposition 2.3 yields

$$\sup\{\kappa_r^{p,\delta} \mid \delta < \gamma\} \leq \sup\{\kappa_0^{p,x,y} \mid x, y < \gamma\}. \tag{4}$$

The theorem follows from (1), (3) and (4). $\square$

2.6 RE Cofinality. As in α-recursion theory, cofinality considerations figure prominently in E-recursively enumerable degree constructions. Lemma 2.7 is needed for the adaptation of Shore's blocking method (3.2.VIII) to E-recursion. For B a set of ordinals, let

$$|B| = \text{ordertype of } B.$$

Assume $L(\kappa)$ is E-closed. Define r.e. cf(λ), the *E-recursively enumerable-on-$L(\kappa)$ cofinality* of λ, to be

$$\begin{aligned}(1)\qquad & \mu\gamma(\mathrm{E}B)[\sup B = \lambda \quad \& \quad |B| = \gamma \quad \& \\ & B \text{ is } E\text{-recursively enumerable on } L(\kappa)]\end{aligned}$$

The last clause of (1) means that B is enumerated via a parameter and computations in $L(\kappa)$. Note that r.e. cf(κ) is the same as $\sigma 1\mathrm{cf}(\kappa)$.

Suppose B satisfies (1). B can be transformed into a B^* that is enumerated in increasing fashion. By "increasing" is meant:

$$B^* = \{x \mid \{e^*\}(p^*, x)\downarrow\};$$
$$|x| = |\{e^*\}(p^*, x)| \quad \text{if} \quad \{e^*\}(p^*, x)\downarrow;$$
$$x, y \in B^* \quad \& \quad x < y \to |x| < |y|.$$

Define:

$$x \in B_0^* \leftrightarrow x \in B \quad \& \quad (y)[y \in B \quad \& \quad |y| < |x| \to y < x];$$
$$x \in B^* \leftrightarrow x \in B_0^* \quad \& \quad (y)[y \in B_0^* \quad \& \quad |y| = |x| \to y \geq x].$$

If no unbounded in λ subset of B belongs to $L(\kappa)$, then the computations necessary for the enumeration of B are unbounded in height below κ, and so the above definition of B^* succeeds. Otherwise, B can be trivially altered to the desired B^*.

2.7 Lemma. *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Then*

$$\text{r.e. cf}(\rho^\kappa) = \text{r.e. cf}(\kappa).$$

Proof. To show r.e. cf$(\rho^\kappa) \geq$ r.e. cf(κ), let B satisfy the matrix of 2.6(1) with ρ^κ in place of λ. Suppose $p \in L(\kappa)$ and $\{e\}(x, p)$ is a partial map of ρ^κ onto $L(\kappa)$.

Define:

$$x^r = \sup\{\kappa_r^{p,y} \mid y \le x\} \ (x < \rho^\kappa);$$
$$B^r = \{x^r \mid x \in B\}.$$

By Theorem 2.5, $B^r \subseteq \kappa$. Let $f(y)$ be the value of $\{e\}(y, p)$ when the latter converges. In that event

(1) $$\kappa_r^{f(y)} \le \kappa_r^{p,y}$$

follows from Lemma 5.3.X. p can be effectively encoded by a set of ordinals, so it is safe to assume p is such a set. According to 5.3.X, divergence witnesses for computations from p, y can defined over $L(\kappa_r^{p,y})$. If (1) is false, these witnesses can be found below $\kappa_r^{f(y)}$, and so can be computed from p, y. But then the standard complete E-recursively enumerable in p, y subset of κ would be E-recursive in p, y.

It follows from (1) that $\sup B^r = \kappa$ and that the ordertype of B^r is at most that of B.

Lemma 5.5.X implies that $\kappa_r^{p,y}$ is the least γ such that every computation from p, y is seen to converge or to have a divergence witness by level γ of $L(\kappa)$. It follows that B^r is E-recursive on $L(\kappa)$.

To show r.e. $\mathrm{cf}(\rho^\kappa) \le$ r.e. $\mathrm{cf}(\kappa)$, let B be an unbounded subset of κ, E-recursively enumerable on $L(\kappa)$ and of ordertype r.e. $\mathrm{cf}(\kappa)$. As shown in subsection 2.6, B can be assumed to be enumerated in increasing fashion. Replace B by

$$B' = \{\delta \mid (\mathrm{Ex})(x \in B \quad \& \quad |x| = \delta)\}.$$

($|\delta|$ is the length of the computation that puts x in B.) B' and B have the same ordertype, and B' is E-recursive on $L(\kappa)$.

Let f be a partial map, E-recursive on $L(\kappa)$, from ρ^κ onto κ. Let

$$t\colon \text{r.e. } \mathrm{cf}(\kappa) \to \kappa$$

enumerate B' in increasing fashion. Assume $t(0) = 0$. For $x \in \operatorname{dom} f$, let $|x|$ be the length of the computation that puts x in $\operatorname{dom} f$. Define:

$$(\operatorname{dom} f)_\gamma = \{x \mid x \in \operatorname{dom} f \quad \& \quad t(\gamma) \le |x| < t(\gamma) + 1\};$$

$$s(\gamma) = \mu x[x \in (\operatorname{dom} f)_\gamma].$$

Since $(\operatorname{dom} f) \notin L(\kappa)$, $s(\gamma)$ is defined on unboundedly many γ's below ρ^κ. Let C be the range of s. The ordertype of C is at most that of B'. Note that s^{-1} is E-recursive on $L(\kappa)$, because t is. Consequently C is E-recursively enumerable on $L(\kappa)$.

The idea behind the above argument is to map B into ρ^κ by a computable function whose inverse is computable. In general the inverse of a one-one partial E-recursive function is not E-recursive, so some care had to be exercised above. $\square$

Theorem 2.5 and Lemma 2.7 are the only structural facts needed for the solution of Post's problem in the next section. Some further facts are needed for Slaman's splitting and density theorems. For Post's problem ordinals less than ρ^κ, the greater E-recursively enumerable projectum (of κ), suffice to index requirements. For splitting, ρ^κ tends to be too large and has to be replaced by $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ. The notion of tame Σ_2 projectum figured prominently in α-recursion theory (cf. Section 2.VIII).

2.8. The Tame Σ_1 Projectum. Assume κ is a limit ordinal. $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ, is

$$\mu\gamma_{\gamma\leq\kappa}(\mathrm{Ef})\,[f\colon \gamma \xrightarrow{\text{onto}} \kappa \quad \& \quad f\in\Sigma_1 \quad \& \quad f \text{ is tame}].$$

To say f is *tame* means:

$$(x)_{x<\mathrm{dom} f}(E\beta)_{\beta<\kappa}(z)_{z<x}[L(\beta)\models (Ey)[f(z)=y]].$$

In short, for a proper initial segment x of f, there is a bound below κ for the range of $f\restriction x$ and the associated existential witnesses.

If $L(\kappa)$ is E-closed and admits Moschovakis witnesses, then according to Exercise 2.16

$$t\sigma 1p(\kappa)\leq\rho^\kappa.$$

Let C be an E-recursively-enumerable-on-$L(\kappa)$, non-E-recursive, regular subset of κ. C ambiguously denotes the characteristic function of C. Define:

$$\{v\}(y,p)\simeq\{e\}(y,p,q) \quad \text{if} \quad v=\langle e,q\rangle; \tag{1}$$

$$\ell^C_p(v)=\sup\{z\mid(y)_{y<z}(C(y)=\{v\}(y,p))\}; \tag{2}$$

$$\delta_C=\mu\gamma_{\gamma\leq\kappa}(\mathrm{Ep})_{p<\kappa}[\kappa=\sup\{\ell^C_p(v)\mid v<\gamma\}]. \tag{3}$$

$\ell^C_p(v)$ measures the length of agreement between $\lambda y\mid C(y)$ and $\lambda y\mid\{v\}(y,p)$. It is commonplace in splitting and density arguments for recursively enumerable sets (cf. Section 1.IX). δ_C is called the *C-recovery parameter*. It plays a central role in the proof of Slaman's splitting theorem. The larger δ_C is, the better, since difficulties occur as $\ell^C_p(x)$ approaches κ. More precisely, δ_C should be larger than the length of any proper initial segment of preservation-type requirements.

The remainder of this section is devoted to showing $\delta_C\geq t\sigma 1p(\kappa)$.

Suppose $A\subseteq\kappa$. A is said to be *tame* $\Sigma_1^{L(\kappa)}$ if A is Σ_1 definable (boldface) over $L(\kappa)$, and if for each $\gamma<\sup A$, the existential witnesses need to establish $A\cap\gamma$ all lie below some $\delta<\kappa$.

2.9 Proposition. *If A is tame $\Sigma_1^{L(\kappa)}$ and* $\sup A<t\sigma 1p(\kappa)$, *then* $A\in L(\kappa)$.

Proof. The idea is to form a tame Σ_1 hull H over which A is Σ_1 definable. To say H is tame means H is the range of a tame Σ_1 function applied to an ordinal. The collapse of H is isomorphic to some $L(\beta)$. If $\beta < \kappa$, then $A \in L(\beta + 1) \in L(\kappa)$. If $\beta = \kappa$, then there exists a tame Σ_1 map from sup A onto $L(\kappa)$.

Form H as follows. Put each $\delta <$ sup A into H. For each $\delta \in A$, put the L-least existential witness that establishes $\delta \in A$ into H. Let H_0 be H so far. If H_0 is bounded below κ, then $A \in L(\kappa)$.

Suppose H_0 is unbounded. Some "bounded" existential witnesses must now be added to H_0. Suppose

$$b_0, b_1, \ldots, b_{n-1} < b < \kappa,$$

$$\mathscr{F}_e(x, y_0, \ldots, y_{n-1}) \text{ is } \Delta_0, \quad \text{and}$$

(1) $$L(\kappa) \models (\mathrm{Ex})_{x \in \underline{b}} \mathscr{F}(x, \underline{b}_0, \ldots, \underline{b}_{n-1}).$$

Let $g(e, b_0, \ldots, b_{n-1}, b)$ be the L-least $x \in b$ that satisfies $\mathscr{F}_e(x, \underline{b}_0, \ldots, b_{n-1})$. If (1) is false, give g the value 0. g is Σ_1, and tame in the sense that any bound on the arguments of g gives rise to a bound on the values of g. Now close H_0 under g. Let

$$H = H_0 \cup g[H_0] \cup gg[H_0] \cup \ldots$$

A slight alteration of g, denoted by g below, produces a tame $\Sigma_1^{L(\kappa)}$ map from $(\sup A)^{<\omega}$ onto H. The remainder of the proof requires a tame $\Sigma_1^{L(\kappa)}$ function f from sup A, or from some ordinal less than $tp\sigma 1(\kappa)$, onto H. Obtaining such an f from g is troublesome because $tp\sigma 1(\kappa)$ can be less than κ without being an $L(\kappa)$-cardinal. Let μ be the $L(\kappa)$-cardinality of sup A. Then

$$\sup A = \mu \cdot \eta + \beta$$

for some $\beta < \mu$. The proof of the proposition proceeds by induction on sup A. If $\beta > 0$, then

$$A = (A \cap \mu \cdot \eta) \cup (A \cap [\mu \cdot \eta, \mu \cdot \eta + \beta))$$

and so $A \in L(k)$ by induction. Thus $\beta = 0$. On similar grounds η is a limit or 1.

Since μ is a cardinal, there is a one-one map r of μ onto $\mu^{<\omega}$. For each $\delta < \eta$, let t_δ be a one-one map of μ onto $\mu \cdot \delta$. t_δ induces a one-one map v_δ of $\mu^{<\omega}$ onto $(\mu \cdot \delta)^{<\omega}$. Define

$$f(x) = g \circ v_{\delta+1} \circ r \circ t_{\delta+1}^{-1}(x) \quad \text{when} \quad x \in [\mu \cdot \delta, \mu \cdot (\delta + 1)).$$

f is a tame $\Sigma_1^{L(\kappa)}$ map from sup A onto H.

Since H is closed under g, H is extensional, hence isomorphic to a transitive set S via the collapsing map. The unboundedness of H, and its closedness under g, imply

$$z \in H \to O(z) \in H.$$

Consequently $S = L(\beta)$ for some $\beta \leq \kappa$. If $\beta < \kappa$, then $A \in L(\beta + 1) \subseteq L(\kappa)$.

Suppose $\beta = \kappa$. Then $\bar{f}$, the collapse of f, is a tame $\Sigma_1^{L(\kappa)}$ map from sup A onto $L(\kappa)$, an impossibility since sup $A < t\sigma 1p(\kappa)$. □

The next result is the principal structural fact needed in Slaman's splitting and density theorems.

2.10 Lemma (Slaman 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Let $C \subseteq \kappa$ be regular, not E-recursive on $L(\kappa)$, and E-recursively enumerable on $L(\kappa)$. Then*

$$\delta_C \geq t\sigma 1p(\kappa).$$

Proof. Suppose $\delta_C < t\sigma 1p(\kappa)$ with the hope of showing C is E-recursive on $L(\kappa)$.

Recall the definition of δ_C, the recovery parameter, from subsection 2.8. Let p be an ordinal that serves in 2.8(3) to define δ_C. Thus

$$\kappa = \sup\{\ell_p^C(x) \mid x < \delta_C\}. \tag{1}$$

Define "γ establishes (Ey)[$C(y) \neq \{x\}(y, p)$]" to mean: the computations and Moschovakis witnesses in $L(\gamma + 1)$ show there is a y such that

$$\{x\}(y, p)\uparrow \quad \text{or}$$
$$[\{x\}(y, p)\downarrow \quad \& \quad C(y) \neq \{x\}(y, p)].$$

For each $x < \delta_C$, define

$$f(x) = \mu\gamma[\gamma \text{ establishes (Ey)}(C(y) \neq \{x\}(y, p))]. \tag{2}$$

f will help compute C. A reflection argument will supply bounds on $f(x)$. Fix $v < \delta_C$, and let

$$w = \sup{}^+ \{\ell_p^C(x) \mid x < v\}.$$

For each $x < v$, $f(x)$ establishes an inequality in at least one of three ways.

Case 1: $\{x\}(y, p)\uparrow$ for some $y < w$. Let y_0 be the least such y. y_0 is the least element of a set co-E-recursively enumerable in x, p, w. Exercise 5.17.X, a corollary of Kechris's basis theorem (5.1.X), implies

$$\kappa_r^{x, p, w, y_0} \leq \kappa_r^{x, p, w}.$$

Hence $f(x) \leq \kappa_r^{x, p, w}$ by Lemmas 5.3.X and 5.6.X.

Case 2: case 1 does not hold, and there is a $y < w$ such that $C(y) = 0 \neq \{x\}(y, p)$. (Recall that "$C(y) = 0$" means $y \notin C$.) Let q be an ordinal parameter needed to E-recursively enumerate C on $L(\kappa)$. Let

$$S_2 = \{y \mid y < w \quad \& \quad \{x\}(y, p) = 1\}.$$

Then $S_2 \leq_E x, p, w$, and, as in case 1, $f(x) \leq \kappa_r^{x,p,q,w}$. In this case $f(x)$ establishes a Moschovakis witness for some y not in C.
Case 3: case 1 does not hold, and there is a $y < w$ such that $C(y) = 1 \neq \{x\}(y, p)$. Let

$$S_3 = \{y \mid y < w \quad \& \quad \{x\}(y, p) = 0\}.$$

Since C is regular, there is an ordinal $\theta < \kappa$ such that the computations in $L(\theta)$ suffice to enumerate $C \cap w$. Hence $f(x) \leq \kappa_0^{x,p,w} + \theta$.

The bounds provided in the above cases imply $\{f(x) \mid x < v\}$ is bounded below κ by Theorem 2.5.

Consequently f is a total, tame $\Sigma_1^{L(\kappa)}$ function from δ_C into κ. Note that the range of f is unbounded by (1). Since f is tame, it is safe to assume f is strictly increasing.

Define E to be

$$\{\langle x, z\rangle \mid x, z < \delta_C \quad \& \quad (\mathrm{E}y)_{y < f(z)}[C(y) \neq \{x\}(y, p)]\}.$$

E is tame $\Sigma_1^{L(\kappa)}$ because f is, or more precisely, because of the above proof that f is tame $\Sigma_1^{L(\kappa)}$. By Proposition 2.9, $E \in L(\kappa)$.

With the aid of E, C is E-recursive on $L(\kappa)$ as follows. Fix $y < \kappa$. There is a unique z_y such that

$$y \in [f(z_y), f(z_y + 1)).$$

The definition of f in (2) via the least ordinal operator implies

$$\{\langle x, f(x)\rangle \mid f(x) \leq y\} \leq_E y, p$$

uniformly in y. Thus z_{y+1} can be computed from y, and

$$(y')_{y' < f(z_{y+1})}[C(y') = \{x\}(y', p)]$$

for some x. Such an x can be extracted from E. $\square$

2.11 Tame Σ_2 Cofinality. Tame Σ_2 functions are needed for the adaptation of Shore's blocking method to the proof of Slaman's splitting and density theorems. They arise in the course of approximating regular, E-recursively-enumerable-on-$L(\kappa)$ sets, when $L(\kappa)$ is E-closed. Suppose C is such a set. Let

$$g(x) = C \cap x \quad \text{and} \quad g_0(\sigma, x) = C^{<\sigma} \cap x$$

for all $x, \sigma < \kappa$. ($C^{<\sigma}$ is that part of C enumerated via computations of height less than σ.) g_0 is E-recursive on $L(\kappa)$, and the regularity of C implies

(1) $$\lim_{\sigma} g_0(\sigma, x) = g(x).$$

In fact

(2) $$(a)_{a < \kappa}(E\tau)(x)_{x < a}(\sigma)_{\sigma > \tau}[g_0(\sigma, x) = g(x)].$$

Let h: dom $h \to \kappa$, where dom $h \leq \kappa$. h is said to be E_2 if there exists an E-recursive-on-$L(\kappa)$ function h_0 such that

$$(x)_{x \in \operatorname{dom} h}(E\tau)(\sigma)_{\sigma \geq \tau}[h_0(\sigma, x) = h(x)].$$

Thus g is E_2 by (1). h is said to be *tame* E_2 if there exists an E-recursive-on-$L(\kappa)$ function h_0 such that

$$(a)_{a < \operatorname{dom} h}(E\tau)(x)_{x < a}(\sigma)_{\sigma > \tau}[h_0(\sigma, x) = h(x)].$$

Thus g is tame E_2 by (2).

Suppose $\lambda \leq \kappa$. The *tame E_2 cofinality* of λ (in symbols, *te2cf*(λ)) is

$$\mu\gamma_{\gamma \leq \lambda}(\mathrm{Ef})\,[f\colon \gamma \to \lambda \quad \& \quad \sup \operatorname{range} f = \lambda \quad \& \quad f \text{ is tame } E_2].$$

According to Exercise 2.15,

$$te2cf(t\sigma 1p(\kappa)) = te2cf(\kappa).$$

This last fact will be quoted in the proof of Slaman's splitting theorem (4.1), where requirements are indexed by ordinals less than $t\sigma 1p(\kappa)$ and the number of blocks of requirements is $te2\mathrm{cf}(\kappa)$.

2.12–2.16 Exercises

2.12. Define rec cf(λ), the E-recursive-on-$L(\kappa)$ cofinality of λ, to be

$$\begin{aligned}\mu\gamma(\mathrm{EB})[\sup B = \lambda \quad & \& \quad \text{ordertype of } B = \gamma \\ & \& \quad B \text{ is } E\text{-recursive on } L(\kappa)].\end{aligned}$$

Assume $L(\kappa)$ is E-closed and show r.e. cf(κ) = rec cf(κ).

2.13. Assume $L(\kappa)$ is E-closed. Show that there exists a function g from r.e. cf(ρ^κ) into ρ^κ such that:

(i) graph of g is E-recursively enumerable on $L(\kappa)$;
(ii) sup range $g = \rho^\kappa$;
(iii) g is strictly increasing;
(iv) $\tau_1 < \tau_2 <$ r.e. cf$(\rho^\kappa) \to |\langle \tau_1, g(\tau_1)\rangle| < |\langle \tau_2, g(\tau_2)\rangle|$.

($|\langle x, g(x)\rangle|$ is the length of the computation that enumerates $\langle x, g(x)\rangle$ in the graph of g.)

2.14. Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Show $\delta_C \leqslant \rho^\kappa$.

2.15. Assume $L(\kappa)$ is E-closed. Show *te2*cf$(t\sigma 1p(\kappa))$ = *te2*cf(κ).

2.16. Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Show $t\sigma 1p(\kappa) \leq \rho^{\kappa}$.

3. van de Wiele's Theorem

van de Wiele [1982] found an important link between Σ_1 admissibility and E-recursiveness. His theorem explains why some Σ_1 functions are E-recursive and others are not. Intuitively, if f is Σ_1 and the search for existential witnesses needed to evaluate f does not extend too far, then f is E-recursive.

More precisely, let f be a total function from V into V. f is said to be *uniformly* Σ_1 *definable on every* Σ_1 *admissible set* if there exists a lightface Σ_1^{ZF} formula $\mathscr{F}(x, y)$ such that for every Σ_1 admissible set A:

$$f[A] \subseteq A;$$

$$f \upharpoonright A = \{\langle a, b\rangle \mid \langle A, \varepsilon\rangle \models \mathscr{F}(\underset{\sim}{a}, \underset{\sim}{b})\}.$$

"lightface" means that all parameters in $\mathscr{F}(x, y)$ are finite ordinals.

van de Wiele's proof is an application of proof-theoretic methods originated by Girard. Subsequently S. Simpson found a proof based on the compactness theorem for first order logic. The argument below is in the spirit of E-recursion and is extracted from Slaman [1981]. The latter approach appears to give more information than any of the others.

Theorem 3.1 (van de Wiele 1982). *Let f be a total function from V into V. Then* (*i*) *and* (*ii*) *are equivalent.*

(*i*) *f is E-recursive.*

(*ii*) *f is uniformly Σ_1 definable on every Σ_1 admissible set A.*

Proof. (i) implies (ii), since the relation, c is a computation of $\{e\}$ (b), is Δ_0^{ZF}, and so Δ_0 bounding implies A is closed under forming computations.

Assume (ii) holds. Fix x. The argument takes place within

$$L(\kappa_r^x, TC(\{x\})). \tag{1}$$

If $f(x) \in (1)$, then $f(x) \in L(\kappa_0^x, TC(x))$ by reflection, and so $f(x) \leq_E x$ by Gandy selection. $f(x)$ will be located in (1) by means of a hull $Z \subseteq (1)$ that collapses to a Σ_1 admissible set.

A curious technical step precedes the definition of Z. Choose $p \in (1)$ so that

$$\kappa_r^{x,p} = \min\{\kappa_r^{x,q} \mid q \in (1)\}. \tag{2}$$

For the moment assume $TC(x)$ is countable. Z will be of the form $\{z_i \mid i < \omega\}$. The z_i's are chosen one at a time from

$$L(\kappa_r^{x,p}, TC(\{x, p\})) \tag{3}$$

so that for all $i < \omega$:

(4) $$z_0 = \langle x, p \rangle;$$

(5) $$\kappa_r^{z_0, \ldots, z_i} = \kappa_r^{z_0};$$

(6) $$Z \text{ is } E\text{-closed; and}$$

(7) $$(\mathrm{E}w)[w \in (3) \quad \& \quad \{e\}(z_i, w)\uparrow] \rightarrow (\mathrm{E}j)[\{e\}(z_i, z_j)\uparrow].$$

In order to see that (6) and (7) can be honored without violating (5), fix i and assume $z_0, \ldots, z_i$ have been chosen in accord with (5). If $\{e\}(z_0, \ldots, z_i) = w$, then $\langle z_0, \ldots, z_i, w \rangle \equiv_E \langle z_0, \ldots, z_i \rangle$, and so

$$\kappa_r^{z_0, \ldots, z_i, w} = \kappa_r^{z_0, \ldots, z_i}.$$

If the antecedent of (7) holds, then Kechris's basis theorem (5.1.X) supplies a w such that

$$\kappa_r^{z_0, \ldots, z_i, w} \leq \kappa_r^{z_0, \ldots, z_i}.$$

It follows from (2) and (4) that $\kappa_r^{z_0, \ldots, z_i, w} = \kappa_r^{z_0}$.

By (7), Z is extensional, hence isomorphic to a transitive set $\bar{Z}$ via a collapsing map t. For each $z \in Z$ let $\bar{z}$ be $t(z) \in \bar{Z}$. To see that $\bar{Z}$ is Σ_1 admissible, let $\mathscr{G}(u, v, y)$ be a Δ_0^{ZF} formula such that

(8) $$\bar{Z} \models (u)_{u \in \bar{a}}(\mathrm{E}v)\mathscr{G}(u, v, \bar{c})$$

for some $\bar{a}, \bar{c} \in \bar{Z}$. To obtain a bound on v in (8) it has to be shown that:

(9) $$(3) \models (u)_{u \in a}(\mathrm{E}v)\mathscr{G}(u, v, c).$$

The derivation of (9) from (8) is based on the following technical fact.

(10) $$[u \in (3) \quad \& \quad (3) \models (\mathrm{E}v)\mathscr{G}(u, v, c)] \rightarrow (\mathrm{E}v)(E\gamma)[\gamma \leq_E x, p, c, u$$
$$\& \quad v \in L(\gamma, TC(\{x, p, c, u\})) \quad \& \quad (3) \models \mathscr{G}(u, v, c)].$$

Assume the antecedent of (10) holds with v_0 as the existential witness. Then

$$v_0 \in L(\gamma, TC(\{x, p\})) \quad \text{for some} \quad \gamma < \kappa_r^{x, p, c, u}$$

by (2), and so the consequent of (10) follows by reflection.

Suppose (9) is false. Then U, the set of all u such that

$$u \in a \quad \& \quad (3) \models (v) \sim \mathscr{G}(u, v, c),$$

is nonempty. It follows from (10) that U is co-E-recursively enumerable in x, p, c, a. Then (7) implies some $u \in U$ belongs to Z. Hence

$$Z \models (\mathrm{Eu})_{u \in a}(v) \sim \mathscr{G}(u, v, c),$$

contrary to (8). Thus (9) follows from (8).

Let (10r) be the result of replacing "$u \in (3)$" by "$u \in a$", and "x, p, c, u" by "x, p, c, a, u", in (10). The proof of (10r) is the same as that of (10). Gandy selection makes it possible to construe the γ of (10r) as an E-recursive function of u on a. Let

$$\delta = \sup\{\gamma(u) \mid u \in a\}.$$

Then $\delta \leq_E x, p, c, a$, and $\delta \in Z$ by (6). Thus (9) implies

$$Z \models (u)_{u \in a}(\mathrm{Ev})_{v \in b}\mathscr{G}(u, v, c),$$

where b is $L(\delta, TC(\{x, p, c, a\}))$. But then $\bar{b}$ bounds v in (8) and the proof of the Σ_1 admissibility of $\bar{Z}$ is complete.

Let $\mathscr{F}(u, v)$ be the Σ_1^{ZF} formula that defines f in every Σ_1 admissible set according to (ii). Then

$$(11) \qquad \bar{Z} \models (\mathrm{Ev})\mathscr{F}(\bar{x}, v).$$

In (11) $\bar{Z}$ can be replaced first by Z and then by (1). Finally by reflection

$$(13) \qquad L(\beta, TC(x)) \models (\mathrm{Ev})\mathscr{F}(x, v)$$

for some $\beta \leq_E x$, and the unique v $(=f(x))$ of (13) can be computed from β.

Thus $f(x)$ is computed by using Gandy selection to extract it from $L(\kappa_0^x, TC(x))$. The presence of $f(x)$ in $L(\kappa_0^x, TC(x))$ was established above by assuming $TC(x)$ was countable. That assumption is eliminable by the Levy–Shoenfield absoluteness theorem. Let $\{e\}$ be the procedure established above for computing $f(x)$ from x when $TC(x)$ is countable. Suppose there is an x_0 such that

$$(14) \qquad f(x_0) \neq \{e\}(x_0).$$

By virtue of $\mathscr{F}$, computations and Moschovakis witnesses, (14) is a Σ_1 statement about x_0 true in V. Let H be a countable Σ_1 substructure of V with x_0 as an element. Let $\bar{H}$ be the transitive collapse of H and $\overline{x_0}$ the image of x_0 in $\bar{H}$. Then

$$f(\overline{x_0}) \neq \{e\}(\overline{x_0})$$

is a Σ_1 statement true in $\bar{H}$. But $TC(\overline{x_0})$ is countable. $\square$

3.2 Exercise. What is the role of the curious parameter p in the proof of Theorem 3.1?

4. Post's Problem for E-Recursion

The theorem to be proved is: assume $L(\kappa)$ E-closed; then there exist two subsets of $L(\kappa)$, each E-recursively enumerable on $L(\kappa)$, such that neither is E-reducible to the other. In short, there exist incomparable E-recursively enumerable degrees. A weird feature of the argument is the absence of injuries. Priorities are assigned to requirements, but only to insure that requirements are met before time runs out. Conflicts between requirements are avoided with the help of Moschovakis witnesses. Injuries occur in the next section in the proof of Slaman's splitting theorem.

The conceptual differences between the solutions to Post's problem for E-recursion and α-recursion arise from the differences between the notion of recursively enumerable (r.e.) set as domain of partial recursive function and as Σ_1 definable set. Classical approaches to Post's problem rely on a dynamic view of r.e. sets: an r.e. set is enumerated by stages. An α-recursively enumerable set A is defined by a Σ_1 formula $\mathscr{F}$. A is enumerated by limiting the existential quantifier in $\mathscr{F}$. For each $\sigma < \alpha$, $A^{<\sigma}$ is the subset of $L(\sigma)$ definable by $\mathscr{F}$. Then $A = \cup\{A^{<\sigma} | \sigma < \alpha\}$, and $\lambda^{\sigma} | A^{<\sigma}$ is α-recursive. Suppose B, the domain of a partial E-recursive function, is to be defined dynamically by sifting through all computations in $L(\kappa)$. At stage σ, elements are to be added to B based on computations of height at most σ. If x is added, then the computation that puts x in B must be E-recursive in x, p, where p is a fixed parameter independent of x, e.g. the parameter needed to enumerate B. It follows that x cannot wait around forever. Either x is put in B prior to stage $\kappa_0^{x,p}$ or it never gets into B. In contrast, in α-recursion it is never to late to add x. Normally $\kappa_0^{x,p} < \kappa$, so it will eventually be too late to add x. To make matters worse (or more interesting) there is no effective way of recognizing $\kappa_0^{x,p}$. Thus x's time may have expired without any awareness of the expiration affecting the enumeration of B.

It may now seem immensely difficult to satisfy positive requirements in the setting of E-recursion. It turns out to be simpler than first supposed, because the presence of Moschovakis witnesses makes it easier than usual to satisfy negative requirements.

Suppose $L(\kappa)$ admits Moschovakis witnesses. A Friedberg–Muchnik incompatibility requirement is handled by waiting for a computation to converge. If it converges, then measures are taken to create and preserve an inequality. If it diverges, then by some stage the divergence is evident. At that stage a Moschovakis witness is chosen, and its role as a divergence witness is preserved forever.

If $L(\kappa) = E(\omega_1)$, then the requirements are indexed by countable ordinals, and each can be satisfied in its turn. For an arbitrary Σ_1 inadmissible $L(\kappa)$ the requirements are indexed by ordinals less than ρ^{κ}. By Theorem 2.5

$$\text{(1)} \qquad \text{if } x < \rho^{\kappa} \quad \text{and} \quad p \in \kappa, \text{ then } \sup\{\kappa_r^{p,y} | y < x\} < \kappa.$$

It will follow from (1) that any set of requirements indexed by ordinals less than $x < \rho$ can be satisfied by stage σ for some $\sigma < \kappa$.

4.1 Theorem (Sacks 1985). *Suppose $L(\kappa)$ is E-closed. Then there exist two subsets of $L(\kappa)$, each E-recursively enumerable on $L(\kappa)$, such that neither is E-reducible to each other on $L(\kappa)$ (Post's problem).*

Proof. First suppose $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$. According to Proposition 1.2 the solution to Post's problem for $L(\kappa)$ provided by α-recursion theory, Theorem 2.6.VIII, is also a solution in the sense of E-recursion theory. So assume $L(\kappa)$ is either Σ_1 inadmissible or the limit of E-closed structures. In either case $L(\kappa)$ admits Moschovakis witnesses.

For $y \in L(\kappa)$, let

$$\{y\}^A(x) \quad \text{be} \quad \{(y)_0\}^A(x, (y)_1)$$

with the proviso that $\{(y)_0\}$ is null when $(y)_0 \geq \omega$. Let f be a partial E-recursive-on-$L(\kappa)$ map of ρ^κ onto $L(\kappa)$. The requirements are:

$$\begin{array}{lll} \text{req. } 2x\text{:} & A \neq \{f(x)\}^B, \text{ and} & \\ \text{req. } 2x+1\text{:} & B \neq \{f(x)\}^A. & \end{array} \qquad (x < \rho^\kappa)$$

Each requirement belongs to a block labeled by an ordinal less than

$$\gamma_0, \text{ the } E\text{-recursively enumerable-on-}L(\kappa) \text{ cofinality of } \rho^\kappa.$$

By Exercise 2.13 there exists a map g of γ_0 into ρ^κ such that:

(1a) the graph of g is E-recursively enumerable on $L(\kappa)$;
(1b) sup (range g) $= \rho^\kappa$;
(1c) g is strictly increasing;
(1d) $\tau_1 < \tau_2 < \gamma_0 \rightarrow |\langle \tau_1, g(\tau_1)\rangle| < |\langle \tau_2, g(\tau_2)\rangle|$,

where $|\langle x, g(x)\rangle|$ is the length of the computation that enumerates $\langle x, g(x)\rangle$ in the graph of g.

In addition $g(0)=0$. Define

$$(\text{req. } 2x) \in \text{block } 2\tau, \quad \text{and} \quad (\text{req. } 2x+1) \in \text{block } 2\tau+1,$$

$$\text{if } g(\tau) \leq x < g(\tau+1).$$

The construction of A and B has the form of an effective transfinite recursion on κ. At stage $\sigma < \kappa$ decisions are made by examining all computations in $L(\sigma+1)$. For convenience it is assumed that the length of a computation c, denoted as usual by

$|c|$, has been redefined so that $|c|$ exceeds all ordinals mentioned in c. Let

$$A^{<\sigma} = \{x \mid x \text{ put in } A \text{ prior to stage } \sigma\},$$
$$\text{and} \quad A^{\sigma} = A^{<\sigma+1} - A^{<\sigma}.$$

Each requirement is acted on at most once!

Req. $2x$ is *acted on* at stage σ if (2a)–(2d) hold. Assume (req. $2x$) $\in$ block 2τ.

(2a) Req. $2x$ has not been acted on prior to stage σ.

(2b) $|\langle \delta, g(\delta)\rangle| < \sigma$ for all $\delta \leq \tau$, and $|f(x)| < \sigma$.

(2c) Every requirement in block y, for all $y < 2\tau$, has been acted on prior to stage σ.

(2d) Let $\sigma(2\tau)$ be the first stage after all the activity of (2b) and (2c) has been completed. Then either (2da) or (2db) holds:

(2da) $\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle\sigma(2\tau), f(x)\rangle)\downarrow$ via a computation in $L(\sigma+1)$. ($\langle\sigma(2\tau), f(x)\rangle$ is an ordinal that encodes the ordered pair $(\sigma(2\tau), f(x))$ and exceeds both of its components.)

(2db) A Moschovakis witness to

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle\sigma(2\tau), f(x)\rangle)\uparrow$$

belongs to $L(\sigma+1)$ via a computation in $L(\sigma+1)$. (Recall that the relation, x witnesses $\{e\}(y)\uparrow$, is E-recursively enumerable.)

Suppose req. $2x$ is acted on at stage σ. If (2da) holds and

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle\sigma(2\tau), f(x)\rangle) = 0,$$

then $\langle\sigma(2\tau), f(x)\rangle$ is put in A at stage σ. Thus

$$A^{\sigma}(\langle\sigma(2\tau), f(x)\rangle) = 1.$$

Req. $2x+1$ is handled by swapping A and B, odd and even, above.

End of construction.

4.2 Proposition. *Suppose every requirement in block z is acted on. Then all such acts occur prior to stage σ for some $\sigma < \kappa$.*

Proof. Suppose $z = 2\tau$ and (req. $2x$) $\in$ block 2τ. All activity with respect to block y, for all $y < 2\tau$, occurs prior to stage $\sigma(2\tau)$. Let $p \in \kappa$ encode $B^{<\sigma(2\tau)}$, $\sigma(2\tau)$, q, where q is the parameter needed to compute f. Then req. $2x$ is acted on by stage $\kappa_r^{p,x}$; by that stage either convergence (2da) is evident or a Moschovakis witness (2db) has appeared. Theorem 2.5 yields

$$\sup\{\kappa_r^{p,v} \mid 2v \in \text{block } 2\tau\} < \kappa,$$

since $\sup(\text{block } 2\tau) < \rho^{\kappa}$. $\square$

4.3 Proposition. *Assume $z <$ r.e. $\mathrm{cf}(\rho^\kappa)$. Suppose every requirement in block y, for all $y < z$, is acted on. Then all such acts occur prior to stage σ for some $\sigma < \kappa$.*

Proof. If $u < v$, then no requirement in block v is acted on until every requirement in block u has been acted on. Hence if z is a successor, then the desired σ exists by Proposition 4.2.

Assume z is a limit. Recall $\sigma(y)$ from the definition of "acted on". By Proposition 4.2

$$\sigma(y) < \kappa \quad \text{for all} \quad y < z.$$

Let S be $\{\sigma(y) | y < z\}$. S is E-recursively enumerable on $L(\kappa)$, because the computations and Moschovakis witnesses that put $\sigma(y)$ in S all lie inside $L(\sigma(y)+1)$. The ordertype of S is at most z, since $\lambda y | \sigma(y)$ is non-decreasing. It follows from Lemma 2.7 that $z <$ r.e. $\mathrm{cf}(\kappa)$. Hence $\sup(S) < \kappa$. □

4.4 End of Proof of 4.1. An induction on block numbers shows every requirement is acted on. Suppose (req. $2x$) $\in$ block 2τ. Assume every requirement in block y, for all $y < 2\tau$ has been acted on. By Proposition 4.3, $\sigma(2\tau) < \kappa$. ($\sigma(2\tau)$ was defined in clause (2d) of the definition of "acted on".) As in the proof of Proposition 3.2, req. $2x$ is acted on by stage $\kappa_r^{p,x}$, where p encodes $B^{<\sigma(2\tau)}$, $\sigma(2\tau)$, q. End of induction.

Suppose req. $2x$ is acted on at stage σ, and

$$\{f(x)\}^{B^{<\sigma(2\tau)}}(\langle\sigma(2\tau), f(x)\rangle) \tag{1}$$

converges. Then clause (2da) holds at stage σ,

$$(1) \neq A^\sigma(\langle\sigma(2\tau), f(x)\rangle) = A(\langle\sigma(2\tau), f(x)\rangle), \quad \text{and}$$

$$B^{<\sigma(2\tau)} = B^{<\sigma} = B \cap \sigma \tag{2}$$

Consequently $A \neq \{f(x)\}^B$.

Suppose (1) diverges. Then clause (2db) holds at stage σ, and some Moschovakis witness w to the divergence of (1) belongs to $L(\sigma+1)$. It follows from (2) that w also witnesses the divergence of

$$\{f(x)\}^B(\langle\sigma(2\tau), f(x)\rangle).$$

Thus again $A = \{f(x)\}^B$.

The most curious feature of the above version of Post's problem is the preservation of divergence witnesses in addition to the usual preservation of inequalities.

A final word on the E-recursive enumerability on $L(\kappa)$ of A and B. Suppose z is put in A at stage σ. Then z is of the form $\langle\sigma(2\tau), f(x)\rangle$ and (1) converges. σ is computable from $B^{<\sigma}$, z and some parameters r independent of z and σ. $B^{<\sigma}$ is computable from $L(\sigma(2\tau))$, r. Thus $\sigma \leq_E z, r$ (uniformly in z), and so A is E-recursively enumerable on $L(\kappa)$. □

4.5 Other Results. The proof of Theorem 4.1 shows:

(1) Assume $\langle L(\kappa; A), A\rangle$ is E-closed. If $\eta^{\kappa;A} = \rho^{\kappa;A}$, and $\kappa_r^{x;A} < \kappa$ for all $x \in L(\kappa; A)$, then $\langle L(\kappa; A), A\rangle$ admits a positive solution to Post's problem.

It is unlikely that $\eta^{\kappa;A} = \rho^{\kappa;A}$ in general. The selection theorem of Griffor and Normann, 5.3.XII, implies:

(2) If $L(\kappa; A) \models$ [there is a greatest cardinal and it is regular], then $\eta^{\kappa;A} = \rho^{\kappa;A}$.

The assumption in (1) on κ_r is related to inadmissibility, as in Lemma 5.6.X. To be precise:

If $\langle L(\kappa; A), A\rangle$ is E-closed, but not Σ_1 admissible, then $\kappa_r^{x;A} < \kappa$ for all $x \in L(\kappa; A)$.

The proof of Theorem 4.1 is applicable to some other transitive E-closed structures $\mathscr{E}$. Let κ be the least ordinal not in $\mathscr{E}$. $\mathscr{E}$ is said to be *effectively wellorderable* if there exists a partial E-recursive on $\mathscr{E}$ map f from κ onto $\mathscr{E}$ such that the only parameter needed to compute f is an ordinal.

(3) Assume $\mathscr{E}$ is E-closed and effectively wellorderable. If $\eta = \rho$ for $\mathscr{E}$, and $\mathscr{E}$ admists Moschovakis witnesses, then $\mathscr{E}$ admits a positive solution to Post's problem.

A consequence of (3) is:

(4) Suppose x is a set of ordinals and $L(\kappa, x)$ is not Σ_1 admissible. If $\eta = \rho$ for $L(\kappa, x)$, then $L(\kappa, x)$ admits a positive solution to Post's problem.

Normann's ground-breaking result [1975] on Post's problem for recursion in ${}^3E$ can be phrased as follows:

(5) If $E(2^\omega) \models [2^\omega$ is wellorderable and of regular cardinality], then $E(2^\omega)$ admits a positive solution to Post's problem.

The quest for Theorem 4.1 was initiated by a desire to circumvent the assumption of regularity in (5). Normann needed it primarily to show $\eta = \rho$ for $E(2^\omega)$; his argument made use of a selection result, Theorem 1.3.XII. Slaman [1983] has shown:

There exists a model M of ZFC in which $E(2^\omega) \models [2^\omega$ is wellorderable and of singular cardinality] and $\eta = \rho$ for $E(2^\omega)$.

It follows from (4) that Slaman's M admits a positive solution to Post's problem for $E(2^\omega)$.

According to 2.4 the A and B supplied by the proof of Theorem 4.1 are subgeneric. The proof yields a stronger property than subgenericity when r.e. $\mathrm{cf}(\rho^\kappa)=\rho^\kappa$. The simplest example is:

(6) $E(\omega_1)$ admits a positive solution (A, B) to Post's problem such that for all $\delta < \omega_1$,

(7)
$$\kappa_0^{\omega_1, \delta; A} = \kappa_0^{\omega_1, \delta; B} = \kappa_0^{\omega_1, \delta}.$$

A set A with property (7) is said to preserve the κ_0-spectrum of $E(\omega_1)$. It is not known if the κ_0-spectrum can be preserved when ρ fails to be r.e. regular.

At this writing nothing is known about situations where Post's problem has a negative solution, or where $\eta \neq \rho$.

(8) Does there exist a set x of ordinals such that $L(\kappa, x)$ is E-closed, is not Σ_1 admissible, and does not admit a positive solution to Post's problem for E-recursion theory?

(9) Does there exist a set x of ordinals such that $L(\kappa, x)$ is E-closed, is not Σ_1 admissible, and $\eta \neq \rho$ for $L(\kappa, x)$?

A negative answer for (8) implies one for (9) by (4). It might be simpler, however, to deal with (9) directly. Both questions are good candidates for forcing arguments, but none of the techniques of Chapter XI appear to be of any use. Perhaps the most pertinent open question about Post's problem is:

(10) Does there exist a model of ZFC in which

$$E(2^\omega) \models [2^\omega \text{ is wellorderable}]$$

and $E(2^\omega)$ does not admit a positive solution to Post's problem?

Exercises 4.6–4.8

4.6. Verify 4.5(1).

4.7. Show that the A and B defined in the proof of Theorem 4.1 are subgeneric.

4.8. Verify 4.5(6)–(7).

5. Slaman's Splitting and Density Theorems

Unlike the solution to Post's problem given in Section 4, the proof of Slaman's splitting theorem is a fullblown "finite" injury argument. A typical requirement is injured repeatedly, and the set of stages at which it is injured is a member of $L(\kappa)$. Splitting in E-recursion bears a superficial resemblance to splitting in α-recursion. Both utilize Shore blocking and preservation of initial segments of equalities, but in vastly different ways. In α-recursion the blocks were defined by a Σ_2 cofinality function, tame by reason of Σ_1 admissibility. Since $L(\kappa)$ may not be Σ_1 admissible, the blocks in E-recursion are defined by a Σ_2 cofinality function chosen to be tame.

The presence of Moschovakis witnesses drives away some of the uncertainties that led to injuries in α-recursion. As in the solution to Post's problem for E-recursion, some preservations are needed to develop Moschovakis witnesses. Otherwise there might be relevant, convergent computations beyond $L(\kappa)$.

Assume C is E-recursively enumerable, but not E-recursive, on $L(\kappa)$. C is to be split into A and B. The requirements are indexed by ordinals less than $t\sigma 1p(\kappa)$, the tame Σ_1 projectum of κ. Let f be a tame Σ_1 function from $t\sigma 1p(\kappa)$ onto κ. For each $x < t\sigma 1p(\kappa)$,

$$\text{req. } 2x \text{ concerns preservation of } \{f(x)\}^{A^{<\sigma}}, \text{ and}$$

$$\text{req. } 2x+1 \text{ concerns preservation of } \{f(x)\}^{B^{<\sigma}}.$$

Let λ_0 be the tame E_2 cofinality of $t\sigma 1p(\kappa)$, and g a tame E_2 function with domain λ_0 and range an unbounded subset of $t\sigma 1p(\kappa)$. In other words, there exists an recursive-on-$L(\kappa)$ function $g_0(\sigma, r)$ such that

$$(a)_{a<\lambda_0}(E\tau)(r)_{r<a}(\sigma)_{\sigma>\tau}[g_0(\sigma, r) = g(r)].$$

Define

$$\begin{array}{lll} (\text{req. } 2x) \in \text{block } 2r & & \text{and} \\ (\text{req. } 2x+1) \in \text{block } 2r+1, & \text{if} & g(r) \leq x < g(r+1). \end{array}$$

As usual let $C^{<\sigma}$ be the subset of C enumerated via computations of height less than σ. For convenience accept a computation of height σ only if all ordinals it mentions are less than σ. C^σ is $C^{<\sigma+1} - C^{<\sigma}$. Assume C^σ has at most one element; if it exists, denote it by v. At stage σ, v is put in A, or in B, but not in both. Suppose adding v to A would injure a requirement in block $2p$ but none in block $2r$ for any $r < p$. Further suppose adding v to B would injure a requirement in block $2q+1$ but none in block $2r+1$ for any $r < q$. Then v is put in A if $2q+1 < 2p$, and put in B if $2p < 2q+1$.

All events at stage σ are determined by computations and witnesses in $L(\sigma+1)$.

5.1 Theorem (Slaman 1985). *Assume $L(\kappa)$ is E-closed. Suppose $C \subseteq L(\kappa)$ is regular, E-recursively enumerable, and not E-recursive, on $L(\kappa)$. Then there exist A and B, each E-recursively enumerable on $L(\kappa)$, such that*

$$C = A \cup B,\ A \cap B = \varnothing,$$

and C is not E-reducible to either A or B on $L(\kappa)$.

Proof. First suppose $L(\kappa)$ is Σ_1 admissible and of the form $E(x)$. According to Proposition 1.2, Shore's splitting theorem for α-recursion theory, Theorem 1.1.IX, is also a splitting theorem in the sense of E-recursion theory. Now assume $L(\kappa)$ is either Σ_1 inadmissible or the limit of E-closed structures. In either event $L(\kappa)$ admits Moschovakis witnesses. A preservation requirement on A at stage σ is a promise to keep some elements of $\kappa - A^{<\sigma}$ out of A. Each such requirement is attached to (indexed by) an even ordinal. Req. $2x$ is *injured* at stage σ if v is added to A at stage σ contrary to some promise made on behalf of req. $2x$ at stage τ for some $\tau \leq \sigma$.

Req. $2x$ is *active* at stage σ if a preservation requirement is attached to $2x$ at stage σ, or if req. $2x$ is injured at stage σ, or if the block assigned to req. $2x$ changes at stage σ. A change in block assignment at stage σ means: a change from stage $\sigma - 1$ if σ is a successor; or the assignment was not constant on some upper segment of ordinals less than σ if σ is a limit.

It will be shown that req. $2x$ is inactive for all sufficiently large σ.

Preservation requirements on A at stage σ arise as follows.

Let $\{x\}_\sigma$ be $\{x\}$ restricted to computations of height at most σ. Define

$$\ell_\sigma^A(x) = \mu y_{y<\sigma}[C^\sigma(y) \neq \{f(x)\}_\sigma^{A^{<\sigma}}(y)].$$

For each $y \leq \ell_\sigma^A(x)$, promise to keep z out of A if

$$\{f(x)\}_\sigma^{A^{<\sigma}}(y) \tag{1}$$

converges and "$z \notin A^{<\sigma}$" is used in the computation of (1). If

$$\{f(x)\}_\sigma^{A^{<\sigma}}(\ell_\sigma^A(x)) \tag{2}$$

diverges, and if no Moschovakis witness to the divergence exists in $L(\sigma+1)$, then promise to keep all elements of $\sigma - A^{<\sigma}$ out of A.

Certain conventions must be kept in mind. $C^{<\sigma}$, $A^{<\sigma}$, and $B^{<\sigma}$ are subsets of σ and are defined by computations in $L(\sigma)$. Thus $A^{<\sigma} \in L(\sigma+1)$. The relation,

$$w \text{ is a witness to the divergence of (2),} \tag{3}$$

is E-recursive with x and σ as parameters. To say w exists in $L(\sigma+1)$ means: there is a sequence c (of computations) in $L(\sigma+1)$ that makes (3) true.

Suppose (2) diverges, and some w witnesses the divergence via $c \in L(\sigma+1)$. Fix w and c, and promise to keep z out of A if "$z \in A^{<\sigma}$" is needed in c. Once w and c are chosen, they remain fixed until some promise associated with them is broken.

B and the odd-numbered requirements are handled similarly.

Adding v (from C) to A or B is described above. The block assignments at stage σ are supplied by g_0, the tame E-recursive approximation of g.

The next lemma is the major part of the proof of Theorem 5.1.

5.2 Lemma. *Fix $b <$ dom g $(=\lambda_0)$. Suppose block z is inactive after stage σ_0 for all $z < b$. Then there exists a stage σ after which block b is inactive.*

Proof. Since C is regular, there is a $\sigma_1 > \sigma_0$ such that

$$C^{<\sigma_1} \cap \sigma_0 = C \cap \sigma_0.$$

Assume $g_0(\tau, z) = g(z)$ for all $\tau \geq \sigma_1$ and $z \leq b$. Suppose b is $2r$. Further assume $f(x)$ is defined prior to stage σ_1 for all $x < g(r+1)$. At stage $\sigma \geq \sigma_1$ all requirements in $\cup\{\text{block } z \mid z < b\}$ are safe from injury, since every relevant negative requirement is bounded above by σ_0. A typical requirement in block b $(= 2r)$ is req. $2x$: $C \neq \{f(x)\}^A$. It follows that req. $2x$ is not injured at stage σ for any $\sigma \geq \sigma_1$.

Consider the behavior of

$$\{f(x)\}_\sigma^{A^{<\sigma}}(y) \tag{1}$$

for $\sigma \geq \sigma_1$. If $y < \ell_\sigma^A(x)$ for some $\sigma \geq \sigma_1$, then (1) is preserved forever after. Define

$$t(x, y) \simeq \mu\sigma_{\sigma \geq \sigma_1}(y < \ell_\sigma^A(x)),$$
$$\{h(x)\}(y) \simeq \{f(x)\}_{t(x,y)}^{A^{<t(x,y)}}(y).$$

The length of block $2r$ is less than $t\sigma 1p(\kappa)$. Consequently Lemma 2.10 implies there is an $\ell < \kappa$ such that

$$(x)_{g(r) \leq x < g(r+1)}(\mathrm{E}y)_{y<\ell}[\{h(x)\}(y) \neq C(y)].$$

Define

$$y_x = \begin{array}{ll} \ell \text{ if } \{h(x)\}(y)\downarrow, & \text{for all } y < \ell, \\ \mu y[\{h(x)\}(y)\uparrow] & \text{otherwise.} \end{array}$$

Choose $\sigma_2 > \sigma_1$ so that $C \cap \ell = C^{<\sigma_2} \cap \ell$.

Suppose $y_x = \ell$. Then this fact is seen to be true by some ordinal less than $\kappa_0^{x,\ell,p}$ (that is, witnessed in $L(\kappa_0^{x,\ell,p})$. p is an ordinal independent of x that encodes several parameters such as $b, \sigma_0, \sigma_1, \sigma_2$ and others occurring in the definitions of g_0, f and C. Let.

$$w_x = \mu y[\{h(x)\}(y)\downarrow \quad \text{and is} \neq C(y)].$$

If $w_x \in C$, then this fact is evident by some ordinal less than $\kappa_0^{x,\ell,p}$. If $w_x \notin C$, then this fact is established by a Moschovakis witness first order definable over $\kappa_r^{w_x,x,\ell,p} < \kappa$ by Lemmas 5.3.X and 5.6.X. Exercise 5.17.X, a corollary of Kechris's basis theorem, implies

$$\kappa_r^{w_x,x,\ell,p} \leq \kappa_r^{x,\ell,p},$$

since w_x is the least member of a subset of ℓ E-co-recursively enumerable in x, ℓ, p.

In short: if $y_x = \ell$, then req. $2x$ is inactive after stage $\kappa_r^{x,\ell,p}$.

Now suppose $y_x < \ell$. Then, as above,

$$\kappa_r^{y_x,x,\ell,p} \leq \kappa_r^{x,\ell,p} \tag{2}$$

by 5.3, 5.6 and 5.17 of Chapter X. Let

$$w_x^* = \mu y_{y<y_x}[\{f(x)\}(y)\uparrow \quad \text{and is} \neq C(y)].$$

Assume $w_x^* < y_x$. If $w_x^* \in C$, then this fact is evident by stage $\kappa_0^{y_x,x,\ell,p}$, hence before $\kappa_r^{x,\ell,p}$ by (2). If $w_x^* \notin C$, then this fact is established by a Moschovakis witness definable over

$$\kappa_r^{w_x^*,y_x,x,\ell,p} \leq \kappa_r^{y_x,x,\ell,p}$$

by 5.17.X.

Thus req. $2x$ is inactive after stage $\kappa_r^{x,\ell,p}$. Hence block b is inactive after stage $\sup\{\kappa_r^{x,\ell,p} | x < g(b)\}$. The latter is less than κ by Theorem 2.5, since $b < t\sigma 1p(\kappa) \leq \rho^\kappa$. □

5.3 Lemma. *Fix $c <$ dom $g = \lambda_0$. Suppose for each $b < c$ there is a σ such that block b is inactive after stage σ. Then there is a σ such that for all $b < c$, block b is inactive after stage σ.*

Proof. A tame E_2 function i: $c \to \kappa$ is defined so that block b is inactive after stage $i(b)$. According to Exercise 2.15, $\lambda_0 = te2\mathrm{cf}(\kappa)$, hence sup (range i) $< \kappa$.

Fix $b < c$. The procedure for E-recursively approximating $i(b)$ is derived from the proof of Lemma 5.2. Notation from 5.2 is used below.

Keep checking on activity in block z for all $z < b$. If any such activity occurs at stage σ, then the current guess at $i(b)$ has to be increased to σ. Eventually stage σ_0 is reached. And after that, stage σ_1 where

$$C^{<\sigma_1} \cap \sigma_0 = C \cap \sigma_0.$$

Meanwhile computations and Moschovakis witnesses are sought to establish inequalities for requirements in block b. Some additional waiting for C to settle down may be necessary as in the definition of σ_2. □

Note that the function i of the proof of Lemma 5.2 is tame E_2, rather than tame Σ_1, because it was necessary to wait for C to settle down on some proper initial segment of κ. This is why tame Σ_1 blocking was adequate for Post's problem but not for splitting.

The proof of Theorem 5.1 is readily completed. By construction $C = A \cup B$ and $A \cap B = \varnothing$. A and B are E-recursively enumerable on $L(\kappa)$, because the decision to put v in A or B at stage σ is based on effective consideration of $L(\sigma)$. Lemma 5.2 and 5.3 imply the desired inequalities are realized.

5.4 Theorem (Slaman 1985). *Assume $L(\kappa)$ is E-closed and admits Moschovakis witnesses. Let $C, D \subseteq \kappa$ be regular and E-recursively enumerable on $L(\kappa)$. Suppose $D <_{L(\kappa)} C$. Then there exist A and B, each E-recursively enumerable on $L(\kappa)$, such that $A \cap B = \varnothing$, $C = A \cup B$, $C \not\leq_\kappa D, A$ and $C \not\leq_\kappa D, B$.*

It follows from Theorem 5.4 that the E-recursively enumerable degrees are dense for every E-closed $L(\kappa)$. If $L(\kappa)$ does not admit Moschovakis witnesses, then $L(\kappa)$ is of the form $E(x)$ for some set x of ordinals, and is Σ_1 admissible, by Theorem 5.8.X. Consequently Shore's density Theorem (5.1.IX) for α-recursion theory applies to $L(\kappa)$ by Proposition 1.2.

The assumption in 5.4 that $L(\kappa)$ admit Moschovakis witnesses is necessary, because Lachlan (197?) has shown that the conclusion of 5.4 fails when $\kappa=\omega$.

The proof of Theorem 5.4 is left as the final exercise. Most, but not all, of the difficulties that arise, as the proof of Theorem 5.1 is extended to cover 5.4, were anticipated in Section 2. Note 2.2.

A final methodological point and question. Post's problem for inadmissible $L(\kappa)$'s was solved without injuries with the aid of Moschovakis witnesses. Priorities were necessary, but at most one attempt was made to satisfy each requirement. Splitting an E-recursively enumerable set C produced a κ-finite set of injuries for each requirement, only because of the need to guess repeatedly at proper initial segments of C; each requirement had to have true knowledge of some such segment before it could be satisfied. Since 5.1 is so close in nature to 5.4, the proof of density for E-closed structures that admit Moschovakis witnesses is a "finite" injury argument. At this writing there are no "infinite" injury arguments in E-recursion, possibly because the search for an inequality can always be resolved below κ by a computation or a Moschovakis witness.

Does there exist an "infinite" injury argument in the setting of E-recursion theory?

5.5 Exercise. Complete the proof of Theorem 5.4.