Chapter IX
Splitting, Density and Beyond

This last chapter on α-recursion theory focuses on priority arguments more difficult than those of Chapters VII and VIII. Shore's splitting theorem relies heavily on his method of Σ_2 blocking. His density theorem, the first instance of α-infinite injury, requires further fine structure results, and consequently its proof is not entirely dynamic. Its nondynamic features support consideration of recursion theory on inadmissible structures, the concluding topic of the chapter.

1. Shore's Splitting Theorem

Let A be a regular α-recursively enumerable set, not α-recursive. The object is to split A into two sets, B_0 and B_1, so that each is of lower α-degree than A. Thus

$$A = B_0 \cup B_1,\ B_0 \cap B_1 = \varnothing, \quad \text{and} \quad A \not\leq_\alpha B_i (i < 2).$$

Superficially the strategy is the same as that followed by Sacks 1963b when $\alpha = \omega$. In the ω-case the splitting theorem makes stronger use of Σ_2 replacement than the Friedman–Muchnik theorem does. In general terms the former is a full-blown Σ_2 recursion while the latter is tame in the sense of Theorem 4.4.VIII. In specific terms the difference arises from the urgency of splitting. At stage σ some x is enumerated in A. That x must be put in either B_0 or B_1 immediately. The force of the positive requirements is so great that numerous negative requirements are unavoidably injured.

The negative requirements are indexed by ordinals less than α^*:

$$\text{Req } 2\varepsilon: \quad A \neq \{\varepsilon\}^{B_0},$$

$$\text{Req } 2\varepsilon + 1: \quad B \neq \{\varepsilon\}^{B_1}.$$

$\{\varepsilon\}$ means $\{f^{-1}\varepsilon\}$ for some one-one α-recursive f from α into α^*.
Req u has higher priority than req v if $u < v$. Thus req 0 is never injured. As in the ω-case, if A and $\{\varepsilon\}^{B_i}$ agree on an initial segment of α at stage σ, then $\{\varepsilon\}^{B_i}$ is commmited to preservation on that initial segment, if the priorities allow it. The preservations associated with req 0 must be bounded in both time and space, since otherwise A would be α-recursive. To compute $A(z)$, unfold the construction until

$\{0\}^{B_0}(z)$ is committed to preservation. At that stage $A(z) = \{0\}^{B_0}(z)$ and the latter equation holds forever after. So there must be a z such that $A(x) = \{0\}^{B_0}(x)$ for all $x < z$, and either $\{0\}^{B_0}(z)$ is undefined, or is unequal to $A(z)$, at the end of the construction. Since the preservations associated with req 0 are bounded, and since A is regular, eventually it will be safe to put every element of A that might injure req 1 into B_0. Thus req 1 is injured only α-finitely often, and so req 1 is met. And so on. Let $f(r)$ be the least σ after which no new preservation commitments are made for the sake of req r. Since f is Σ_2^α, and not tame, it is unlikely that there will be time to meet all requirements.

If $\alpha^* \leq \sigma 2\text{cf}(\alpha)$ then all is well. If not, then Shore's idea is to divide α^* into $\sigma 2\text{cf}(\alpha^*)$ many blocks. The negative requirements are divided as follows.

g: $\sigma 2\text{cf}(\alpha) \to \alpha$ is strictly increasing, Σ_2^α and unbounded.

Block 2δ: all requirements of the form $A \neq \{\varepsilon\}^{B_0}$ for all $\varepsilon \in [g(\delta), g(\delta+1))$.

Block $(2\delta+1)$: all requirements of the form $A \neq \{\varepsilon\}^{B_1}$ for all $\varepsilon \in [g(\delta), g(\delta+1))$.

Within each block there is no conflict between requirements. Hence each block can be regarded as a single requirement. It turns out that the supremum of all stages at which block b is active is a Σ_2^α function of b. Lemma 3.3.VIII says $\sigma 2\text{cf}(\alpha^*) = \sigma 2\text{cf}(\alpha)$, so there is enough time to meet all requirements.

1.1 Theorem (Shore 1975). *Let A be α-recursively enumerable and regular. Then there exists α-recursively enumerable B_0 and B_1 such that $A = B_0 \cup B_1$, $B_0 \cap B_1 = \varnothing$ and $A \not\leq_\alpha B_i$ $(i < 2)$.*

Proof. Assume the terminology of the preliminary discussion of splitting immediately above. Let $g(\sigma, \delta)$ be an α-recursive approximation of the Σ_2^α cofinality function $g(\delta)$. Thus

$$\lim_\sigma g(\sigma, w) = g(\delta)$$

for all $\delta < \sigma 2\text{cf}(\alpha)$. Note that $g \restriction \sigma 2\text{cf}(\alpha)$ is tame Σ_2^α. The key function in the creation of negative requirements will be

$$t_i(\sigma, \varepsilon) = \mu y[A^{<\sigma}(y) \neq \{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)].$$

In order to construe each block of requirements as a single requirement, let

$$t(\sigma, 2\delta + i) = \sup{}^+ \{t_i(\sigma, \varepsilon) | g(\sigma, \delta) \leq \varepsilon < g(\sigma, \delta+1)\}.$$

($\sup^+$ is the strict supremum; thus $\sup^+\{\beta\} = \beta + 1$.) t measures the length of initial segments to be preserved. m (below) measures the restraint on B_i needed to preserve those segments. The definition of $m(\sigma, 2\delta + i)$ has an all-important monotonicity clause, (1a). (1a) insures that increases in m are accompanied by increases in

t. It would be senseless to impose greater and greater negative restraints on B_i merely to preserve a fixed initial segment.

The definition of $m(\sigma, 2\delta + i)$ has two cases.

(1a) If $t(\sigma, 2\delta + i) > \sup^+_{\gamma < \sigma} t(\tau, 2\delta + i)$, then $m(\sigma, 2\delta + i)$ is the max of the right side of (2) and $\sup^+ \{z \mid$ "$z \notin B_i^{<\sigma}$" is needed for the computation of $\{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)$ for some $\varepsilon \in [g(\sigma, \delta), g(\sigma, \delta + 1))$ and some $y < t_i(\sigma, \varepsilon)\}$.

(1b) If $t(\sigma, 2\delta + i) \leq \sup_{\tau < \sigma} t(\tau, 2\delta + 1)$, then

(2) $$m(\sigma, 2\delta + i) = \sup_{\tau < \sigma} m(\tau, 2\delta + i).$$

The equality $A^{<\sigma}(y) = \{\varepsilon\}^{B_i^{<\sigma}}(y)$ is *committed to preservation* at stage σ if (1a) holds and $y < t_i(\sigma, \varepsilon)$. Let x be an element of A enumerated at stage σ. Block $2\delta + i$ is *injured* at stage σ if x is put in B_i and $x < m(\sigma, 2\delta + i)$. Let $2u_j + j$ be the least w such that block w would be injured if x were put in B_j at stage σ.

Put x in B_1 if $u_0 \leq u_1$, and put x in B_0 if $u_1 < u_0$.

Block $2\delta + 1$ is *stable* at stage τ if

$$(\sigma)_{\sigma \geq \tau}[m(\sigma, 2\delta + i) = m(\tau, 2\delta + i)].$$

1.2 Proposition. *Fix $\gamma < \sigma 2\mathrm{cf}(\alpha^*)$. Suppose for each $w < \gamma$, there is a τ such that block w is stable at stage τ. Then there is a τ such that for all $w < \gamma$, block w is stable at stage τ.*

Proof. Let $f(w)$ be the least τ such that w is stable at stage τ. Then f is $\mu - \Pi_1^\alpha$, hence Σ_2^α as in Exercise 2.12.VII. According to Lemma 3.3.VII, $\sigma 2\mathrm{cf}(\alpha^*) = \sigma 2\mathrm{cf}(\alpha)$. Hence $\sup f[\gamma] < \alpha$. □

The next lemma is the combinatoric essence of the proof of Theorem 1.1.

1.3 Lemma. *Fix δ and i. Suppose there is a τ such that block w is stable at stage τ for all $w < 2\delta + i$. Then there is a ρ such that block $2\delta + i$ is stable at stage ρ.*

Proof. Let $m = \sup\{m(\tau, w) \mid w < 2\delta + i\}$. Since A is regular, there is a $\sigma_0 > \tau$ such that $(A - A^{<\sigma_0}) \cap m$ is empty. Thus from stage σ_0 on, it is impossible to injure block w for any $w < 2\delta + i$. Hence block $2\delta + i$ will not be injured at, or after, stage σ_0. Assume σ_0 is so large that

$$(\sigma)_{\sigma \geq \sigma_0}[g(\sigma, \delta) = g(\delta) \quad \& \quad g(\sigma, \delta + 1) = g(\delta + 1)].$$

Let K be the set of all $\varepsilon \in [g(\delta), g(\delta + 1)]$ such that:

$$(\mathrm{Ep})_{\sigma_0 \leq p}(\mathrm{Eq})_{p < q}(\mathrm{Ey})\ [\text{the equality } A^{<p}(y) = \{\varepsilon\}_p^{B_i^{<p}}(y)$$

is committed to preservation at stage p & $A^{<p}(y) = 0$ & $A^{<q}(y) = 1$]. Since $\varepsilon \in K$, the choice of σ_0 implies for some y, $\{\varepsilon\}^{B_i}(y)$ is defined but unequal to $A(y)$. K is α-finite, since K is an α-recursively enumerable set bounded by an ordinal less than α^*. Let $\sigma_1 > \sigma_0$ be so large that every $\varepsilon \in K$ is established by the appearance of a suitable p, q and y prior to stage σ_1. Then for each $\varepsilon \in K$, there is a permanent inequality that prevents any increase in $t_i(\sigma, \varepsilon)$ after stage σ_1. Thus $t_i(\sigma, \varepsilon) \leq t_i(\sigma_1, \varepsilon)$ for all $\sigma \geq \sigma_1$ and $\varepsilon \in K$.

Consider $\varepsilon \in [g(\delta), g(\delta)] - K$. Any equality of the form $A^{<\sigma}(y) = \{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)$ committed to preservation at stage $\sigma \geq \sigma_1$ is permanent. For such a σ and y,

$$A(y) = \{\varepsilon\}^{B_i}(y).$$

It follows that the set of all such y's, as ε ranges over $[g(\delta), g(\delta + 1)] - K$, is bounded below α. Otherwise A would be α-recursive. Let b bound all such y's.

For each $\sigma \geq \sigma_1$ let b_σ be the sup of all y such that $A^{<\sigma}(y) = \{\varepsilon\}_\sigma^{B_i^{<\sigma}}(y)$ is committed to preservation at stage σ for some $\varepsilon \in [g(\delta), g(\delta+1)]$. b_σ is a non-decreasing function of σ bounded by b. The set of stages at which b_σ increases is α-recursively enumerable. The enumeration of an α-finite set in increasing order must finish in α-finitely many steps. Thus $b = b_{\sigma_2}$ for some $\sigma_2 \geq \sigma_1$. Now the monotonicity hypothesis of clause (1a) exerts its power. After stage σ_2, clause (1b) holds, and so block $2\delta + i$ is stable. ☐

End of proof of Theorem 1.1: By 1.2 and 1.3 every block is eventually stable. Suppose $A = \{\varepsilon\}^{B_i}$ for some $\varepsilon \in [g(\delta), g(\delta + 1))$. Then $t_i(\delta, \varepsilon)$ converges to α, and block $2\delta + 1$ is never stable.

2. *Further Fine Structure*

The proof of Shore's splitting theorem was entirely dynamic and so holds in a variety of Σ_1 admissible structures that are not L-like (cf. Exercise 2.13). Shore's density theorem, IX.5.1, is tied strongly to $L(\alpha)$ by some fine structure facts based on collapsing arguments, in particular Lemma 2.2 below.

2.1 Proposition. *Assume $P(x, y)$ is Σ_2^α. Then there exists a partial Σ_2^α function f such that*

$$(x)[(\mathrm{E}y)P(x, y) \leftrightarrow f(x) \text{ is defined } \& \ P(x, f(x))].$$

(Uniformization of Σ_2^α by Σ_2^α.)

Proof. Let $P(x, y)$ be $(\mathrm{E}u)(v)Q(u, v, x, y)$ for some $\Delta_0^\alpha\, Q$. It suffices to uniformize $(v)Q((y)_0, v, x, (y)_1)$ by some partial Σ_2^α function f. Thus it is safe to assume P is Π_1^α. Say $P(x, y)$ is $(v)R(v, x, y)$ for some $\Delta_0^\alpha R$. Define

$$P_1(x, y) \text{ by } P(x, y) \quad \& \quad (z)_{z<y}(\mathrm{E}v) \sim R(v, x, z).$$

$P_1(x, y)$ defines the graph of f. The predicate $z < y$ is α-recursive since it refers to the natural enumeration of $L(\alpha)$. The Σ_1 admissibility of $L(\alpha)$ implies

$$(z)_{z<y}(\mathrm{Ev}) \sim R(v, x, z) \leftrightarrow (\mathrm{Ew})(z)_{z<y}(\mathrm{Ev})_{v<w} \sim R(v, x, z).$$

Thus P_1 and f are Σ_2^α. □

The Σ_2 projectum of α, denoted by $\sigma 2p(\alpha)$, is the least $\gamma \leq \alpha$ such that some partial Σ_2^α function maps γ onto α. The next lemma is analogous to Proposition 2.1.VII, and can be proved in a similar fashion, that is, dynamically, when α is Σ_2 admissible. Otherwise a collapsing argument is necessary. Jensen has proved Lemma 2.2 with n in place of 2 for all $n \geq 1$ and all α without any admissibility assumptions.

2.2 Lemma. *Assume α is Σ_1 admissible. If $\gamma < \sigma 2p(\alpha)$ and $Y \subseteq \gamma$ is Σ_2^α, then Y is α-finite.*

Proof. As in the proof of the enumeration theorem (1.9.VII), the natural enumeration of $L(\alpha)$ gives rise to an α-recursive enumeration of all Δ_0 facts about elements of $L(\alpha)$. Thus there is a Δ_1^α formula $Q(u, v, e, x, y)$ such that

$$L(\alpha) \models Q(u, v, e, x, y) \quad \text{iff} \quad L(\alpha) \models F_e(u, v, x, y).$$

F_e is the e-th Δ_0^{ZF} formula. Q is lightface Δ_1^α because the definition of the natural enumeration of $L(\alpha)$ does not require any parameters from $L(\alpha)$.

It follows that $(\mathrm{Eu})(v)Q(u, v, e, x, y)$ is a universal, lightface Σ_2^α formula. By Proposition 2.1 $(\mathrm{Eu})(v)Q(u, v, e, x, y)$ can be uniformized by a partial Σ_2^α function $h(e, x)$. If $P(x, y)$ is the e-th Σ_2^α formula with free variables x and y, and if

$$L(\alpha) \models (\mathrm{Ey})P(\underset{\sim}{a}, y) \text{ for some } a \in L(\alpha),$$

then $h(e, a)$ is defined and $L(\alpha) \models P(a, \underline{h(e, a)})$.

If e is the Gödel number of a Σ_2^α formula $P(x_1, \ldots, x_n)$, let $\langle e \rangle$ be the Gödel number of a Σ_2^α formula $Q(x)$ with the property that,

$$P(x_1, \ldots, x_n) \leftrightarrow Q(\langle x_1, \ldots, x_n \rangle).$$

Let $h_{\langle\rangle}(e, x_1, \ldots, x_n)$ be $h(\langle e \rangle, \langle x_1, \ldots, x_n \rangle)$.

$z^{<\omega}$ is the set of all finite sequences of elements of z. Define $H_2(z)$, the Σ_2^α Skolem hull of z, to be $h[\omega \times z^{<\omega}]$. Then $H_2(z) \prec_2 L(\alpha)$ by Exercise 2.8. Hence $[V = L]$, as described in the proof of Lemma 2.5.VII, is true in $H_2(z)$. Define the collapse of $H_2(z)$ as in the proof of Lemma 2.6.VII. For each $x \in H_2(z)$,

$$t(x) = \{t(y) \mid y \in x \quad \& \quad y \in H_2(z)\}.$$

$t[H_2(z)]$ is transitive, and t maps $H_2(z)$ isomorphically onto $t[H_2(z)]$.

$Y \subseteq \gamma$ has a Σ_2 definition over $L(\alpha)$ with parameter $p \in L(\alpha)$. Let $z = \gamma \cup \{p\}$. Then $Y \subseteq \gamma$ has a Σ_2 definition over $H_2(z)$ with parameter $p \in H_2(z)$, since $H_2(z) \prec_2 L(\alpha)$. Now collapse the hull. $t[Y] \subseteq t[\gamma]$ is Σ_2 definable over $t[H_2(z)]$ with parameter $t(p) \in t[H_2(z)]$. By Lemma 2.5.VII, $t[H_2(z)] = L(\gamma_0)$ for some $\gamma_0 \leq \alpha$. Thus Y is Σ_2 definable over $L(\gamma_0)$. If $\gamma_0 < \alpha$, then Y must be α-finite.

Suppose $\gamma_0 = \alpha$. Then h gives rise to a partial map of $\omega \times z^{<\omega}$ onto $H_2(z)$; in addition the map is Σ_2 over $H_2(z)$. Since $\gamma < \sigma 2p(\alpha)$, the α-cardinality of $\omega \times z^{<\omega}$ equals some $\rho < \sigma 2\mathrm{cf}(\alpha)$. But then there is a partial map Σ_2^α from ρ onto $L(\alpha)$, an impossibility. □

Assume $A \subseteq L(\alpha)$. The Σ_1 projectum of α relative to A, denoted by $\sigma 1 p_A(\alpha)$, is the least $\gamma \leq \alpha$ such that there exists a partial map Σ_1 over $\langle L[A, \alpha], A \rangle$ from γ onto $L(\alpha)$. In short

$$\alpha_A^* = \mu\gamma(\mathrm{E}f)\,[\mathrm{dom}\, f \subseteq \gamma \quad \& \quad f \in \Sigma_1^{\alpha, A} \quad \& \quad \mathrm{rng}\, f = L(\alpha)].$$

In general "$f \in \Sigma_1^{\alpha, A}$" means the graph of f is defined by a Σ_1 formula whose existential quantifier ranges over $L[A, \alpha]$, whose parameters belong to $L[\alpha, A]$, and whose atomic subformulas may include "$x \in A$". If A is regular, then $L[A, \alpha] = L(\alpha)$, and "$f \in \Sigma_1^{\alpha, A}$" is equivalent to "$f \leq_{w\alpha} A$".

The next theorem is a rare combination of fine structure and recursive approximation.

2.3 Theorem (Shore 1976). *Let A be regular and α-recursively enumerable. If $\gamma < \alpha_A^*$ and $Y \subseteq \gamma$ is $\Sigma_1^{\alpha, A}$, then Y is α-finite.*

Proof. Since A is regular the natural enumeration of $L(\alpha)$ leads to a $\Delta_1^{\alpha, A}$ enumeration of T^A, the set of all Δ_0 sentences true in $\langle L(\alpha), A \rangle$. The truth value of each such sentence depends not on A, but only on $A \cap L(\delta)$ for some $\delta < \alpha$. Regularity was assumed so that $A \cap L(\delta)$ would be an element of A. Thus $T^A \subseteq L(\alpha)$; and

$$\langle e, \langle b \rangle \rangle \in T^A \leftrightarrow \langle L(\alpha), A \rangle \models F_e(\langle \underline{b} \rangle),$$

where $F_e(\langle x \rangle)$ is the e-th Δ_0 formula of ZF with $x \in A$ as an additional atomic formula, and $\langle b \rangle \in L(\alpha)$.

A universal $\Sigma_1^{\alpha, A}$ predicate can be obtained from T^A. That predicate can be uniformized by a partial $\Sigma_1^{\alpha, A}$ function thanks to the regularity of A. Thus there exists a universal partial $\Sigma_1^{\alpha, A}$ function h. If $P(\langle x \rangle, y)$ is the e-th Δ_0 formula of ZF with $x \in A$ as an extra atomic formula and $\langle x \rangle, y$ as free variables, and

$$L(\alpha) \models (\mathrm{E}y)\, P(\langle \underline{a} \rangle, y)$$

for some $\langle a \rangle \in L(\alpha)$, then $h(e, \langle a \rangle)$ is defined and

$$L(\alpha) \models P(\langle \underline{a} \rangle, \underline{h(e, \langle a \rangle)}).$$

Let $z = \gamma \cup \{p\}$, where p encodes the parameters needed for the $\Sigma_1^{\alpha, A}$ definition of $Y \subseteq \gamma$, and the Σ_1^α definition of g_0 below. Form $H = h[\omega \times z^{<\omega}]$, the $\Sigma_1^{\alpha, A}$ hull of z. Then $\langle H, A \rangle \prec_1 \langle L(\alpha), A \rangle$. The last assertion means that every $\Sigma_1^{\alpha, A}$ sentence with parameters in H is true in $\langle L(\alpha), A \rangle$ iff it is true in $\langle H, A \rangle$.

Suppose H is bounded, that is, $H \subseteq L(\beta)$ for some $\beta < \alpha$. Then

$$\langle H, A \rangle \prec_1 \langle L(\beta), L(\beta) \cap A \rangle.$$

Y is Σ_1 over $\langle H, A \rangle$, hence Σ_1 over $\langle L(\beta), L(\beta) \cap A \rangle$, hence α-finite, since $L(\beta) \cap A$ is α-finite.

Suppose $\gamma < \sigma 2p(\alpha)$. Since Y is $\Sigma_1^{\alpha, A}$, and A is regular and Σ_1^α, it follows that Y is Σ_2^α (cf. Exercise 2.9). Then Y is α-finite by Lemma 2.2.

Assume H is unbounded and $\gamma \geq \sigma 2p(\alpha)$ with the intention of showing $H = L(\alpha)$. It suffices to show $\alpha \subseteq H$ since the f of 1.8.VII is lightface. Let

$$O(c) = \mu\beta[c \in L(\beta)] \quad (c \in L(\alpha)).$$

$O(c)$ is Σ_1^α, hence $O[H] \subseteq H$, and so H contains arbitrarily large ordinals less than α. Let g be a partial Σ_2^α function from $\sigma 2p(\alpha)$ onto α. By Proposition 2.2.VIII there is an α-recursive $g_0(\sigma, x)$ such that

$$\lim_\sigma g_0(\sigma, x) = g(x)$$

for all $x \in \operatorname{dom} g$. Fix $u < \alpha$. Choose $x < \sigma 2p(\alpha)$ so that $g(x) = u$. Then for a sufficiently large σ in H, $g_0(\sigma, x) = u$; and so $u \in H$, since $\sigma 2p(\alpha) \leq \gamma$.

The equality of H and $L(\alpha)$ implies there is a partial $\Sigma_1^{\alpha, A}$ map, namely h, from $\omega \times z^{<\omega}$ onto $L(\alpha)$. An impossibility since the α-cardinality of $\omega \times z^{<\omega}$ is less than $\sigma 1 p_A(\alpha)$. □

2.4 Corollary (R. Shore 1976). *Suppose A is α-recursively enumerable, regular and incomplete. Then $\sigma 1 \mathrm{cf}_A(\alpha) \geq \sigma 1 p_A(\alpha)$.*

Proof. Assume $\sigma 1 \mathrm{cf}_A(\alpha) < \sigma 1 p_A(\alpha)$ with the intention of showing A complete. Let C be a regular, complete, α-recursively enumerable set, and $\{C^\sigma \mid \sigma < \alpha\}$ an α-recursive enumeration of C. Let $f\colon \sigma 1 \mathrm{cf}_A(\alpha) \to \alpha$ be an unbounded $\Sigma_1^{\alpha, A}$ function. Since C is regular, for each x there is a y such that

$$C \cap f(x) \subseteq C^{f(y)}. \tag{1}$$

As a relation on x and y, (1) is a $\Pi_1^{\alpha, A}$ subset of $(\sigma 1 \mathrm{cf}_A(\alpha))^2$, hence α-finite by Theorem 2.3. Let $g(x)$ be the least y that satisfies (1). It follows that $g\colon \sigma 1 \mathrm{cf}_A(\alpha) \to \sigma 1 \mathrm{cf}_A(\alpha)$ is α-finite. Then

$$H \subseteq cC \leftrightarrow (\mathrm{Ex})[H \subseteq f(x) \quad \& \quad H \cap C^{f(g(x))} = \phi]$$

for all α-finite H, and so $C \leq_\alpha A$. (Keep in mind that the regularity of A implies every $\Sigma_1^{\alpha, A}$ function, in particular f, is weakly α-recursive in A.) $\square$

2.5 Weak Σ_1 Admissibility. Let β be a limit ordinal. $L(\beta)$ need not be Σ_1 admissible, but it is closed under pairing, and consequently the fundamentals of recursion theory lift to $L(\beta)$. β is said to be *weakly Σ_1 admissible* if $\sigma 1\mathrm{cf}(\beta) \geq \sigma 1 p(\beta)$. Thus Corollary 2.4 becomes: if A is α-recursively enumerable, regular and incomplete, then the structure $\langle L(\alpha), A\rangle$ is weakly Σ_1 admissible. Some of the solutions to Post's problem given in Chapter VIII can be adapted to weakly Σ_1 admissible structures. The main obstacle is the limited time in which to meet incomparability requirements. There are α requirements but only $\sigma 1\mathrm{cf}_A(\alpha)$ stages of construction. This difficulty is overcome by the next lemma, which is essential to the proof of density in Section 5.

2.6 Lemma. *Suppose $\langle L[A, \alpha], A\rangle$ is weakly Σ_1 admissible. Then there exists a one-one $\Sigma_1^{\alpha, A}$ map from $\sigma 1\mathrm{cf}_A(\alpha)$ onto α.*

Proof. $c = \sigma 1\mathrm{cf}_A(\alpha)$ and $p = \sigma 1 p_A(\alpha)$. Let f be a strictly increasing $\Sigma_1^{\alpha, A}$ map from c into α, unbounded in α. Let g be a one-one $\Sigma_1^{\alpha, A}$ map from α into p.

Define h, a partial, one-one $\Sigma_1^{\alpha, A}$ map from $c \times c$ onto α as follows: $h(u, v) = w$ if:

(a) $g(w) = u$; and
(b) the existential witness needed to show "$g(w) = u$", and the parameters in the $\Sigma_1^{\alpha, A}$ definition of g belong to $L[A, f(v)]$; and
(c) (b) is false when v is replaced by $v_0 < v$.

To check that h is $\Sigma_1^{\alpha, A}$, note that the existential witnesses needed to define $f(v_0)$ for all $v_0 \leq v$ are bounded below α, since $v < c$. Range $h = \alpha$, because $\alpha = \text{range } g^{-1}$ and domain $g^{-1} \subseteq p \subseteq c$. The domain of h is a $\Delta_1^{\alpha, A}$ subset of $c \times c$. Extend the domain of h to all of $c \times c$ by setting $h(u, v) = 0$ for those $\langle u, v\rangle$'s not covered by (a), (b), (c).

Thus h is a $\Sigma_1^{\alpha, A}$ map from $c \times c$ onto α. Let $t \in L[\alpha, A]$ map c onto $c \times c$, and set $h_1 = h \circ t$. Define $h_2(x)$ by recursion on $x < c$.

$$m(x) = \mu z[h_1(z) \notin h_2[x]].$$

$$h_2(x) = h_1(m(x)).$$

Since $x < c$, the existential witnesses needed to establish $m(x)$ are bounded below α. Thus h_2 is a one-one, $\Sigma_2^{\alpha, A}$ map from c onto α. $\square$

Note that the Σ_1 admissibility of $L(\alpha)$ was not used in the proof of Lemma 2.6.

2.7 Corollary (Shore 1976). *Suppose A is α-recursively enumerable, regular and incomplete. Then there exists a one-one, $\Sigma_1^{\alpha, A}$ map from $\sigma 1\mathrm{cf}_A(\alpha)$ onto α.*

Proof. By Corollary 2.4 and Lemma 2.6.

An early version of Corollary 2.7, for $\alpha = \omega_1^{CK}$, occurs in the proof of Driscoll's (1968) density theorem for metarecursion theory.

2.8–2.12 Exercises

2.8. Let $H_2(z)$ be the Σ_2^α Skolem hull of z as defined in Lemma 2.2. Show $H_2(z) \prec_2 L(\alpha)$, that is, each Σ_2^{ZF} sentence with parameters in $H_2(z)$ is true in $H_2(z)$ if it is true in $L(\alpha)$.

2.9. Let A be a regular, α-recursively enumerable set. Suppose Y is $\Sigma_1^{\alpha, A}$. Show Y is Σ_2^α.

2.10. Let β be a weakly Σ_1 admissible ordinal. Show there exists a one-one Σ_1^β map from $\sigma 1 \mathrm{cf}(\beta)$ onto β.

2.11. Let β be weakly Σ_1 admissible. Reformulate and prove the combinatoric lemma, 2.3.VII, for $L(\beta)$.

2.12. Let β be weakly Σ_1 admissible. $A \subseteq L(\beta)$ is said to be β-recursively enumerable if A is Σ_1^β. Define "weakly β-recursive in" as in subsection 3.2.VII with β in place of α. Show there exist two β-recursively enumerable sets such that neither is weakly β-recursive in the other.

2.13. Let A be a Σ_1 admissible structure of the form $\langle L[B, \alpha], \varepsilon, B \rangle$. Prove Shore's splitting theorem for A.

3. *Density for* ω

The following sketch of the original proof of the density of the recursively enumerable degrees will prove helpful in the proof of density for all α given in Sections 4 and 5.

Let A and C be recursively enumerable subsets of ω such that $A <_T C$. The objective is a recursively enumerable B such that $A <_T B <_T C$. $A \leq_T B$ is accomplished by planting A in the even coordinates of B. The remaining action takes place on the odd coordinates of B. The strategy for realizing $B \not\leq_T A$ is positive in nature; bits of C are planted in B. If the strategy fails, then $C \leq_T A$. If it succeeds, then the bits planted in B add up to something infinite but manageable with respect to the negative requirements.

The strategy for realizing $C \not\leq_T B$ is negative in nature; initial segments of $\{e\}^B$ are preserved. If the strategy fails, then $C \leq_T A$.

Positive requirements associated with $B \neq \{e_1\}^A$ have higher priority than negative requirements associated with $C \neq \{e_2\}^B$ if $e_1 < e_2$. To meet positive requirement e_1, it must be shown that the obstacles raised by negative requirement e $(e \leq e_1)$ drop back simultaneously, and infinitely often, to some fixed $\ell(e_1)$. To meet negative requirement e_2, it must be shown that injury set I_{e_2} is recursive in A. I_{e_2} is the set of all elements added to B for the sake of positive requirement e ($e < e_2$) together with A.

The term "infinite injury" refers to the fact that I_{e_2} is infinite (rather than finite). The operation "lim inf" is also helpful, because recursive sequences generated during the construction tend to have limit infimums rather than limits.

The planting strategy for $B \neq \{e\}^A$ is as follows. Let A^s be that part of A enumerated by the end of stage s. Let $\{e\}_s$ be $\{e\}$ restricted to the first s computations. (The complexity of planting arises from the possibility that both (i) and (ii) may be true.

(i) $\{e\}^A(n)$ is undefined.
(ii) $\{e\}_s^{A^s}(n)$ is defined for infinitely many s.

In that event there is no hope of finding an s such that

$$B^s(n) \neq \{e\}_s^{A^s}(n),$$

and then preserving the above inequality forever after.) Assume that the odd part of B is divided, in an effective manner depending on e, into infinitely many infinite rows. Let $\langle e, n, i \rangle$ be the i-th space in the n-th row. The idea is to plant $C(n)$ in the n-th row of B. $C \not\leq_T A$, so if enough of C is planted in B, then $B \not\leq_T A$. On the other hand, the planting must not go too far, because $C \not\leq_T B$ is also desired.

$\ell(e, n, s)$ is the proposed location for $C(n)$ at stage s. If $\ell(e, n, s) = -1$, then there is no location.

$\ell(e, n, 0) = -1$.

Case 1: $\ell(e, n, s-1) = -1$. If

$$(x)_{x \leq n}\,[B^{s-1}(x) = \{e\}_s^{A^{s-1}}(x)], \tag{1}$$

then $\ell(e, n, s) = s$. Otherwise $\ell(e, n, s) = -1$.

Case 2: $\ell(e, n, s-1) = t \neq -1$. The value t was chosen at stage $t < s$ when (1) held with t in place of s. If for all $x \leq n$, $B^{s-1}(x) = B^{t-1}(x)$,

$$\{e\}_s^{A^{s-1}}(x) = \{e\}_s^{A^{t-1}}(x), \tag{2}$$

and the *same* computation is used for both sides of (2), then $\ell(e, n, s) = \ell(e, n, s-1)$. Otherwise $\ell(e, n, s) = -1$.

Initially there is no location for $C(n)$. A location is defined at the first stage (1) is true. That location remains fixed unless a relevant change in B occurs or some change in A occurs that invalidates a computation needed for (1). In that event there is again no location. Thus as s increases, the location may come and go. Each time it returns, it is a bit further to the right. If B and $\{e\}^A$ agree on $[0, n]$, then eventually a permanent location develops, and conversely. If eventually there is no stage at which B and $\{e\}^A$ appear to agree on $[0, n]$, then the location eventually disappears forever. If they appear to agree infinitely often, without agreeing in the limit, then the location moves steadily off to infinity on the right.

P_e^s, the e-th set of positive requirements at stage s, consists of all sentences of the form:

$$\text{if} \quad n \in C, \quad \text{then} \quad \langle e, n, \ell(e, n, s)\rangle \in B.$$

Of course the above makes sense only if $\ell(e, n, s) \neq -1$.

If $\lim_s \ell(e, n, s)$ exists and $\neq -1$, call it $\ell(e, n)$.

P_e, the e-th set of positive requirements, consists of all sentences of the form:

$$\text{(3)} \qquad \text{if} \quad n \in C, \quad \text{then} \quad \langle e, n, \ell(e, n)\rangle \in B.$$

Matters are arranged so that the only way $\langle e, n, \ell(e, n)\rangle$ can land in B is via (3).

Suppose all but finitely much of P_e is met with the intent of showing $B \neq \{e\}^A$. For a reductio ad absurdum, assume $B = \{e\}^A$. Then $\ell(e, n)$ exists for all n and is computable from A. But then $C \leq_T A$, since

$$\langle e, n, \ell(e, n)\rangle \in B \leftrightarrow n \in C$$

for all but finitely many n.

Let P_e^* be the set added to B for the sake of P_e^s. Again assume all but finitely much of P_e is met in order to determine P_e^*. As above $B \neq \{e\}^A$. Let n_0 be the least n such that $\ell(e, n)$ does not exist. For each $n < n_0$, there is only a finite amount of activity on row n. If $n \geq n_0$, the location either (i) eventually disappears forever, or (ii) moves off to infinity on the right.

Consequently P_e^* is recursive. Suppose $n \geq n_0$. To decide if $\langle e, n, i\rangle$ is in B, run the construction until a stage s is found such that $s > i$, and $\ell(e, n, s)$ is either -1 or greater than i. Then $\ell(e, n, t) \neq i$ for all $t \geq s$, and so $\langle e, n, i\rangle$ was put in B only if it was put in before stage s.

If the recursive determination of P_e^* is valid for all $e < e_2$, then the injury set

$$I_{e_2} = A \cup \bigcup \{P_e^* \mid e < e_2\}$$

is recursive in A.

The e-th set of *negative requirements*: Define

$$r(e, s) = \mu z_{z<s}[C^{s-1}(z) \neq \{e\}_s^{B^{s-1}}(z)].$$

Let $p(e, x, s)$ be the $\sup^+$ of all negative facts about B^{s-1} used in the computation of $\{e\}_s^{B^{s-1}}(x)$. Define

$$p(e, s) = \sup \{p(e, x, s) \mid x < r(e, s)\}.$$

Keeping numbers less than $p(e, s)$ out of B will preserve the value of $\{e\}_s^{B^{s-1}}(z)$ for all $z < r(e, s)$.

Stage $s > 0$ begins with the definition of $r(e, s)$ and $p(e, s)$ for all e followed by the addition of the s-th member of A to B. Then attempts to satisfy P_e^s alternate with revisions of r and p. Fix e. Let w_e^s be the least element added to B for the sake of A or $P_{e'}^s$ $(e' < e)$. Define

$$r_0(e, s) = \mu z_{z < r(e,s)} \left[w_e^s < p(e, z, s) \right].$$

$$p_0(e, s) = \sup \{ p(e, x, s) \mid x < r_0(e, s) \}.$$

Now add to B all elements not less than $\inf\{p_0(e', s) \mid e' < e\}$ required by P_e^s.

Suppose I_e is recursive in A to show

(4) $$\liminf_s r_0(e, s) < \infty.$$

The negation of (4) implies $C \leq_T A$. Fix n to see how $C \restriction n$ is computed from A. If (4) fails, then

(5) $$(\mathrm{Et})(s)_{s \geq t}[r_0(e, s) \geq n].$$

At the beginning of stage t,

$$(x)_{x \leq n}[C^{t-1}(x) = \{e\}_t^{B^{t-1}}(x)],$$

since $r(e, s) \geq r_0(e, s)$. It follows from (5) that the computation of $\{e\}_t^{B^{t-1}})\restriction n$ is permanent, and that

$$C \restriction n = \{e\}^B \restriction n = \{e\}_t^{B^{t-1}}) \restriction n.$$

To find t run the construction until a stage t is reached where $r(e, t) \geq n$ and an appeal to I_e makes clear that the t-th approximation of $\{e\}^B \restriction n$ is permanent.

Similarly $I_e \leq_T A$ implies $\liminf_s p_0(e,s)$ exists. $\lambda s \mid p_0(e, s)$ behaves as follows. There is an n_e and an s_e such that the computation of

(6) $$\{e\}_{s_e}^{B^{s_e}} \restriction n_e$$

is permanent. In addition (6) equals $C \restriction n_e$ and $\{e\}^B(n_e)$ is either undefined or unequal to $C(n_e)$. Hence $\liminf_s p_0(e, s)$ is the $\sup^+$ of the negative facts about B used in the computation of (6).

The barrier to meeting P_{e_0} is defined by

(7) $$\sup_{e < e_0} \liminf_s p_0(e, s).$$

The drama of infinite injury is at its highest pitch when it is revealed that

(8) $$(E_\infty s)\,(e)_{e \leq e_0}\,[p_0(e, s) = \liminf_s p_0(e, s)].$$

("$(E_\infty s)$" means "there exist infinitely many s".)

The most direct proof of (8) is due to Lachlan. For each s let x_s be the least element put in B at stage s. Let

$$D = \{s \mid (t)_{t>s}(x_t > x_s)\}.$$

The value of $p_0(e, s)$ is a downward revision of the value of $p(e, s)$ caused by the addition of w_e^s to B. $x_s \leq w_e^s$. Hence for all sufficiently large s in D, $p_0(e, s)$ is a non-decreasing function of s, and

(9) $$\liminf_s p_0(e, s) = \lim \{p_0(e, s) \mid s \in D\}.$$

(8) is a consequence of (9). (8) implies I_{e0} is recursive.

$B \leq_T C$ is yet to be shown. B is the disjoint recursive sum of A and $P_e^*(e \geq 0)$. $A \leq_T C$ by hypothesis. P_e^* is recursive in C uniformly in e. There is a recursive function z such that $P_e^* = \{z(e)\}^C$ for all e. Procedure $\{z(e)\}$ is defined by recursion on e. Fix $\langle e, n, i\rangle$. Run the construction and keep an eye on $\ell(e, n, s)$. If $\ell(e, n)$ does not exist, then there is a stage $s \geq i$ such that $\ell(e, n, s) > i$ or $\ell(e, n, s) = -1$. Either way $\langle e, n, i\rangle$ gets into B only if it does before stage s. Suppose $\ell(e, n)$ does exist. Then $\ell(e, n) = \ell(e, n, t)$ for some t that can be recognized by referring to A; at stage t the computation from A underlying the value of $\ell(e, n, t)$ was correct.

If $\ell(e, n) < i$, then $\langle y, n, i\rangle \notin B$. If $\ell(e, n) > i$, then running the construction will settle "$\langle e, n, i\rangle \in B$?". Finally suppose $\ell(e, n) = i$. If $n \notin C$, then $\langle e, n, i\rangle \notin B$. Assume $n \in C$. Then $\langle e, n, i\rangle$ is not put in B iff some permanent negative requirement keeps it out. Such a requirement, if it exists, can be found by running the construction. Its permanence is established by some negative facts about $A \cup \{P_{e'}^* \mid e' < e\}$ computed from C via $\{z(e')\}$ $(e' < e)$. □

4. *Preliminaries to α-Density*

Suppose A and C are regular α-r.e., sets such that $A <_\alpha C$. The following parameters will be used in the construction of an α-recursively enumerable B such that $A <_\alpha B <_\alpha C$:

$$c_1^A = \sigma 1\text{cf}_A(\alpha);$$
$$c_2^A = \sigma 2\text{cf}_A(\alpha);$$
$$\alpha_A^* = \sigma 1 p_A(\alpha).$$

Define $c_2^A(x) = \sigma 2\text{cf}_A^\alpha(x)$, that is, the Σ_2 cofinality of x in the structure $\langle L[A, \alpha], A\rangle$.

4.1 Lemma

(*i*) $c_1^A \geq \alpha_A^*$.
(*ii*) (Ek) [*k*: $c_1^A \to \alpha$ *is one-one, onto and* $\Sigma_1^{\alpha, A}$].

(iii) If $Z \subseteq \delta < \alpha_A^*$ *and* Z *is* $\Sigma_1^{\alpha, A}$*, then* Z *is α-finite.*
(iv) $c_2^A = c_2^A(\alpha_A^*) = c_2^A(c_1^A)$.

Proof. (i) is Corollary 2.4. (ii) is Corollary 2.7. (iii) is Theorem 2.3. (iv) is a relativization of Lemma 3.3.VIII. The relativization succeeds with the aid of (i), (ii) and (iii). □

4.2 Cofinality Function *g*. According to Lemma 4.1(iv) there exists a $\Sigma_2^{\alpha, A}$ function h: $c_2^A \to c_1^A$ with range unbounded in c_1^A. Thus

$$h(x) = y \leftrightarrow (\mathrm{E}u)\,(v) D^A(u, v, x, y)$$

for some $\Delta_0^{\alpha, A}$ formula D^A. u can be construed as less than c_1^A with the aid of a $\Sigma_1^{\alpha, A}$ map k from c_1^A onto α provided by Lemma 4.1(ii).

$$h(x) = y \leftrightarrow (\mathrm{E}u)_{u < c_A}\,(v) D^A\,(k(u), v, x, y).$$

Define

$$g(x) = \mu z (v) D^A(k(z_0), v, x, z_1).$$

(For simplicity, z_i instead of $(z)_i$.)
Note that $D^A(k(z_0), v, x, z_1)$ is $\Pi_1^{A, \alpha}$ because it is equivalent to

$$(t)[k(z_0) = t \to D^A(t, v, x, z_1)].$$

Thus g: $c_2^A \to c_1^A$ is $\Sigma_2^{A, \alpha}$ and has range unbounded in c_1^A. g is more suitable than h for approximation. Let

$$g_\sigma^A(x) = \mu z_{z < \sigma}\,(v)_{v < \sigma}\,D^A\,(k(z_0), v, x, z_1).$$

Then g_σ^A is $\Sigma_1^{\alpha, A}$. Consider

(1) $\quad g(x) = z \leftrightarrow (v) D^A(k(z_0), v, x, z_1) \quad \& \quad (w)_{w < z}(\mathrm{E}v) \sim D^A(k(w_0), v, x, w_1).$

For each $w < g(x)$ let $\ell(x, w)$ bound x, w and all the quantifiers in the $\Sigma_1^{\alpha, A}$ formula

(2) $$(\mathrm{E}v) \sim D^A(k(w_0), v, x, w_1).$$

Thus $L[A, \alpha] \models (2)$ iff $L[A \cap \ell(x, w),\ \ell(x, w)] \models (2)$. Let $\ell(x)$ be the $\sup^+$ of $\{\ell(x, w) \mid w < g(x)\}$. $\ell(x) < \alpha$ because $g(x) < c_1^A$. Then

$$g(x) = \mu z (v)_{v < \ell(x)}\, D^A(k(z_0), v, x, z_1),$$
$$\sigma \geq \ell(x) \to g_\sigma^A(x) = g(x).$$

Define g_σ by replacing A by $A^{<\sigma}$ in the definition of g_σ^A. g_σ is Σ_1^α.
Call σ *A-correct* if $A^{<\sigma} = A \cap \sigma$.

4.3 Proposition. *If $c_1^A > \omega$, then there exist arbitrarily large A-correct stages.*

Proof. Choose σ_0. Let σ_{n+1} be the least $\sigma \geq \sigma_n$ such that $A \cap \sigma_n = A^{<\sigma}$. Then $\lambda_n | \sigma_n$ is $\Sigma_1^{\alpha,A}$. Let $\tau = \lim_n \sigma_n$. $\tau < \alpha$, since $\omega < c_1^A$. $A^{<\tau} = A \cap \tau$. □

4.4 Lemma. *Assume $b < c_2^A$.*
(i) $(x)_{x<b}[g_\sigma(x) \geq g(x)]$ for all sufficiently large σ.
(ii) $g_\sigma \upharpoonright b = g \upharpoonright b$ for all sufficiently large A-correct σ.

Proof. (1) The function ℓ: $c_2^A \to \alpha$ of subsection 4.2 is $\Sigma_2^{\alpha,A}$. Hence $\ell[b]$ is bounded by some $\ell_b < \alpha$. Consider $x < b$.

$$g(x) = \mu z(v)_{v<\ell_b} D^A(k(z_0), v, x, z_1). \tag{1}$$

The right side of (1) is evaluated using only $A \cap \ell_b$, hence $g \upharpoonright b$ is α-finite thanks to the regularity of A. Suppose $A \cap \ell_b \subseteq A^{<\sigma_0}$ for some $\sigma_0 \geq \ell_b$. Let $\sigma \geq \sigma_0$. Then

$$(z)_{z<g(x)} (\mathrm{E}v)_{v<\ell_b} \sim D^{A^{<\sigma}}(k(z_0), v, x, z_1)$$

because D is Δ_0 and

$$(z)_{z<g(x)} (\mathrm{E}v)_{v<\ell_b} \sim D^{A\cap\ell_b}(k(z_0), v, x, z_1).$$

Hence $g_\sigma(x) \geq g(x)$.
(ii) Suppose $\sigma \geq \sigma_0$ is A-correct. Then for $x < b$,

$$g_\sigma(x) = g_\sigma^{A<\sigma}(x) = g_\sigma^{A\cap\sigma}(x) = g_\sigma^A(x). \quad \square$$

g will be used in the next section to define blocks of requirements, It will follow from Lemma 4.4(i) that for all sufficiently large σ, if a negative requirement lands in block x at stage σ, then it is never discarded. g will be approximated by g_σ. On the surface g is Σ_3^α, and in general Σ_3^α blocking functions are intractable in the presence of Σ_2 inadmissibility. But g is workable because it is $\Sigma_2^{A,\alpha}$ for an A such that $\langle L[\alpha, A], A\rangle$ is weakly Σ_1 admissible, and Σ_2 blocking needs only Σ_1 admissibility to succeed.

5. Shore's Density Theorem

The density theorem for ω, as sketched in Section 3, appears to rely on the fact that $L(\omega)$ satisfies Σ_3 replacement. In fact it uses only Σ_2^A replacement, where A is an incomplete, recursively enumerable set. Very little is known about making a Σ_1 admissible α do the work of Σ_3 replacement. The proof of the density theorem for α makes α do the work of Σ_2^A replacement for an incomplete, regular, α-recursively enumerable set. For such an A, $\langle L[A,\alpha], A\rangle$ is weakly Σ_1 admissible, and so some of the thinking behind the α-finite injury method is applicable. New difficulties arise because A has to be guessed at.

5.1 Theorem (Shore 1976). *Let A and C be α-recursively enumerable sets such that $A <_\alpha C$. Then there exists an α-recursively enumerable B such that $A <_\alpha B <_\alpha C$.*

Proof. As usual A and C are assumed to be regular. The following construction yields a regular B. The density sketch given in Section 3 will be relied on heavily. The principal difference between the argument below and that of Section 3 is the use of blocking. Let g and g_σ be as in Section 4. Let f^A be a one-one $\Sigma_1^{\alpha,A}$ map of α into α_A^*. *Block* y is $[0, f^A g(y))$ for each $y < c_2^A$. Each ε in block y is associated with a reduction procedure $\{(f^A)^{-1}(\varepsilon)\}$, written simply as $\{\varepsilon\}$. $f^A g[c_2^A]$ is unbounded in α_A^* by Theorem 2.3. During the construction $(f^A)^{-1}(\varepsilon)$ is approximated by $f_\sigma^{-1}(\varepsilon)$, abbreviated as ε_σ. f_σ is the result of replacing A by $A^{<\sigma}$ in the $\Sigma_1^{\alpha,A}$ definition of f. *Note Well*: Let $D^A(a,b,z)$ be a $\Delta_0^{\alpha,A}$ formula such that

$$f^A(a) = b \leftrightarrow (\mathrm{E}z) D^A(a,b,z).$$

Define $f_\sigma(a)$ to be $\mu b_{b<\sigma} (\mathrm{E}z)_{z<\sigma} D^{A^{<\sigma}}(a,b,z)$. If $f_\sigma(a) < \sigma$ for some A-correct σ, then $f_\sigma(a) = f^A(a)$. ε_σ is not defined at stage σ unless $\varepsilon < \sigma$ and $f_\sigma^{-1}(\varepsilon) < \sigma$. If $f_\sigma^{-1}(\varepsilon)$ has more than one value, then the least is used.

The even part of B is reserved for A. All the remaining action takes place on the odd part of B. *Positive requirements* are elements of

$$\{\langle y,n,i\rangle \mid y < c_2^A \quad \& \quad n < c_1^A \quad \& \quad i < \alpha\}.$$

They are added to B in order to insure that $B \neq \{\varepsilon\}^A$ for all ε in block y.

Let $k^A\colon c_1^A \to \alpha$ be an onto, $\Sigma_1^{\alpha,A}$ map as in Lemma 4.1 (ii). Define k_σ by replacing A by $A^{<\sigma}$ in the $\Sigma_1^{\alpha,A}$ definition of k^A. Then

$$k^A(x) = \lim_\sigma k^\sigma(x)$$

for all $x < c_1^A$, and k^A is tame Σ_2^α (tame via the α-recursive approximation k_σ).

The planting of C in B is less troublesome if C is replaced by C_0 provided by Proposition 3.4.VII. The proof of 3.4 implies C_0 is regular if C is. C_0 is α-recursively enumerable, $C_0 \equiv_\alpha C$ and

(1) $$(X)[C_0 \leq_{w\alpha} X \leftrightarrow C \leq_\alpha X].$$

$\ell(y,n,\sigma)$ is the proposed location of $C_0(k^A(n))$ in B at the beginning of stage σ. $\ell(y,n,0) = -1$ (no location).

Case 1: $\sup^+\{\tau \mid \tau < \sigma \,\&\, \ell(y,n,\tau) = -1\} = \sigma$. Then $\ell(y,n,\sigma) = \sigma$ if

(2) $(E\varepsilon)_{\varepsilon < f_\sigma g_\sigma(y)}\, [B^{<\sigma} \restriction k_\sigma[n+1] = \{\varepsilon_\sigma\}_\sigma^{A^{<\sigma}} \restriction k_\sigma[n+1]]$; otherwise $\ell(y,n,\sigma) = -1$.

Case 2: $\lim_{\tau<\sigma} \ell(y,n,\tau) = \gamma$ for some $\gamma < \sigma$. The value γ was chosen at stage γ when (2) held with γ in place of σ. If

$$B^{<\gamma} \upharpoonright k_\gamma[n+1] = B^{<\sigma} \upharpoonright k_\sigma[n+1] \quad \text{and} \quad (\gamma - A^{<\gamma}) \cap A^{<\sigma} = \varnothing,$$

then $\ell(y,n,\sigma) = \gamma$; otherwise $\ell(y,n,\sigma) = -1$.

Initially there is no location. A location is created at stage σ if B appears to equal $\{\varepsilon\}^A$ on $k[n+1]$. It is lost (i.e., $= -1$) if σ turns out to be A-incorrect or if $B^{<\sigma} \upharpoonright k_\sigma[n+1]$ changes.

The y-th set of *positive requirements at stage* σ is

$$P_y^\sigma = \{\langle y, n, \ell(y,n,\sigma)\rangle \mid k_\sigma(n) \in C_0\}.$$

If $\lim_\sigma \ell(y,n,\sigma)$ exists and $\neq -1$, then it defines a permanent location denoted by $\ell(y,n)$.

The y-th set of *positive requirements* is

$$P_y = \{\langle y, n, \ell(y,n)\rangle \mid k^A(n) \in C_0\}.$$

From now on assume

$$(3) \qquad c_1^A > \omega.$$

According to Proposition 4.3 there are arbitrarily large A-correct stages.

The first thing to show is (4).

(4) If P_y is met for all sufficiently large $n < c_1^A$, then $B \neq \{\varepsilon\}^A$ for every ε in block y.

To check (4) fix ε in block y and assume $B = \{\varepsilon\}^A$ for a contradiction. Then $\ell(y,n)$ exists for all $n < c_1^A$. $\lambda n \mid \ell(y,n)$ is weakly α-recursive in A as follows. Run the construction until an A-correct stage σ is found that satisfies the matrix of (2). (Note that

$$\{\varepsilon\}^A \upharpoonright k^A[n+1]$$

is determined by an α-finite initial segment of A because $n < c_1^A$). But then $C \leq_\alpha A$ by (1). (Similar to the argument following (3) of Section 3.)

A shade more intricate than (4) is:

(5) If P_y is met for all sufficiently large $n < c_1^A$, then $\ell(y,n)$ does not exist for some n.

The proof of (5) takes into account the details of the approximation of blocking. Suppose (5) fails with the intent of showing $C_0 \leq_{w\alpha} A$. As in the proof of (4) it suffices to check that $(\lambda n \mid \ell(y,n)) \leq_{w\alpha} A$ and $B \leq_{w\alpha} A$. Choose σ_0 with the aid of

Lemma 4.4 so that

(6a) $$g_\sigma(y) \geq g(y) \quad \text{for all} \quad \sigma \geq \sigma_0, \text{ and}$$
$$g_\sigma(y) = g(y) \quad \text{for all} \quad A\text{-correct } \sigma \geq \sigma_0.$$

Hence any ε that participates in (2) at an A-correct stage after σ_0 is less than $f^A g(y)$.
Put ε in J if: $\varepsilon < f^A g(y)$; and

(6b) at some A-correct stage σ after σ_0, ε participates in (2) and thereby compels the value of $\ell(y,n,\sigma)$ to be σ;

and $B^{<\sigma} \restriction k[n+1] \neq B^{<\tau} \restriction k[n+1]$ for some $\tau > \sigma$.

If ε belongs to J, then a permanent inequality between B and $\{\varepsilon\}^A$ develops at stage τ of clause (ii), and subsequent to τ, ε cannot participate in (2). J is $\Sigma_1^{\alpha, A}$, bounded below α_A^*, hence α-finite by Lemma 4.1 (iii). Furthermore all activity associated with the definition of J is α-finite. Consequently the associated values of $k[n+1]$ are bounded below α. Moreover the associated values of $n+1$ are bounded below c_1^A.

Let $\bar{J}$ be $f^A g(y) - J$. Eventually only ε's in $\bar{J}$ participate in (2). Thus for all sufficiently large $n < c_1^A$, the value of $\ell(y,n)$ is determined by some ε in $\bar{J}$. Hence $(\lambda n \mid \ell(y,n)) \leq_{w\alpha} A$. Also $B \leq_\alpha A$, because for all sufficiently large $n < c_1^A$, there is an ε in $\bar{J}$ such that $\{\varepsilon\}^A \restriction k^A[n+1]$ is defined, and for all such ε, the computation from A equals $B \restriction k^A[n+1]$. So (5) is proved.

The hypothesis of (5) has a further consequence. Let

$$P_y^* = \{\langle y,n,i\rangle \mid n < c_1^A \quad \& \quad i < \alpha\} \cap B.$$

Then P_y^* is α-recursive. Let n_0 be the least n that satisfies the conclusion of (5). The total of all activity on row n for all $n < n_0$ is α-finite, since $n_0 < c_1^A$ and $\ell(y,n)$ $(n < n_0)$ can be computed from A and $B \restriction k[n_0]$. If $n \geq n_0$, then the location for row n either (i) eventually disappears or (ii) moves off to $\infty (= \alpha)$. If (i) holds for some $n \geq n_0$, then (i) holds for all $n' \geq n$, and the total of all activity on all rows from n on is α-finite.

Consequently P_y^* is α-recursive if the conclusion of (5) holds. Suppose $n \geq n_0$. To decide if $\langle y,n,i\rangle \in B$, run the construction until some σ is found such that $\sigma > i$, and $\ell(y,n,\sigma)$ is either -1 or greater than i. Then $\ell(y,n,\tau) \neq i$ for all $\tau \geq \sigma$, and so $\langle y,n,i\rangle$ is put in B only if it is put in before stage σ.

A *negative requirement* is an ordinal denoted by $p(\sigma,\varepsilon,x)$. Its purpose is to preserve the value of

(7) $$\{\varepsilon_\sigma\}_\sigma^{B^{<\sigma}}(k_\sigma(x));$$

its value is the supremum$^+$ of all negative facts about $B^{<\sigma}$ used in the computation of (7). It is added to the y-th block of negative requirements at stage σ if certain conditions hold. Once added to block y at stage σ it remains there forever or until removed at stage $\tau \geq \sigma$. Removal is caused by injuries or changes in A. $p(\sigma, \varepsilon, x)$ is injured if some $w < p(\sigma, \varepsilon, x)$ is added to B at stage $\tau \geq \sigma$. σ is seen to be A-incorrect at stage $\tau > \sigma$ if

$$A^{<\sigma} \neq A^{<\tau} \cap \sigma.$$

Define

$$q(\sigma, \varepsilon) = \mu x[C_0^{<\sigma}(k_\sigma(x)) \neq \{\varepsilon_\sigma\}_\sigma^{B^{<\sigma}}(k_\sigma(x))],$$

$$r(\sigma, y) = \sup\{q(\sigma, \varepsilon) \mid \varepsilon < f_\sigma g_\sigma(y)\}.$$

Removal of negative requirements at the beginning of stage σ: If ρ is seen to be A-incorrect at stage σ, then remove all negative requirements added at stage ρ. If $p(\rho, \varepsilon, x)$ is removed, then also remove $p(\rho, \varepsilon, x')$ for all $x' > x$.

Addition of negative requirements at the beginning of stage σ: Suppose $\varepsilon < f_\sigma g_\sigma(y)$ and $x < q(\sigma, \varepsilon)$. Add $p(\sigma, \varepsilon, x)$ to block y if

(8) $r(\sigma, y) > \sup^+\{x' \mid (E\rho)_{\rho<\sigma}(E\varepsilon')[p(\rho, \varepsilon', x')$ added to block y at stage ρ and not yet removed$]\}$.

Clause (8) limits the addition of negative requirements to blocks. It is necessitated by blocking, and is similar to the monotonicity clause, (1a), in the proof of Shore splitting, Theorem 1.1.

Note: the above addition step is performed after the preceding removal step.

Construction of B. Add the σ-th member of A to B. Next comes a recursion on $y < c_2^A$ that alternates between removing negative requirements from block y and moving positive requirements from P_y^σ into B.

Fix $y < c_2^A$. Let w_y^σ be the least element of $A \cup \bigcup\{P_{y'}^\sigma \mid y' < y\}$ added to B at stage σ. If w_y^σ is less than some negative requirement in block y, then remove that requirement.

If $p(\rho, \varepsilon, x)$ is removed, then also remove $p(\rho, \varepsilon, x')$ for all $x' > x$. The removal of negative requirements injured by w_σ^y makes it easier to add elements of P_y^σ to B. Define

$$p(\sigma, y) = \text{sup of negative requirements still in block } y.$$

Add to B all members of P_y^σ not less than

$$\sup\{p(\sigma, y) \mid y' \leq y\}.$$

End of recursion on y and stage σ of construction of B.

Behavior of Negative Requirements. Suppose $p(\sigma, \varepsilon, x)$ is put in block y at the beginning of stage σ and is never removed. Thus σ is A-correct and all computations based on $A^{<\sigma}$, and performed at stage σ, are correct. In particular $k_\sigma \upharpoonright x = k \upharpoonright x$ and $\varepsilon_\sigma = (f^A)^{-1}(\varepsilon)$. Also $p(\sigma, \varepsilon, x')$ has not been removed for any $x' < x$. So

$$\{\varepsilon_\sigma\}^{B^{<\sigma}} \upharpoonright k_\sigma[x] = \{\varepsilon\}^B \upharpoonright k[x].$$

It may happen that

$$C_0^{<\sigma} \upharpoonright k[x] \neq C_0^{<\tau} \upharpoonright k[x]$$

for some $\tau > \sigma$. In that event a permanent inequality between C_0 and $\{\varepsilon\}^B$ arises at stage τ; and so at all subsequent stages, no new negative requirement associated with ε is added to any block. In the absence of a permanent inequality the situation is more complicated.

For $y < c_2^A$ define the y-th injury set to be

$$I_y = A \cup \bigcup \{P_{y'}^* \mid y' < y\}.$$

for any $t\colon \alpha \to \alpha$, define

$$\liminf_{\sigma} t(\sigma) \text{ to be } \mu\beta(\sigma)\,(E\tau)_{\tau>\sigma}\,[t(\tau) \leq \beta].$$

An induction on y shows:

(9a) $\liminf_{\sigma} p(\sigma, y) < c_1^A$;

(9b) $I_y \leq_\alpha A$ and I_y is regular.

The induction on y is organized as follows: (9b) is true when $y = 0$ because $I_0 = A$; (9b) implies (9a); if (9a), with y' in place of y, holds for all $y' < y$, then (9b) holds.

Assume (9b) to prove (9a). As in the proof of Proposition 4.3, the regularity of I_y and assumption (3) imply the existence of arbitrarily large I_y-correct σ, that is,

$$I_y \cap \sigma = I_y^{<\sigma} = \{z \mid z \in I_y \quad \& \quad z \text{ put in } B \text{ before stage } \sigma\}.$$

I_y-correctness entails A-correctness since $A \subseteq I_y$.

Let σ_0 be as in (6a). Then on all I_y-correct stages σ beyond $\sigma_0, f_\sigma g_\sigma(y) = f^A g(y)$. More precisely, the y-th block, $[0, f_\sigma g_\sigma(y))$, is constant on all sufficiently large A-correct stages. Beyond σ_0, if a negative requirement is added to block y at an I_y-correct stage, then it is never removed; if it is added before some I_y-correct stage τ, and is not removed before stage τ, then it is never removed.

Put ε in K if:

$\varepsilon < f^A g(y)$;
at some I_y-correct $\sigma > \sigma_0$, clause (8) holds;
and for some $x < q(\sigma, \varepsilon)$ & $\tau > \sigma$, $C_0^{<\sigma}(k(x)) \neq C_0^{<\tau}(k(x))$.

If ε is in K, then some permanent inequality develops between C_0 and $\{\varepsilon\}^B$ and $q(\sigma, \varepsilon)$ is permanently bounded. K is Σ_1^A because $I_y \leq_\alpha A$ by (9b). K, and all activity associated with the development of K, are α-finite, because K is bounded below α_A^*.

Let $\bar{K}$ be $K - f^A g(y)$. Eventually only ε's in $\bar{K}$ participate in the addition of negative requirements to block y at A-correct stages. Any such addition is permanent if it is made at an I_y-correct stage or if it is not removed prior to the next I_y-correct stage. Thus the set of negative requirements in block y, viewed only at all sufficiently large I_y-correct σ, is a non-decreasing function of σ. If (9a) is false, then $C_0 \leq_{w\alpha} A$ (hence $C \leq_\alpha A$) as follows. To compute $C_0(k^A(n))$ from A, look for ε in $\bar{K}$ and $\sigma > \sigma_0$ such that

$$(10) \qquad n < q(\sigma, \varepsilon) \quad \& \quad \sigma \text{ is } I_y\text{-correct.}$$

Then $C_0(k^A(n)) = \{\varepsilon\}_\sigma^{B^{<\sigma}}(k_\sigma^A(n))$, since $\varepsilon \notin K$. The existence of ε and σ satisfying (10) follows from the monotonicity clause, (8). To verify the last claim, focus on the sufficiently large I_y-correct stages. On those stages $p(\sigma, y)$ is nondecreasing. The falsity of (9a) implies $p(\sigma, y)$ increases unboundedly often. All such increases are associated with ε's in $\bar{K}$. Clause (8) implies that $r(\sigma, y)$ is non-decreasing on all sufficiently large I_y-correct stages and increases unboundedly often on such stages. Each increase in r is the result of an increase in some $q(\sigma, \varepsilon)$ for some ε in $\bar{K}$, an increase beyond the previous value of r.

Thus (9b) implies (9a). The next task is to draw a further consequence of (9b), namely

(11) all sufficiently large members of P_y are put in B.

Consider the behavior of $p(\sigma, y)$ on all sufficiently large I_y-correct stages. As described immediately above, $p(\sigma, y)$ is nondecreasing and bounded. Thus

$$(12) \qquad \lim_\sigma \inf p(\sigma, y) = \lim_\sigma \{p(\sigma, y)) \mid \sigma\ I_y\text{-correct}\} < \alpha.$$

The ineluctable barrier to adding elements of P_y to B is

$$(13) \qquad \sup_{y' \leq y} \lim \inf_\sigma p(\sigma, y').$$

Suppose $y' < y$. Then every $I_{y'}$-correct stage is also I_y-correct, since $I_{y'} \subseteq I_y$. Also (9b) implies $I_{y'} \leq_\alpha A$ and $I_{y'}$ is regular. Hence the derivation of (9a) from (9b) also shows

$$(14) \qquad \lim_\sigma \inf p(\sigma, y') = \lim_\sigma \{p(\sigma, y') \mid \sigma\, I_y\text{-correct}\} < \alpha$$

for all $y' < y$. It follows from (14) that

$$\lim \inf_\sigma p(\sigma, y')\ (y' \leq y)$$

is $\Sigma_2^{\alpha, A}$, because $I_y \leq_\alpha A$. $y < c_2^A$, so (13) $< \alpha$. Thus any member of P_y larger than (13) can be added to B at any sufficiently large I_y-correct stage.

The last part of the induction on y is devoted to proving (9b) under the assumption that (9a), with y' in place of y, holds for all $y' < y$. By induction (9b), hence (11), holds with y' in place of y for all $y' < y$. It follows from (5) that

(15) $\ell(y', n)$ does not exist for some $n \quad (y' < y)$.

Let $n_0(y')$ be the least n that satisfies (15). Recall the proof of the α-recursiveness of $P^*_{y'}$ that immediately follows the proof of (5). A Gödel number for the α-recursive set $P^*_{y'}$ can be obtained effectively from the values of $n_0(y')$ and $\ell(y', n)$ $(n < n_0(y'))$. Hence $I_y \leq_\alpha A$ if the functions

$$(16) \qquad \begin{array}{ll} n_0(y') & (y' < y) \\ \ell(y', n) & (y' < y \;\&\; n < n_0(y')) \end{array}$$

are α-finite. It suffices to show these functions are $\Sigma_2^{\alpha, A}$ because $y < c_2^A$. (If h is any $\Sigma_2^{\alpha, A}$ function on c_2^A, then h is tame $\Sigma_2^{\alpha, A}$, and so $h \restriction y$ is $\Sigma_1^{\alpha, A}$, hence α-finite by the regularity of A.) The definition of $\ell(y', n)$ is $\Sigma_2^{\alpha, A}$, because it says some computations from A exist for an initial segment of arguments shorter than c_1^A and the results agree with B (cf. "$\ell(x) < \alpha$" in Section 4.2). The definition of $n_0(y')$ has a clause concerning the non-existence of a computation from A, a $\Pi_1^{\alpha, A}$ statement.

So ends the proof of (9a) and (9b) by induction on y. All that remains is the proof of $B \leq_\alpha C$ and the disposal of assumption (3). The recovery of B from C is controlled by the permanent negative requirements. A simultaneous recursion on y defines α-recursive functions $\varepsilon_1(y)$ and $\varepsilon_2(y)$ such that for all $y < c_2^A$:

(17a) the set of permanent negative requirements in block y is α-recursively enumerable in C via Gödel number $\varepsilon_1(y)$;

(17b) $P^*_y \leq_{w\alpha} C$ via Gödel number $\varepsilon_2(y)$.

Fix $y < c_2^A$ and assume $\varepsilon_1(y')$ is already defined for all $y' \leq y$. Consider $\langle y, n, i\rangle$ in order to see how $\{\varepsilon_2(y)\}$ works. Suppose $\ell(y, n)$ does not exist. Then running the construction will produce a stage $\sigma > i$ at which $\ell(y, n, \sigma) > i$ or $= -1$. By then the question, $\langle y, n, i\rangle \in B$?, will have been resolved. Suppose $\ell(y, n)$ does exist. If $\ell(y, n) > i$, then running the construction will resolve the matter. If $\ell(y, n) < i$, then an appeal to A establishes that the computations from A in (2) are correct, and that consequently the location will never move out to i. If $\ell(y, n) = i$, then an appeal to C_0 is needed. If $k^A(n) \in C_0$, then $\langle y, n, i\rangle$ is kept out of B by a permanent negative requirement in block y' (for some $y' \leq y$) enumerated from C via Gödel number $\varepsilon_1(y')$.

Now fix y and assume $\varepsilon_2(y')$ is defined for all $y' < y$. To see how $\varepsilon_1(y)$ works, run the construction. A negative requirement put in block y at stage σ is permanent if σ is I_y-correct. (More precisely, a negative requirement put in block y at stage σ is permanent if none of the computations underlying that requirement use a negative membership fact contradicted by a positive fact about I_y.) A full account of the

I_y-correctness of σ can be ascertained from C via $\varepsilon_2(y')$ $(y' < y)$. This last claim is delicate and has to be supported by careful examination of the description of ε_2 given in the previous paragraph. Note that appeals to C_0 and A are made by $\{\varepsilon_2(y')\}$ only if $n < n_0(y')$. By (16) and remarks subsequent, $n_0(y')$ $(y' < y)$ is bounded below c_1^A. Consequently $k^A(n)$ $(n < n_0(y')$ & $y' < y)$ is bounded below α. Therefore the set of appeals to C_0 made by $\varepsilon_2(y')(y' < y)$ is bounded below α, hence is α-finite by the regularity of C_0. The appeals to A concern computations from A on an initial segment of arguments bounded by $\sup\{n_0(y') \mid y' < y\}$, which, as just noted, is less than c_1^A.

Thus the I_y-correctness of σ can be ascertained from C via $\varepsilon_2(y')$ $(y' < y)$, since the procedures $\{\varepsilon_2(y')\}$ $(y' < y)$ will draw only on α-finitely much of C and A as they compute $P_{y'}^*$ from C. That ends the definitions of ε_1 and ε_2. Something a bit stronger than (17a) and (17b) has been proved.

(18) The set of all permanent negative requirements is α-recursively enumerable in C. For each $y < c_2^A$, the enumeration of requirements in the blocks below block y draws only on a bounded part of C determined by y.

(18) is what is needed to see $B \leq_\alpha C$. Suppose H is an α-finite subset of $\alpha - B$. "$H \subseteq cB$" is established by an α-finite set of facts about C as follows. First a $y_0 < c_2^A$ has to be found so that

$$\sup_{y' < y_0} \lim_{\sigma} \inf p(\sigma, y') > \sup H.$$

By (18), y_0 can be established by α-finitely much of C. Suppose $\langle y, n, i \rangle \in H$. The question, $\langle y, n, i \rangle \in B$?, is dealt with as it was in the definition of ε_2. If $y \geq y_0$, then some permanent negative requirements involved in the definition of y_0 will serve, if needed, as the reason that $\langle y, n, i \rangle \notin B$. If $y < y_0$, then (18) implies a bound on the amount of C needed to establish a permanent negative requirement that keeps $\langle y, n, i \rangle$ out of B.

Assumption (3) is disposed of in the next subsection.

End of proof of density.

5.2 Assumption (3). The above account of Theorem 5.1 relied heavily on assumption (3), namely $c_1^A > \omega$. Now suppose $c_1^A = \omega$. Then

$$c_2^A = \alpha_A^* = c_1^A = \omega,$$

since $\alpha_A^* \leq c_1^A$ by Lemma 4.1. Many difficulties disappear. There is no blocking. The only complication left is the use of k^A, the $\Sigma_1^{\alpha, A}$ map from c_1^A onto α. The density construction given above is greatly simplified. It still works and is similar to one given by Driscoll (1968) for $\alpha = \omega_1^{\mathrm{CK}}$. Proposition 4.3 is lost. There may not be any A-correct stages. Of course there is less need for them with blocking gone. Instead of A-correct stages, non-deficiency stages are used as in the ω-case. In short the density argument for $c_1^A = \omega$ is close to that of classical recursion theory (cf. Exercise 5.5).

5.3 $\Sigma_2^{\alpha, A}$ versus Σ_3^α. A look backward reveals that the functions developed in the proof of density are at worst $\Sigma_2^{\alpha, A}$, hence Σ_3^α. The knowledge they are Σ_3^α would have been of little use with only Σ_1 admissibility available. But since they are $\Sigma_2^{\alpha, A}$, and since $\langle L(\alpha), A \rangle$ is weakly Σ_1 admissible, it was possible to apply the methods of Chapters VIII and IX for making Σ_1 admissibility do the work of Σ_2 admissibility. The idea of blocking was important. Blocking makes possible the full utilization of the limited Σ_2 admissibility properties possessed by every Σ_1 admissible $L(\alpha)$.

It is not clear how far Σ_1 admissibility can be stretched. α-recursively enumerable sets A and B are said to form a minimal pair if every set α-recursive i n both A and B is α-recursive. The existence of a minimal pair when $\alpha = \omega$ is a well known result of classical recursion theory (Lachlan 1966, Yates 1966). For $\alpha > \omega$ some positive partial results have been obtained by Lerman & Sacks 1972, Maass 1977a and Shore 1975. The problem remains open for most α because the classical proof, and its α-variations, appear to need strong forms of Σ_2 replacement that do not seem manageable by Σ_2 blocking.

It is quite possible that some of the constructions of the classical theory of recursively enumerable sets make essential use of Σ_2 replacement. Evidence for this view is provided by a result of Shore 1976. He showed that Lachlan's non-splitting theorem 1975 fails when $\alpha = \omega_\omega$.

5.5–5.6 Exercises

5.5. Prove Shore's density theorem when $\sigma 1\mathrm{cf}_A(\alpha) = \omega$.

5.6. Assume A is α-recursively enumerable and incomplete. Prove

$$\sigma 2\mathrm{cf}_A^\alpha(\alpha) = \sigma 2\mathrm{cf}_A^\alpha(\alpha_A^*) = \sigma 2\mathrm{cf}_A^\alpha(\sigma 1\mathrm{cf}_A(\alpha)).$$

6. *β-Recursion Theory*

β-recursion theory was introduced by S. Friedman and Sacks 1977. Its purpose is to see how far recursion theory can be developed without Σ_1 admissibility. Some of the technical problems that arise are similar to those discussed above when α is Σ_1 admissible, but not Σ_2 admissible, and a Σ_2 construction is attempted. The proper setting for β-recursion theory is Jensen's J hierarchy, a reformulation of Gödel's L hierarchy. For the brief sketch given here L will suffice. From now on let β be a limit ordinal. Thus $L(\beta)$ need not be Σ_1 admissible, but it will be closed under the operations of pairing and union, and it will satisfy Δ_0-separation.

The fundamental definitions of β-recursion are in essence the same as those of α-recursion. Let $A \subseteq L(\beta)$. A is β-recursively enumerable if A is Σ_1^β. A is β-recursive if A is Δ_1^β. The β-finite sets are simply the sets in $L(\beta)$. $\leq_{w\beta}$ (weakly β-recursive in) and $\leq_\beta$ (β-recursive in) are defined in precisely the same manner as their counterparts in α-recursion theory. The failure of Σ_1 admissibility makes possible new distinctions among the β-recursive sets. For example it can happen that A is β-recursive, but not β-recursive in $\varnothing$, the empty set.

For each $\gamma \leq \beta$, the Σ_n^β cofinality of γ, and the Σ_n^β projection of γ, are defined as in α-recursion theory. β^*, the Σ_1^β projection of β, is of special interest. The loss of Σ_1 admissibility shifts the burden of many proofs from the dynamic approach to that of fine structure. An example is the proof of: if A is Σ_1^β and $A \subseteq \delta < \beta^*$, then A is β-finite (cf. Exercise VII.2.10). Let $\hat{\beta}$ be the least γ such that there exists a one-one, β-recursive map of γ onto β. $\hat{\beta} < \beta$ iff β is not Σ_1 admissible. S. Friedman observed that $\hat{\beta} = \max(\beta^*, \sigma 1\mathrm{cf}^\beta(\beta))$. If β is not Σ_1 admissible, then there is a greatest β-cardinal; also $\beta^* < \beta$.

An extremely useful distinction made by Maass is: call β weakly admissible if $\sigma 1\mathrm{cf}^\beta(\beta) \geq \beta^*$; otherwise call β strongly inadmissible. It turns out that some, but not all, of the ideas and results of α-recursion theory carry over to β when β is weakly admissible. The truth of this was evident in the proof of Shore's density theorem, which exploited the weak admissibility of $\langle L(\alpha), A\rangle$, a consequence of the α-recursive enumerability and incompleteness of A. If β is weakly admissible, then there is a one-one, β-recursive correspondence between $\sigma 1\mathrm{cf}^\beta(\beta)$ and β, and β^* behaves in a familiar manner. It is then not surprising that the solution to Post's problem comes over from α-recursion theory. There exist β-recursively enumerable sets B and C such that $B \not\leq_{w\beta} C$ and $C \not\leq_{w\beta} B$. On the other hand the regular sets theorem can fail. Maass 1977b gives a complete description of those β-recursively enumerable sets that have the same degree as some regular β-recursively enumerable set when β is weakly admissible.

A stronger assumption than weak admissibility is: $\langle L(\beta), A\rangle$ is weakly admissible for every regular β-recursive A. Another way to put it is: $\sigma 1\mathrm{cf}^\beta(\beta) \geq \beta^*$ and $\sigma 2\mathrm{cf}^\beta(\beta) \geq \sigma 2p^\beta(\beta)$, the Σ_2^β projectum of β. If β satisfies the stronger assumption, then the regular sets theorem holds (Maass 1977b) and the density theorem (for β-recursively enumerable sets) holds (Homer & Sacks 1983). It is not known if density holds for every weakly admissible β. Some complex partial results have been obtained by Bailey 1984.

The central problem of the subject is Post's. A strong solution to Post's problem consists of two β-recursively enumerable sets such that neither is weakly β-recursive in the other. S. Friedman 1979 has shown: if β^* is regular (in the sense that cardinals are regular) with respect to all Σ_1^β functions, then β has a strong solution to Post's problem. His argument is based on an effective version of Jensen's diamond principle. He has also found a β such that β does not have a strong solution to Post's problem. It is still possible that every β has a weak *solution, a pair of β-recursively enumerable sets such that neither is β-recursive in* the other.

Maass 1977 points out that the methods of β-recursion theory are applicable to α-recursion theory. An excellent example is his proof that: α is Σ_2 admissible iff every Σ_2^α set, in which ϕ' is α-recursive, is of the same α-degree as the α-jump of some incomplete α-recursively enumerable set.

6.1–6.2 Exercises

6.1. Suppose $\lambda \leq \beta^*$ and λ is a successor β-cardinal. Show λ is regular with respect to all Σ_1^β functions.

6.2. Suppose $\sigma 1 \mathrm{cf}^\beta(\beta) \leq \beta^*$. Show there exists a one-one, α-recursive map of β^* onto β.