Chapter VI
Hyperregularity and Priority

The notion of hyperregularity is introduced and studied with the help of priority arguments. Two solutions of Post's problem are obtained, as well as Simpson's dichotomy, for the metarecursively enumerable degrees.

1. Hyperregular Sets

Let $A \subseteq \omega_1^{\mathrm{CK}}$. A is said to be hyperregular if for every $f \leq_w A$ and γ, $f[\gamma]$ is bounded. The concept of hyperregularity, suitably generalized, figures prominently in the solutions of Post's problem offered in Part C. Its relation to Σ_1 admissibility is made clear in Chapter VII. (For those who cannot wait: Assume A is regular. Then A is hyperregular iff $\langle L(\omega_1^{\mathrm{CK}}), A \rangle$ is Σ_1 admissible.)

$$f(m) = \begin{cases} |m| & m \in O \\ 1 & m \notin O, \end{cases}$$

is weakly metarecursive in O and maps ω onto ω_1^{CK}. The main result of this section is the existence of a metarecursively enumerable set that is hyperregular but not metarecursive.

In general a hyperregular set need not be regular, but the next lemma says otherwise for the case of maximum interest.

1.1 Lemma. *If A is metarecursively enumerable and hyperregular, then A is regular.*

Proof. Suppose $A \cap \gamma$ is not metafinite. Let g metarecursively enumerate $A \cap \gamma$ without repetitions. Define

$$f(\delta) = \begin{cases} 0 & \text{if} \quad \delta \geq \gamma \quad \text{or} \quad \delta \notin A, \\ \sigma & \text{if} \quad \delta < \gamma \quad \text{and} \quad g(\sigma) = \delta. \end{cases}$$

Then $f \leq_w A$ and $f[\gamma] = \omega_1^{\mathrm{CK}}$. $\square$

If a set is both regular and hyperregular, then it behaves as tamely as possible with respect to reducibility, as is expressed by the next lemma, and by Exercise 1.7.

1.2 Lemma. *Suppose A is regular and hyperregular.*

(i) *If $f \leq_w A$ and $\gamma < \omega_1^{CK}$, then $f \upharpoonright \gamma$ is metafinite.*
(ii) *If $f \leq_w A$, then $f \leq_M A$.*
(iii) *If $B \leq_w A$, then B is regular and hyperregular.*

Proof
  (i) Let ϕ be metarecursive and such that

$$f(\beta) = \delta \leftrightarrow (E\rho)(EH)(EJ)[\phi(\rho,\beta) = \langle H, J, \delta \rangle \ \& \ H \subseteq A \ \& \ J \subseteq cA].$$

Let $g(\beta)$ be

(1) $\mu\rho(EH)(EJ)(E\rho)[\phi(\rho,\beta) = \langle H, J, \delta \rangle \ \& \ H \subseteq A \ \& \ J \subseteq cA].$

$g(\beta)$ is the "least computation" of $f(\beta)$ from A via ϕ. To see that $g \leq_w A$, let $\phi(\rho, \beta)$ be $\langle H_\beta^\rho, J_\beta^\rho, \delta_\beta^\rho \rangle$ and define

$$t(\beta) = \sup{}^+ \{H_\beta^\rho \cup J_\beta^\rho \,|\rho \leq g(\beta)\}.$$

($\sup^+$ is the strict least upper bound.) Since A is regular, $A \cap t(\beta)$ is metafinite. The value of $g(\beta)$ can be obtained by substituting $A \cap t(\beta)$ for A in (1). Thus the value of $g(\beta)$ is determined by a metafinite set of facts about A, and so $g \leq_w A$.
   It follows from the hyperregularity of A that $g[\gamma]$ is bounded, and consequently that $t[\gamma]$ is bounded by some $s(\gamma)$. The values of $f(\beta)$ ($\beta < \gamma$) are computable from the metafinite set $A \cap s(\gamma)$ via ϕ, hence $f \upharpoonright \gamma$ is metafinite.
   (ii) The proof of (i) showed that $f \upharpoonright \gamma$, regarded as a function of γ, is weakly metarecursive in A. It follows that $f \leq_M A$.
   (iii) If $h \leq_w B$ and $B \leq_w A$, then $h \leq_w A$ by (ii). So B is hyperregular because A is. Fix γ and define

$$k(\delta) = \begin{cases} \delta + 1 & \text{if} \quad \delta \in B \cap \gamma \\ 0 & \text{otherwise.} \end{cases}$$

By (i) $k[\gamma]$ is metafinite, hence $B \cap \gamma$ is.   □

   The next result is needed for Simpson's dichotomy (Section 3), and is proved by a *recursive approximation* technique.

1.3 Lemma (Simpson 1971). *If A is metarecursively enumerable but not hyperregular, then some non-hyperarithmetic Π_1^1 C is weakly metarecursive in A.*

Proof. By Owings' ω-sets theorem (4.6) it suffices to find a metarecursively enumerable ω-set $D \leq_w A$. Since A is not hyperregular, there exists an $f: \omega \to \omega_1^{CK}$ such that $f \leq_w A$, f is strictly increasing, and $f[\omega]$ is unbounded. Let D be the complement of the range of f. The fact that f is strictly increasing on ω implies

$D \leq_w A$. A metarecursive approximation of f is needed to show D is metarecursively enumerable.

Let ϕ be metarecursive and such that

$$f(n) = \delta \leftrightarrow (\mathrm{EH})(\mathrm{EJ})(E\sigma)[\phi(n,\sigma) = \langle H, J, \delta \rangle \quad \& \quad H \subseteq A \quad \& \quad J \subseteq cA]$$

for all $n < \omega$. f is approximated by guessing at A. Suppose g is metarecursive and $g[\omega_1^{\mathrm{CK}}] = A$. Let

$$A^\delta = \{g(\beta) | \beta < \delta\}.$$

Thus A is the union of a nondecreasing, metarecursive sequence of metafinite sets. Define

$$f(n,\tau) = \delta \leftrightarrow (\mathrm{EH})(\mathrm{EJ})(E\sigma)_{\sigma < \tau}[\phi(n,\sigma) = \langle H, J, \delta \rangle \quad \& \quad H \subseteq A^\tau \quad \& \quad J \subseteq cA^\tau].$$

As defined above, $\lambda n \tau | f(n,\tau)$ is a many-valued, partial metarecursive function. Make it total on $\omega \times \omega_1^{\mathrm{CK}}$ by setting $f(n,\tau) = 0$ when no $\sigma < \tau$ satisfies the defining condition. Make it single-valued by insisting on the least such σ when there is one. By the regular sets theorem (4.3) A may be taken to be regular. It follows that

$$(\gamma)(E\sigma)(\tau)_{\tau \geq \sigma}[A^\tau \cap \gamma = A \cap \gamma].$$

Consequently

(1) $$(n)(E\sigma)(\tau)_{\tau \geq \sigma}[f(n,\tau) = f(n)].$$

In short $\lim_{\tau} f(n,\tau) = f(n)$. The function $f(n,\tau)$ is said to metarecursively approximate the function $f(n)$.

For the moment assume (2):

(2a) $\qquad\qquad$ for each n, $\lambda\tau | f(n,\tau)$ is nondecreasing;

(2b) $\qquad (n)(\sigma)(\tau)[\sigma > \tau \quad \text{and} \quad f(n,\sigma) > f(n,\tau) \rightarrow f(n,\sigma) \geq \sup_{m < \omega} f(m,\tau)].$

If (3) holds, then D is metarecursively enumerable.

(3) $\qquad \delta \in D \leftrightarrow (E\tau)(\mathrm{Em})[f(m,\tau) > \delta \quad \text{and} \quad (n)_{n < m}(\delta \neq f(n,\tau))]$

To verify (3) first assume $\delta \in D$. Thus $\delta \neq f(n)$ for any n. Hence there exist m and τ such that

$$f(m) > \delta \quad \text{and} \quad (n)_{n \leq m}(f(n) = f(n,\tau)).$$

It follows from (2a) that m and τ satisfy the right side of (3). Now fix n and suppose $f(n)$ substituted for δ satisfies the right side of (3). Thus there exist τ and m such that

$$f(m,\tau) > f(n) \quad \text{and} \quad (p)_{p < m}(f(n) \neq f(p,\tau)).$$

Since f is strictly increasing, it follows from (2a) that $m > n$. Hence

$$f(m, \tau) > f(n) \neq f(n, \tau).$$

But then (2) implies $f(n, \sigma) \geq f(m, \tau)$ for some $\sigma > \gamma$; hence $f(n, \sigma) > f(n)$, an impossibility according to (1) and (2a).

For the sake of (2) define

$$h(n, \tau) = \begin{cases} \sup\limits_{\delta < \tau} h(n, \delta) & \text{if} \quad f(n, \tau) \leq \sup\limits_{\delta < \tau} h(n, \delta), \\[2ex] f(n, \tau) \cup \sup\limits_{\substack{\delta < \tau \\ m < \omega}} h(m, \delta) & \text{otherwise.} \end{cases}$$

Let $h(n) = \lim\limits_{\tau} h(n, \tau)$. $h(n, \tau)$ obeys (2). $h \leqslant_w A$ and $h([\omega]$ is unbounded. $\quad\square$

1.4–1.10 Exercises

1.4. Suppose every set weakly metarecursive in A is regular. Show A is hyperregular.

1.5. Find a set that is hyperregular but not regular. Find such a set metarecursive in O.

1.6. (Macintyre 1969). Find a set that does not have the same metadegree as any regular set. Find such a set metarecursive in O.

1.7. Recall the definitions of $C(E,B)$ and $C_m(E,B)$ given in Exercise 3.8 of Chapter V. B is said to be *subgeneric* if

$$C(\omega_1^{CK} + 1, E, B) = C(\omega_1^{CK}, E, B).$$

The computations from B via E of nonrecursive ordinal height add nothing to $C(E, B)$. (In Chapter VII it will be seen that B is subgeneric iff $\langle L[B, \omega_1^{CK}], B \rangle$ is Σ_1 admissible.) Show every subgeneric set is hyperregular.

1.8. Find a subgeneric set that is not regular.

1.9. Assume B is regular and hyperregular. Show B is subgeneric. Show $C(E,B) = C_m(E,B)$.

1.10. Find a hyperregular set that is not subgeneric.

2. Two Priority Arguments

The main result of this section is the existence of a hyperregular, metarecursively enumerable set A that is not metarecursive. It follows from Lemma 2.1 that A is not metacomplete and hence constitutes a positive solution to Post's problem for the

metarecursively enumerable degrees. The construction of A is a priority argument in the sense originated by Friedberg 1957b and Muchnik 1956. It suggests that the generalization of finite to metafinite, and r.e. to meta r.e., is solid enough to support the combinatorics associated with nontrivial dynamic arguments of classical recursion theory.

A secondary result of this section is the construction by means of a priority argument of a non-hyperarithmetic Π_1^1 set B of lower metadegree than Kleene's O. The construction of B is intended to suggest that metarecursion theory can be applied to prove theorems about Π_1^1 sets analogous to classical results about recursively enumerable sets. One such is the existence of a maximal Π_1^1 set (Exercise 2.9). It has the virtue of being expressible without any reference to notions of metarecursion theory.

2.1 Theorem (Sacks 1966). *There exists a hyperregular, metarecursively enumerable set that is not metarecursive.*

Proof. Recall the partial metarecursive function $\phi_1(e, x)$ of Theorem 1.4.V. Let R_e be the range of $\lambda x |\phi_1(e, x)$. Then $\{R_e | e < \omega\}$ is an enumeration of the metarecursively enumerable sets. Recall that

(1) $$\phi_1(e, x) \simeq y \leftrightarrow (f)(Eu) \sim T(\bar{f}(u), g(e), n(x), n(y)).$$

Since the right side of (1) is Π_1^1, it is equivalent to

$$h(g(e), n(x), n(y)) \in O$$

for some recursive function h. Define

$$y \in R_e^\sigma \quad \text{by} \quad (Ex)_{x < \sigma}[|h(g(e), n(x), n(y))| < \sigma].$$

Then R_e^σ is metafinite, $\lambda \sigma e | R_e^\sigma$ is metarecursive, and $R_e = \bigcup_\sigma R_e^\sigma$. $\lambda \sigma e | R_e^\sigma$ provides a simultaneous metarecursive enumeration of the metarecursively enumerable sets.

Say $\{e\}^A(\gamma)$ is defined and equal to δ iff

(2) $$(EH)(EJ)[\langle H, J, \gamma, \delta \rangle \in R_e \; \& \; H \subseteq A \; \& \; J \subseteq cA.]$$

$\{e\}^A$ is a many-valued, partial function. If $\{e\}^A$ is single-valued and total, then $\{e\}^A \leq_w A$. Conversely, if $f \leq_w A$, then $f = \{e\}^A$ for some $e < \omega$.

The theorem is proved by metarecursively enumerating a set A with three objectives in mind.

(3) cA is unbounded.

(4)   If R_e is unbounded, then $R_e \cap A \neq \varnothing$

(5)   If $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$, then $\lambda n |\{e\}^A(n)$ is metafinite.

(3) and (4) combine to show cA is not metarecursively enumerable. Hence A is not metarecursive, as in Post's simple set construction in classical recursion theory. (5) implies A is hyperregular, since any infinite recursive ordinal can be put into a matafinite, one-to-one correspondence with ω.

A will be the union of a non-decreasing, metarecursive sequence A^σ $(\sigma < \omega_1^{CK})$ of metafinite sets. Let $A^{<\sigma}$ denote $\cup\{A^\delta | \delta < \sigma\}$. Say $\{e\}_\sigma^{A^{<\sigma}}(\gamma)$ is defined and equal to δ iff

$$(6) \qquad (EH)(EJ)[\langle H, J, \gamma, \delta\rangle \in R_e^\sigma \quad \& \quad H \subseteq A^\sigma \quad \& \quad J \subseteq cA^{<\sigma}].$$

$\{e\}^{A^{<\sigma}}(\gamma)$ is a many-valued, partial metarecursive function of e, σ and γ.

If $H \subseteq A$, then $H \subseteq A^\sigma$ for all sufficiently large σ. Hence any $\langle H, J, \gamma, \delta\rangle$ that satisfies the matrix of (2) also eventually satisfies the matrix of (6). Thus $\{e\}^A(\gamma) = \delta$ implies $\{e\}_\sigma^{A^{<\sigma}}(\gamma) = \delta$ for all sufficiently large σ.

For the sake of objectives (4) and (5) attempts are made during the enumeration of A to satisfy the following *requirements*.

Req $2e$: If R_e is unbounded, then there is a σ such that $R_e \cap A^\sigma \neq \varnothing$.

Req $2e + 1$: If $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$, then

$$(7) \qquad \lambda n | \{e\}^A(n) = \lambda n | \{e\}^{A^{<\sigma}}(n)$$

for all sufficiently large σ.

An even-numbered requirement is said to be positive, because it is met by adding an element to A. An odd-numbered requirement is said to be negative, because it is met by keeping every member of some metafinite set out of A. To elaborate, suppose a σ can be found such that the right side of (7) is defined for all $n < \omega$. Then req $2e + 1$ can be met by not adding any element of $cA^{<\sigma}$ to A at stage τ for any $\tau \geq \sigma$. This strategy, if successful, preserves forever after computations established at the beginning of stage σ. Consequently an odd-numbered requirement is sometimes called a preservation requirement.

Positive and negative requirements have a potential for conflict. Suppose δ is added to A at stage τ to meet requirement $2e_0$. Suppose further that $\delta \notin A^{<\sigma}$ is a negative fact needed for some computation developed at stage $\sigma < \tau$ and being preserved in the hope of meeting req $2e + 1$. Such an event is termed an *injury* to req $2e + 1$ for the sake of req $2e_0$. The frequency of injury is minimized by Friedberg–Muchnik priorities. Req $2e_0$ is said to be of higher *priority* than req $2e + 1$ if $e_0 \leq e$. An injury to one requirement for the sake of another is allowed to happen only if the injured requirement has lower priority than the other. Thus adding an element to A for the sake of req $2e_0$ never injures req $2e + 1$ when $e < e_0$.

A recursion on σ simultaneously defines metarecursive functions $\lambda\sigma | A^\sigma$, $\lambda\sigma e | m(\sigma, e)$ and $\lambda\sigma e | r(\sigma, e)$. m guides the preservation requirements, and r insures that cA is unbounded.

Stage σ. If $\{e\}_\sigma^{A^{<\sigma}}(n)$ is defined, let $\langle H_{e,n}^\sigma, J_{e,n}^\sigma\rangle$ be the least $\langle H, J\rangle$ that satisfies

$$(E\delta)[\langle H, J, n, \delta\rangle \in R_e^\sigma \quad \& \quad H \subseteq A^{<\sigma} \quad \& \quad J \subseteq cA^{<\sigma}].$$

("least $\langle H, J \rangle$" is given a precise meaning by referring to some metarecursive one-to-one correspondence between ω_1^{CK} and the set of all metafinite $\langle H, J \rangle$'s.) Define

$$n_e = \mu m_{m \le \omega}(n)_{n < m}[\{e\}_\sigma^{A < \sigma}(n) \text{ is defined}],$$

$$m(\sigma, e) = \mu \beta[(\delta)_{\delta < \sigma}(\beta \ge m(\delta, e) \quad \& \quad \beta \ge \sup\{J_{e,n}^\sigma | n < n_e\})]$$

(When possible, $\delta \le m(\sigma, e)$ is kept out of $A^\tau(\tau \ge \sigma)$ in order to preserve $\{e\}_\sigma^{A < \sigma}(n)$ for all $n < n_e$.)

Let $p: \omega_1^{CK} \to \omega$ be a one-one, into, metarecursive function. Define

$$r(\sigma, e) = \mu \beta[\beta \notin A^{< \sigma} \quad \& \quad \beta \ge \sup_{\delta < \sigma} r(\delta, e) \quad \& \quad (e \in p[\sigma] \to \beta \ge p^{-1}(e))].$$

Stage σ is completed by attending to req $2e_0$, where $e_0 = h(\sigma)$ and h is a metarecursive function that enumerates every finite ordinal unboundedly often. If

$$(8) \qquad\qquad\qquad R_{e_0}^\sigma \cap A^{< \sigma} = \varnothing,$$

and there is a β such that

$$(9) \qquad\qquad \beta \in R_{e_0}^\sigma \quad \& \quad (e)_{e < e_0}[m(\sigma, e) < \beta \quad \& \quad r(\sigma, e) < \beta],$$

then add the least such β to A. Otherwise $A^\sigma = A^{< \sigma}$.

Condition (8) is all important. It implies:

(10)   For each e_0, there is at most one stage σ at which an ordinal is added to A for the sake of req $2e_0$.

If β is added to A at stage σ and $\beta \le m(\sigma, e)$, then that act is said to *injure* req $2e + 1$. (9) and (10) imply:

(11)   For each e the set of stages at which req $2e + 1$ is injured has cardinality at most $e + 1$.

To prove (5) fix e and assume $\{e\}^A(n)$ is defined and single-valued for all $n < \omega$. It follows that for each $m < \omega$, and all sufficiently large σ,

$$(12) \qquad\qquad (n)_{n < m}[\{e\}_\sigma^{A < \sigma}(n) \text{ is defined}].$$

By (11) there is a stage σ_e after which req $2e + 1$ is never injured. For each $m < \omega$, let $\sigma(m)$ be the least $\sigma > \sigma_e$ that satisfies (12). The "least" computations that make (12) true when $\sigma = \sigma(m)$ are preserved forever after. $\lambda m | \sigma(m)$ is metarecursive, hence $\{\sigma(m) | m < \omega\}$ is bounded above by some τ. Consequently,

$$(n)_{n < \omega}[\{e\}_\tau^{A < \tau}(n) = \{e\}^A(n)],$$

and (5) follows.

To prove (3), fix e and consider the behavior of $r(\sigma, e)$ as σ increases. $r(\sigma, e)$ is non-decreasing. It increases at at most $e + 2$ stages: once if $e \in p[\omega_1^{CK}]$, and at most once for the sake of req $2e_0$ for each $e_0 \leq e$. Let

$$(13) \qquad\qquad r(e) = \lim_\sigma r(\sigma, e).$$

The set $\{r(e) | e < \omega\}$ is unbounded, since $r(p(\gamma)) \geq \gamma$ for all $\gamma < \omega_1^{CK}$. For all σ and e, $r(\sigma, e) \notin A^{<\sigma}$. Hence for all e, $r(e) \notin A$.

To prove (4), fix e_0 and assume R_{e_0} is unbounded. To find a β that satisfies (9), it suffices by (13) to show

$$(14) \qquad\qquad \lim_\sigma m(\sigma, e) \text{ exists}$$

for each $e < e_0$. Fix e. Let d_e be the strict upper bound of all m such that $\sigma(m)$, as defined above in the proof of (5), exists. Thus $d_e \leq \omega$. The set $\{\sigma(m) | m < d_e\}$ is either finite, or metafinite as in the proof of (5), hence bounded above by some τ. It follows that $m(\sigma, e) = m(\tau, e)$ for all $\sigma \geq \tau$. $\square$

2.2 The Projectum. Recall the function $p: \omega_1^{CK} \to \omega$ from the proof of Theorem 2.1. It is metarecursive, one-one and into. It injects ω_1^{CK} into ω in a manner effective enough to allow requirements to be indexed by finite ordinals rather than recursive ordinals. It is conventional to sum up the situation by calling ω the projectum of ω_1^{CK}. The role of the projectum in the proof of Theorem 2.2 is seen by considering I_e, the set of stages at which req $2e + 1$ is injured. The construction of A is such that I_e is metarecursively enumerable (uniformly in e). To make the proof of 2.2 work it was necessary to bound I_e. The bound followed from the finiteness of I_e, which in turn followed from the finiteness of e. It would have been enough for I_e to have been metafinite. If requirements had been indexed by recursive ordinals, then I_δ, when $\delta \geq \omega$, would have had little chance of being finite or even metafinite, although it would have still been metarecursively enumerable.

The notion of projectum is central in the priority arguments of Part C, where ω_1^{CK} is replaced by an arbitrary Σ_1 admissible ordinal.

2.3 Corollary. *There exists a metarecursively enumerable set that is neither meta-recursive nor metacomplete.*

Proof. Let A be an instance of Theorem 2.1. A is not metacomplete by Proposition 4.5.V and Lemma 1.2(iii). $\square$

The proof of Corollary 2.3 provides a hyperregular solution to Post's problem for the metarecursively enumerable degrees. The next theorem supplies a Π_1^1 solution B. The construction of B draws on priorities as in the proof of Theorem 2.2, but there is an additional difficulty. Since $B \subseteq \omega$, a metafinite computation of B can involve unboundedly much of B. As a result the positive and negative requirements can crowd each other more severely than they did above.

Theorem 2.4 should be compared with Spector's Theorem (7.2.II): Kleene's O is hyperarithmetic in every non-hyperarithmetic Π_1^1 set.

2.4 Theorem (Sacks 1966). *There exists a non-hyperarithmetic Π_1^1 set B such that Kleene's O is not weakly metarecursive in B.*

Proof. Two Π_1^1 sets, B and C, are simultaneously metarecursively enumerated with the following objectives in mind.

(1) $\omega - B$ is infinite.

(2)   If Q is Π_1^1 and infinite, then $Q \cap B \neq \varnothing$.

(3) $C \not\leq_w B$.

(1) and (2) imply B is not hyperarithmetic. It follows from (3) that $O \not\leq_w B$, since C is many-one reducible to O via a recursive function.

Recall R_e and R_e^σ from the beginning of the proof of Theorem 2.1. Let $Q_e = R_e \cap \omega$ and $Q_e^\sigma = R_e^\sigma \cap \omega$. Then $\{Q_e | e < \omega\}$ is an enumeration of the Π_1^1 sets; $\lambda \sigma e | Q_e^\delta$ is metarecursive; and $Q_e = \bigcup_\sigma Q_e^\sigma$. Define $\{e\}^B(n)$ as in Theorem 2.1, formula (2), with A replaced by B and γ by n. B will be the union of a non-decreasing, metarecursive sequence $B^\sigma (\sigma < \omega_1^{CK})$ of metafinite subsets of ω. Define $\{e\}_\sigma^{B^{<\sigma}}(n)$ as in Theorem 2.1, formula (6), with A replaced by B and γ by n.

Objectives (2) and (3) lead to the following requirements.

Req 2e: If Q_e is infinite, then $Q_e^\sigma \cap B^\sigma \neq \phi$ for some σ.

Req 2e + 1: There exists an n such that if $\{e\}^B(n)$ is defined and singlevalued, then $C(n) \neq \{e\}^B(n)$. ($C(n)$ is the characteristic function of C.) The construction is a recursion on σ that simultaneously defines metarecursive functions $\lambda \sigma | B^\sigma$, $\lambda \sigma | C^\sigma$, $\lambda \sigma e | r(\sigma, e)$, $\lambda \sigma e | n(\sigma, e)$ and $\lambda \sigma e | J(\sigma, e)$. For each e, $r(\sigma, e)$ converges to the e-th member of $\omega - B$, and $n(\sigma, e)$ converges to an n that satisfies req 2e + 1. $J(\sigma, e)$ is a finite subset of $cB^{<\sigma}$ associated with the preservation of $\{e\}^{B^{<\sigma}}(n(\sigma, e))$.

Let $\lambda mn | z(m, n)$ be a one-one recursive function that maps ω^2 onto ω.

Stage $\sigma = 0$. Set $B^0 = C^0 = \varnothing$, $J(0, e) = \varnothing$, $r(0, e) = e$ and $n(0, e) = z(0, e)$.

Stage $\sigma > 0$. (For any $\lambda \sigma x | f(\sigma, x)$, define

$$f(<\sigma, x) = \sup\{f(\delta, x) | \delta < \sigma\}.)$$

Assume

(4) $\omega - B^{<\sigma}$ is infinite & $r(<\sigma, e) < \omega$ & $n(<\sigma, e) < \omega$

for all e. (Assumption (4) will be proved by induction on $\delta \leq \sigma$ in a moment.) Define by induction on e

(5) $r(\sigma, e) = \mu m [m \in (\omega - B^{<\sigma})$ & $m \geq r(<\sigma, e)$

$$\& \quad (i)_{i < e}(m > r(\sigma, i))].$$

Let h be a metarecursive function that enumerates all finite ordinals unboundedly often. Stage σ splits into two cases corresponding to even and odd numbered requirements.

Case 0: $h(\sigma) = 2e$. If $Q_e^\sigma \cap B^{<\sigma} = \varnothing$ and there is a p such that

$$(6) \qquad p \in Q_e^\sigma \quad \& \quad (i)_{i<e}[p \notin J(<\sigma, i) \quad \& \quad p > r(\sigma, i)],$$

then add the least such p to $B^{<\sigma}$ to obtain B^σ, and define

$$(7) \qquad n(\sigma, i) = n(<\sigma, i) \text{ for all } i < e;$$

$$n(\sigma, i) = ux[x > n(<\sigma, i) \quad \& \quad (\text{Em})(x = z(i, m))]$$

for all $i \geq e$.

Otherwise $B^\sigma = B^{<\sigma}$ and $n(\sigma, i) = n(<\sigma, i)$ for all i.

Case 1: $h(\sigma) = 2e + 1$. Let $B^\sigma = B^{<\sigma}$ and $n(\sigma, i) = n(<\sigma, i)$ for all i. Define

$$(8) \qquad \{e\}_{\sigma, f}^{B^{<\sigma}}(n)$$

as in the proof of Theorem 2.1, formula (6), with A replaced by B, γ by n, and $cA^{<\sigma}$ by $\omega - B^{<\sigma}$, and with the restriction that J be finite. The restriction on J is important because a typical metafinite computation from $B^{<\sigma}$ might mention all of ω and prevent any p from satisfying (6). Of course there is now the fear that (8) is not an adequate approximation of $\{e\}^B(n)$. The fear will become groundless, after it is shown that B satisfies the even-numbered requirements. If

$$(9) \qquad n(\sigma, e) \notin C^{<\sigma} \quad \& \quad \{e\}_{\sigma, f}^{B^{<\sigma}}(n(\sigma, e)) = 0,$$

then add $n(\sigma, e)$ to $C^{<\sigma}$ to obtain C^σ, and let $J(\sigma, e)$ be the "least" finite subset of $\omega - B^{<\sigma}$ that includes $J(<\sigma, e)$ and is needed for the computation of the second half of (9).

If (9) is false, then $C^\sigma = C^{<\sigma}$ and $J(\sigma, e) = J(<\sigma, e)$. End of construction. In both cases it is understood that no function changes unless otherwise indicated. For example, in case (1), $r(\sigma, e) = r(<\sigma, e)$.

The first order of business is to prove (4). Assume $\omega - B^{<\delta}$ is infinite $\& r(<\delta, e) < \omega \& n(<\delta, e) < \omega$ for all $\delta < \sigma$. Fix e and trace the behavior of $r(\delta, e)$ as δ increases from 0 to σ. Each change corresponds to the addition of an element to B for the sake of req. $2e_0$ for some $e_0 \leq e$. At most one such addition occurs on behalf of req. $2e_0$ thanks to the clause, "$Q_{e_0}^\delta \cap B^{<\delta} = \phi$", which precedes any such addition at stage δ. Thus $r(\delta, e)$ charges at most $e + 1$ times, and so $r(<\sigma, e) < \omega$. Then $\omega - B^{<\sigma}$ is infinite, since the range of $\lambda e | r(<\sigma, e)$ is infinite and is contained in $\omega - B^{<\sigma}$. The behavior of $n(\delta, e)$ as δ increases from 0 to σ is similar to that of $r(\delta, e)$, so $n(<\sigma, e) < \omega$.

The above reasoning also shows $r(\sigma, e)$ changes only finitely often as σ approaches ω_1^{CK}. Let

$$r(e) = \lim_\sigma r(\sigma, e)$$

Then $\omega - B$ is infinite, since the range of $\lambda e|r(e)$ is infinite and is contained in $\omega - B$.

$J(\sigma, e)$ changes only finitely often as σ approaches ω_1^{CK}, since each such change is occasioned by an attempt to satisfy req $2e_0$ for some $e_0 \leq e$. Since each $J(\sigma, e)$ is finite, $\lim_\sigma J(\sigma, e)$ must be finite, and so there is no substantial obstacle to satisfying an even-numbered requirement. Consequently $\omega - B$ contains no infinite metafinite set, and so (8) is an excellent approximation of $\{e\}^B(n)$.

Fix e to show req $2e + 1$ is met. Let n be $\lim_\sigma n(\sigma, e)$, and let σ_0 be the stage at which the final value of $n(\sigma, e)$ is attained. If (9) holds for some $\sigma \geq \sigma_0$ such that $h(\sigma) = 2e + 1$, then $C(n) = 1$. Also
$$J(\sigma, e) \subseteq \omega - B,$$

because otherwise (7) would force an increase in $\lambda\sigma|n(\sigma, e)$ at some stage after stage σ. Hence
$$\{e\}^{B <\sigma}_{\sigma, f}(n) = \{e\}^B(n) = 0.$$

Suppose (9) fails for all $\sigma \geq \sigma_0$. Then $n \notin C$, because $n(\sigma_0, e) \notin C^{<\sigma_0}$, and because the one-one-ness of $\lambda jm|z(j, m)$ prevents n from being put in C for the sake of req $2j + 1$ when $j \neq e$. It follows that $\{e\}^B(n)$ cannot equal 0, since otherwise (9) would hold for all sufficiently large σ.   □

The next result, promised in subsection 3.1.V., is the failure of $\leq_w$ to be transitive for Π_1^1 sets. Note that Lemmas 1.1 and 1.2 imply $\leq_w$ is transitive on hyperregular, metarecursively enumerable sets.

2.5 Corollary (Driscoll 1968). *There exist Π_1^1 sets A, B and C such that $A \leq_M B$ and $B \leq_w C$, but $A \not\leq_w C$.*

Proof. By Theorem 2.4 there are Π_1^1 sets A and C such that $A \not\leq_w C$ and C is not hyperarithmetic. Let g metarecursively enumerate C without repetitions, and let Dg be the deficiency set of g. Dg is an ω-set, as defined in subsection 4.4.V. By Theorem 4.6.V there is a metacomplete Π_1^1 set B such that $B \leq_f D_g$. Hence $A \leq_M B$. According to Lemma 4.2.(v).V, $D_g \leq_M C$, so $B \leq_w C$.   □

2.6 Further Results. The priority arguments of Theorems 2.3 and 2.4 can be easily modified to obtain a familiar form of the solution to Post's problem, the existence of two metarecursively enumerable sets (and two Π_1^1 sets) such that neither is weakly metarecursive in the other. The e-th negative requirement is injured at most 2^e times, rather than $e + 1$ times as in 2.3.

Call a Π_1^1 set B *maximal* if $\omega - B$ is infinite and every Π_1^1 superset of B differs finitely from either B or ω. The Friedberg maximal set construction of classical recursion theory lifts to the Π_1^1 case via metarecursion theory. As in the proof of Theorem 2.4 requirements are indexed by finite ordinals. Unlike the classical case B

will be infinite at intermediate stages of the construction, and so some care is needed to keep $\omega - B$ infinite.

Driscoll 1968 showed that the metarecursively enumerable degrees are dense. His argument is modeled on the classical case, but he makes several departures. Suppose B and C are metarecursively enumerable, and $B <_M C$. By Theorem 4.3.V it is safe to assume B and C are regular. Without regularity it seems difficult to begin looking for D such that $B <_M D <_M C$. If B is hyperregular, then Driscoll's argument is close to the classical case. If B is not hyperregular, then he observes there is an $f \leq_w B$ such that f is a one-one map of ω onto ω_1^{CK}. Then f is used to make ω, rather than ω_1^{CK}, the domain of preservation requirements associated with D. The regularity of B is used to develop a suitable metarecursive approximation of f. Driscoll's theorem is a special case of Shore's density theorem proved at the end of Part C.

2.7–2.12 Exercises

2.7. Find two hyperregular, metarecursively enumerable sets such that neither is weakly metarecursive in the other.

2.8. Find two Π_1^1 sets such that neither is weakly metarecursive in the other.

2.9. Find a maximal Π_1^1 set. (A Π_1^1 set A is *maximal* iff $\omega - A$ is infinite and for all Π_1^1 B, if $A \subseteq B \subseteq \omega$, then $B - A$ or $\omega - B$ is finite.)

2.10. For each $B \subseteq \omega_1^{CK}$ define $C^+(E, B)$ as $C(E, B)$ was in Exercise 3.8.V. save that computations of arbitrary height are allowed. Kreisel calls A *computable from B* (in symbols $A \leq_c B$) if there is an E with principal function letter f such that

$$\delta \in A \leftrightarrow (f(\underline{\delta}) = \underline{0}) \in C^+(E, B)$$
$$\delta \notin A \leftrightarrow (f(\underline{\delta}) = \underline{1}) \in C^+(E, B)$$

for all $\delta < \omega_1^{CK}$. Show $\leq_c$ is transitive. Show there are only two c-degrees of Π_1^1 sets.

2.11. Suppose B is regular and hyperregular, and $A \leq_c B$. Show $A \leq_w B$.

2.12. Find two metarecursively enumerable sets such that neither is computable from the other.

3. Simpson's Dichotomy

Let $\mathscr{RE}$ be the set of all metarecursively enumerable degrees and $\mathscr{H}$ the set of all members of $\mathscr{RE}$ with hyperregular representatives. Lemmas 1.1 and 1.2 imply $\mathscr{H}$ is an initial segment of $\mathscr{RE}$. $\mathscr{RE} - \mathscr{H}$ is non-empty, since it contains the meta-degrees

of all non-hyperarithmetic Π_1^1 sets. Simpson's dichotomy says it contains nothing else.

3.1 Proposition. *Suppose B is a non-hyperarithmetic Π_1^1 set. Then there exists a simple Π_1^1 set C such that $B \equiv_f C$.*

Proof. f is a one-one, metarecursive function whose range is B, and $\{H_\sigma | \sigma < \omega_1^{CK}\}$ is a metarecursive enumeration of all infinite, hyperarithmetic sets. C will be the union of a non-decreasing, metarecursive sequence $\{C^\sigma | \sigma < \omega_1^{CK}\}$ of hyperarithmetic sets.

$C^0 = \varnothing$ and $C^\lambda = \cup \{C^\sigma | \sigma < \lambda\}$ for λ a limit. To define $C^{\sigma+1}$, assume $\omega - C^\sigma$ is infinite, and let

$$\{a_0^\sigma < a_1^\sigma < a_2^\sigma < \ldots\} = \omega - C^\sigma,$$

$$b = \mu a[a \in H_\sigma \ \& \ a \geq a_{f(\sigma)}^\sigma],$$

$$C^{\sigma+1} = C^\sigma \cup \{b, a_{f(\sigma)}^\sigma\}.$$

The assumption that $\omega - C^\sigma$ is infinite is proved by induction on σ. Suppose it is true for all $\sigma < \lambda$. Fix m to show $\omega - C^\lambda$ has at least m members. Let τ be such that

$$(\sigma)_{\tau \leq \sigma < \lambda}[f(\sigma) \geq m].$$

Then $(i)_{i<m}(\sigma)_{\tau \leq \sigma < \lambda}[a_i^\tau = a_i^\sigma = a_i^\lambda]$. The argument is valid when $\lambda = \omega_1^{CK}$, hence $\omega - C$ is infinite.

Let $a_i = \lim_\sigma a_i^\sigma$. If C were not simple, then there would be an infinite hyperarithmetic $H \subseteq \omega - C$, and a σ such that $H = H_\sigma$ and $H_\sigma \cap C^{\sigma+1} \neq \varnothing$. Finally

$C \equiv_f B$ because

$$x \notin C \leftrightarrow (E\sigma)[x \notin C^\sigma \ \& \ f[\sigma] \supseteq B \cap (x+1)],$$

$$x \notin B \leftrightarrow (E\sigma)[x \notin f[\sigma] \ \& \ a_x^\sigma = a_x]. \quad \square$$

3.2 Theorem (Simpson 1971). *If A is a non-hyperregular, metarecursively enumerable set, then there exists a Π_1^1 set B of the same metadegree as A.*

Proof. By 1.3 and 3.1 there is a simple Π_1^1 set $C \leq_w A$. Let f metarecursively enumerate C without repetitions. Define $B = f[A]$.

Clearly $A \leq_M B$. To see $B \leq_M A$ consider a typical metafinite $K \subseteq cB$. $f^{-1}[K]$ is a meta r.e. subset of cA. By Exercise 4.8.V, A can be assumed to be simple. Hence $f^{-1}[K]$ is bounded above strictly by some δ, and $K \cap C = K \cap f[\delta]$. Thus $K \cap C$ and $K - C$ are metafinite. Since C is simple $K - C$ is finite. In short, for metafinite K,

$$K \subseteq cB \rightarrow K - C \text{ is finite.}$$

Let F range over all finite subsets of ω_1^{CK}. Then

$$K \subseteq cB \leftrightarrow (EF)(EH)[K = F \cup H \ \& \ F \subseteq cC$$
$$\& \ H \subseteq C \ \& \ f^{-1}[H] \subseteq cA].$$

(H ranges over the metafinite sets.) It follows that $B \leq_M A$, since $C \leq_w A$. $\square$