Chapter IV
Measure and Forcing

The measure-theoretic analysis of Π_1^1 sets begun in Section 6.II is continued. It is shown that every Π_1^1 set of positive measure has a hyperarithmetic member. Forcing over the hyperarithmetic hierarchy is developed in order to construct a minimal hyperdegree and to prove Louveau's separation theorem.

1. Measure-Theoretic Uniformity

The goal of this section is to show $\mathcal{M}(\omega_1^{CK}, T)$ is a model of Δ_1^1 comprehension for almost all T. It follows that $\omega_1^{CK} = \omega_1^T$ for almost all T. Recall the notions of ordinal rank (subsection 4.1.III) and full ordinal rank (subsection 4.4.III) for formulas of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$. Define the number quantifier rank of a formula $\mathcal{F}$ to be the number of occurrences of (Ex), (y), ... in $\mathcal{F}$.

Let $\mathcal{F}$ be a ranked sentence of $\mathcal{L}(\omega_1^{CK}, T)$. By Lemma 4.6.II the set

$$\{T \mid \mathcal{M}(\omega_1^{CK}, T) \models \mathcal{F}\}$$

is Δ_1^1, hence Borel and measurable according to subsection 6.1.II. Denote its measure by $p(\mathcal{F})$, the probability that $\mathcal{F}$ is true. The next task is to show the graph of $p(\mathcal{F})$, as $\mathcal{F}$ ranges over ranked sentences, is Π_1^1.

1.1 Proposition. *Let $\mathcal{F}(X^\alpha)$ be a ranked formula whose only free variable is X^α. Let $\mathcal{G}_i(x)$ $(i \leq n)$ be formulas of rank at most α whose only free variable is x. Then*

$$\bigvee_{i \leq n} \mathcal{F}(\hat{x}\mathcal{G}_i(x))$$

is logically equivalent to a sentence of full ordinal rank less than that of $(EX^\alpha)\mathcal{F}(X^\alpha)$.

Proof. Put $\bigvee_{i \leq n} \mathcal{F}(\hat{x}\mathcal{G}_i(x))$ into prenex normal form without changing the pattern of set quantifiers. Then contract like quantifiers as in the proof of Theorem 1.5 of Chapter I. The result will have at least one less occurrence of (EX^α) than does $(EX^\alpha)\mathcal{F}(X^\alpha)$. □

1.2 Proposition. *Let $\mathscr{F}(x)$ be a ranked formula whose only free variable is x. Then $\bigvee_{i\leq n} \mathscr{F}(\underline{i})$ is logically equivalent to a formula of the same full ordinal rank but lower number quantifier rank.*

1.3 Theorem. *The predicate $p(\mathscr{F})\geq r$, restricted to ranked $\mathscr{F}$ and rational r, is Π_1^1.*

Proof. An application of Theorem 1.6(i).I. similar to that made in the proof of Lemma 4.5.III. There exists a Σ_1^1 formula $A(X)$ such that

(1) $$p(\mathscr{F})\geq r \leftrightarrow (X)[A(X)\rightarrow \langle \mathscr{F}, r\rangle \in X].$$

The only conceptual difficulty arises from the impossibility of computing $p(\mathscr{G} \,\&\, \mathscr{H})$ from $p(\mathscr{G})$, $p(\mathscr{H})$. The difficulty is avoided by resort to prenex normal form. This logical trick is equivalent to construing a Borel set as a monotone union (or intersection) of Borel sets of lower rank.

Consider $(\mathrm{E}X^\alpha)F(X^\alpha)$. Let $\mathscr{G}_i(x)$ $(i<\omega)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x.

$$p((\mathrm{E}X^\alpha)\mathscr{F}(X^\alpha)) = \sup_n p(\bigvee_{i\leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x))).$$

Hence $p((\mathrm{E}X^\alpha)\mathscr{F}(X^\alpha))\geq r$ is equivalent to

$$(\delta)(\mathrm{E}n)[p(\bigvee_{i\leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x)))\geq r-\delta],$$

where δ ranges over the positive rationals. By Proposition 1.2, the needed reduction in full ordinal rank has occurred.

$A(X)$ in full is the conjunction of (2) through (6).

(2) $\mathscr{F}$ is quantifierless & $p(\mathscr{F})\geq r\rightarrow\langle\mathscr{F}, r\rangle\in X$.

(3) $$b\in O \;\&\; (\delta)(\mathrm{E}n)\left[\left\langle \bigvee_{i\leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x)), r-\delta \right\rangle \in X\right] \rightarrow \langle (\mathrm{E}X^{|b|})F(X^{|b|}), r\rangle \in X.$$

(4) $$b\in O \;\&\; (n)\left[\left\langle \bigwedge_{i\leq n} \mathscr{F}(\hat{x}\mathscr{G}_i(x)), r \right\rangle \in X\right] \rightarrow \langle (X^{|b|})\mathscr{F}(X^{|b|}), r\rangle \in X.$$

(5) $$(\delta)(\mathrm{E}n)\left[\left\langle \bigvee_{i\leq n} \mathscr{F}(\underline{i}), r-\delta \right\rangle \in X\right] \rightarrow \langle (\mathrm{E}x)\mathscr{F}(x), r\rangle \in X.$$

(6) $$(n)\left[\left\langle \bigwedge_{i\leq n} \mathscr{F}(\underline{i}), r \right\rangle \in X\right] \rightarrow \langle (x)\mathscr{F}(x), r\rangle \in X.$$

(1) is proved by induction on the full ordinal rank and number quantifier rank of F. □

1.4 Lemma. *Let $\mathscr{F}(x, Y)$ be a formula of $\mathscr{L}(\omega_1^{CK}, \mathscr{T})$ whose only unranked set variable is the free variable Y. Assume $p((x)(EY)\mathscr{F}(x, Y)) \geq r$. Then for some $\alpha < \omega_1^{CK}$, $p((x)EY^\alpha)\mathscr{F}(x, Y) \geq r$.*

Proof. Recall (1) from the proof of Theorem 1.3.I. The assumption yields

$$(1) \qquad (m)(\delta)(\mathrm{E}a)_{a \in O}\left[p\left((\mathrm{E}Y^{|a|})\left(\bigwedge_{n \leq m} \mathscr{F}(\underline{n}, (Y^{|a|})_n) \right) \right) \geq r - \delta \right].$$

According to Theorem 1.3 the matrix of (1) is Π_1^1. It follows from Theorem 2.6.II that a can be regarded as a hyperarithmetic function of m and δ. Spector's boundedness theorem (5.6.I) implies

$$(m)(\delta)[|a(m, \delta)| < \alpha]$$

for some $\alpha < \omega_1^{CK}$. Consequently

$$(m)\left[p\left((\mathrm{E}Y^\alpha) \bigwedge_{n \leq m} \mathscr{F}(\underline{n}, (Y^\alpha)_n) \right) \geq r \right],$$

and so

$$p((x)(\mathrm{E}Y^\alpha)\mathscr{F}(x, Y^\alpha)) \geq r. \qquad \square$$

1.5 Theorem. *If $\mathscr{F}$ is an instance of the Δ_1^1 comprehension theorem, then $p(\mathscr{F}) = 1$.*

Proof. Let $A(x, Y)$ and $B(x, Z)$ be arithmetic, and let K be the set of all T such that

$$(x)[(\mathrm{E}Y)A(x, Y) \leftrightarrow (Z)B(x, Z)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. Thus for each $T \in K$,

$$(1) \qquad (x)(\mathrm{E}Y)[A(x, Y) \vee \sim B(x, Y)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. By Lemma 1.4, for almost every T that satisfies (1), there is an $\alpha < \omega_1^{CK}$ such that

$$(x)(\mathrm{E}Y^\alpha)[A(x, Y^\alpha) V \sim B(x, Y^\alpha)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. But then for almost all $T \in K$,

$$(\mathrm{E}X)(x)[x \in X \leftrightarrow (\mathrm{E}Y)A(x, Y)]$$

holds in $\mathscr{M}(\omega_1^{CK}, T)$. The "$X$" is $\hat{x}(\mathrm{E}Y^\alpha)A(x, Y^\alpha)$. $\square$

1.6 Corollary (Sacks 1969, Tanaka 1968). *For almost all T, $M(\omega_1^{CK}, T) = \mathrm{HYP}(T)$ and $\omega_1^T = \omega_1^{CK}$.*

Proof. Lemma 4.16.III. □

The next proposition says that $\mathscr{L}(\omega_1^{\mathrm{CK}}, T)$ is closed under formation of hyperarithmetic conjunctions.

1.7 Proposition. *Let $\mathscr{F}_i\ (i < \omega)$ be a hyperarithmetic sequence of ranked sentences. Then there exists a ranked sentence $\mathscr{F}$ such that for all T,*

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F} \quad \textit{iff} \quad (i)[\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}_i].$$

Proof. By Lemma 4.6.III, the set of all T such that

$$(1) \qquad (i)[\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}_i]$$

is Δ_1^1, hence hyperarithmetic according to subsection 5.6.II. It suffices to show: for each $b \in O$ there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, T)$ such that

$$n \in H_b^T \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}(\underline{n})$$

for all n and T. This last is managed by the same recursion on $<_O$ used in the proof of Lemma 4.7.III to show H_b is representable by a formula of $\mathscr{L}(\omega_1^{\mathrm{CK}}, T)$. The recursion step relies on a relativization of Corollary 4.4.II to T. □

1.8 Lemma. *Let $\mathscr{F}$ be a ranked sentence and δ a positive rational. Then there exists a ranked sentence $\mathscr{G}$ such that*

(*i*) $p(\mathscr{F}\ \&\ \sim \mathscr{G}) < \delta$, and

(*ii*) $\{T \mid \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G}\}$ *is a closed subset of* $\{T \mid \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}\}$.

Proof. Let $P(\mathscr{F}, \delta, \mathscr{G})$ be the conjunction of (i) and (ii). It follows from Lemma 4.5.III and Theorem 1.3 that $P(\mathscr{F}, \delta, \mathscr{G})$ is Π_1^1. ("Closedness" is Π_1^1.)

$(\mathscr{F})(\delta)(\mathrm{E}\mathscr{G})P(\mathscr{F}, \delta, \mathscr{G})$ is proved by induction on the full ordinal rank and logical complexity of $\mathscr{F}$.

Suppose $\mathscr{F}$ is $(Y^\alpha)\mathscr{F}_1(Y^\alpha)$. Let $\mathscr{G}_i(x)\ (i < w)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x. By induction

$$(i)(\mathrm{E}\mathscr{H})P\left(\mathscr{F}_1(\hat{x}\mathscr{G}_i(x)), \frac{\delta}{2^{i+1}}, \mathscr{H}\right).$$

By Lemma 2.6.II there is a hyperarithmetic sequence $\mathscr{H}_i\ (i < \omega)$ such that

$$(i)\ P\left(\mathscr{F}_1(\hat{x}\mathscr{G}_i(x)), \frac{\delta}{2^{i+1}}, \mathscr{H}_i\right).$$

Let $\mathscr{H}$ be equivalent to the conjunction of the $\mathscr{H}_i$'s as provided by Proposition 1.7. Then $P(\mathscr{F}, \delta, \mathscr{H})$.

Suppose $\mathscr{F}$ is $(\mathrm{E}Y^{\alpha})\mathscr{F}_1(Y^{\alpha})$. The countable additivity of p implies there is an n such that

$$(1) \qquad p\left(\bigvee_{i\leq n} \mathscr{F}_1(\hat{x}\mathscr{G}_i(x))\right) \geq p(\mathscr{F}) - \frac{\delta}{2}.$$

By Proposition 1.1 the finite disjunction occurring in (1) is equivalent to a formula of lower full ordinal rank than $\mathscr{F}$. Hence by induction there is an $\mathscr{H}$ such that

$$P\left(\bigvee_{i\leq n} \mathscr{F}_1(\hat{x}\mathscr{G}_i(x)), \frac{\delta}{2}, \mathscr{H}\right).$$

Again, $P(\mathscr{F}, \delta, \mathscr{H})$. □

1.9–1.11 Exercises

1.9. Show the measure of a hyperarithmetic set of reals is a hyperarithmetic real.

1.10. Show the measure of a Π_1^1 set A of reals is the supremum of the measures of the hyperarithmetic subsets of A.

1.11. Show the predicate, $\mu(A) > \delta$, is Π_1^1, where A ranges over Π_1^1 sets and δ over rationals.

2. *Measure-Theoretic Basis Theorems*

The main result of this section is: if A is a Π_1^1 set of positive measure, then A has a hyperarithmetic member. In addition the concept of measure-theoretic bounding is formulated and proved for the hyperarithmetic hierarchy.

2.1 Lemma. *Suppose $\mathscr{F}$ is ranked. If $p(\mathscr{F}) > 0$; then $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$ for some $T \in \mathrm{HYP}$.*

Proof. By Lemma 1.8 $\mathscr{F}$ can be replaced by a "closed subset" $\mathscr{G}$ such that $p(\mathscr{G}) > 0$. Define $\mathscr{H}_n (n < \omega)$

$$\mathscr{H}_n \text{ is } \begin{cases} \underline{n} \in \mathscr{T} & \text{if } p\left(\mathscr{G} \;\&\; \bigwedge_{i<n} \mathscr{H}_i \;\&\; \underline{n} \in \mathscr{T}\right) > 0, \\ \underline{n} \notin \mathscr{T} & \text{otherwise.} \end{cases}$$

It follows from Theorem 1.3 that $T = \{n \mid \mathscr{H}_n \text{ is } \underline{n} \in \mathscr{T}\}$ is hyperarithmetic. Since $\mathscr{G}$ is "closed", $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G}$. □

2.2 Theorem (Sacks 1969, Tanaka 1968). *If $P(X)$ is Π_1^1 and $\hat{X}P(X)$ has positive measure, then $P(X)$ has a hyperarithmetic solution.*

Proof. The relativization of Theorem 3.5.III to X yields an arithmetic predicate $A(X, Y)$ such that

$$P(X) \leftrightarrow (\mathrm{E}Y)_{Y \leq_h X} A(X, Y).$$

By Corollary 1.6

$$P(T) \leftrightarrow \mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y) A(\mathcal{T}, Y)$$

for almost all T. Consequently $P((\mathrm{E}Y)A(\mathcal{T}, Y)) > 0$, and by the countable additivity of p, $p((\mathrm{E}Y^\alpha)A(\mathcal{T}, Y^\alpha)) > 0$ for some $\alpha < \omega_1^{\mathrm{CK}}$. By Lemma 2.1 there is an $H \in \mathrm{HYP}$ such that

$$\mathcal{M}(\omega_1^{\mathrm{CK}}, H) = (\mathrm{E}Y)\mathcal{A}(\mathcal{T}, Y).$$

Since $\mathcal{M}(\omega_1^{\mathrm{CK}}, H) = \mathrm{HYP}$, $P(H)$ holds. □

2.3 Measure-Theoretic Bounding. Assume $P(T, x, y)$ is Π_1^1. It follows from Kreisel's uniformization theorem (2.3.II) relativized to T, that

(1) $$(x)(\mathrm{Ey})P(T, x, y) \rightarrow (\mathrm{Ef})_{f \leq_h T}(x)(\mathrm{Ey})_{y \leq f(x)} P(T, x, y).$$

For almost all T the bounding function f of (1) can be taken to be hyperarithmetic, that is

(2) $$(x)(\mathrm{Ey})P(T, x, y) \rightarrow (\mathrm{Ef})_{f \in \mathrm{HYP}}(x)(\mathrm{Ey})_{y \leq f(x)} P(T, x, y).$$

Fix a rational $\delta > 0$. Suppose $f \in \mathrm{HYP}$ is sought so that

$$(x)(\mathrm{Ey})_{y \leq f(x)} P(T, x, y)$$

holds for all T, save those in a set of measure less than δ, for which the left side of (2) holds. Choose a rational m such that

$$m \leq \mu((x)(\mathrm{Ey})P(T, x, y)) < m + \frac{\delta}{2}.$$

Then

(3) $$(x)(\mathrm{Ez})\left[\mu((\mathrm{Ey})_{y \leq z} P(T, x, y)) > m - \frac{\delta}{2^{x+2}}\right].$$

The matrix of (3) is Π_1^1 by Exercise 1.11. By Kreisel's selection Lemma (2.6.II), z can be taken to be $f(x)$ for some hyperarithmetic f. Hence

$$\mu((x)(\mathrm{Ey})_{y \leq f(x)} P(T, x, y)) > m - \frac{\delta}{2}. \quad \square$$

For a formulation of measure-theoretic bounding over a countable model of ZF & $V=L$, see Sacks 1969.

2.4–2.6 Exercises

2.4. Suppose $A \in 2^{\omega} - \text{HYP}$. Show $\{X \mid A \leq_h X\}$ has measure 0.

2.5. Call T Δ_1^1-random if T belongs to every Δ_1^1 set of measure 1. Define Σ_1^1 and Π_1^1 randomness similarly. Show Π_1^1-randomness is equivalent to Δ_1^1-randomness, but not to Σ_1^1-randomness.

2.6. Show Σ_1^1 dependent choice holds in $\mathscr{M}(T_0, T_1, T_2, \ldots)$ for almost all $\langle T_0, T_1, T_2, \ldots\rangle$. The language $\mathscr{L}(\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \ldots)$ and the structure $\mathscr{M}(T_0, T_1, T_2, \ldots)$ are defined in virtually the same way as $\mathscr{L}(\mathscr{T})$ and $\mathscr{M}(\mathscr{T})$ were in Section 4.III. The only difference occurs in the definition of atomic formula, where $\mathscr{T}_i (i < w)$ takes the place of $\mathscr{T}$.

3. *Cohen Forcing*

The method of Cohen forcing over the hyperarithmetic hierarchy was developed by S. Feferman 1965. It has many applications, the simplest of which is the direct construction of two hyperarithmetically incomparable reals, each recursive in Kleene's O.

A Cohen forcing condition is a consistent, finite conjunction of formulas of the form $\underline{m} \in \mathscr{T}$ or $\underline{n} \notin \mathscr{T}$. It helps to think of them as nonempty subbasic open subsets of 2^{ω}. They are denoted by $p, q, r, \ldots,$; $T \in p$ means T satisfies p, or T is a member of $p \subseteq 2^{\omega}$. $p \geq q$ (p is extended by q) means $(T)[T \in q \rightarrow T \in p]$.

3.1 Cohen's Forcing Relation $\Vdash$. Let $\mathscr{F}$ be a sentence of $\mathscr{L}(\omega_1^{\text{CK}}, \mathscr{T})$, as defined in subsection 4.1.III. Define $p \Vdash \mathscr{F}$ (p forces $\mathscr{F}$) by recursion on the full ordinal rank and logical complexity of $\mathscr{F}$ as follows.

$p \Vdash \underline{n} \in \mathscr{T}$	iff	$(T)[T \in p \rightarrow n \in T]$.
$p \Vdash (\text{E}X^{\alpha})\mathscr{F}(X^{\alpha})$	iff	$p \Vdash \mathscr{F}(\hat{x}\mathscr{G}(x))$
for some	$\mathscr{G}(x)$	of rank at most α.
$p \Vdash (\text{E}X)\mathscr{F}(X)$	iff	$(E\alpha)[p \Vdash (\text{E}X^{\alpha})\mathscr{F}(X^{\alpha})]$.
$p \Vdash \mathscr{F}$ & $\mathscr{G}$	iff	$p \Vdash \mathscr{F}$ and $p \Vdash \mathscr{G}$.
$p \Vdash (\text{Ex})\mathscr{F}(x)$	iff	$(\text{En})[p \Vdash \mathscr{F}(\underline{n})]$.
$p \Vdash \sim \mathscr{F}$	iff	$(q)_{p \geq q} \sim [q \Vdash \mathscr{F}]$.
$p \Vdash t_1 = t_2$	iff	$\text{val}(t_1) = \text{val}(t_2)$.
$p \Vdash t \in \mathscr{T}$	iff	$\text{val}(t) = n$ and $p \Vdash \underline{n} \in \mathscr{T}$.

t_1, t_2 and t are closed number theoretic terms as in the proof of Lemma 4.5.III. Note that at the ground level, forcing and truth coincide. After that, forcing respects the logical connectives. If one clause of the definition can be said to give the essence of forcing, it is the clause that defines the forcing of a negation.

One way to make sense out of the definition of Cohen forcing for ranked sentence of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$ is to consider how to approximate a hyperarithmetic set of reals by an open set. At the ground level a subbasic open set is approximated by itself. At higher levels the approximation is built up from approximations available at lower levels. It turns out that $\cup\{p|p \Vdash \mathcal{F}\}$ is a good approximation of $\{T|\mathcal{M}(\omega_1^{CK}, T)\models \mathcal{F}\}$, and that $\cup\{p|p \Vdash \sim\sim \mathcal{F}\}$ is slightly better. Some calculation is needed to show that the latter open set is hyperarithmetic and regular, and that the error of approximation is meager (cf. Exercise 3.10).

3.2 Lemma. *The predicate $p \Vdash \mathcal{F}$, restricted is Σ_1^1 $\mathcal{F}$'s, is Π_1^1.*

Proof. Same as that of Lemma 4.5.III. The clauses of the definition of $\Vdash$, when $\mathcal{F}\in\Sigma_1^1$, correspond to closure conditions whose conjunction is some Σ_1^1 formula $A(X, T)$. An induction on full ordinal rank and logical complexity shows

$$\begin{aligned} p \Vdash \mathcal{F} \ &\text{iff} \ (X)[A(X,T)\to\langle p,\mathcal{F},0\rangle\in X] \\ &\text{iff} \ \sim(X)[A(X,T)\to\langle p,\mathcal{F},1\rangle\in X]. \end{aligned}$$

3.3 Genericity. T is generic (in the sense of Cohen) with respect to a sentence $\mathcal{F}$ if there exists a p such that

$$T\in p \quad \text{and} \quad [p \Vdash \mathcal{F} \text{ or } p \Vdash \sim\mathcal{F}].$$

T is *generic* (over $\mathcal{M}(\omega_1^{CK})$) if T is generic with respect to every sentence $\mathcal{F}$ of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$.

3.4 Proposition. *(i)* $(p)(\mathcal{F})\sim\{p \Vdash \mathcal{F} \,\&\, p \Vdash \sim\mathcal{F}]$.

(ii) $(p)(\mathcal{F})(\mathrm{Eq})_{p\geq q}\, [q \Vdash \mathcal{F} \vee q \Vdash \sim\mathcal{F}]$.

(iii) $(p)(q)\,(\mathcal{F})\,[p \Vdash \mathcal{F} \,\&\, p\geq q \to q \Vdash \mathcal{F}]$.

Proof. (i) and (ii) follow from the definition of $p \Vdash \sim\mathcal{F}$. (iii) is proved by induction on the full ordinal rank and logical complexity of $\mathcal{F}$. □

It follows from Proposition 3.4(ii) that generic T's exist. Let $\mathcal{F}_i$ $(i<\omega)$ be a list of all sentences of $\mathcal{L}(\omega_1^{CK}, \mathcal{T})$. Choose p_i $(i<\omega)$ so that for all i,

$$p_i \geq p_{i+1} \quad \text{and} \quad [p_i \Vdash \mathcal{F}_i \vee p_i \Vdash \sim\mathcal{F}_i].$$

Since every sentence of the form $\underline{n}\in\mathcal{T}$ appears among the $\mathcal{F}_i$'s, it must be that $\bigcap_{i<\omega} p_i$ has a unique member, call it T. Clearly T is generic. According to Exercise 3.18, there exists a generic T hyperarithmetic in O.

3.5 Lemma. *Let T be generic. Then*

$$\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{F} \quad \textit{iff} \quad (\mathrm{E}p)\,[T \in p \;\&\; p \Vdash \mathcal{F}]$$

for every sentence $\mathcal{F}$ of $\mathcal{L}(\omega_1^{\mathrm{CK}}, \mathcal{T})$.

Proof. By induction on the full ordinal rank and logical complexity of $\mathcal{F}$.

Suppose $\mathcal{F}$ is $\mathcal{G}_0 \& \mathcal{G}_1$ and $\mathcal{M}(\omega^{\mathrm{CK}}, T) \models \mathcal{F}$. By induction there exist p_i $(i < z)$ such that $p_i \Vdash \mathcal{G}_i$ and $T \in p_i$. Note that p_0 & p_1 is a forcing condition, because $T \in p_0 \cap p_1$. By Proposition 3.4(iii) p_0 & $p_1 \Vdash \mathcal{G}_0$ & $\mathcal{G}_1$.

Suppose $\mathcal{F}$ is $(\mathrm{E}X^\alpha)\,\mathcal{H}(X^\alpha)$ and $p \Vdash \mathcal{F}$. Then $p \Vdash \mathcal{H}(\hat{x}\mathcal{G}(x))$ for some $\mathcal{G}(x)$ of rank at most α. The full ordinal rank of $\mathcal{H}(\hat{x}\mathcal{G}(x))$ is less than that of $\mathcal{F}$, so $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{H}(\hat{x}\mathcal{G}(x))$. Hence $\mathcal{F}$ is true in $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$.

Suppose $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \sim\mathcal{G}$. $\mathcal{G}$ is less complex than $\sim\mathcal{G}$, so there is no p such that $T \in p$ and $p \Vdash \mathcal{G}$. But T is generic, so there is a p such that $T \in p$ and $p \Vdash \sim\mathcal{G}$.

Suppose $T \in p$ and $p \Vdash \sim\mathcal{G}$. For the sake of a contradiction suppose $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{G}$. By induction there is a q such that $T \in q$ and $q \Vdash \mathcal{G}$. Then Lemma 3.4(iii) implies p & $q \Vdash \mathcal{G}$ and p & $q \Vdash \sim\mathcal{G}$, an impossibility according to 3.4(i). □

3.6 Theorem (Feferman 1965). *If T is generic, then Δ_1^1 comprehension holds in $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$.*

Proof. Suppose $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies

$$(x)\,[(\mathrm{E}Y)A(x, Y) \leftrightarrow (Z)B(x, Z)]$$

for some arithmetic A and B. Then

$$(1) \qquad (x)(\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)]$$

holds. Since T is generic, there is a p such that $T \in p$ and $p \Vdash (1)$. The definition of $\Vdash$ implies:

$$p \Vdash \sim(\mathrm{E}x)\sim(\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)],$$

$$(q)_{p \geq q} \sim [q \Vdash (\mathrm{E}x) \sim (\mathrm{E}Y)\,[A(x, Y) \vee B(x, Y)]]$$

$$(q)_{p \geq q}\,(n) \sim [q \Vdash \sim(\mathrm{E}Y)\,[A(\underline{n}, Y) \vee \sim B(\underline{n}, Y)]],$$

$$(2) \qquad (q)_{p \geq q}\,(n)(\mathrm{E}r)_{q \geq r}[r \Vdash (\mathrm{E}Y)\,[A(\underline{n}, y) \vee \sim B(\underline{n}, Y)]].$$

The matrix of (2) is equivalent to

$$(3) \qquad (\mathrm{E}a)_{a \in O}\,[r \Vdash (\mathrm{E}Y^{|a|})\,[A(\underline{n}, Y^{|a|}) \vee B(\underline{n}, Y^{|a|})]].$$

(3) is Π_1^1 by Lemma 3.2. Kreisel's selection Lemma (2.6.II) yields r and a as hyperarithmetic functions of q and n. By Spector's boundedness theorem (5.6.I)

there is a recursive upper bound γ on $|a(q,n)|$. Thus

$$(n)(q)(\mathrm{E}r)_{q \geq r}\, [r \Vdash (\mathrm{E}Y^{\gamma})\, [A(\underline{n}, Y^{\gamma}) \vee \sim B(\underline{n}, Y^{\gamma})]],$$

and so p forces

$$(4) \qquad (n)(\mathrm{E}Y^{\gamma})[A(\underline{n}, Y^{\gamma}) \vee \sim B(\underline{n}, Y^{\gamma})]].$$

Since T is generic and a member of p, $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies (4). But then $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies

$$(\mathrm{E}X)(x)[x \in X \leftrightarrow (\mathrm{E}Y)A(x, Y)].$$

(The "X" is $\hat{x}(\mathrm{E}Y^{\gamma})\,A(\underline{n}, Y^{\gamma})$). □

3.7 Theorem. *Assume T is generic and $(\mathrm{E}X)\mathscr{F}(X)$ has no unranked variables save X. If*

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}X)\mathscr{F}(X),$$

then $\mathscr{M}(\omega_1^{\mathrm{CK}}) \models (\mathrm{E}X)\mathscr{F}(X)$.

Proof. Suppose $p \Vdash (\mathrm{E}X)\mathscr{F}(X)$. Then $p \Vdash (\mathscr{F}(\hat{x}\mathscr{G}(x))$ for some $\mathscr{G}(x)$ of rank $\alpha < \omega_1^{\mathrm{CK}}$. The relation $p \Vdash \mathscr{K}$, restricted to sentences $\mathscr{K}$ of rank at most α, is Δ_1^1 by Exercise 3.14. Recall the construction of a generic T following Proposition 3.4. Repeat that construction with the $\mathscr{F}_i$'s replaced by a hyperarithmetic enumeration of all formulas of rank at most α. Then the constructed set, call it H, can be taken to be hyperarithmetic, since it can be defined by recursion on ω relative to a hyperarithmetic predicate. H obeys Lemma 3.5 with respect to all sentences of rank at most α. $H \in p$, since the construction of H can start with p. Hence $\mathscr{M}(\omega_1^{\mathrm{CK}}, H) \models \mathscr{F}(\hat{x}\mathscr{G}(x))$. $\mathscr{M}(\omega_1^{\mathrm{CK}}, H)$ is $\mathscr{M}(\omega_1^{\mathrm{CK}})$ by Lemma 4.16(ii).III. □

3.8 Corollary. *If T is generic, then O is Σ_1^1 definable over $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$.*

Proof. By Lemma 3.5.III,

$$x \in O \leftrightarrow (\mathrm{E}Y)_{Y \in \mathrm{HYP}}\, A(x, Y)$$

for some arithmetic A. Theorem 3.7 implies

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y)A(\underline{n}, Y). \qquad \square$$

3.9 Category Versus Measure. There is an analogy between the results of Sections 3 and 2 based on a standard analogy between category and measure. Let A be a subset of 2^{ω}. (Recall the topology assigned to 2^{ω} in subsection 6.1.II.) A is said to be *nowhere dense* if it has an empty interior. A is said to be *meager* if it is contained in a countable union of nowhere dense closed sets. Baire's category theorem states: the complement of a meager set is dense. Consequently meager sets are thought to be small and analogous to sets of measure 0.

It follows from Proposition 3.4(ii) that the set of T's such that T is generic with respect to a given sentence $\mathscr{F}$ is dense open, and the set of all generic T is co-meager. Thus Theorem 3.6 is analogous to Theorem 1.5, and Exercise 3.11 to Theorem 2.2.

3.10–3.18 Exercises

3.10. An open subset of 2^ω is said to be regular if it equals the interior of its closure. Let A be a Δ_1^1 subset of 2^ω. Find a Δ_1^1 B such that B is regular open and $(A-B) \vee (B-A)$ is meager.

3.11. (Hinman, Thomason). Let A be a non-meager, Π_1^1 subset of 2^ω. Show A has a hyperarithmetic element.

3.12. If T is generic, then $T \notin \text{HYP}$.

3.13. If T is generic, then $O^T \leq_h T, O$. (O^T is the hyperjump of T, defined in Section 7.II.)

3.14. Use 4.1(2).III and the proof of 3.2 to show $p \Vdash \mathscr{F}$, restricted to $\mathscr{F}$ of rank at most α, is Δ_1^1 when $\alpha < \omega_1^{\text{CK}}$.

3.15. Assume T is generic. Let $T_i (i<2)$ be $\{n | 2n+i \in T\}$. Show T_i is generic. Show $T_0 \not\leq_h T_1$ and $T_1 \not\leq_h T_0$.

3.16. (H. Friedman). Find a T which is generic with respect to all ranked sentences (of $\mathscr{L}(\omega_1^{\text{CK}}, T)$), but which is not generic.

3.17. (S. Feferman). Cohen forcing for $\mathscr{L}(\omega_1^{\text{CK}}, \mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \ldots)$ is defined by taking as forcing conditions consistent finite conjunctions of formulas of the form $m \in \mathscr{T}_i$, $n \notin \mathscr{T}_j$ $(i, j < \omega)$. Show $\mathscr{M}(T_0, T_1, T_2, \ldots)$ satisfies Σ_1^1 dependent choice when $\langle T_0, T_1, T_2, \ldots \rangle$ is generic (cf. Exercise 2.6).

3.18. Show there exists a generic $T \leq_h O$.

3.19. Find a hyperarithmetic set that is not an arithmetic singleton.

4. Perfect Forcing

Let $P, Q, R, \ldots$ denote perfect subsets of 2^ω as in subsection 6.1.III. In this section Cohen's forcing method is extended from finite conditions to perfect ones. The resulting generic reals differ vastly from those of Section 3. In particular a real generic in the sense of the present section has minimal hyperdegree. According to Exercise 3.15, a real generic in the sense of subsection 3.3 lies above two hyperarithmetically incomparable reals.

Let p, q, r, ... be sequence numbers that encode finite initial segments (of characteristic functions) of subsets of ω. As in subsection 6.1.III, a perfect subset of 2^ω can be encoded by the set of codes for the initial segments of its members. The natural homeomorphism between 2^ω and a perfect set P gives rise to a standard encoding of P denoted by $\lambda_{ij}|p_{ij}$ $(i<\omega \;\&\; j<2^i)$. $p_{i+1,2j}$ and $p_{i+1,2j+1}$ are incomparable extensions of $p_{i,j}$.

$$P=\hat{X}((i)\,(\mathrm{E}j)_{j<2^i}\,(X\in p_{ij})).$$

P is *hyperarithmetically encodable* if $\lambda_{ij}|p_{ij}$ is a hyperarithmetic function. From now on P ambiguously denotes a hyperarithmetically encodable perfect set and its standard code. Thus the set of all P's is Π_1^1.

4.1 The Perfect Forcing Relation. Let $\mathscr{F}$ be a sentence of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ as defined in subsection 4.1.III. The perfect forcing relation, $P \Vdash \mathscr{F}$, is defined by six closure conditions.

(1) If $\mathscr{F}$ is ranked and $(X)\,[X\in P \rightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, X)\models \mathscr{F}]$, then $P \Vdash \mathscr{F}$.

(2) If $\mathscr{F}(Y^\alpha)$ is unranked and $P \Vdash \mathscr{F}(\hat{x}\mathscr{G}(x))$ for some $\mathscr{G}(x)$ of rank at most α, then $P \Vdash (\mathrm{E}Y^\alpha)\,\mathscr{H}(Y^\alpha)$.

(3) If $P \Vdash (\mathrm{E}Y^\alpha)\,\mathscr{F}(Y^\alpha)$ for some $\alpha<\omega_1^{\mathrm{CK}}$, then $P \Vdash (\mathrm{E}Y)\,\mathscr{F}(Y)$.

(4) If $\mathscr{F}(x)$ is unranked and $P \Vdash \mathscr{F}(\underline{n})$ for some n, then $P \Vdash (\mathrm{E}x)\,\mathscr{F}(x)$.

(5) If $\mathscr{F} \;\&\; \mathscr{G}$ is unranked, $P \Vdash \mathscr{F}$ and $P \Vdash \mathscr{G}$, then $P \Vdash \mathscr{F} \;\&\; \mathscr{G}$.

(6) If $\mathscr{F}$ is unranked and $(Q)_{P\supseteq Q} \sim [Q \Vdash \mathscr{F}]$, then $P \Vdash \sim \mathscr{F}$.

Clause (1) may seem unorthodox in that it treats all ranked sentences as if they were at the ground level. (1) makes it easy to show that the perfect forcing relation, restricted to ranked sentences, is Π_1^1, but obscures the fact that

$$(\mathscr{F})(P)\,(\mathrm{E}Q)_{P\supseteq Q}\,[Q \Vdash \mathscr{F} \text{ or } Q \Vdash \sim\mathscr{F}].$$

4.2 Lemma. *The relation* $P \Vdash \mathscr{F}$, restricted to Σ_1^1 $\mathscr{F}$'s, is Π_1^1.

Proof. Clause (3) of the definition of H makes it safe to assume $\mathscr{F}$ is ranked. Hence

(1) $$P \Vdash \mathscr{F} \leftrightarrow (X)\,[X\in P \rightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, X)\models \mathscr{F}]$$

The right side of (1) is Π_1^1 by Lemma 4.5.II. □

The next lemma establishes the so-called fusion (or splitting) property of perfect forcing in the hyperarithmetic case.

4.3 Lemma. *Let $\mathscr{F}_i (i<\omega)$ be a hyperarithmetic sequence of Σ_1^1 sentences. Suppose*

$$(i)\ (Q)_{P\supseteq Q}(\mathrm{E}R)_{Q\supseteq R}[R \Vdash \mathscr{F}_i].$$

Then $(\mathrm{E}Q)_{P\supseteq Q}(i)[Q \Vdash \mathscr{F}_i]$.

Proof. By lemma 4.2 the predicate

$$Q \supseteq R \;\&\; R \Vdash \mathscr{F}_i$$

is Π_1^1. Kreisel's uniformization theorem (2.6.II) implies R can be construed as a partial Π_1^1 function of Q and i. $R(Q, i)$ is iterated below to produce $\lambda ij/Q_{i,j}$, a hyperarithmetic function as in Exercise 4.16.

A hyperarithmetic family $\{Q_{i,j} \mid i < \omega \;\&\; j < 2^i\}$ is defined by recursion on i with the aid of $\lambda iQ \mid R(Q, i)$ and effective splitting.

$$Q_{0,0} = P.$$

(1) $$Q_{i+1,2j} \subseteq Q_{i,j} \quad\text{and}\quad Q_{i+1,2j+1} \subseteq Q_{i,j}.$$

(2) $$Q_{i+1,2j} \cap Q_{i+1,2j+1} = \varnothing.$$

$$Q_{i+1,2j} \Vdash \mathscr{F}_i \quad\text{and}\quad Q_{i+1,2j+1} \Vdash \mathscr{F}_i.$$

Let Q be $\bigcap_i \bigcup_{j<2^i} Q_{i,j}$. Then $Q \in \mathrm{HYP}$ and $Q \subseteq P$. Q is perfect because of the splitting in (1) and (2). Note that if $X \in Q$, then for each i, there is a unique j such that $X \in Q_{i,j}$.

Fix i to check $Q \Vdash \mathscr{F}_i$. Let $\mathscr{F}_i$ be $(\mathrm{E}X)\mathscr{G}_i(x)$. Q is contained in $\cup\{Q_{i,j} \mid j < 2^i\}$, and for each $j < 2^i$,

$$Q_{i+1,j} \Vdash (\mathrm{E}X^{\alpha_j})\mathscr{G}_i(X^{\alpha_j})$$

for some $\alpha_j < \omega_1^{\mathrm{CK}}$. Let α be $\sup_{j<2^i} \alpha_j$. Then $Q \Vdash (\mathrm{E}X^\alpha)_i(X^\alpha)$. □

Lemma 4.3 is the central fact of perfect forcing over the hyperarithmetic hierarchy. Its counterpart in the set theoretic case (Sacks 1967) is applied to show aleph-one is preserved in generic extensions of L.

4.4 Lemma. $(\mathscr{F})(P)(\mathrm{E}Q)_{P\supseteq Q}[Q \Vdash \mathscr{F}$ or $Q \Vdash \sim \mathscr{F}]$.

Proof. Suppose $\mathscr{F}$ is unranked. Then the lemma follows from clause (6) of the definition of H.

Suppose $\mathscr{F}$ is ranked. Proceed by induction on the full ordinal rank and logical complexity of $\mathscr{F}$ as defined in subsection 4.4.III. For example, suppose $\mathscr{F}$ is $(\mathrm{E}X^\alpha)\mathscr{H}(X^\alpha)$. Let $\mathscr{G}_i(x)$ $(i < \omega)$ be an effective enumeration of all formulas of rank at most α whose sole free variable is x. First assume

(1) $$(\mathrm{E}i)(\mathrm{E}Q)_{P\supseteq Q}[Q \Vdash \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

Then $Q \Vdash \mathscr{F}$. Now assume (1) fails. By induction

$$(i)(Q)_{P \supseteq Q}(\mathrm{E}R)_{Q \supseteq R}(R \Vdash \sim \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

Lemma 4.3 yields

$$(\mathrm{E}Q)_{P \supseteq Q}(i)[Q \Vdash \sim \mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

But then $Q \Vdash \sim \mathscr{F}$. □

Lemma 4.4 has roughly the same content as: each Δ_1^1 set or its complement contains a perfect Δ_1^1 set. Note that an uncountable Δ_1^1 set need not contain a perfect Δ_1^1 set.

4.5 Perfect Genericity. Suppose $T \subseteq \omega$. T is said to be *generic* (in the sense of perfect forcing) if for each sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$, there is a P such that $T \in P$ and either $P \Vdash \mathscr{F}$ or $P \Vdash \sim \mathscr{F}$. It follows from Lemma 4.4 that generic T's exist inside every P. Let $\mathscr{F}_i$ $(i < \omega)$ be an enumeration of all sentences of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$. Define $P_i (i < \omega)$ so that $P_0 = P$, $P_i \supseteq P_{i+1}$, and

$$P_{i+1} \Vdash \mathscr{F}_i \quad \text{or} \quad P_{i+1} \Vdash \sim \mathscr{F}_i.$$

Then $\cap\{P_i | i < \omega\} = \{T\}$ and T is generic.

The next proposition is a technicality needed for the proof of the truth lemma (4.8).

4.6. Proposition. *Suppose T is generic with respect to all ranked sentences, and $T \in P \cap Q$. Then there exists an R such that $T \in R \subseteq P \cap Q$.*

Proof. The statement $T \in P$ is equivalent to

$$(i)(\mathrm{Ej})[j < 2^i \quad \& \quad T \in p_{ij}],$$

where $\lambda_{ij} | p_{ij}$ is the standard hyperarithmetic code for P. It follows from Lemma 4.7.III. that there exists a ranked sentence $\mathscr{G}$ such that

$$(T)[T \in P \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G}].$$

Let $\mathscr{H}$ be a ranked sentence that bears the same relation to Q that $\mathscr{G}$ does to P.

By supposition $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G} \ \& \ \mathscr{H}$. Since T is generic, there is an R such that $T \in R$, and

$$R \Vdash \mathscr{G} \ \& \ \mathscr{H} \quad \text{or} \quad R \Vdash \sim (\mathscr{G} \ \& \ \mathscr{H}).$$

If $R \Vdash \sim (\mathscr{G} \ \& \ \mathscr{H})$, then clause (1) of the definition of $\Vdash$ implies $T \notin P \cap Q$. Hence $R \Vdash \mathscr{G} \ \& \ \mathscr{H}$. And so $R \subseteq P \cap Q$. □

4.7 Proposition. *If $P \Vdash \mathcal{F}$ and $P \supseteq Q$, then $Q \Vdash \mathcal{F}$.*

Proof. Clause (1) of the definition of H disposes of the matter if $\mathcal{F}$ is ranked. For unranked $\mathcal{F}$ the proof proceeds by induction on the full ordinal rank and logical complexity of $\mathcal{F}$. □

4.8 Lemma. *Suppose T is generic. Then*

$$\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{F} \quad \textit{iff} \quad (\mathrm{E}P)_{T \in P}[P \Vdash \mathcal{F}]$$

for every sentence $\mathcal{F}$ of $\mathcal{L}(\omega_1^{\mathrm{CK}}, \mathcal{T})$.

Proof. Clause (1) of the definition of $\Vdash$ takes care of ranked $\mathcal{F}$'s. For unranked $\mathcal{F}$'s the proof is by induction on logical complexity as in Lemma 3.5. The only difficulty occurs when $\mathcal{F}$ is $\mathcal{G}\ \&\ \mathcal{H}$. Suppose

$$\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{G}\ \&\ \mathcal{H}.$$

By induction there exist P and Q such that $T \in P \cap Q$, $P \Vdash \mathcal{G}$ and $Q \Vdash \mathcal{H}$. Lemma 4.6 supplies an $R \subseteq P \cap Q$ such that $T \in R$. By Proposition 4.7 $R \Vdash \mathcal{G}\ \&\ \mathcal{H}$. □

4.9–4.15 Exercises

4.9. Let A be a Δ_1^1 subset of 2^ω. Find a perfect Δ_1^1 set P such that $P \subseteq A$ or $P \subseteq 2^\omega - A$.

4.10. Show there exists a generic $T \leq_h O$.

4.11. Assume $\mathcal{F}$ is Σ_1^1. Show

$$P \Vdash \mathcal{F} \quad \text{iff} \quad (X)[X \in P \to \mathcal{M}(\omega_1^{\mathrm{CK}}, X) \models \mathcal{F}].$$

4.12. If T is generic (in the sense of perfect forcing) and $S \equiv_h T$, then S is generic. (Refute this assertion for Cohen genericity.)

4.13. If T is generic, then $T \notin \mathrm{HYP}$.

4.14. If T is generic, then Kleene's O is Σ_1^1 definable over $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$. (First show an arithmetic predicate with a solution in $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$ has a hyperarithmetic solution.)

4.15. Define $\Vdash_1$ by using the perfect conditions of subsection 4.1 and the recursion of subsection 3.1. For example $P \Vdash_1 \sim \mathcal{F}$ iff $(Q)_{P \supseteq Q} \sim Q \Vdash_1 \mathcal{F}$ whether or not $\mathcal{F}$ is ranked. Call T generic in the sense of $\Vdash_1$ if for each sentence $\mathcal{F}$ of $\mathcal{L}(\omega_1^{\mathrm{CK}}, \mathcal{T})$, there is a P such that $T \in P$ and either $P \Vdash_1 \mathcal{F}$ or $P \Vdash_1 \sim \mathcal{F}$. Show T is generic in the sense of $\Vdash_1$ iff T is generic as defined in subsection 4.4.

4.16. Let $g(0) = 0$, and $g(n+1) \simeq f(g(n))$ for all n, where f is partial Π_1^1. If g is total, then g is Δ_1^1 by 1.7.I.

5. *Minimal Hyperdegrees*

T is said to be of minimal hyperdegree of $T \notin \mathrm{HYP}$ and $(S)[S \leq_h T \rightarrow S \in \mathrm{HYP} \vee T \leq_h S]$. The existence of minimal Turing degrees was proved by Spector 1956. The hyperdegree case combines Spector's ideas with perfect forcing over the hyperarithmetic hierarchy. The principal result of this section is that every perfectly generic T is of minimal hyperdegree.

5.1. $\mathscr{M}$-Definability. Suppose $I \subseteq \omega$. I is said to be *$\mathscr{M}$-definable* if there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$, in which $\mathscr{T}$ does not occur, such that

$$(n)[n \in I \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}) \models \mathscr{F}(\underline{n})].$$

I is said to be *generically definable* if there is a formula $\mathscr{F}(x)$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ such that

$$(n)[n \in I \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, \mathscr{T}) \models \mathscr{F}(\underline{n})]$$

for all generic T (generic in the sense of perfect forcing).

5.2 Lemma. *The following are equivalent*
(*i*) *I is arithmetic in O.*
(*ii*) *I is $\mathscr{M}$-definable.*
(*iii*) *I is generically definable.*

Proof. By Theorems 3.5.III and 4.8.III, there is an arithmetic $A(x, Y)$ such that for all n,

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}) \models (\mathrm{E}Y)A(\underline{n}, Y).$$

It follows from the parenthetical remark in Exercise 4.14 that

$$n \in O \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}Y)A(\underline{n}, Y).$$

for all generic T. Thus O is $\mathscr{M}$-definable and generically definable. Consequently every set arithmetic in O is.

Suppose I is $\mathscr{M}$-definable. Then Lemma 4.5.III implies I is arithmetic in O.

Finally suppose I is generically definable via the formula $\mathscr{F}(x)$. It suffices, by Lemma 4.2, to see

$$n \in I \leftrightarrow (P)(\mathrm{E}Q)_{P \supseteq Q}[Q \Vdash \mathscr{F}(\underline{n})]$$

in order to conclude I is arithmetic in O. First let $n \in I$, T be generic, and $T \in P$. Then

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}(\underline{n}).$$

By Lemma 4.8 there is an R such that $T \in R$ and $R \Vdash \mathscr{F}(\underline{n})$. Proposition 4.6 provides a $Q \subseteq P \cap R$ such that $T \in Q$. But then $Q \Vdash \mathscr{F}(\underline{n})$ by Proposition 4.7.

Now let $n \notin I$. Choose a generic T and a P such that

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \sim \mathscr{F}(\underline{n}),$$

$T \in P$, and $P \Vdash \sim \mathscr{F}(\underline{n})$. Then

$$\sim(\mathrm{E}Q)_{P \supseteq Q}[Q \Vdash \mathscr{F}(\underline{n})]. \quad \square$$

Let K be a set of P's and Q's, that is a set of hyperarithmetically encodable perfect conditions. K^i is the set of indices of members of K. $2^b \cdot 3^e$ is an index for P if $b \in O$, $\{e\}^{H_b}$ is total, and $\{e\}^{H_b}$ $(= \lambda ij | p_{ij})$ is the standard code for P.

K is *dense* if $(P)(\mathrm{E}Q)[P \supseteq Q \in K]$.

5.3 Theorem. *T is generic iff $T \in \cup\{P | P \in K\}$ for every dense set K arithmetic in O.*

Proof. Suppose T satisfies the density hypothesis. Fix $\mathscr{F}$. The set

$$\{P | P \Vdash \mathscr{F} \quad \text{or} \quad P \Vdash \sim \mathscr{F}\}$$

is dense by Lemma 4.4, and arithmetic in O by Lemma 4.2. Hence T is generic with respect to $\mathscr{F}$.

Now suppose K is dense and arithmetic in O. By Lemma 5.2 there is an $\mathscr{F}(x)$ such that

$$n \in K^i \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}(\underline{n})$$

for all generic T. Let H be the arithmetic formula of Theorem 4.2.II. Define $\mathscr{G}$ by

$$(\mathrm{E}x)(\mathrm{E}Y)[\mathscr{F}(x) \ \& \ H((x)_0, Y)$$
$$\& \ (i)(\mathrm{E}j)_{j<2^i}(\mathrm{E}z)(\{(x_1)\}^Y(i,j) = z = (\ell h(z)))].$$

$\mathscr{G}$ says: $(\mathrm{E}P)[P \in K \ \& \ T \in P]$. ($z$ is a sequence number that encodes an initial segment of the characteristic function of $\mathscr{T}$.) In fact

$$\text{(1)} \qquad T \in \cup\{P | P \in K\} \leftrightarrow \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{G}$$

holds for all generic T. It need only be shown that the right side of (1) holds for all generic T. If not, there is a P and a generic $T \in P$ such that $P \Vdash \sim \mathscr{G}$. Since K is dense, $P \supseteq Q$ for some $Q \in K$. Choose a generic $T^* \in Q$. It follows from (1) that

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T^*) \models \mathscr{G},$$

and so $T^* \in R$ for some R that forces $\mathscr{G}$. By Proposition 4.6 there is an $S \subseteq Q \cap R$ such that $T^* \in S$. $S \Vdash \mathscr{G}$ since $R \Vdash \mathscr{G}$ and $S \Vdash \sim \mathscr{G}$ since $S \subseteq P$. $\square$

5.4 Theorem. *If T is generic, then $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies Δ_1^1 comprehension.*

Proof. Similar to the proof of Theorem 3.6. It is enough to show: if $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models (x)$ $(\mathrm{E}Y)A(x, Y)$ for some arithmetic A, then $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models (x)(\mathrm{E}Y^\alpha)A(x, Y)$ for some $\alpha < \omega_1^{\mathrm{CK}}$. Let K be the set of all P such that

$$P \Vdash \sim(x)(\mathrm{E}Y)A(x, Y)$$
$$\text{or } (\mathrm{Eb})_{b \in O}[P \Vdash (x)(\mathrm{E}Y^{|b|})A(x, Y^{|b|})].$$

By Lemma 4.2 K is arithmetic in O. To see K is dense, fix P. If P contracts to some Q that forces $\sim(x)(\mathrm{E}Y)A(x, Y)$, then all is well. If not, then P contracts to some Q that forces $(x)(\mathrm{E}Y)A(x, Y)$. It follows, as in the derivation of (2) from (1) in the proof of Theorem 3.6, that

$$(n)(R)_{Q \supseteq R}(\mathrm{E}S)_{R \supseteq S}[S \Vdash (\mathrm{E}Y)A(\underline{n}, Y)].$$

The fusion lemma (4.3) yields $S \subseteq Q$ such that

$$(n)[S \Vdash (\mathrm{E}Y)A(\underline{n}, Y)],$$

or in more detail,

$$(n)(\mathrm{Ec})_{c \in O}[S \Vdash (\mathrm{E}Y^{|c|})A(\underline{n}, Y^{|c|})].$$

Kreisel's selection Lemma (2.5.II) implies c can be constructed as a hyperarithmetic function of n. Then Spector's boundedness theorem (5.6.I) provides a recursive bound α on c: $|c(n)| < \alpha$ for all n. Thus $P \supseteq S \in K$.

By Theorem 5.3 T belongs to some member of K. $\square$

Let $\mathcal{G}(x)$ be a ranked formula of $\mathcal{L}(\omega_1^{\mathrm{CK}}, \mathcal{T})$ whose sole free variable is x. For each $T \subseteq \omega$, define

$$\mathcal{G}^T \quad \text{by} \quad \{n | \mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{G}(\underline{n})\}.$$

Thus for each T the $\mathcal{G}^T$'s are the members of $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$.

The essential content of the next lemma is: let $f: 2^\omega \to 2^\omega$ be Δ_1^1; then there exists a perfect Δ_1^1 $A \subseteq 2^\omega$ such that f, restricted to A, is either constant or 1-1.

5.5 Lemma. *For each P and $\mathcal{G}(x)$ there are $Q \subseteq P$ and $H \in \mathrm{HYP}$ such that either (i) or (ii) holds.*

(*i*) $(T)[T \in Q \to \mathcal{G}^T = H]$.
(*ii*) $(T)[T \in Q \to T$ *recursive in* $\mathcal{G}^T, H]$.

Proof. Case I: there is an $R \subseteq P$ such that

$$(Q_0)_{Q_0 \subseteq R}(Q_1)_{Q_1 \subseteq R}(n) \sim [Q_0 \Vdash \mathcal{G}(\underline{n}) \quad \text{and} \quad Q_1 \Vdash \sim \mathcal{G}(\underline{n})].$$

Define H by

$$n \in H \leftrightarrow (\mathrm{E}Q)_{Q \subseteq R}[Q \Vdash \mathcal{G}(\underline{n})].$$

Observe that $n \notin H \leftrightarrow (\mathrm{E}Q)_{Q \subseteq R}[Q \Vdash \sim \mathscr{G}(\underline{n})]$. It follows from Lemma 4.2 that $H \in \mathrm{HYP}$. The fusion Lemma (4.3) supplies a $Q \subseteq R$ such that for all n,

$$n \in H \to Q \Vdash \mathscr{G}(\underline{n})$$
$$\text{and} \quad n \notin H \to Q \Vdash \sim \mathscr{G}(\underline{n}).$$

Then $\mathscr{G}^T = H$ for all $T \in Q$.

Case II: Case I fails. Hence

$$(R)_{P \supseteq R}(\mathrm{E}Q_0)_{Q_0 \subseteq R}(\mathrm{E}Q_1)_{Q_1 \subseteq R}(\mathrm{E}n)[Q_0 \Vdash \mathscr{G}(\underline{n}) \quad \text{and} \quad Q_1 \Vdash \sim \mathscr{G}(\underline{n})].$$

The present situation is similar to that found in the proof of Lemma 4.3. Q_0, Q_1 and n can be construed as partial Π_1^1 functions of R. They are applied in a recursion of length ω to obtain hyperarithmetic functions $\lambda ij | Q_{ij}$ and $\lambda ij | f(i,j)$ such that:

$$Q_{00} = P, \quad Q_{i+1,2j} \cap Q_{i+1,2j+1} = \varnothing,$$
$$Q_{i+1,2j} \subseteq Q_{ij} \quad \text{and} \quad Q_{i+1,2j+1} \subseteq Q_{ij},$$
$$Q_{i+1,2j} \Vdash \mathscr{G}(\underline{f(i,j)}), \quad \text{and}$$
$$Q_{i+1,2j+1} \Vdash \sim \mathscr{G}(\underline{f(i,j)}).$$

Let Q be $\bigcap_i \bigcup_{j<2^i} Q_{ij}$. Q is hyperarithmetic and perfect.

Fix $T \in Q$ to see why T is recursive in $\mathscr{G}^T$, H, where H is $\langle Q, f \rangle$. The idea is: $\mathscr{G}^T$ is 1-1 on Q, so T can be recovered from $\mathscr{G}^T$ with the help of Q and f. Let $t(i)$ be the unique j such that $T \in Q_{i,j}$. The construction of Q implies

$$t(i+1) = 2t(i) \quad \text{if} \quad f(i, t(i)) \in \mathscr{G}^T$$
$$2t(i)+1 \quad \text{if} \quad f(i, t(i)) \notin \mathscr{G}^T.$$

Thus t, hence T, is recursive in $\mathscr{G}^T$, Q, f. □

5.6 Theorem (Gandy & Sacks 1967). *If T is generic in the sense of perfect hyperarithmetic forcing, then T is of minimal hyperdegree.*

Proof. By Exercise 4.13 $T \notin \mathrm{HYP}$. Suppose $X \leq_h T$. By Lemma 5.4 $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies Δ_1^1 comprehension, and so by Lemma 4.16.III, $X \in \mathscr{M}(\omega_1^{\mathrm{CK}}, T)$. Thus $X = \mathscr{G}^T$ for some ranked $\mathscr{G}(x)$. Let K be the set of all Q that satisfy (1) or (2) of Lemma 5.5 for some hyperarithmetic H. Then K is dense, and K is arithmetic in O; in fact K is Π_1^1. Theorem 5.3 implies $T \in Q$ for some $Q \in K$. Hence

$$X \in \mathrm{HYP} \quad \text{or} \quad T \leq_h X. \quad \square$$

5.7–5.9 Exercises

5.7. Suppose $f: 2^\omega \to 2^\omega$ is Δ_1^1. Find a perfect Δ_1^1 set A such that the restriction of f to A is either constant or 1-1.

5.8 (Local Cohen Forcing). Let Q be a hyperarithmetically encodable, perfect subset of 2^ω. The Cohen forcing method of Section 3 is localized to Q as follows. In essence Q replaces 2^ω. $p \Vdash_Q \mathscr{F}$ is defined as in subsection 3.1 save that $p, q, r, \ldots$ now encode finite initial segments of members of Q. Generic is defined as in subsection 3.3, and it follows that all generic T's belong to Q. Recall the trick of the proof of Theorem 3.7. Use local Cohen forcing to give proofs of Lemmas 4.4 and 5.5.

5.9. Try exercises 4.12 and 4.15 again.

6. Louveau Separation

The Kleene separation theorem for subsets of ω states: if $A, B \subseteq \omega$ are Σ_1^1 and disjoint, then there exists a Δ_1^1 $C \subseteq \omega$ such that $A \subseteq C$ and $C \cap B = \varnothing$. It was obtained as a corollary to a reduction theorem (3.7.II) for Π_1^1 subsets of ω. Kleene separation for subsets of ω^ω (Exercise 5.11.II) is obtained similarly by relativizing Theorem 3.7.II in the manner of Section 5.II.

Let A and B be disjoint subsets of ω^ω. As in subsection 5.4.II, there are n_A and n_B such that

$$(f)[f \in A \leftrightarrow n_A \in O^f], \quad \text{and}$$
$$(f)[f \in B \leftrightarrow n_B \in O^f].$$

Let $f \in A_0 \leftrightarrow f \in A \;\;\&\;\; |n_A|_{O^f} \leq |n_B|_{O^f}$, and

$f \in B_0 \leftrightarrow f \in B \;\;\&\;\; |n_B|_{O^f} < n_A|_{O^f}$,

Lemma 2.1.II, relativized to f, implies A_0 and B_0 are Π_1^1. By construction $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$.

As in subsection 3.6.II, reduction for Π_1^1 implies separation for Σ_1^1. Kleene separation for subsets of ω^ω is: if $A, B \subseteq \omega^\omega$ are Σ_1^1 and disjoint, then there exists a Δ_1^1 C such that $A \subseteq C$ and $B \cap C = \varnothing$. Louveau separation is concerned with the complexity of the separating set C. In order to make the notion of complexity precise, a hierarchy of subsets of ω^ω is defined. Let $D \subseteq \omega^\omega$. D is *subbasic* if D is of the form $\{f \mid f(m) = n\}$. D is *clopen* if D is a finite Boolean combination of subbasic sets.

$D \in \mathbf{\Pi}_0^o$ if D is clopen.

$D \in \mathbf{\Pi}_\gamma^o$ if D is a countable intersection of complements of sets in $\mathbf{\Pi}_{<\gamma}^o = \cup\{\mathbf{\Pi}_\delta^o \mid \delta < \gamma\}$.

D is *open* if it is a union of clopen sets. D is *Borel* if it belongs to the least σ-algebra containing the open sets (cf. subsection 6.1.II). According to Exercise 6.10, D is Borel iff D belongs to $\mathbf{\Pi}^o_{\omega_1}$. If $D \in \mathbf{\Pi}^o_\gamma - \mathbf{\Pi}^o_{<\gamma}$, then D is said to have rank γ. One measure of the complexity of D is its rank. Another is the way D is put together from sets of rank less than D. The "putting together" is described by a code for D. The code for a clopen set is obtained from a fixed, effective one-one correspondence between $\mathbf{\Pi}^o_o$ and $\{5^{n+1} | n < \omega\}$. Each clopen set has a code of the form $\{5^{n+1}\}$. Suppose

$$D = \cap \{\omega^\omega - D_m | m < \omega\},$$
$$D \in \mathbf{\Pi}^o_\gamma, \quad D_m \in \mathbf{\Pi}^o_{\gamma_n} \quad (\gamma_n < \gamma),$$

and c_m is a code for $D_m (m < \omega)$. Then $\{2^m \cdot 3^n | n \in c_m\}$ is a code for D. According to Exercises 6.7 and 6.8: if D has a Δ^1_1 code, then $D \in \mathbf{\Pi}^o_{\omega_1^{\mathrm{CK}}}$; D is Δ^1_1 iff D has a Δ^1_1 code. D is said to be a $\mathbf{\Pi}^o_\gamma(\Delta^1_1)$ set if $D \in \mathbf{\Pi}^o_\gamma$ and D has a Δ^1_1 code.

6.1 Theorem (Louveau 1980). *If $A, B \subseteq \omega^\omega$ are Σ^1_1, and A is separable from B by a $\mathbf{\Pi}^o_\gamma$ set, then A is separable from B by a $\mathbf{\Pi}^o_\gamma(\Delta^1_1)$ set.*

6.2 Corollary (Louveau 1980). *If $A \subseteq \omega^\omega$ is Δ^1_1 and $\mathbf{\Pi}^o_\gamma$, then A is $\mathbf{\Pi}^o_\gamma(\Delta^1_1)$.*

Corollary 6.2 is remarkable. It implies, for example, if A is Δ^1_1 and an intersection of open sets, then A is the intersection of a Δ^1_1 sequence of Δ^1_1 open sets. The proof of Theorem 6.1 is a combination of bounding arguments and forcing. The role of forcing is to approximate certain sets by unions of forcing conditions in the manner discussed at the end of subsection 3.1. The conditions are nonempty Σ^1_1 subsets of ω^ω. Σ^1_1 forcing was invented by Gandy circa 1964. One difficulty of Σ^1_1 forcing is that Σ^1_1 conditions are not closed. A contracting sequence of Σ^1_1 conditions can shrink down to nothing. Consequently it is necessary to pay close attention to the existential witnesses that establish membership in Σ^1_1 sets.

A set is open in the *Gandy topology* for ω^ω if it is a union of Σ^1_1 sets. The next result says that the Baire category theorem holds for the Gandy topology. It is needed for the "approximation" aspect of the proof of Louveau's separation theorem.

6.3 Theorem. *Let ω^ω have the Gandy topology. If O_i $(1 \leq i < \omega)$ is a sequence of dense open sets, then $\bigcap_i O_i$ is dense.*

Proof. Let P, P_1, P_2, etc. denote nonempty Σ^1_1 subsets of ω. Fix P with the intent of finding an $f \in P \cap \bigcap_i O_i$. Recall Seq and $>$ from subsection 5.1.I. The construction below defines P_i, f and g_{ij} $(1 \leq i, j < \omega)$ such that:

(a) $P \supseteq P_i \supseteq P_{i+1}$ and $P_i \subseteq O_i$.
(b) $f_i \in \mathrm{Seq}$, $\ell h(f_i) = i$, and $f_i > f_{i+1}$.

(c) $g_{ij} \in \underline{\mathrm{Seq}}$, $\ell h(g_{ij}) = j$, and $g_{ij} > g_{i,j+1}$.

(d) Let $\bar{f}(i) = f_i$ and $\bar{g}_i(j) = g_{ij}$. g_i is an existential witness to the membership of f in P_i.

It is immediate from (a)–(d) that $f \in P \cap \bigcap_i P_i$.

Suppose Q is $\hat{f}(\mathrm{E}g)(x)R(\bar{f}(x), \bar{g}(x))$ from some recursive R. $Q(s, t)$, a notation needed for the construction, is defined by:

$$\hat{f}[\bar{f}(\ell h(s)) = s \ \& \ (\mathrm{E}g)(\bar{g}(\ell h(t)) = t \ \& \ (x)R(\bar{f}(x), \bar{g}(x)))].$$

The construction proceeds as follows.

(1) Choose $P_1 \subseteq O_1 \cap P$. P_1 exists by density of O_1.
(2) Choose f_1 and g_{11} so that $P_1(f_1, g_{11}) \neq \varnothing$.
(3) Choose $P_2 \subseteq P_1(f_1, g_{11}) \cap O_2$.
(4) Choose f_2 and g_{21} so that $f_1 > f_2$ and $P_2(f_2, g_{21}) \neq \varnothing$.
(5a) Note that $P_1(f_2, g_{11}) \cap P_2(f_2, g_{21}) \neq \varnothing$, because $P_1(f_1, g_{11}) \supseteq P_2(f_2, g_{21})$.
(5b) $P_1(f_2, g_{11}) = \cup\{P_1(f_2, x) \mid g_{11} > x\}$.
(6) It follows from (5a)–(5b) that g_{12} can be chosen so that $P_1(f_2, g_{12}) \cap P_2(f_2, g_{21}) \neq \varnothing$.
(7) Choose $P_3 \subseteq P_1(f_2, g_{12}) \cap P_2(f_2, g_{21}) \cap O_3$.

Steps (3)–(6) define stage 2 of the construction.
Stage i is similar and begins with the choice of P_i inside

$$P_1(f_{i-1}, g_{1,i-1}) \cap P_2(f_{i-1}, g_{2,i-2}) \cap \dots \cap P_{i-1}(f_{i-1}, g_{i-1,1}) \cap O_i.$$

Then f_i and g_{i1} are chosen so that $f_{i-1} > f_i$ and $P_i(f_i, g_{i1}) \neq \varnothing$. Next, as in step (6), g_{1i} is chosen so that

$$P_1(f_i, g_{1i}) \cap P_2(f_i, g_{2,i-2}) \cap \dots \cap P_{i-1}(f_i, g_{i-1,1}) \cap P_i(f_i, g_{i1}) \neq \varnothing.$$

Then in the same manner $g_{2,i-1}, g_{3,i-2}, \dots, g_{i-1,2}$ are chosen in succession. □

Louveau separation (Theorem 6.1) is proved by induction on γ, referred to below as the *main induction*. It is safe to assume $\gamma < \omega_1^{\mathrm{CK}}$ by Exercise 5.11.II and Exercise 6.7 below. The proof of 6.1 is broken into three lemmas (6.4–6.6), each of which makes use of the main induction.

Suppose $H \subseteq \omega^\omega$. The γ-closure of H, denoted by $\bar{H}^\gamma$, is defined by:

$$x \notin \bar{H}^\gamma \leftrightarrow (\mathrm{E}Y)[x \in Y \in \mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1 \ \& \ Y \cap H = \varnothing].$$

Note that γ-closure is the usual notion of closure for subsets of ω^ω with respect to the topology generated by $\mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1$.

6.4 Lemma. *If $A, B \subseteq \omega^\omega$ are Σ^1_1, and A is separable from B by $\bar{A}^\gamma$, then A is separable from B by some $\mathbf{\Pi}^o_\gamma(\Delta^1_1)$ set.*

Proof. The hypothesis of the lemma yields

(1) $$(X)_{X \in B}(\mathrm{E}Y)[X \in Y \in \mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1 \ \& \ Y \cap A = \emptyset].$$

The Y of (1) is a $\mathbf{\Pi}^o_{<\gamma}$ set that separates the Σ^1_1 sets Y and A. The main induction implies that Y and A can be separated by a $\mathbf{\Pi}^o_{<\gamma}(\Delta^1_1)$ set. Thus the matrix of (1) can be replaced by

(2) $$X \in Y \in \mathbf{\Pi}^o_{<\gamma}(\Delta^1_1) \quad \& \quad Y \cap A = \emptyset.$$

Let C be the set of all hyperarithmetic codes for $\mathbf{\Pi}^o_{<\gamma}(\Delta^1_1)$ sets. C is Π^1_1. Let Y_z be the set coded by $z \in C$. (1) becomes

(3) $$(X)_{X \in B}(\mathrm{E}Z)_{Z \in C}[x \in Y_Z \quad \& \quad Y_Z \cap A = \emptyset].$$

The predicates, $Z \in C \ \& \ X \in Y_Z$ and $Z \in C \ \& \ X \notin Y_C$, are Π^1_1 (as in Lemma 2.1.II), hence the matrix of (3) is Π^1_1. It follows from Exercise 2.11.II, a variant of Kreisel selection (Lemma 2.6.II), that the Z of (3) can be taken to be a hyperarithmetic function of X, call it $m(X)$. Let

$$B_0 \text{ be } \{m(X) | X \in B\}, \text{ and}$$
$$B_1 \text{ be } \{Z | Z \in C \ \& \ Y_Z \cap A = \emptyset\}.$$

Then $B_0 \in \Sigma^1_1$, $B_0 \subseteq B_1$ and $B_1 \in \Pi^1_1$. By Exercise 5.12.II, a variant of Exercise 3.9.II, there is a Δ^1_1 D such that $B_0 \subseteq D \subseteq B_1$. (Note that Kleene separation for Σ^1_1 sets was used to obtain D.)

Let E be $\cup\{Y_Z | Z \in D\}$. Then $B \subseteq E$, $E \cap A = \emptyset$ and $(\omega^\omega - E) \in \mathbf{\Pi}^o_\gamma(\Delta^1_1)$. □

The proof of Lemma 6.4 was a bounding argument typical of hyperarithmetic theory. The next lemma is an approximation result inspired by Gandy forcing. Let ω^ω have the Gandy topology as in Theorem 6.3. Suppose $H \subseteq \omega^\omega$. As usual H is said to be *nowhere* dense if the closure of H has an empty interior, and *meager* if H is contained in a countable union of nowhere dense sets. Theorem 6.3 is equivalent to: if H is meager and open, then H is empty.

6.5 Lemma. *If $\gamma < \omega_1^{\mathrm{CK}}$ and $H \in \mathbf{\Pi}^o_\gamma$, then there exists an L such that $L = \bar{L}^\gamma$ and $(H - L) \cup (L - H)$ is meager (in the sense of the Gandy topology).*

Proof. By induction on γ. Not to be confused with the main induction. Let H be $\omega^\omega - \bigcup_n \{H_n | n < \omega\}$, where for each n, $H_n \in \mathbf{\Pi}^o_{\gamma_n}$ for some $\gamma_n < \gamma$. By induction there is an L_n such that

$$\bar{L}_n^{\gamma_n} = L_n \quad \text{and} \quad (H_n - L_n) \cup (L_n - H_n) \text{ is meager.}$$

Define $L = -\cup\{Y \mid Y \in \mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1 \ \&\ Y \cap H \text{ is meager}\}$.

$H - L$ is meager, since a countable union of meager sets is meager.

$L - H = \bigcup_n [(L \cap H_n \cap L_n) \cup (L \cap (H_n - L_n))]$. So the meagerness of $L - H$ will follow from that of $L \cap L_n$. $L \cap L_n$ is closed. Suppose C is Σ^1_1 and $C \subseteq L \cap L_n$ with the intent of showing $C = \varnothing$. Since $C \in \Sigma^1_1$, the main induction implies (as in the passage from (1) to (2) in the proof of Lemma 6.4):

$$\bar{C}^{\gamma_n} = -\cup\{Y \mid Y \in \mathbf{\Pi}^o_{<\gamma_n}(\Delta^1_1) \ \&\ Y \cap C = \varnothing\}.$$

The set of codes for members of $\mathbf{\Pi}^o_{<\gamma_n}(\Delta^1_1)$ is a Π^1_1 set of hyperarithmetic reals. Consequently $\bar{C}^{\gamma_n}$ is Σ^1_1.

$$\bar{C}^{\gamma_n} \subseteq \bar{L}^{\gamma_n}_n = L_n; \quad \text{so} \quad \bar{C}^{\gamma_n} \cap H \subseteq L_n \cap H \subseteq L_n - H_n.$$

Since $L_n - H_n$ is meager, $\bar{C}^{\gamma_n} \cap H$ must also be meager. Clearly $\bar{C}^{\gamma_n} \in \mathbf{\Pi}^o_{<\gamma}$. It follows from the definition of L that $\bar{C}^{\gamma_n} \cap L = \varnothing$. Hence $C \cap L = \varnothing$ and so $C = \varnothing$.

If $Y \in \mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1$ and $Y \cap H$ is meager, then $Y \cap L = \varnothing$. So $L \supseteq \bar{L}^\gamma$. □

6.6 Lemma. *If A, $B \subseteq \omega^\omega$ are Σ^1_1, and A is separable from B by a $\mathbf{\Pi}^o_{<\gamma}$ set, then $\bar{A}^\gamma \cap B = \varnothing$.*

Proof. Suppose $A \subseteq H$ and $H \cap B = \varnothing$ for some $H \in \mathbf{\Pi}^o_\gamma$. As was noted just before Lemma 6.4, it is safe to assume $\gamma < \omega_1^{\mathrm{CK}}$. Lemma 6.5 provides an L such that $L = \bar{L}^\gamma$ and $(H - L) \cup (L - H)$ is meager. $A \subseteq H$, so $A - L$ is meager. L is closed and A is open, so $A - L$ is open. By Theorem 6.3 $A - L = \varnothing$. Hence $\bar{A}^\gamma \subseteq \bar{L}^\gamma = L$.

$H \cap B = \varnothing$, so $L \cap B$ is meager, and $\bar{A}^\gamma \cap B$ is meager. The main induction implies $\bar{A}^\gamma$ is open, as $\bar{C}^{\gamma_n}$ was open in the proof of Lemma 6.5. Thus $\bar{A}^\gamma \cap B$ is a meager, open set, hence empty. □

Louveau's separation theorem (6.1) is an immediate consequence of Lemmas 6.4 and 6.6. The proof of 6.1 in outline is as follows. Suppose A and B are separated by some $H \in \mathbf{\Pi}^o_\gamma$ for some $\gamma < \omega_1^{\mathrm{CK}}$. H can be well approximated by some L with the property that $\bar{L}^\gamma = L$. In essence the complement of L is the union of all Σ^1_1 forcing conditions of boldface rank less than γ that force generic reals out of H. A comparison of $\bar{A}^\gamma$ with L shows $\bar{A}^\gamma$ separates A and B. Baire's category theorem for the Gandy topology shows the error introduced by substituting L for H amounts to nothing. A bounding argument, together with the Kleene separation result, shows $\bar{A}^\gamma$ can be replaced by a $\mathbf{\Pi}^o_\gamma(\Delta^1_1)$ set. Induction on γ is used throughout to replace "$Y \in \mathbf{\Pi}^o_{<\gamma} \cap \Sigma^1_1$" by "$Y \in \mathbf{\Pi}^o_{<\gamma}(\Delta^1_1)$". The effect is, in 6.4, to make the matrix of (1) Π^1_1, and, in 6.5–6.6, to show the δ-closure of a Σ^1_1 set is Σ^1_1 $(\delta \leq \gamma)$. Less precisely, the effect of the induction on γ is to show: with respect to the forcing of rank γ sentences with Σ^1_1 conditions of rank less than γ, there is no difference between lightface and boldface conditions.

6.7–6.11 Exercises

6.7. If $D \in \mathbf{\Pi}^o_{\omega_1}$ and D has a Δ^1_1 code, then $D \in \mathbf{\Pi}^o_{\omega_1^{\mathrm{CK}}}$.

6.8. Suppose $D \in \mathbf{\Pi}^o_{\omega_1}$. Show $D \in \Delta^1_1$ iff D has a Δ^1_1 code.

6.9. Find a Σ^1_1 set $D \subseteq \omega^\omega$ such that $D \in \mathbf{\Pi}^o_{\omega_1^{\mathrm{CK}}} - \mathbf{\Pi}^o_{<\omega_1^{\mathrm{CK}}}$.

6.10. D is Borel iff $D \in \mathbf{\Pi}^o_{\omega_1}$.

6.11. Use Gandy forcing, as Harrington did, to prove Silver's theorem: Let E be a Π^1_1 equivalence relation whose field is ω^ω. Suppose there are uncountably many E-equivalence classes; show there are continuum many.