Chapter III
Σ_1^1 Predicates of Reals

This chapter concentrates on basis theorems for Σ_1^1 and Π_1^1 predicates. In addition hyperarithmetic quantifiers are classified, and closure points of Π_1^0, and monotonic Π_1^1, inductive definitions are computed.

1. Basis Theorems

Let B be a set of functions and $\mathscr{P}$ a family of predicates of the form $P(f)$. B is said to be a basis for $\mathscr{P}$ if for each $P(f) \in \mathscr{P}$,

$$(\mathrm{Ef})P(f) \to (\mathrm{Ef})_{f\in B} P(f).$$

A basis B for $\mathscr{P}$ is of maximal interest when the complexity of a typical member of B is minimal relative to that of a typical member of $\mathscr{P}$. Basis theorems permeate the literature of higher recursion theory, possibly because they clarify the scope of certain quantifiers. Thus Kleene's basis theorem implies that the scope of an existential function quantifier, when applied to a Σ_1^1 predicate, can be reduced to those functions recursive in Kleene's O. A result of Shoenfield states that the Δ_2^1 reals are a basis for the Σ_2^1 predicates. The Kleene and Shoenfield arguments have in common the analysis of quantifiers by ordinals as in Section 5.I. In general basis theorems are proved by exploiting an appropriate assignment of ordinals to finite partial functions.

Every Σ_1^1 predicate $P(f)$ can be put in the form

$$(\mathrm{Eg})(x)R(\bar{f}(x), \bar{g}(x))$$

for some recursive R. Therefore finding a solution f of $P(f)$ is equivalent to finding a solution $\langle f, g\rangle$ of $(x)R(\bar{f}(x), \bar{g}(x))$. Thus the basis problem for Σ_1^1 predicates is the same as that for Π_1^0 predicates. The recursive predicates have a trivial basis (Exercise 1.5), but the next result indicates the surprising complexity introduced by a single universal number quantifier.

1.1 Theorem (Kleene). *The hyperarithmetic functions are not a basis for the Π_1^0 predicates.*

Proof. Let $P(f)$ be $f \notin \text{HYP}$. $P(f)$ is Σ_1^1 by Corollary 1.4(ii).II. Put $P(f)$ in the form $(\text{E}g)(x)R(\bar{f}(x), \bar{g}(x))$. Then $(x)R(\bar{f}(x), \bar{g}(x))$ has no hyperarithmetic solution $\langle f, g\rangle$. □

1.2 Branches of Trees. According to Theorem 1.1 there is a recursive predicate R such that no hyperarithmetic f satisfies $(x)R(\bar{f}(x))$, but some f does. R corresponds to a recursive tree T_R with the property that f is a solution of $(x)R(\bar{f}(x))$ if and only if f is an infinite branch of T_R. A tree, by definition, is a partial ordering with a greatest element and such that every pair of incomparables has no lower bound. T_R is the partial ordering of sequence numbers (as in subsection 5.1.I) consisting of those $\bar{f}(x)$ that satisfy $(i)_{i<x}R(\bar{f}(i))$. (Visualize the tree upside down; the greatest element is on top and the branches go down.) f satisfies $(x)R(\bar{f}(x))$ iff $\bar{f}(0) > \bar{f}(1) > \bar{f}(2) > \ldots$ is an infinite branch.

The next theorem says that a recursive tree with an infinite branch must have one recursive in Kleene's O. If it still seems hard to believe that such a tree need not have a hyperarithmetic branch, pay attention to the difference between finite branching and infinite branching. According to Exercise 1.7, a recursive tree with finite branching and an infinite branch has one recursive in $0'$. Later it will be shown that a recursive tree with less than 2^ω branches has only hyperarithmetic branches.

1.3 Theorem (Kleene). *The functions recursive in O are a basis for the Σ_1^1 predicates.*

Proof. Let R be recursive, and assume $(x)R(\bar{f}(x))$ holds for some f. Define $Q(s)$ to be: s is a sequence number in the field of T_R (defined in subsection 1.2) and the part of T_R below s is not wellfounded. Recall: $>$ orders T_R as in 5.1.I.

$Q(s)$ is Σ_1^1, hence recursive in O by Theorem 5.4.I. Define f recursively in Q so that

$$\bar{f}(n+1) = \mu s[Q(s) \quad \& \quad \ell h(s) = n+1 \quad \& \quad \bar{f}(n) > s].$$

f is welldefined because

$$(s)(\text{E}t)[Q(s) \rightarrow Q(t) \quad \& \quad s > t].$$

In short the step-by-step definition of f avoids all wellfounded subtrees of T_R. □

The recursive ordinals play a significant, if hidden, role in the proof of Kleene's basis theorem. For each s in the field of T_R, let T_R^s be the part of T_R below s. If T_R^s is wellfounded, then it has an ordinal height denoted by $|T_R^s|$ defined in subsection 4.2.I. Let $|s|$ be $|T_R^s|$ when T_R^s is wellfounded. Since R is recursive, $|s|$, when defined, is a recursive ordinal. $Q(s)$ is equivalent to: $|s|$ is not defined. The infinite path f defined in the proof of Theorem 1.2 is the "leftmost" branch that omits all s such that $|s|$ is defined. According to Exercise 1.8, the "leftmost" branch has the same hyperdegree as Kleene's O or is hyperarithmetic. H. Friedman and D. A. Martin have independently shown: if a recursive tree has a nonhyperarithmetic branch, then it has one in each hyperdegree greater than or equal to that of Kleene's O.

1.4 Theorem (Gandy). *$\{f \mid f <_h O\}$ is a basis for the Σ_1^1 predicates.*

Proof. Suppose $P(f)$ is Σ_1^1 and $(\mathrm{E}f)P(f)$. Then

(1) $$P(f) \quad \& \quad g \not\leq_h f$$

is Σ_1^1 according to subsection 5.6.II.
(1) has a solution since only countably many g's are hyperarithmetic in each f. By Theorem 1.3, (1) has a solution $\langle f, g\rangle \leq_T O$. Thus

$$P(f) \quad \& \quad f \leq_h O \quad \& \quad g \leq_h O \quad \& \quad g \not\leq_h f.$$

Hence $f <_h O$. $\cdot\ \square$

1.5 Corollary *The Σ_1^1 set $\{f \mid \omega_1^f = \omega_1^{\mathrm{CK}}\}$ is a basis for the Σ_1^1 predicates.*

Proof. Suppose $P(f)$ is Σ_1^1 and $(\mathrm{E}f)P(f)$. By Theorem 1.4 there is an f such that $P(f)$ and $f <_h O$. Corollary 7.7.II implies $\omega_1^f = \omega_1^{\mathrm{CK}}$. $\square$

References to model theory are in order. The proof of Theorem 1.3 is analogous to a Henkin-style construction of a countable, consistent set of sentences. In the setting of first order logic, a Henkin construction can be regarded as a step-by-step development of an infinite path through a tree with finite branching. The infinite branching allowed in the proof of Theorem 1.2 corresponds to sentences of countably infinite length having the form of a disjunction. In the same vein the finite proofs of first order logic correspond to infinite proofs encoded by hyperarithmetic reals. This vein will be mined further later, but before leaving it, observe that Corollary 1.5 has the appearance of a type-omitting result. The type omitted is: x is a non-recursive ordinal recursive in f.

Another basis result of the type-omitting sort will be proved later in this Chapter. If states: if $A \notin \mathrm{HYP}$, then $\{f \mid A \not\leq_h f\}$ is a basis for the Σ_1^1 predicates. Still another will come up in Chapter IV. Theorem 2.2.IV states: if a Π_1^1 set has positive measure, then it has a hyperarithmetic member.

1.6–1.9 Exercises

1.6. Let F be the set of all f unequal to 0 on only finitely many coordinates. Show F is a basis for the recursive predicates.

1.7. Suppose T is a recursive tree with finite branching and an infinite branch. Show T has an infinite branch recursive in $0'$. ($0'$ is the complete recursively enumerable subset of ω.)

1.8. Suppose T is a recursive tree with infinite branching. Let f and g be infinite branches. f is said to be to the left of g if for some $x, \bar{f}(x) = \bar{g}(x)$ and $f(x) < g(x)$. Assume T has an infinite branch. Show the leftmost branch of T is either hyperarithmetic or has the same hyperdegree as Kleene's O. Show T has a branch of hyperdegree less than that of O.

1.9. (H. Friedman). Use Corollary 1.4 to show there is a model M of set theory such that the ordinals of M are a proper end extension of the recursive ordinals, and such that $\sup\{\delta \mid \delta \in M \;\&\; \delta \text{ is standard}\} = \omega_1^{CK}$. (Thus M omits ω_1^{CK}.)

2. *Unique Notations for Ordinals*

A is said to be a set of unique notations for the constructive ordinals if $A \subseteq O$ and the restriction of $<_O$ to A^2 is a linear ordering whose ordertype is ω_1^{CK}. In brief A is a branch that goes all the way through O. Clearly each constructive ordinal has just one notation in A. It will soon be proved that there exists a Π_1^1 set O_1 of unique notations. O_1 will be useful in the analysis of hyperarithmetic quantifiers in the next section, and in the definitions of the basic concepts of metarecursion theory in Part B.

2.1 Lemma (Kleene). *There exists a recursive linear ordering with an infinite descending sequence, but no hyperarithmetic such sequence.*

Proof. As noted in subsection 1.2, there is a recursive R such that no hyperarithmetic f satisfies $(x)R(\bar{f}(x))$ but some f does. T_R, the partial ordering of sequence numbers $\bar{g}(x)$ such that $(i)_{i<x}R(\bar{g}(x))$, is a recursive tree with an infinite branch but no hyperarithmetic such branch. T_R can be linearized by ordering sequence numbers in the Kleene–Brouwer fashion: $\bar{g}(u) <_{KB} \bar{h}(v)$ if

$$(1)\qquad \begin{array}{l} \text{(i) } u > v \quad \& \quad (i)_{i<v}(g(i) = h(i)), \quad \text{or} \\ \text{(ii) } (\mathrm{E}k)[k < \min(u, v) \quad \& \quad g(k) < h(k) \quad \& \quad (i)_{i<k}(g(i) = h(i))]. \end{array}$$

Let $<_R$ be the restriction of $<_{KB}$ to the sequence numbers in the field of T_R. Each solution f of $(x)R(\bar{f}(x))$ defines an infinite descending sequence of $<_R$ according to (1)(i). Suppose $<_R$ has a hyperarithmetic infinite descending sequence $x_0 >_R x_1 >_R x_2 \ldots$ with the intent of showing $(x)R(\bar{f}(x))$ has a hyperarithmetic solution. Note that any set of sequence numbers of bounded length is wellordered by $<_{KB}$. Consequently the lengths of the x_i's are unbounded. Hence it is safe to assume that $\ell h(x_i)$ is a strictly increasing function of i, since some hyperarithmetic subsequence of the x_i's must have that property. An induction on n shows

$$(n)(\mathrm{E}j)(i)_{i \geq j}[(x_i)_n = (x_j)_n].$$

Define $t(n)$ by $\mu j(i)_{i \geq j}[(x_i)_n = (x_j)_n]$, and $f(n)$ by $(x_{t(n)})_n - 1$. Then f is a hyperarithmetic solution of $(x)R(\bar{f}(x))$. $\square$

Model theory yields another view of Lemma 2.1. Suppose M is a model of a suitable fragment of set theory and the ordinals of M are a proper end extension of the recursion ordinals. Thus the recursive ordinals in the sense of M include some nonstandard ordinals as well as the standard recursive ordinals. Let δ be a nonstandard recursive ordinal of M, and let $<_R \in M$ be a recursive wellordering whose ordertype is δ. Then $<_R$ has an infinite descending sequence outside M but not inside M. If $<_R$ had a hyperarithmetic, infinite descending sequence, then that sequence would belong to M, since all standard recursive ordinals belong to M, hence all standard H-sets belong to M. The next lemma computes the ordertype of a nonstandard recursive ordinal.

2.2 Lemma. *Let $<_R$ be a recursive linear ordering with an infinite descending sequence but no hyperarithmetic such sequence.*

(*i*) (*Gandy*) *Let Q be the maximum, wellordered initial segment of $<_R$. Then Q is Π_1^1 but not Σ_1^1, and the restriction of $<_R$ to Q has ordertype ω_1^{CK}.*

(*ii*) (*Harrison*) *The ordertype of $<_R$ is $\omega_1^{\mathrm{CK}}(1+\eta)+\gamma$, where η is the ordertype of the rationals, and γ is a recursive ordinal.*

Proof

(i) Q is Π_1^1 since

$$x \in Q \leftrightarrow x \in <_R \quad \& \quad \sim(\mathrm{Ef})(n)[f(n+1) <_R f(n) <_R x].$$

To see that Q is not Σ_1^1, choose b in the field of $<_R$ but not in Q. Then there is an infinite descending sequence below b and above Q. Suppose Q is Σ_1^1. Then the Π_1^1 relation

$$x <_R b \quad \& \quad y <_R x \quad \& \quad y \notin Q$$

can be uniformized by some Π_1^1 $P(x, y)$ according to Theorem 2.3.II. Let $g(O)$ be some x below b and above Q, and $g(n+1)$ the unique y such that $P(g(n), y)$. Then g is a hyperarithmetic, infinite descending sequence in $<_R$.

Suppose the ordertype of $<_R$ restricted to Q were less than ω_1^{CK}. Then there would be an $a \in O$ such that for all x,

$$x \in Q \leftrightarrow x \in \text{field of } <_R \ \& \ (\mathrm{Ef})[f \text{ is a one-one orderpreserving map of } <_R^x \text{ into } W_{q(a)}]. \tag{1}$$

$<_R^x$ is the initial segment of $<_R$ below x, and $W_{q(a)}$ is the recursively enumerable initial segment of $<_O$ below a, as in Theorem 3.5.I. (1) implies Q is Σ_1^1.

If the ordertype of $<_R$ restricted to Q were greater than ω_1^{CK}, then ω_1^{CK} would be a recursive ordinal.

(ii) Let x, y belong to the field of $<_R$. Call $x \sim y$ if $[x, y]$ or $[y, x]$ is a closed interval of $<_R$ that contains no infinite descending sequences. $\sim$ is a Π_1^1 equivalence relation that partitions $<_R$ into intervals. Each such interval is Π_1^1, since a typical interval is $\{y \mid y \sim x\}$.

Each interval has an R-least member. Otherwise

$$(u)(\mathrm{E}v)[u \sim x \to v \sim x \quad \& \quad v <_R u].$$

Π_1^1 uniformization (Theorem 2.3.I) makes it possible to construe v as a hyperarithmetic function of u, and so yields a hyperarithmetic, infinite descending sequence in $<_R$.

Since $\{y|y \sim x\}$ has a least member, it follows from the definition of $\sim$ that the restriction of $<_R$ to $\{y|y \sim x\}$ is a wellordering. To determine the ordertype of $\{y|y \sim x\}$, consider $U_x = \{y|x <_R y\}$. If U_x contains no infinite descending sequence, then the ordertype of U_x, hence of $\{y|y \sim x\}$ is a recursive ordinal. If U_x contains an infinite descending sequence, then (i) implies that the ordertype of the maximal, wellfounded initial segment of U_x, hence of $\{y|y \sim x\}$, is ω_1^{CK}. Note there is at most one $\{y|y \sim x\}$ whose ordertype is less than ω_1^{CK}.

If $x <_R y$ and $x \not\sim y$, then there is a z such that $x <_R z <_R y$, $x \not\sim z$ and $z \not\sim y$; otherwise ω_1^{CK} would be a recursive ordinal. $\square$

2.3 Proposition. *There exists a recursive g such that for all e: if $W_e \subseteq O$ and the restriction of $<_O$ to W_e is linear, then*

$$g(e) \in O \quad \& \quad (a)[a \in W_e \to a <_O g(e)].$$

Proof. Let $W_e^n = \{u|(\mathrm{E}y)_{y<n} T(e, u, y)\}$. Then $W_e = \cup\{W_e^n|n < \omega\}$, and each W_e^n is finite. Let q be the recursive function of Theorem 3.5.I. Define

$$\langle u, v\rangle \in W^* \quad \text{by} \quad \langle u, v\rangle \in W_{q(2^v)}.$$

Then W^* is recursively enumerable, and the restriction of W^* to O^2 is $<_O$.

Let $\theta(e, i)$ be a partial recursive function with the following property. If $\theta(e, i)$ is defined for all $i < n$, and if the restriction of W^* to

$$W_e^n \cup \{\theta(e, i)|i < n\} \cup \{1\} \tag{1}$$

is linear, then $\theta(e, n)$ is defined and equal to 2^m, where m is the greatest member of (1) with respect to W^*.

There is a recursive function s such that $\{s(e)\}(i) \simeq \theta(e, i)$ for all e and i. Define $g(e)$ to be $3 \cdot 5^{s(e)}$.

Suppose $W_e \subseteq O$ and the restriction of $<_O$ to W_e is linear. By induction on i, $\theta(e, i)$ is defined and belongs to O, and the restriction of $<_O$ to $W_e \cup \{\theta(e, j)|j < i\} \cup \{1\}$ is linear. In addition if $a \in W_e$, then $a <_O \theta(e, i)$ for some i, and so $a <_O g(e)$. $\square$

2.4 Theorem (Feferman & Spector 1962). *There exists a Π_1^1 set O_1 of unique notations for the constructive ordinals, that is $O_1 \subseteq O$ and the restriction of $<_O$ to O_1 is linear.*

Proof. As in Gandy 1960. Let $<_R$ be the recursive linear ordering of Lemma 2.1. $<_R$ has a maximum wellordered initial segment Q of ordertype ω_1^{CK} according to Lemma 2.2(i). The function g of Proposition 2.3 will be used to imbed Q in O. The range of the imbedding will be cofinal in O_1.

There exist recursive functions j and I such that

$$W_{j(e,x)} = \{\{e\}(a) \mid a <_R x\}, \text{ and}$$
$$\{I(e)\}(x) = g(j(e, x)).$$

Fix c so that $\{I(c)\} \simeq \{c\}$. Let f be $\{c\}$. Then

$$W_{j(c,x)} = \{f(a) \mid a <_R x\}, \text{ and}$$
$$f(x) = g(j(c, x)).$$

An induction on $<_R$ shows

$$[x, y \in Q \quad \& \quad x <_R y] \rightarrow f(x) <_O f(y).$$

The range of f has gaps easily filled. Define

$$a \in O_1 \leftrightarrow (\mathrm{Ex})[x \in Q \quad \& \quad a \in W_{p(f(x))}],$$

where p is the recursive function of Theorem 3.5.I. $\square$

The proof of Theorem 2.4 given above is quite different from the one presented by Feferman and Spector 1962. Their approach is recursion theoretic in appearance and model theoretic in reality. They study O^*, a nonstandard version of O. The elements of O^* are notations for ordertypes of recursive linear orderings without hyperarithmetic, infinite descending sequences. They show O^* is a proper end extension of O (cf. Exercise 2.6). Then for each $e \in O^* - O$, the set

$$\{a \mid a <_{O^*} e \ \& \ a \in O\}$$

is a Π_1^1 set of unique notations.

2.5–2.7 Exercises

2.5. Simplify the proof of Proposition 2.3.

2.6. (Feferman & Spector 1962). Let $A(X)$ be the arithmetic closure condition of subsection 2.1.I used to define $<_O$. Define $<_{O^*}$ to be $\cap\{X \mid \langle 1, 2\rangle \in X \ \& \ A(X) \ \& \ X \in \mathrm{HYP}\}$. Show $<_{O^*}$ is a proper end extension of $<_O$. Let $e \in O^* - O$. Show $\{a \mid a <_{O^*} e \ \& \ a \in O\}$ is a Π_1^1 set of unique notations.

2.7. Let M be a model of set theory in which the integers are standard and ω_1^{CK} is not. Let δ be a nonstandard recursive ordinal. What is the ordertype of δ in the real world?

3. *Hyperarithmetic Quantifiers*

Sometime ago A. Mostowski conjectured that each Π_1^1 predicate $P(x)$ could be put in the form

$$(1) \qquad (\mathrm{E}f)_{f \in \mathrm{HYP}} A(f, x)$$

for some arithmetic A. Earlier Kleene had shown that (1) is Σ_1^1 for every arithmetic A. Consequently Mostowski's conjecture states that with respect to arithmetic predicates, the effect of applying an universal function quantifier is the same as applying an existential hyperarithmetic function quantifier. The conjecture becomes plausible in the light of model theory. Let $\mathscr{F}$ be a sentence of $\mathscr{L}\omega_1^{\mathrm{CK}}, \omega$, an extension of first order logic, in which quantifier prefixes are finite in length but conjunctions and disjunctions can be countably infinite. Assume $\mathscr{F}$ is encoded by a hyperarithmetic real. Thus $\mathscr{F}$ is built up in the same manner as a Borel set of recursive ordinal rank and hyperarithmetic complexity. According to an appropriate completeness theorem, (2) and (3) are equivalent.

$$(2) \qquad \mathscr{F} \text{ is true in every countable structure.}$$

$$(3) \qquad \mathscr{F} \text{ has a proof in } \mathscr{L}\omega_1^{\mathrm{CK}}, \omega.$$

On its face (2) is a Π_1^1 predicate of $\mathscr{F}$. (3) is clearly existential. Less clear is that proofs in $\mathscr{L}\omega_1^{\mathrm{CK}}, \omega$ are hyperarithmetically encodable. Perhaps it helps to note that finite sentences have finite proofs, and to recall claims made above that hyperarithmetic is analogous to finite.

3.1 Lemma (Kleene). *If $A(x,f)$ is arithmetic, then $(\mathrm{E}f)_{f \in \mathrm{HYP}} A(x,f)$ is Π_1^1.*

Proof. An immediate consequence of the proof of Corollary 1.4(ii).II. □

The proof of Mostowski's conjecture, or the Spector–Gandy theorem as it is now called, is based on properties of nonstandard end extensions of the hyperarithmetic hierarchy expressed by the next two lemmas.

3.2 Lemma (Enderton & Putnam). *There exists a recursive function g such that for all e: if $3 \cdot 5^e \in O$ and*

$$(n)[H_{\{e\}(n)} \leq_T X],$$

then $H_{3 \cdot 5^e} = \{g(e)\}^{X''}$.

Proof. Let $H(a, X)$ be the Π_2^0 predicate of Theorem 4.2.II. Define $P(a, z, X)$ by $\{z\}^X$ is total & $H(a, \{y \mid \{z\}^X(y) = 1\})$.

If $a \in O$, then $P(a, z, X)$ is equivalent to $H_a = \{z\}^X$. $P(a, z, X)$ is Π_2^0, hence recursive in X'' uniformly in X. In other words, for some e_0,

$$P(a, z, X) \leftrightarrow \{e_0\}^{X''}(a, z)$$

for all a, z and X. Let $\phi(e, y, X)$ be the partial function defined by

$$\{\mu z P(\{e\}((y)_1), z, X)\}^X((y)_0).$$

For some e_1, $\phi(e, y, X) \simeq \{e_1\}^{X''}(e, y)$ for all e, y and X. Let g be a recursive function such that

$$\{g(e)\}^{X''}(y) \simeq \phi(e, y, X).$$

Suppose $3 \cdot 5^e \in O$ and $H_{\{e\}(n)} \leq_T X$ for all n. Then

$$\begin{aligned} y \in H_{3 \cdot 5^e} &\leftrightarrow (y)_0 \in H_{\{e\}((y)_1)} \\ &\leftrightarrow (\mathrm{E}z)[P(\{e\}((y)_1), z, X) \quad \& \quad (y)_0 \in \{z\}^X] \\ &\leftrightarrow \{g(e)\}^{X''}(y) = 1. \quad \square \end{aligned}$$

3.3 Lemma. *Suppose $\{A_n \mid n < \omega\}$ is a sequence of sets such that $A'_{n+1} \leq_T A_n$ for all n. Then $X \leq_T A_0$ for all hyperarithmetic X.*

Proof. An induction on $<_O$ shows $(n)[H_b \leq_T A_n]$ for all $b \in O$.

If b is 2^m, then $H_m \leq_T A_{n+1}$ by induction, and so $H_{2^m} \leq_T A'_{n+1} \leq_T A_n$.

Suppose b is $3 \cdot 5^e$. Then by Lemma 3.2,

$$H_b \leq_T A''_{n+2} \leq_T A'_{n+1} \leq_T A_n. \quad \square$$

The existence of a sequence $\{A_n \mid n < \omega\}$ such that $A'_{n+1} \leq_T A_n$ for all n is a consequence of the fact that $O \notin \Sigma_1^1$ (cf. exercise 3.6). Model theory also provides such sequences. Let M be a nonstandard model of set theory in which the integers are standard and ω_1^{CK} is not. If δ is a nonstandard recursive ordinal, then there is an infinite descending sequence (outside M) $\delta = \delta_0 > \delta_1 > \delta_2 > \ldots$ of nonstandard recursive ordinals.

The associated H-sets satisfy the hypothesis of Lemma 3.3. $H'_{\delta_{n+1}} \leq_T H_{\delta_n}$, since H-sets are always separated by at least one jump.

Assume $<$ is a linear ordering of a subset of ω. X is said to be a *Turing jump hierarchy* on $<$ if

(1) $$(m)\,(n)\,[m < n \rightarrow (X)'_m \leq_T (X)_n].$$

If (1) holds, $<$ is said to *support* X. Note that (1) is an arithmetic statement about $<$ and X. Not much is known about those linear orderings that support Turing jump hierarchies. Obviously every recursive wellordering does. H. Friedman has found a recursive linear ordering of ordertype $\omega_1^{\mathrm{CK}}(1+\eta)$ that does not, although another recursive linear ordering of the same ordertype does.

The Spector–Gandy theorem is a consequence of the next lemma, which characterizes recursive wellorderings by an unexpected appeal to definability notions.

3.4 Lemma (Spector, Gandy). *Let $<$ be a recursive linear ordering. $<$ is a wellordering iff $<$ supports some hyperarithmetic Turing jump hierarchy.*

Proof. If $<$ is a wellordering, then it supports some proper initial segment of the H-sets.

Suppose $<$ supports some hyperarithmetic X and has an infinite descending sequence $m_0 > m_1 > \dots$. Then $(X)'_{m_{i+1}} \leq_T (X)_{m_i}$ for all i, and Lemma 3.3 implies X is recursive in $(X)_{m_1}$. But then

$$X' \leq_T (X)'_{m_1} \leq_T (X)_{m_0} \leq_T X. \quad \square$$

3.5 Theorem (Spector 1959, Gandy 1960). *Each Π_1^1 predicate $P(x)$ can be put in the form*

$$(\mathrm{E}Y)_{Y \in \mathrm{HYP}} A(Y, x)$$

for some arithmetic A.

Proof. According to Proposition 5.3 of Chapter I, there is a partial ordering $S(x)$ of sequence numbers, recursive in x uniformly, such that

$$P(x) \leftrightarrow S(x) \text{ is wellfounded.}$$

$S(x)$ can be linearized as in the proof of Lemma 2.1. Let $S_{KB}(x)$ be the Kleene–Brouwer ordering of $S(x)$. Then

$$P(x) \leftrightarrow S_{KB}(x) \text{ is wellordered.} \tag{1}$$

By Lemma 3.4 the right side of (1) is equivalent to: $S_{KB}(x)$ supports some hyperarithmetic, Turing jump hierarchy. $\square$

3.6–3.12 Exercises

3.6. Find a sequence $\{A_n | n < \omega\}$ of sets such that $A'_{n+1} \leq_T A_n$ for all n. (Cf. Exercise 2.6.)

3.7. Find a sequence $\{A_n | n < \omega\}$ of sets such that $A'_{n+1} \equiv_T A_n$ for all n.

3.8. (Enderton & Putnam). For each $n \geq 0$, let $X^{(n+1)}$ be $(X^{(n)})'$, and let $X^{(0)}$ be X. Suppose $X^{(n)} \leq_T Y$ for all n. Let $X^{(\omega)}$ be $\{\langle u, v\rangle | u \in X^{(v)}\}$. Show $X^{(\omega)} \leq_T Y''$.

3.9. Let M be a nonstandard model of set theory in which the integers are standard and ω_1^{CK} is not. Suppose δ is a nonstandard recursive ordinal. Let $\delta = \delta_0 > \delta_1 > \delta_2 > \ldots$ be an infinite descending sequence not in M. Show $H'_{\delta_{i+1}} \leq_T H_{\delta_i}$ for all i.

3.10. Formulate and prove: $<$ is a wellordering iff $<$ supports a Turing jump hierarchy X such that X is hyperarithmetic in $<$.

3.11. Show $(\mathrm{E}B)[B \in 2^{\omega}\ \&\ B \in \mathrm{HYP}\ \&\ R(B, x, Y)]$ is Π_1^1 if R is Π_1^1.

3.12. Suppose $P(X, Y)$ is Δ_1^1 and $(X)[(\mathrm{E}Y)P(X, Y) \rightarrow (E_1 Y)P(X, Y)]$. Show $\hat{X}(\mathrm{E}Y)P(X, Y)$ is Δ_1^1.

3.13. Show each Π_1^1 predicate $P(X)$ can be put in the form

$$(\mathrm{E}Y)_{Y \leq_h X} A(X, Y)$$

for some arithmetic A by relativizing 3.5 to X.

4. The Ramified Analytic Hierarchy

The ramified analytic hierarchy was introduced by Kleene, possibly to show the hyperarithmetic sets could be defined by an iterative process couched in the language of analysis. The length of the process is of course ω_1^{CK}, and a typical stage adds all sets (of numbers) that have analytical definitions over the universe of sets already defined. The equivalence of ramified analytic and hyperarithmetic definability is closely related to the existence of upward persistent Δ_1^1 definitions of H-sets, as in Corollary 4.4.II. The equivalence was clarified by Kreisel, who showed HYP is the least ω-model of the Δ_1^1 comprehension axiom schema.

The following account of the Kleene–Kreisel results is influenced by the needs of Chapter IV, which studies various sideways extensions, generic and otherwise, of the ramified analytic hierarchy.

One caution must be observed. Gandy and Putnam have extended the ramified analytic hierarchy to its natural endpoint, β_0, an ordinal far greater than ω_1^{CK}. Any reference to RAH in this book is a reference to RAH through the recursive ordinals only.

4.1. The Language $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$. Feferman 1965 introduced the language $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ to study Cohen-generic extensions of the ramified analytic hierarchy (equivalent to the hyperarithmetic hierarchy), as in Chapter IV below. $\mathscr{T}$ is a set constant that denotes a subset of ω. The other primitive symbols of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ are: $x, y, z, \ldots$ (number variables); $\underline{0}, \underline{1}, \underline{2}, \ldots$ (numerals); $X^{\beta}, Y^{\beta}, Z^{\beta}, \ldots$ (set variables of rank β for each $\beta < \omega_1^{\mathrm{CK}}$); $X, Y, Z, \ldots$ (unranked set variables); $+$ (plus), $\cdot$ (times) and $'$ (successor); and $\in$ (membership).

The number theoretic terms are: numerals, number variables. $t + s$, $t \cdot s$, and t', where t and s are number theoretic terms.

The atomic formulas are $t = s$ and $t \in U$, where t and s are number theoretic terms and U is $\mathscr{T}$ or a set variable.

Formulas are built up from atomic formulas, propositional connectives, and quantifiers over all three types of variables. A formula $\mathscr{F}$ is *ranked* if all of its set variables are ranked. The *ordinal rank* of a ranked $\mathscr{F}$ is the least α such that $\beta \leq \alpha$ for all X^β occurring freely in $\mathscr{F}$ and $\beta < \alpha$ for all bound X^β in $\mathscr{F}$.

4.2. Gödel Numbers. An assignment of Gödel numbers to formulas is needed to classify relations between formulas such as truth or forcing. The Gödel numbering is routine save for the complication caused by the use of recursive ordinals as superscripts of ranked variables. Since the correspondence between primitive symbols and their Gödel numbers should be as effective as possible, it follows that the Gödel number of ranked variable X^β should resemble a notation for β. Let O_1 be the Π_1^1 set of unique notations for the recursive ordinals provided by Theorem 2.4. Then the Gödel number of X^β is $\langle 2, a \rangle$, where $a \in O_1$ and $|a| = \beta$. The remaining primitive symbols are numbered routinely with the result that:

(1) The set of Gödel numbers of formulas of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ is Π_1^1.
(2) For each $\alpha < \omega_1^{\mathrm{CK}}$ the set of Gödel numbers of ranked formulas of rank less than α is recursively enumerable uniformly in the unique notation for α in O_1.

(2) is a consequence of Theorem 3.5(i).I, since the restriction of $<_O$ to O_1 is linear.

4.3. The Structure $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$. Assume $T \subseteq \omega$. The structure $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ consists of those sets ramified analytic in T via ordinals less than ω_1^{CK}. It will be of interest chiefly when $\omega_1^{\mathrm{CK}} = \omega_1^T$, and will prove useful in the study of such T's.

A simultaneous recursion on $\beta < \omega_1^{\mathrm{CK}}$ defines the structure $\mathscr{M}(\beta, T)$ and truth in $\cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}$ for sentences of rank at most β.

(1) Suppose $\mathscr{F}$ is a sentence of rank at most β. Thus if X^α is a bound variable of $\mathscr{F}$, then $\alpha < \beta$. $\mathscr{F}$ is true in $\cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}$ if $\mathscr{F}$ is true when $\mathscr{T}$ is interpreted as T; $X^\alpha, Y^\alpha, \ldots$ are restricted to $\mathscr{M}(\alpha, T)$; $x, y, \ldots$ range over ω; and $+, \cdot, '$, the numerals and $\in$ have their usual meaning.

(2) $\mathscr{M}(\beta, T)$ is the collection of all sets defined as follows. Let $\mathscr{G}(x)$ be a ranked formula of rank at most β whose only free variable is x. Define $\hat{x}\mathscr{G}(x)$ to be the set of all n such that

$$\mathscr{G}(\bar{n}) \text{ is true in } \cup\{\mathscr{M}(\alpha, T) | \alpha < \beta\}.$$

$\hat{x}\mathscr{G}(x)$ is a typical member of $\mathscr{M}(\beta, T)$.

Define $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ to be $\cup\{\mathscr{M}(\alpha, T) | \alpha < \omega_1^{\mathrm{CK}}\}$. Finally define $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$ ($\mathscr{F}$ is true in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$) for an arbitrary sentence $\mathscr{F}$ of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ by allowing the unranked set variables to range over $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ and interpreting the remaining symbols as in (1).

A sentence of $\mathscr{L}(\omega_1^{\mathrm{CK}}, \mathscr{T})$ is said to be Σ_1^1 (or existential) if it is ranked, or of the form $(\mathrm{E}X_1) \ldots (\mathrm{E}X_n)\mathscr{G}$ for some formula $\mathscr{G}$ with no unranked bound variables. Lemma 4.5 says that $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$, restricted to Σ_1^1 $\mathscr{F}$'s, is Π_1^1. This means that the predicate, e is the Gödel number of an existential sentence true in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$, is Π_1^1 with e and T as its free variables.

4.4 Full Ordinal Rank. The notion of ordinal rank defined in subsection 4.1 is not fine enough for some of the inductions that lie ahead.

Suppose $\mathscr{F}$ is a ranked sentence. The full ordinal rank of $\mathscr{F}$ is a function

$$r: \quad [-1, \omega_1^{\mathrm{CK}}) \to \omega$$

such that for each $\beta \in \omega_1^{\mathrm{CK}}$, $r(\beta)$ is the number of occurrences of $(\mathrm{E}X^\beta)$, (X^β), $(\mathrm{E}Y^\beta), \ldots$ in $\mathscr{F}$, and such that $r(-1)$ is the number of occurrences of (x), $(\mathrm{E}y)$, $(y), \ldots$ in $\mathscr{F}$. Clearly r is 0 for all but finitely many arguments. Suppose r_0 and r_1 are full ordinal ranks. r_0 is said to be less than r_1 (in symbols $r_0 < r_1$) if there is a $\beta \in [-1, \omega_1^{\mathrm{CK}})$ such that

$$(1) \qquad r_0(\beta) < r_1(\beta) \quad \& \quad (\alpha)_{\alpha > \beta}(r_0(\alpha) = r_1(\alpha)).$$

If $r_0 < r_1$, then at most one β satisfies (1), namely the greatest β such that $r_0(\beta) < r_1(\beta)$. $<$ is a wellordering of height ω_1^{CK}.

The key property of $<$ is the reduction of full ordinal rank occasioned by the substitution of an appropriate set term for a bound variable. Consider a sentence of the form $(\mathrm{E}X^\beta)\mathscr{F}(X^\beta)$. Let $\mathscr{G}(x)$ be a ranked formula of rank at most β whose only free variable is x. The result of substituting $\hat{x}\mathscr{G}(x)$ for X^β in $\mathscr{F}(X^\beta)$ is denoted by $\mathscr{F}(\hat{x}\mathscr{G}(x))$, and is accomplished by substituting $\mathscr{G}(b)$ for each occurrence in $\mathscr{F}(X^\beta)$ of $b \in X^\beta$, where b is a number theoretic term. The full ordinal rank of $\mathscr{F}(\hat{x}\mathscr{G}(x))$ is less than that of $(\mathrm{E}X^\beta)\mathscr{G}(X^\beta)$ since all the quantified set variables occurring in $\mathscr{G}(x)$ have superscripts less than β.

It is often convenient to assume an arbitrary ranked sentence $\mathscr{F}$ is in prenex normal form. The standard syntactical manipulations that put a sentence in prenex normal form do not increase full ordinal rank.

4.5 Lemma. *The predicate $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$, restricted to Σ_1^1 $\mathscr{F}$'s, is Π_1^1.*

Proof. It is enough to show $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$, restricted to ranked $\mathscr{F}$'s, is Π_1^1, since

$$\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}X)\mathscr{G}(X)$$

is equivalent to

$$(\mathrm{E}b)[b \in O_1 \;\&\; \mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models (\mathrm{E}X^{|b|})\mathscr{G}(X^{|b|})].$$

The definition of $\mathscr{M}(\omega_1^{\mathrm{CK}}, T) \models \mathscr{F}$ for ranked $\mathscr{F}$'s is a recursion on the full ordinal rank of $\mathscr{F}$ and logical complexity of $\mathscr{F}$. To see that the definition is Π_1^1 consider the following Σ_1^1 closure conditions with Theorem 1.6(i). in mind. (0 is true and 1 is false.)

$$(1) \qquad \langle \mathscr{F}, 0 \rangle \in X \;\&\; \langle \mathscr{G}, 0 \rangle \in X \to \langle \mathscr{F} \;\&\; \mathscr{G}, 0 \rangle \in X.$$

(1) is a Σ_1^1 closure condition because the intended meaning of (1) is:

(1*) If c and d are Gödel numbers of ranked sentences, both of which are true, then their conjunction is true.

Since the set of Gödel numbers of ranked sentence is Π_1^1, it follows that (1*) is Σ_1^1. The same applies to (2)–(12).

(2) $\langle \mathscr{F}, 1\rangle \in X \vee \langle \mathscr{G}, 1\rangle \in X \rightarrow \langle \mathscr{F} \ \& \ \mathscr{G}, 1\rangle \in X$.

(3) $\langle \mathscr{F}, 0\rangle \in X \rightarrow \langle \sim \mathscr{F}, 1\rangle \in X$.

(4) $(\mathscr{F}, 1\rangle \in X \rightarrow \langle \sim \mathscr{F}, 0\rangle \in X$.

(5) $\langle \mathscr{F}(\underline{n}), 0\rangle \in X \rightarrow \langle (\mathrm{Ex})\mathscr{F}(x), 0\rangle \in X$.

(6) $(n)[\langle \mathscr{F}(\underline{n}), 0\rangle \in X] \rightarrow \langle (x)\mathscr{F}(x), 0\rangle \in X$.

(7) $b \in O_1 \ \& \ \mathrm{rank}(\mathscr{G}(x)) \leq |b|$
$\quad \& \ \langle \mathscr{F}(\hat{x}\mathscr{G}(x)), 0\rangle \in X \rightarrow \langle (\mathrm{E}X^{|b|})\mathscr{F}(X^{|b|}), 0\rangle \in X$.

(8) $b \in O_1 \ \& \ [\langle \mathscr{F}(\hat{x}\mathscr{G}(x)), 1\rangle \in X$ whenever rank of $\mathscr{G}(x) \leq |b|] \rightarrow \langle (\mathrm{E}X^{|b|})\mathscr{F}(X^{|b|}), 1\rangle \in X$.

In (9)–(12) s and t are closed number theoretic terms, and $\mathrm{val}(s)$ is the numerical value of s.

(9) $\mathrm{val}(s) \in T \rightarrow \langle t \in \mathscr{T}, 0\rangle \in X$.

(10) $\mathrm{val}(s) \notin T \rightarrow \langle t \in \mathscr{T}, 1\rangle \in X$.

(11) $\mathrm{val}(t_1) = \mathrm{val}(t_2) \rightarrow \langle t_1 = t_2, 0\rangle \in X$.

(12) $\mathrm{val}(t_1) \neq \mathrm{val}(t_2) \rightarrow \langle t_1 = t_2, 1\rangle \in X$.

Let $A(X, T)$ be the conjunction of (1)–(12). $A(X, T)$ is Σ_1^1. An induction on the full ordinal rank and logical complexity of $\mathscr{F}$ shows

$$\begin{aligned} \mathscr{M}(\omega_1^{\mathrm{CK}}, T) &\models \mathscr{F} \\ &\leftrightarrow (X)[A(X, T) \rightarrow \langle \mathscr{F}, 0\rangle \in X] \\ &\leftrightarrow \sim (X)[A(X, T) \rightarrow \langle \mathscr{F}, 1\rangle \in X]. \end{aligned} \tag{13}$$

A sample of the reasoning encountered in the induction is as follows. Suppose $\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle \in X$ for all X such that $A(X, T)$ holds. Suppose further there is an X_0 such that $A(X_0, T)$ and either $\langle \mathscr{F}, 0\rangle$ or $\langle \mathscr{G}, 0\rangle$ fails to belongs to X_0. Clause (1) is the only clause that can require $\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle$ to be in X. Therefore $X_0 - \{\langle \mathscr{F} \ \& \ \mathscr{G}, 0\rangle\}$ is an X that satisfies $A(X, T)$, a contradiction. Hence $\langle \mathscr{F}, 0\rangle$ and $\langle \mathscr{G}, 0\rangle$ belong to every X satisfying $A(X, T)$. By induction $\mathscr{F} \ \& \ \mathscr{G}$ is true in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$, since both $\mathscr{F}$ and $\mathscr{G}$ are less complex than $\mathscr{F} \ \& \ \mathscr{G}$.

(13) is a Π_1^1 definition of truth in $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ for ranked sentences.

4.6 Lemma. *For each $\alpha < \omega_1^{CK}$ the predicate $\mathscr{M}(\omega^{CK}, T) \models \mathscr{F}$, restricted to $\mathscr{F}$'s of rank less than α, is Δ_1^1 uniformly in the unique notation for α.*

Proof. Let g be a recursive function such that for each $b \in O_1$, $W_{g(b)}$ is the recursively enumerable set of Gödel numbers of sentences of rank less than $|b|$. The existence of g was noted in subsection 4.1(2). Fix b. The predicate

$$\mathscr{F} \in W_{g(b)} \quad \& \quad \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{F}$$

and its "negative",

$$\mathscr{F} \in W_{g(b)} \quad \& \quad \mathscr{M}(\omega_1^{CK}, T) \models \sim \mathscr{F},$$

are Π_1^1 by Lemma 4.5. □

The next lemma echoes the persistence phenomenon heard in Lemma 1.6.I.

Lemma 4.7. *For each hyperarithmetic set H there is a ranked formula $\mathscr{H}(x)$ such that for all T and n,*

$$n \in \mathscr{H} \leftrightarrow \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{H}(\underline{n}).$$

Proof. There exist $b \in O_1$ and recursive function f such that

$$n \in H \leftrightarrow (\mathrm{E}x)[f(n) = x \ \& \ x \in H_b].$$

by Exercise 2.10.II. The graph of f is representable by a arithmetic formula of $\mathscr{L}(\omega_1^{CK}, T)$ in which T does not appear.

H_b is represented by a formula developed by recursion on $<_O$. For each $a <_O b$ assume there is a ranked formula $\mathscr{H}_a(x)$ such that for all T and n,

$$n \in H_a \leftrightarrow \mathscr{M}(\omega_1^{CK}, T) \models \mathscr{H}_a(\underline{n}))$$

Assume the ordinal rank of $\mathscr{H}_a$ is $|a| + 1$. Then $\mathscr{M}(|b|, T)$ includes $\{H_a | a <_O b\}$ for all T. According to Corollary 4.4.II there is a Σ_1^1 formula, $(\mathrm{E}Y) A(x, Y)$, that defines H_b in any universe containing $\{H_a | a <_O b\}$. Consequently $(\mathrm{E}Y^{|b|}) A(x, Y^{|b|})$ defines H_b in $\mathscr{M}(\omega_1^{CK}, T)$ for all T, and has rank $|b| + 1$.

The *ramified analytic hierarchy* is denoted by $\mathscr{M}(\omega_1^{CK})$ and is defined to be $\mathscr{M}(\omega_1^{CK}, \phi)$.

4.8 Theorem (Kleene). $\mathscr{M}(\omega_1^{CK}) = \mathrm{HYP}$.

Proof. Lemma 4.7 implies HYP $\subseteq \mathscr{M}(\omega^{CK}, T)$ for all T, hence for $T = \phi$.

Suppose $B \in \mathscr{M}(\omega_1^{CK})$. Then there is a ranked formula $\mathscr{G}(x)$ of $\mathscr{L}(\omega_1^{CK}, T)$ such that for all n,

$$(1) \qquad n \in B \leftrightarrow \mathscr{M}(\omega_1^{CK}, \phi) \models \mathscr{G}(\underline{n}).$$

By Lemma 4.6 the right side of (1) is Δ_1^1, hence hyperarithmetic. □

4.9 Δ_1^1 Comprehension. The following sentence is a typical instance of G. Kreisel's Δ_1^1 comprehension axiom scheme.

$$(1) \qquad (x)[(\mathrm{E}Y)A(x,Y) \leftrightarrow (Z)B(x,Z)] \\ \rightarrow (\mathrm{E}X)(x)[x \in X \leftrightarrow (\mathrm{E}Y)A(x,Y)].$$

$A(x,Y)$ and $B(x,Z)$ are arithmetic. If they contain set parameters, (1) is said to be boldface, otherwise lightface. An ω-model N is simply any nonempty subset of 2^ω. Any sentence $\mathscr{G}$ of analysis can be interpreted in N by restricting the set variables of $\mathscr{G}$ to N and the number variables to ω. The study of ω-models of fragments of analysis originated in proof theory, but turned out to have applications to recursion theory as in the next lemma, a characterization of a recursion-theoretic hierarchy as the least ω-model of some fragment of analysis.

4.10 Theorem (Kreisel). *The intersection of all ω-models of Δ_1^1 comprehension is a model of Δ_1^1 comprehension, namely* HYP.

Proof. HYP is an ω-model of Δ_1^1 comprehension, since Theorem 3.1 implies any set that has a Σ_1^1 definition over HYP must be Π_1^1.

Suppose N is an ω-model of Δ_1^1 comprehension to see HYP $\subseteq N$. As in the proof of Lemma 4.7 it suffices to show $H_b \in N$ by induction on $<_O$. By Corollary 4.4 of Chapter II there is a Δ_1^1 definition of H_b that defines H_b in any ω-model containing $\{H_a \mid a <_O b\}$, hence in N by induction. $\square$

4.11 Σ_1^1 Choice. The following sentence is a typical instance of Kreisel's Σ_1^1 axiom of choice scheme.

$$(1) \qquad (x)(\mathrm{E}Y)A(x,Y) \rightarrow (\mathrm{E}Y)(x)A(x,(Y)_x).$$

($m \in (Y)_x$ means $2^x \cdot 3^m \in Y$.) $A(x,Y)$ is an arithmetic predicate in which set parameters may occur. If none do, (1) is said to be lightface.

Long ago Kreisel asked if Δ_1^1 choice followed from Δ_1^1 comprehension. (A well known consequence of the Kondo–Addison Theorem is that Σ_1^1 choice follows from Δ_2^1 comprehension (Chapter IV).) J. Steel devised an appropriate notion of forcing to show boldface Δ_1^1 comprehension does not imply boldface Σ_1^1 choice. Later L. Harrington showed boldface Δ_1^1 comprehension does not imply lightface Σ_1^1 choice by means of a self-referential argument.

4.12 Proposition. *Suppose N is an ω-model of Σ_1^1 choice and is downward closed under many-one reducibility. Then N is a model of Δ_1^1 comprehension.*

Proof. Suppose

$$(x)[(\mathrm{E}Y)A(x,Y) \leftrightarrow (Z)B(x,Z)]$$

is true in N for some arithmetic A and B. Then

$$(x)(\mathrm{E}Y)[(x\in(Y)_0 \quad \& \quad A(x,(Y)_1) \\ \vee (x\notin(Y)_0) \quad \& \quad \sim B(x,(Y)_1)]$$

holds in N. Σ_1^1 choice provides a Y in N such that for all x,

$$(x)[(x\in(Y)_{x,0} \quad \& \quad A(x,(Y)_{x,1}) \\ \vee (x\notin(Y)_{x,0} \quad \& \quad \sim B(x,(Y)_{x,1})].$$

Let X be $\{x|x\in(Y)_{x,0}\}$. Then

$$(x)[x\in X \leftrightarrow (\mathrm{E}Y)A(x,Y)]. \quad \square$$

4.13 Theorem (Kreisel). HYP *is the least ω-model of Σ_1^1 choice downward closed under many-one reducibility.*

Proof. Suppose $(x)(\mathrm{E}Y)_{Y\in\mathrm{HYP}}A(x,Y)$ for some arithmetic $A(x,Y)$ with hyperarithmetic parameters. Then

(1) $$(x)(\mathrm{Ea})(\mathrm{Ee})[a\in O \quad \& \quad A(x,\{e\}^{H_a})].$$

Let $H(a,X)$ be the Π_2^0 predicate of Theorem 4.2.II. Then (1) is equivalent to

(2) $$(x)(\mathrm{Ea})(\mathrm{Ee})[a\in O \quad \& \quad (Z)(H(a,Z)\rightarrow A(x,\{e\}^Z))].$$

Since the matrix of (2) is Π_1^1, Lemma 2.6.II provides hyperarithmetic functions $a(x)$ and $e(x)$ such that

$$(x)[a(x)\in O \quad \& \quad (Z)(H(a(x),Z)\rightarrow A(x,\{e(x)\}^Z))].$$

The range of $a(x)$ is bounded by some $b\in O$ by Corollary 5.6.I. To be precise, $(x)(a(x)\in O_b)$. It follows from Lemma 2.1.II and Theorem 1.3.II that the predicate

(3) $$m\in\{e(x)\}^{H_{a(x)}}$$

is Δ_1^1 hence hyperarithmetic. Thus there is a $Y\in\mathrm{HYP}$ such that $m\in(Y)_x$ is equivalent to (3). Clearly $(x)A(x,(Y)_x)$. $\square$

4.14 Σ_1^1 Dependent Choice. A typical instance of Σ_1^1 dependent choice is

$$(Y)(\mathrm{E}Z)A(Y,Z)\rightarrow(\mathrm{E}Y)(x)A((Y)_x,(Y)_{x+1}).$$

$A(Y,Z)$ is an arithmetic predicate that may contain set parameters. According to Exercise 4.22 HYP is an ω-model of Σ_1^1 dependent choice. Σ_1^1 choice is an immediate consequence of Σ_1^1 dependent choice; H. Friedman found an ω-model of the former in which the latter fails.

In Chapter IV it will be shown that $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$ is a model of Σ_1^1 dependent choice for almost all T, that is a set of T's of measure 1. At that time the equivalences of Lemma 4.16 will prove useful. The proof of 4.16 requires the following Corollary of Lemma 4.6.

4.15 Lemma. *Suppose $X \in \mathcal{M}(\omega_1^{\mathrm{CK}}, T)$. Then there exists a $b \in O$ such that X is recursive in H_b^T. (b depends only on a ramified analytic definition of X from T.)*

Proof. Let $G(x)$ be a ranked formula such that

(1) $$n \in X \leftrightarrow \mathcal{M}(\omega_1^{\mathrm{CK}}, T) \vDash G(\underline{n})$$

for all n. By Lemma 4.6, the right side of (1) is a Δ_1^1 predicate with n and T as free variables. According to subsection 5.6.II, there is a $b \in O$ and an e such that the right side of (1) is $\{e\}^{H_b^T}$ for all T. □

4.16 Lemma. *The following are equivalent.*

(*i*) $\omega_1^{\mathrm{CK}} = \omega_1^T$.
(*ii*) $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) = \mathrm{HYP}(T)$.
(*iii*) $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$ *satisfies* Δ_1^1 *comprehension.*

Proof. (i) → (ii). Lemma 4.15 implies $\mathcal{M}(\omega_1^{\mathrm{CK}}, T) \subseteq \mathrm{HYP}(T)$. Suppose $B \in \mathrm{HYP}(T)$. Let B be H_b^T. $|b|_T < \omega_1^{\mathrm{CK}}$. There is a formula $\mathcal{H}_b$ of rank $|b| + 1$ such that

(1) $$b \in H_b^T \leftrightarrow \mathcal{M}(\omega_1^{\mathrm{CK}}, T) \vDash \mathcal{H}_b(\underline{n})$$

for all n. (1) is proved as was Lemma 4.7 with the aid of Corollary 4.4.II, now relativized to T.

(ii) → (iii). $\mathrm{HYP}(T)$ is an ω-model of Δ_1^1 comprehension by a relativization to T of the proof of Theorem 4.10.

(iii) → (i). By Theorem 4.10 relativized to T, $\mathrm{HYP}(T) \subseteq M(\omega_1^{\mathrm{CK}}, T)$. It follows from Lemma 4.15 that each set hyperarithmetic in T is recursive in some H_b^T, where $|b|_T < \omega_1^{\mathrm{CK}}$. But then there cannot be an $a \in O^T$ such that $|a|_T = \omega_1^{\mathrm{CK}}$. □

4.17–4.21 Exercises

4.17. Prove 4.2(1) and 4.2(2).

4.18. Show $<$, defined in subsection 4.4, is wellfounded.

4.19. Prove 4.13(3) is Δ_1^1.

4.20. Show Σ_1^1 dependent choice implies Σ_1^1 choice.

4.21. Prove there is an $\mathscr{H}_b$ of rank $|b|+1$ that satisfies (1) of 4.16.

4.22. Show HYP is an ω-model of Σ_1^1 dependent choice.

5. Kreisel Compactness

Some of the boundedness properties of the hyperarithmetic hierarchy lead to a compactness theorem for ω-logic discovered by G. Kreisel. His result is: if I is a Π_1^1 set of sentences of ω-logic such that every hyperarithmetic subset of I has a model, then I has a model. Note that "hyperarithmetic" takes the place of "finite" in the compactness theorem for first order logic. Kreisel's theorem paved the way for Barwise's compactness theorem for countable Σ_1 admissible sets. Since this is not a book on model theory, the Kreisel result is presented in set theoretic terms.

Recall what is meant by the compactness of 2^ω. 2 is the two-element space with the discrete topology. 2^ω is the product of countably many copies of 2 and has the product topology. If I is a family of closed subsets of 2^ω such that every finite subfamily of I has a nonempty intersection, then I has a nonempty intersection.

Kreisel compactness is concerned with families of Δ_1^1 subsets of 2^ω. At first this idea seems hopeless, because it is easy to find a family of Δ_1^1 subsets with the finite intersection property whose intersection is empty. Kreisel's insight provides a way out. First ask that every hyperarithmetic subfamily have a nonempty intersection. Second require that the family be Π_1^1. As has been observed earlier, finite bears to recursively enumerable a relation similar to that borne by hyperarithmetic to Π_1^1.

If n is a Δ_1^1-index for a subset of 2^ω, then D_n denotes the subset indexed by n.

5.1 Theorem (Kreisel). *Let I be a Π_1^1 set of indices of Δ_1^1 sets. Suppose*

$$\cap\{D_n \mid n\in H\} \neq \varnothing$$

for every hyperarithmetic $H\subseteq I$. Then

$$\cap\{D_n \mid n\in I\} \neq \varnothing.$$

Proof. Suppose not. Then

$$(X)(\mathrm{E}y)[y\in I \;\&\; X\notin D_y]. \tag{1}$$

The matrix of (1), call it $Q(X,y)$, is Π_1^1. The proof of Theorem 2.3.II shows $Q(X,y)$ can be uniformized by some Π_1^1 $P(X,y)$. The virtue of $P(X,y)$ is its Δ_1^1-ness, since

$$\sim P(X,y)\leftrightarrow(\mathrm{E}z)_{z\neq y}P(X,z).$$

Hence the set J of all n such $(\mathrm{E}X)P(X,n)$ is a Σ_1^1 subset of I, and

$$\cap\{D_n \mid n\in J\} = \varnothing.$$

By the remark following Theorem 3.7.II, there is a hyperarithmetic H separating J and $\omega - I$, that is $J \subseteq H \subseteq I$. But then

$$\cap \{D_n | n \in H\} = \varnothing. \quad \square$$

5.2 Corollary. *Let I be a Π_1^1 set of ranked sentences of $\mathscr{L}(\omega_1^{\mathrm{CK}}, T)$. Suppose for each hyperarithmetic $H \subseteq I$, there is a T such that $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ is a model of H. Then there is a T such that $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ is a model of I.*

Proof. Let $\mathscr{F}$ be a ranked sentence. The set of all T such that $\mathscr{M}(\omega_1^{\mathrm{CK}}, T)$ satisfies $\mathscr{F}$ is Δ_1^1 by Lemma 4.6. $\square$

5.3–5.4 Exercises

5.3. Show the matrix of 5.1(1) is Π_1^1.

5.4. Show that $Q(X,y)$ of the proof of Theorem 5.1 can be uniformized by some Π_1^1 $P(X,y)$.

6. Perfect Subsets of Σ_1^1 Sets

The real line has the same cardinality as 2^ω. Thus one version of Cantor's continuum hypothesis is: if K is an uncountable subset of 2^ω, then K has the same cardinality as 2^ω. A classical result of descriptive set theory states that the continuum hypothesis holds for every boldface Σ_1^1 set. As might be expected the classical result says more, namely that K contains a perfect set. By combining the classical splitting argument with some effectiveness considerations, still more information is obtained. For example every member of a countable Σ_1^1 set is hyperarithmetic.

6.1 Perfect Subsets of 2^ω. $P, Q, R, \ldots$ denote perfect subsets of 2^ω. "Perfect" means closed, nonempty and without isolated points, in short homeomorphic to 2^ω. An $x \in P$ is isolated if $\{x\} = P \cap U$ for some open set U. The usual picture of 2^ω is that of a tree with binary branching. The members of 2^ω correspond to the infinite branches of the tree. An arbitrary perfect P should be pictured similarly. To make the picture precise, let P^* be the set of all sequence numbers that represent initial segments of elements of P. Thus $s \in P^*$ iff $\mathrm{Seq}(s)$, $(i)_{i<2}((s)_i \in \{1,2\})$, and

$$(\mathrm{E}X)\,[X \in P \;\&\; (i)_{i<\mathscr{lh}(s)}(i \in X \leftrightarrow (s)_i = 1)].$$

The elements of P^* are the branchpoints of a tree whose branches are the members of P. The lack of isolated elements in P corresponds to the existence in P^* of incomparable extensions of each member of P^*.

P^* is the standard encoding of P. Assume $V \subseteq \omega$. V encodes a perfect set if $\varnothing \neq V \subseteq \text{Seq}$,

$$(s)(\text{Et})(\text{Eu})[s \in V \rightarrow t, u \in V \;\&\; s \geq t \;\&\; s \geq u \;\&\; t|u],$$

(Recall $s \leq t$ means s is extended by t, and $t|u$ means $[t \ngeq u \;\&\; u \ngeq t]$.)

$$\text{and } (s)\;[s \in V \rightarrow (i)_{i < 1h(s)}((s)_i \in \{1,2\})].$$

The perfect set P_V coded by V consists of all X such that arbitrarily long initial segments of the characteristic function of X are coded by sequence numbers in V. Note that the standard coding of P_V is recursive in V.

It is helpful to allow $P, Q, R, \ldots$ to denote both perfect sets and their standard encodings. Thus "P is hyperarithmetic" means "the standard encoding of P is a hyperarithmetic set of sequence numbers".

6.2 Theorem. *Let $K \subseteq 2^\omega$ be Σ_1^1. Then (i)–(iv) are equivalent.*
(i) K contains a perfect set recursive in O.
(ii) K contains a perfect set.
(iii) K has a nonhyperarithmetic member.
(iv) For each hyperarithmetic real H, there is a member of K not recursive in H.

Proof. Clearly (i) → (ii), (ii) → (iii) since perfect sets are uncountable, (iii) → (iv). It suffices to show (iv) → (iii) and (iii) → (i).

Assume (iii) fails with the intent of refuting (iv). The failure of (iii) implies

$$(1) \qquad (X)(\text{Eb})[X \in K \rightarrow b \in O \quad \& \quad X \leq_T H_b].$$

The matrix of (1) is Π_1^1 and so can be uniformized by some Π_1^1 $Q(X,b)$ as in the proof of Theorem 2.3.II. Since Q defines the graph of a total function, it follows that $Q(X,b)$ is Σ_1^1:

$$\sim Q(X,b) \leftrightarrow (\text{Ec})_{c \neq b} Q(X,c).$$

Thus there is a Δ_1^1 function $b(X)$ such that

$$(X)[X \in K \rightarrow b(X) \in O \quad \& \quad X \leq_T H_{b(X)}].$$

By Spector's boundedness theorem (5.6.I), there is a $c \in O$ such that

$$(X)[|b(X)| < |c|].$$

Consequently every member of K is recursive in H_c by Theorem 4.5.II.

Suppose K has a nonhyperarithmetic member in order to show K has a perfect subset encodable by a real recursive in O. It is safe to assume every element of K is nonhyperarithmetic, since

$$X \in K \quad \& \quad X \notin \text{HYP}$$

is Σ_1^1 by Corollary 1.4.II. The normal form of $X \in K$ is

$$(Eg)(x)R(\bar{X}(x), \bar{g}(x))$$

for some recursive R. $X(x)$ is the characteristic function of X. If $(x)R(\bar{X}(x), \bar{g}(x))$ holds, then g is said to be a witness to the fact that $X \in K$.

The construction of the perfect $P \subseteq K$ includes the construction of a witness for each $X \in P$. $Q(s,t)$ is the Σ_1^1 predicate defined by:

(i) s and t are sequence numbers of the same length.
(ii) there exist X and g such that s encodes an initial segment of X, t encodes an initial segment of g, and g witnesses $X \in K$.

Assume $Q(s,t)$ holds to show there exist $\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ such that:

(iii) $s > s_1$, $s > s_2$, $t > t_1$ and $t > t_2$;
(iv) $Q(s_1, t_1)$ and $Q(s_2, t_2)$;
(v) $s_1 | s_2$.

$\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ are said to be incomparable extensions of $\langle s, t \rangle$. Suppose no such exist. Then there is a *unique* X such that s encodes an initial segment X and t encodes an initial segment of some witness to $X \in K$. That unique X is hyperarithmetic by Theorem 1.6(ii).I, contrary to the assumption no member of K is hyperarithmetic.

The set $\{\langle s_j^i, t_j^i \rangle | i < \omega \ \& \ j < 2^i\}$ is defined by recursion on i. s_o^o and t_o^o are null. Fix i and j. Choose $\langle s_{2j}^{i+1}, t_{2j}^{i+1} \rangle$ and $\langle s_{2j+1}^{i+1}, t_{2j+1}^{i+1} \rangle$ to be incomparable extensions of $\langle s_j^i, t_j^i \rangle$. The choices can be made effectively from O since O is a complete Π_1^1 set (Theorem 5.4.I).

Let P be the perfect set encoded by $\{s_j^i | i < \omega \ \& \ j < 2^i\}$. □

Countable Π_1^1 subsets of 2^ω are much more complicated than their Σ_1^1 counterparts. Their elements tend to be scattered throughout the constructible hierarchy. They will be discussed in Section 9.

The next result is a uniform version of part of Theorem 6.2.

6.3 Theorem. *There exists a recursive function h such that: if c is an index for a Σ_1^1 predicate $C(X)$ with no perfect set of solutions, then $h(c) \in O$ and*

$$(X)[C(X) \rightarrow X \leq_T H_{h(c)}].$$

Proof. Let $P(X,b)$, clearly Π_1^1, be

$$b \in O \ \& \ [C(X) \rightarrow X \leq_T H_b].$$

By Theorem 6.2, $(X)(Eb)\, P(X,b)$ if $C(X)$ has no perfect set of solutions. According to Theorem 5.4.I relativized to X, there is a recursive f such that

$$P(X,b) \leftrightarrow f(b) \in O^X.$$

Let $Q(X,b)$ be Π_1^1 and uniformize $P(X,b)$ as in the proof of Theorem 6.2. An index for Q is easily derived from one for P. $Q(X,b)$ is Σ_1^1, because $(X)Q(X,b)$ and Q defines a function. Let B be the range of that function. Corollary 3.4.II states that a bound for B can be computed from an index for B as a Σ_1^1 set. Thus there is a recursive function h such that $h(c)$ bounds B. Then $X \leq_T H_{h(c)}$ if $C(X)$. $\square$

The proof of Theorem 6.2 is similar to a Henkin construction of a perfect set of countable models of a countable theory. The definition of P can be regarded as the simultaneous construction of a continuum of models of a theory that says: "I am a nonhyperarithmetic member of K". The Henkin approach will be applied in Section 9 to the study of Π_1^1 sets.

7. Kreisel's Basis Theorem

Let K be a set of axioms in the language of analysis, that is the language of subsection 1.2.I. The predicates of analysis are the analytical predicates. Let Z be a real that occurs in every model of K. What can be said about Z? To make sense of the question, some bound on the complexity of K is needed. From a syntactic vantage point Z might correspond to some term in K compelled to denote Z by K and the rules and axioms of ω-logic. From a model theoretic stance Z cannot be omitted from any model of K. Kreisel's result is that Z must be hyperarithmetic if K is Π_1^1.

7.1 Lemma. *Suppose $A \notin \mathrm{HYP}$, $b \in O$ and K is an uncountable Σ_1^1 set. Then for each e there exists an uncountable $\Sigma_1^1 K^* \subseteq K$ such that*

$$A \neq \{e\}^{H_b^X}$$

for all $X \in K^*$.

Proof. Since K is uncountable and HYP is Π_1^1, it is safe to assume $K \cap \mathrm{HYP} = \varnothing$.

Case 1: $(EX)[X \in K \;\&\; \{e\}^{H_b^X}$ is not total]. Let K^* be the set of all $Y \in K$ such that $\{e\}^{H_b^Y}$ is not total. K^* is Σ_1^1 by subsection 5.6.II, and uncountable by Theorem 6.2, since $K \cap \mathrm{HYP} = \varnothing$.
Case 2: Case 1 fails and

$$\{e\}^{H_b^X}(m) \neq \{e\}^{H_b^Y}(m)$$

for some m and X, $Y \in K$. Choose such an m. Let K^* be the set of all $Z \in K$ such that $A(m) \neq \{e\}^{H_b^Z}(m)$. K^* is uncountable by Theorem 6.2.
Case 3: Cases 1 and 2 fail. Thus there is a function h such that

$$(m)(X)[X \in K \to \{e\}^{H_b^X}(m) = h(m)].$$

h is Σ_1^1, hence Δ_1^1, because

$$h(m) = n \leftrightarrow (\mathrm{E}X)(X \in K \;\; \& \;\; \{e\}^{H_b^X}(m) = n).$$

Let K^* be K. □

The above lemma will make more sense after the introduction of Gandy forcing in Section 6.IV. That notion of forcing employs uncountable Σ_1^1 sets as forcing conditions. The proof of Lemma 7.1 shows: if K forces $\{e\}^{H_b^G}$ to be total and constant, then K forces $\{e\}^{H_b^G}$ to be hyperarithmetic.

7.2 Theorem (Kreisel). *Suppose $A \notin \mathrm{HYP}$ and K is a nonempty Σ_1^1 set. Then for some $X \in K$, $A \not\leq_h X$.*

Proof. By Theorem 6.2 and Corollary 1.5 it is safe to assume

$$K \cap \mathrm{HYP} = \varnothing, \; K \text{ is uncountable and}$$
$$(X)[X \in K \to \omega_1^X = \omega_1^{\mathrm{CK}}].$$

A contracting sequence K^i $(i < \omega)$ of uncountable Σ_1^1 sets, and a sequence h_i of reals, are defined simultaneously by recursion on i such that:

$K^0 = K$;
if $b = (i)_0$, $e = (i)_1$ and $b \in O$, then K^{i+1} has the properties attributed to K^* by Lemma 7.1;

$\cap \{K^i | i < \omega\} = \{X\}$;

h_i is an existential witness to $X \in K^i$.

X, K^i and h_i are defined above as f, P_i and g_i are in the proof of Theorem 6.3.IV. (In fact the present theorem follows directly from Lemma 7.1.III and Theorem 6.3.IV; for each $b \in O$ and e, the set of all K^* provided by 7.1 is dense.) □

7.3 Corollary (Kreisel). *Let K be a Π_1^1 set of axioms in the language of analysis. If X belongs to every ω-model of K, then X is hyperarithmetic.*

Proof. The set of all real Z such that Z encodes a countable ω-model of K is Σ_1^1. □

Corollary 7.3 is a special case of a type omitting theorem for ω-logic. If a Π_1^1 set of axioms of ω-logic has a model, then it has one that omits any given nonhyperarithmetic type. The analogous result for first order logic is: if a countable theory has a model, then it has one that omits any given nonprincipal type.

The recursion theorist winding his way through a Σ_1^1 set is a brother to the model theorist threading his way through a Henkin tree.

8. Inductive Definitions

The leading role of inductive definitions in recursion theory has been emphasized by R. Gandy for many years. Two prime examples are the definition of O in Chapter I, and the definition of E-recursion in Part D. Two basic cases dominate the subject, monotonic Π_1^1 and arbitrary Π_1^0.

8.1 Closure Ordinals. Let Γ be a map from 2^ω into 2^ω. Define Γ_α by recursion on α.

$$\Gamma_0 = \varnothing. \quad \Gamma_{\alpha+1} = \Gamma(\Gamma_\alpha) \cup \Gamma_\alpha.$$

$$\Gamma_\lambda = \cup \{\Gamma_\alpha | \alpha < \lambda\} \qquad (\lambda \text{ a limit}).$$

$$\Gamma_\infty = \cup \{\Gamma_\alpha | \alpha \text{ is an ordinal}\}.$$

Γ_∞ is said to be the set inductively defined by Γ. Γ_α is the α-th stage of the inductive definition of Γ_∞. The closure ordinal of Γ, denoted by $|\Gamma|$, is the least α such that $\Gamma_{\alpha+1} = \Gamma_\alpha$. Thus $\Gamma_\infty = \Gamma_{|\Gamma|}$.

Γ is *progressive* if $X \subseteq \Gamma(X)$ for all X. From now on assume all Γ's are progressive, a harmless assumption since $\Gamma^*(X)$, defined by $\Gamma(X) \cup X$, is progressive and differs little from Γ.

Γ is *monotonic* if $X \subseteq Y$ implies $\Gamma(X) \subseteq \Gamma(Y)$. Mathematical definitions tend to be monotonic. If Γ is monotonic, then $\Gamma_\infty = \cap \{X | \Gamma(X) = X\}$.

Γ is classified by classifying the predicate $x \in \Gamma(Y)$.

8.2 Theorem. *Suppose $A \in \Pi_1^1$, $P(X)$ is Π_1^0, and $P(A)$. Then $P(H)$ for some hyperarithmetic $H \subseteq A$. (Π_1^0 reflection.)*

Proof. Let $P(X)$ be $(z)R(\bar{X}(z))$ for some recursive R. Let f be recursive so that

$$n \in A \leftrightarrow f(n) \in O$$

for all n (cf. Theorem 5.4.I). For each $\alpha < \omega_1^{\mathrm{CK}}$ let A_α be $\{n | f(n) \in O_\alpha\}$. Each A_α is hyperarithmetic by Theorem 2.4.II; one of them will be the desired H.

Let $P_0(z,b,c)$ be

$$c \in O \quad \& \quad |b| < |c| \quad \& \quad R(\bar{A}_{|c|}(z)).$$

$P_0(z,b,c)$ is Π_1^1. $P(A)$ implies

$$(z)(b)(\mathrm{E}c)[b \in O \to P_0(z,b,c)].$$

More precisely, $R(\bar{A}(z))$ implies $R(\bar{A}_\alpha(z))$ for all sufficiently large recursive α, because only finitely many membership facts about A are needed to compute the truth-value of $R(\bar{A}(z))$. P_0 can be uniformized by some Π_1^1 Q according to Theorem

2.3 of Chapter II. Thus there is a partial Π_1^1 function $t(z,b)$ such that

$$(z)(b)[b \in O \to P_0(z,b,t(z,b))].$$

A recursive subsequence $\{b_n | n < \omega\}$ of O is defined by recursion. $b_0 = 1$. The set $\{t(z,b_n) | z < \omega\}$ is Σ_1^1 by proposition 1.7.I. Corollary 3.4.II makes it possible to compute b_{n+1}, a strict upper bound for $\{|t(z,b_n)| \, | z < \omega|\}$ in O. Let β, necessarily a recursive limit ordinal, be the sup of the $|b_n|$'s. For each z, β is the limit of ordinals δ such that $R(\bar{A}_\delta(z))$ holds. It follows that $(z)R(\bar{A}_\beta(z))$ holds. Since β is a limit and computations from A are finite, $\sim R(\bar{A}_\beta(z))$ would imply $\sim R(\bar{A}_\delta(z))$ for all sufficiently large $\delta < \beta$. H is A_β. □

Theorem 8.2 was presented to suggest a connection between reflection and inductive definitions, a link developed by Aczel and Richter 1972. Reflection phenomena are crucial in E-recursion theory as will be seen in Part D. The idea behind 8.2, and similar theorems, is: "enumerate" A until some "finite" subset is found that satisfies P.

8.3 Corollary (Gandy). *If* Γ *is* Π_1^0, *then* $|\Gamma| \le \omega_1^{CK}$ *and* $\Gamma_\infty \in \Pi_1^1$.

Proof. An effective transfinite recursion produces a recursive function f such that: if $b \in O$, then $f(b)$ is a hyperarithmetic index for $\Gamma_{|b|}$. Consequently $\Gamma_{\omega_1^{CK}}$ is Π_1^1. It follows from the proof of Theorem 8.2 that

$$n \in \Gamma_{\omega_1^{CK}+1} \to n \in \Gamma_\alpha$$

for some recursive α depending on n. □

One way to remember the proof of 8.2, and hence of 8.3, is to think of it as a downward Skolem–Löwenheim argument. From the viewpoint of metarecursion theory (Part B), the β of 8.2 is a fixed point of a continuous metarecursive Skolem function. The downward Skolem idea is more evident when Γ is monotone Π_1^1 (cf. subsection 8.7).

Γ is said to be *positive* if the predicate $x \in \Gamma(Y)$ is positive with respect to Y. This means it can be written in prenex normal form with a matrix $\bigvee_i \bigwedge_j \mathscr{F}_{ij}$ in disjunctive normal form such that no $\mathscr{F}_{ij}$ is of the form $z \notin Y$. Thus $x \in \Gamma(Y)$ is a consequence of positive facts about Y. x gets into $\Gamma(Y)$ because certain z's already belong to Y. Note that positivity implies monotonicity.

8.4 Proposition. *If* Γ *is monotonic, then* Γ *is positive.*

Proof. $x \in \Gamma(Y)$ is equivalent to

$$(Z)[Y \subseteq Z \to x \in \Gamma(Z)],$$

$$(Z)(Eu)[(u \in Y \;\&\; u \notin Z) \vee x \in \Gamma(Z)]. \quad \square$$

8.5 Theorem (Spector 1955). *Suppose A is Π_1^1 and Γ is monotonic Π_1^1. Then*

$$\Gamma(A) = \cup\{\Gamma(H) \mid H \subseteq A \ \& \ H \in \mathrm{HYP}\}.$$

Proof. $n \in \Gamma(A)$ iff $S_R^A(n)$ is wellfounded, by the relativization of Proposition 5.3.I to A, as in 5.4.II. $S_R^A(n)$ is recursive uniformly in A, n. Fix n and assume $n \in \Gamma(A)$. Then, as in the proof of Proposition 8.4,

$$(Z)(\mathrm{E}u)[(u \in A \quad \& \quad u \notin Z) \vee S_R^Z(n) \text{ is wellfounded}]. \tag{1}$$

Let f be a recursive function such that for all m,

$$m \in A \leftrightarrow f(m) \in O.$$

Define A_α to be $\{m \mid f(m) \in O_\alpha\}$. Then (1) becomes

$$(Z)(\mathrm{E}b)[b \in O^Z \quad \& \quad (A_{|b|} \nsubseteq Z \vee |S_R^Z(n)| = |b|)]. \tag{2}$$

The matrix of (2) is Π_1^1, hence it can be uniformized by a Π_1^1 predicate $Q(Z, b)$ by adapting the proof of Theorem 2.3.II. For each Z, let $t(Z)$ be the unique b satisfying $Q(Z, b)$. Then t is Δ_1^1 and

$$(Z)[t(Z) \in O^Z \quad \& \quad (A_{|t(Z)|} \nsubseteq Z \vee |S_R^Z(n)| = |t(Z)|)]. \tag{3}$$

A slight modification of the proof of Lemma 5.5.II yields a recursive ordinal bound δ on $|t(Z)|$. If there is no such δ, then O is Σ_1^1.

$$\begin{aligned} b \in O \leftrightarrow (\mathrm{E}f)(\mathrm{E}Z)(x)(y)(m)[m = t(Z) \rightarrow (x, y \in S_R(b) \\ \& \quad x > y \rightarrow \langle f(y), f(x) \rangle \in W_{q(m)}^Z)]. \end{aligned}$$

Thus for some $\delta < \omega_1^{\mathrm{CK}}$, $|t(Z)| < \delta$ for all Z. (cf. Exercise 5.10.II). (3) becomes

$$(Z)[A_\delta \subseteq Z \rightarrow |S_R^Z(n)| < \delta],$$

and so $n \in \Gamma(A_\delta)$. $\square$

An intuitive summary of Theorem 8.5 and its proof is: if $n \in \Gamma(A)$, then "enumerate" A until some hyperarithmetic $H \subseteq A$ is developed with the property that $n \in \Gamma(H)$.

8.6 Corollary (Spector 1955). *If Γ is monotonic Π_1^1, then $|\Gamma| \leq \omega_1^{\mathrm{CK}}$ and $\Gamma_\infty \in \Pi_1^1$.*

Proof. If $X \in \Pi_1^1$, then $\Gamma(X)$ is Π_1^1 by Theorem 8.5. An effective transfinite recursion on $<_O$ shows $\Gamma_{|b|}$ is Π_1^1 uniformly in b for all $b \in O$. Hence $\Gamma_{\omega_1^{\mathrm{CK}}}$ is Π_1^1. Another application of 8.5 yields $\Gamma(\Gamma_{\omega_1^{\mathrm{CK}}}) = \Gamma_{\omega_1^{\mathrm{CK}}}$. $\square$

A type-omitting argument in the setting of ω-logic clarifies the roles of monotonicity and Π_1^1-ness in Theorem 8.5. Let A be a finite subset of the axioms of set theory with enough strength to establish the essential properties of the hyperarithmetic hierarchy, and to prove the existence and countability of ω_1^{CK}. The statement, M is a real that encodes an ω-model of A, is arithmetic, hence has a solution N such that $\omega_1^N = \omega_1^{\mathrm{CK}}$ by Corollary 1.4.

Let V be the real world of sets, and let V_N be the model encoded by N. The reals of V_N are a subset of the reals of V. The encoding has the property that each real of V_N is recursive in N. Note that any Π_1^1 sentence true in V is also true in V_N, since the transition from V to V_N simply restricts a universal function quantifier to a countable set.

A ordinal z in the sense of V_N is called standard if it belongs to V, that is z is truly wellordered. Suppose γ is a standard ordinal countable inside V_N. Thus γ is the ordertype of some linear ordering X of ω for some $X \in V_N$. X is recursive in N, so $\omega_1^X = \omega_1^{\mathrm{CK}}$, and so γ is a recursive ordinal. On the other hand each recursive ordinal (of V) is a recursive ordinal in the sense of V_N. Thus the standard recursive ordinals of V_N are the recursive ordinals.

Let $(\omega_1^{\mathrm{CK}})^{V_N}$ denote the least nonrecursive ordinal in the sense of V_N. The above considerations imply $(\omega_1^{\mathrm{CK}})^{V_N}$ is not standard. It follows that $(\omega_1^{\mathrm{CK}})^{V_N}$ is the limit of nonstandard ordinals.

Suppose $A \subseteq w$ is Π_1^1. Let f be a recursive function such that for all m, $m \in A \leftrightarrow f(m) \in O$, and let A_δ be $\{m \mid f(m) \in O_\delta\}$ for each $\delta < \omega_1^{\mathrm{CK}}$. Then $A_\delta^{V_N} = A_\delta$ for each standard recursive δ, and $A_\beta^{V_N} \supseteq A$ for each nonstandard recursive β.

Let Γ be monotonic and Π_1^1. Suppose $n \in \Gamma(A)$. Then for each nonstandard recursive β,

$$(1) \qquad n \in \Gamma(A_\beta^{V_N}) \text{ is true in } V$$

by monotonicity of Γ. Since (1) is a Π_1^1 fact, and V_N is an ω-model, (1) is also true in V_N. In V_N let δ_0 be the least β such that $n \in \Gamma(A_\beta^{V_n})$. δ_0 must be standard, hence recursive, because there is no least nonstandard ordinal in V_N. Therefore $A_{\delta_0}^{V_N} = A_{\delta_0}$ and $n \in \Gamma(A_{\delta_0})$.

The above is called a type-omitting argument because the standard ordinal ω_1^{CK} is omitted from V_N.

8.7 Inducively Defined Sets of Reals. Theorem 8.6 has been lifted by Cenzer 1976 to sets of reals. Let Γ map 2^{2^ω} into itself. Let A be a variable that ranges over subsets of 2^ω. A predicate $R(A, f, X)$ is said to be recursive if there is a Gödel number e such that for all A, f and X,

$$\{e\}^{A,f,X} \text{ is defined, and}$$
$$R(A,f,X) \leftrightarrow \{e\}^{A,f,X} = 0.$$

In the language of Part D: $R(A,f,X)$ is E-recursive in f, X with $r \in A$ as an additional predicate. Only two facts are needed here.

(i) The computation of $\{e\}^{A,f,X}$ uses only countably many membership facts about A.

(ii) Let B be a real that encodes a countable set of reals whose complement relative to 2^ω is denoted by A_B. Then $(f)R(A_B, f, X)$ is a Π_1^1 predicate with free variables B and X.

Both (i) and (ii) should seem plausible. (i) is analogous to the fact that the computation of $\{e\}^X$ uses only finitely many membership facts about X. (ii) is reasonable, since A_B is equivalent to a real, and since a Π_1^1 predicate of sets of reals, restricted to reals, should be Π_1^1 in the usual sense.

8.8 Proposition. *If* $\Gamma\colon 2^{2^\omega} \to 2^{2^\omega}$ *is monotonic* Π_1^1, *then it is positive.*

Proof. Similar to 8.4. Let B be the real variable of 8.7(ii). Then $(f)R(A,f,X)$ implies

(1) $$(B)[A \subseteq A_B \to (f)R(A_B, X, f)].$$

Assume X satisfies (1) to show $X \in \Gamma(A)$. If not, then by 8.7(i) there is an f such that $\sim R(A_B, f, X)$ holds for some B such that $A \subseteq A_B$. □

8.9 Theorem (Cenzer 1976). *Suppose* $\Gamma\colon 2^{2^\omega} \to 2^{2^\omega}$ *is monotonic* Π_1^1, *and* $X \in \Gamma_\infty$. *Then* $X \in \Gamma_\beta$ *for some* $\beta < \omega_1^X$.

Proof. $\Gamma_0 = \varnothing$. By the monotonicity of Γ, $X \in \Gamma_1$ iff $(B)(f)R(A_B, f, X)$. Thus Γ_1 is Π_1^1 by 8.7(ii). Similarly, $X \in \Gamma_2$ iff

$$(B)[\Gamma_1 \subseteq A_B \to (f)R(A_B, f, X)],$$

and so Γ_2 is Π_1^1. In this manner an effective transfinite recursion shows

(1) $$e \in O^X \quad \& \quad X \in \Gamma_{|e|_X}$$

is Π_1^1 uniformly in X and e. It follows that $\Gamma_{\omega_1^X}$ is Π_1^1 uniformly in X.

To complete the argument, fix X and assume $X \in \Gamma_{\omega_1^X + 1}$ with the intent of showing $X \in \Gamma_{\omega_1^X}$. The desired bound on X is obtained as in the proof of Theorem 8.5. Γ is monotonic, so

$$(B)[\Gamma_{\omega_1^X} \subseteq A_B \to (f)R(A_B, f, X)],$$

since $X \in \Gamma_{\omega_1^X + 1}$. Let B_0 be the countable set of reals encoded by B. Then

(2) $$(B)(\mathrm{E}b)[b \in O^{B,X} \quad \& \quad (\Gamma_{|b|} \cap B_0 \neq \varnothing \vee |S_R^{B,X}| = |b|)].$$

$S_R^{B,X}$ is a binary relation recursive in B, X (uniformly) that is wellfounded iff $(f)R(A_B, X, f)$. The latter formula is Π_1^1 in B and X according to 8.7(ii). $S_R^{B,X}$ is the relativization of S_R (in subsection 5.2.I) to B, X. The matrix of (2) is Π_1^1, and so can be uniformized by some Π_1^1 predicate $Q(B, b)$ with parameter X (cf. Theorem 2.3.II). Let $b(B)$ be the unique b satisfying $Q(B, b)$. Then

$$(B)[b(B) \in O^{B,X} \quad \& \quad (\Gamma_{b(B)} \cap B_0 = \varnothing \vee |S^{B,X}| = |b|)].$$

As in the conclusion of the proof of 8.5, there is an ordinal $\delta < \omega_1^X$ such that $|b(B)| < \delta$ for all B. Consequently

$$(B)[\Gamma_\delta \subseteq A_B \rightarrow (f)R(A_B, f, X)],$$

hence $x \in \Gamma_{\delta+1}$. $\square$

8.10 Corollary (Cenzer 1976). *If* $\Gamma: 2^{2^\omega} \rightarrow 2^{2^\omega}$ *is monotonic* Π_1^1, *then* Γ_∞ *is* Π_1^1 *and* $|\Gamma| = \omega_1$.

8.11 Bounding Arguments. The proofs of Theorems 8.5 and 8.9 made use of a bounding principle first stated as Exercise 5.10.II. Let $P(X, b)$ be Π_1^1. Suppose $(X)(\mathrm{E}b)[P(X, b) \ \& \ b \in O^X]$. Then there exists a $\delta < \omega_1^{\mathrm{CK}}$ such that

$$(X)(\mathrm{E}b)\,[P(X, b) \quad \& \quad |b|_X < \delta].$$

A proof of Exercise 5.10 was incorporated into the last part of the proof of Theorem 8.5. The above bounding principle captures a strong closure property of the recursive ordinals. A weaker form (X replaced by x) will be used extensively in the next chapter.

8.12–8.15 Exercises

8.12. Show each Π_1^1 set of numbers is one-one reducible to one inductively defined by a $\Pi_1^0\,\Gamma$, by a monotonic $\Pi_1^1\,\Gamma$.

8.13. Find a Π_1^1 set of numbers not inductively definable by any monotonic $\Pi_1^1\,\Gamma$.

8.14. If A is a set of formulas, let $|A|$ be the supremum of all ordinals γ such that $\gamma = |\Gamma|$ for some Γ definable by a formula in A. Show $|\Pi_1^0| = \omega_1^{\mathrm{CK}}$. Show $|\Sigma_1^1| \neq |\Pi_1^1|$. (S. Aanderaa has shown $|\Pi_1^1| < |\Sigma_1^1|$.)

8.15. Call $P(X)$ monotonic if $P(X) \rightarrow P(Y)$ whenever $X \subseteq Y$. Suppose $P(X)$ is monotonic Π_1^1, A is Π_1^1, and $P(A)$. Show $P(H)$ for some hyperarithmetic $H \subseteq A$.

9. Π_1^1 *Singletons*

The subject of Π_1^1 singletons verges on set theory, but the results of this section have been phrased and proved in a mood of sympathy for those new to constructible sets and forcing. Most of the ideas applied have already been used in the earlier

accounts of Π_2^0 singletons and scattered Σ_1^1 sets. A is a Π_1^1 singleton if A is the unique solution of some Π_1^1 predicate. A set is *scattered* if it contains no perfect subset.
(Review: a set K of reals is *perfect* if K is nonempty, closed and has no isolated points; $p \in K$ is *isolated* if $\{p\} = K \cap V$ for some open set V; a perfect set of reals has the same cardinality as the continuum.)

9.1 Proposition. *If A is a Π_1^1 singleton, then so are O^A and every $B \equiv_h A$.*

Proof. Fix $B \equiv_h A$. Thus

$$B = \{e\}^{H_b^A} \quad \& \quad A = \{c\}^{H_a^B}$$

for some $b \in O^A$ and $a \in O^B$. Then B is the unique solution of

$$P(\{c\}^{H_a^Y}) \quad \& \quad Y = \{e\}^{H_b^{\{c\}^{H_a^Y}}},$$

where $P(Y)$ is a Π_1^1 predicate whose unique solution is A. $<_{O^A}$ is the unique solution of $P(\{e_0\}^Y)$ & [Y satisfies closure condition $A(X)$ of subsection 2.1.I relativized to $\{e_0\}^Y$] & Y is wellfounded. $\{e_0'\}^{O^Z} = Z$ for all Z. $\square$

9.2 Constructibility. The formulas of Zermelo–Fraenkel set theory (ZF) are well-formed combinations of the symbols of first order logic and ε, the membership symbol. The first order variables $x, y, z, \ldots$ range over the universe of sets. Suppose b and c are sets. b is said to be a first order definable subset of c if there is a formula $\mathscr{F}(x)$ of ZF such that

$$(d)[d \in b \leftrightarrow d \in c \quad \& \quad \langle c, \varepsilon \rangle \models \mathscr{F}(\underline{d})]. \tag{1}$$

$\mathscr{F}(x)$ may mention finitely many sets from c. The last part of (1) means $\mathscr{F}(\underline{d})$ is true when $\underline{d}$ is interpreted as d and the quantifiers of $\mathscr{F}$ range over c.

Gödel's hierarchy L of constructible sets is defined by iteration of Q. $Q(c)$ is the set of all first order definable subsets of c.

$$L(0) = \phi. \quad L(\beta + 1) = Q(L(\beta)).$$
$$L(\lambda) = \cup\{L(\beta) \mid \beta < \lambda\} \quad \text{if } \lambda \text{ is a limit.}$$
$$L = \cup\{L(\beta) \mid \beta \in On\}.$$

Gödel showed L is a model of ZF, the axiom of choice and the generalized continuum hypothesis. The central features of L are its downward Skolem–Löwenheim properties. For example, if $2^\omega \cap (L(\beta + 1) - L(\beta))$ is nonempty, then β is countable. These matters are discussed at greater length in Part C and there utilized in the solution to Post's problem for α-recursion theory.

9.3 Lemma. *Each nonempty Π_1^1 set of reals has a member X constructible via an ordinal recursive in X, that is $X \in L(\beta)$ for some $\beta < \omega_1^X$.*

Proof. According to subsection 5.4.II, a typical Π_1^1 predicate has the form, $\{e\}^Y$ is wellfounded, where $\{e\}$ is such that $\{e\}^Y$ is total and a binary relation between numbers for all Y. The Kleene–Brouwer ordering of sequence numbers, invoked in the proof of Lemma 2.1, makes it possible to say:

$$(1) \qquad \{e\}^Y \text{ is wellordered}$$

is a typical Π_1^1 predicate, where $\{e\}^Y$ is total and a linear ordering of numbers for all Y.

The solutions of (1) are associated with the infinite branches of a tree T defined as follows. A node of T is a pair $\langle s, g\rangle$ such that:

(i) s is a sequence number that encodes an initial segment of a subset of ω;
(ii) let $\{e\}_\ell^s$ be the finite linear ordering computed from s via e and computations of length less than s; g is an order-preserving map of $\{e\}_\ell^s$ into ω_1.

Say $\langle s, g\rangle >_T \langle t, h\rangle$ if s is properly extended by t and $g \subseteq h$. Let $\{\langle s_i, g_i\rangle \mid i < \omega\}$ be an infinite branch of T. The union of the s_i's is the characteristic function of some solution Y of (1). The union of the g_i's is a witness to the wellorderedness of $\{e\}^Y$.

Let $\langle t_i, h_i\rangle$ and $\langle s_i, g_i\rangle$ be infinite branches of T. $\langle t_i, h_i\rangle$ is said to be to the *left* of $\langle s_i, g_i\rangle$ if:

(iii) for some i and n, $t_i(n) < s_i(n)$ & $(m)_{m<n}(t_i(m) = s_i(m))$;
(iv) or $(i)(s_i = t_i)$ and for some j and x, $g(x) < h(x)$ & $g(y) = h(y)$ for all y less than x according to the linear ordering $\{e\}_\ell^{s_j}$.

Assume (1) has a solution. Then T has a leftmost infinite branch Z. Z is readily defined by recursion on i. At state $i + 1$, suppose $\langle s_i, g_i\rangle$ has already been chosen, and let $\langle s, g\rangle$ be an arbitrary immediate successor of $\langle s_i, g_i\rangle$ in T. Let $T_{s,g}$ be the subtree of T whose nodes extend $\langle s, g\rangle$. Discard $\langle s, g\rangle$ if $T_{s,g}$ is wellfounded. Among the remaining nodes there is a leftmost one in the same sense of leftness as that defined by (iii) and (iv). It is $\langle s_{i+1}, t_{i+1}\rangle$.

Let Z_0 be the subset of ω whose characteristic function is $\cup\{s_i \mid s_i \in Z\}$, and let δ be $|\{e\}^{Z_0}|$. Define T^δ by restricting T to δ. The nodes of T^δ are the nodes of T that mention only ordinals less than δ. $T^\delta \in L$, because $\delta \in L$ and T^δ is first order definable over $L(\delta)$.

Suppose T^δ had no infinite branch inside L. Then inside L, T^δ is wellfounded and has ordinal height $|T^\delta|$, and there is a map k from T^δ into $|T^\delta|$ such that $k(u) < k(v)$ whenever $u < v$ in T^δ. But then k serves to show T^δ is truly wellfounded and so has no infinite branch anywhere. This last cannot be, because Z is an infinite branch of T^δ.

Let the leftmost branch of T^δ in L be Z^L. That part of T^δ to the left of Z^L is wellfounded inside L, hence truly wellfounded by the argument of the previous paragraph. Hence Z^L is the leftmost path of T^δ not just in L but truly so. It follows that $Z = Z^L$.

All that remains is the location of Z in L. Suppose $Z = \{\langle s_i, g_i\rangle\}$. Let Z_0 be the set encoded by $\{\langle s_i\rangle\}$, and g^Z the function from $\{e\}^{Z_0}$ onto δ encoded by $\{\langle g_i\rangle\}$.

Call a node *radical* if it belongs to T^δ and lies to the left of Z. Let R^{Z_0} be a wellordering of ω of height δ and recursive in Z_0. Then the nodes of T^δ can be construed as elements of $\omega \times \omega$, and g^Z as a function from $\{e\}^{Z_0}$ onto R^{Z_0}. The uniqueness of g^Z as an orderpreserving function implies it is hyperarithmetic in Z_0. It follows that the set of all radical nodes is hyperarithmetic in Z_0. The subtree T_u^δ that issues from a radical node u is wellfounded. Its ordinal height $|T_u^\delta|$ is less than $\omega_1^{Z_0}$, since T_u^δ can be construed as a tree hyperarithmetic in Z_0 (uniformly in u). The matrix of

$$(u)(\mathrm{E}b)[u \in \text{Radical} \rightarrow b \in O^{Z_0} \quad \& \quad |T_u^\delta| < |b|_{Z_0}]$$

is Π_1^1. It follows from Kreisel uniformization (2.3.II) that b can be taken to be a hyperarithmetic function of u. Hence Spector bounding (5.6.I) implies the strict supremum of $|T_u^\delta|$ over all radical u is some $\gamma < \omega_1^{Z_0}$.

The recursion that defined Z from T also defines Z from T^δ. At stage $i + 1$ of the recursion, the leftmost $\langle s, g \rangle$ is chosen such that $\langle s, g \rangle$ is an immediate successor of $\langle s_i, g_i \rangle$ and $T_{s,g}^\delta$ is not wellfounded. The last clause is equivalent to

(1) $$|T_{s,g}^\delta| \text{ is not less than } \gamma.$$

(1) is false iff there is a map f from $T_{s,g}^\delta$ onto $|T_{s,g}^\delta|$ such that $f(u) = |T_u^\delta|$ for all $u \in T_{s,g}^\delta$. If f exists, then it is unique and defined by recursion on the wellfounded relation $T_{s,g}^\delta$. The recursion can be performed at level $\delta + |T_{s,g}^\delta|$ of L. Thus f, if it exists, is first order definable over $L(\delta + |T_{s,g}^\delta|)$. It follows that the recursion that defines Z can be carried out at level $\delta + \gamma$ of L. $\square$

9.4 Corollary (Addison, Kondo). *Each nonempty Π_1^1 set of reals contains a Π_1^1 singleton.*

Proof. Suppose K is Π_1^1 and nonempty. By Lemma 9.3 there is an X in K such that $X \in L(\omega_1^X)$. With each such X, associate a pair $\langle \gamma, \delta \rangle_X$ of ordinals: $X \in L(\gamma + 1) - L(\gamma)$ and $|\{e\}^X| = \delta$. (K is the set of all X such that $\{e\}^X$ is wellfounded.) The formula, $X \in K$ and $X \in L(\omega_1^X)$ and $\langle \gamma, \delta \rangle_X$ has the least possible value, is Π_1^1, and almost has a unique solution. To clinch the uniqueness, minimize the Gödel number of the formula that defines X as a subset of $L(\gamma)$. The Gödel number has to include parameters needed for the definition of X over $L(\gamma)$. The proof of Lemma 9.3 makes it safe to assume those parameters are ordinals. $\square$

9.5 Theorem (Mansfield, Solovay). *If K is Π_1^1, then (i), (ii) and (iii) are equivalent.*

(i) K contains a perfect set.
(ii) K contains a perfect set with constructible code.
(iii) K has a member $X \notin L(\omega_1^X)$.

Proof. (i) $\rightarrow$ (ii). The set of all Z's that encode perfect subsets of K is Π_1^1, hence at least one such is constructible by Lemma 9.3.

(ii) $\rightarrow$ (iii). Let $Z \in L$ encode a perfect subset of K denoted by P_Z as in subsection 6.1. The predicate

$$X \in P_Z \quad \& \quad X \not\leq_h Z$$

is satisfiable by all but countably many $X \in P_Z$, is Σ_1^1 in Z, and so has a solution X such that $\omega_1^{X,Z} = \omega_1^Z$ by Corollary 1.5 relativized to Z. If $X \in L(\omega_1^X)$, then $X \in L(\omega_1^Z)$, and consequently $X \leq_h Z$ by Exercise 9.11.

(iii) $\rightarrow$ (i). The second half of the predicate

$$(1) \qquad X \in K \quad \& \quad X \notin L(\omega_1^X)$$

is Σ_1^1 according to Exercise 9.7. Let R be a recursive predicate such that $(\mathrm{E}f)(x)R(\bar{s}(x), \bar{f}(x))$ holds iff s is the characteristic function of some X such that $X \notin L(\omega_1^X)$. The solutions of (1) are associated with a tree U that combines tree T of the proof of Lemma 9.3 with tree T_R needed in the proof of Kleene's basis Theorem (1.3). T is needed for the first half of (1), and T_R for the second half. A node of U has the form $\langle s, g, w \rangle$, where $\langle s, g \rangle$ is a node of T, w is a sequence number that encodes an initial segment of a function, length of w = length of s, and $R(s, w)$ holds. An infinite branch through U delivers an $X \in K$, a witness to the fact that $X \in K$, and a witness to the fact that $X \notin L(\omega_1^X)$.

As in the proof of 9.3, $X \in K$ iff $\{e\}^X$ is wellordered. Choose X_0, a solution of (1), to minimize $|\{e\}^{X_0}| = \delta$. Define U^δ by restricting U to δ. Thus each node of U^δ mentions only ordinals less than δ. The perfect set construction takes place within U^δ. Let $U^\delta_{s,g,w}$ be the subtree of U^δ issuing from node $\langle s, g, w \rangle$. For the moment assume U^δ has the following *splitting* property: If $U^\delta_{s,g,w}$ has an infinite branch, then there exist $\langle s_i, g_i, w_i \rangle$ $(i < 2)$ such that $\langle s, g, w \rangle$ is extended by $\langle s_i, g_i, w_i \rangle$, $U^\delta_{s_i,g_i,w_i}$ has an infinite branch, and s_0 is incomparable with s_1.

The construction proceeds as in the proof of Theorem 6.2. $\{\langle s_i^j, g_i^j, w_i^j \rangle \mid i < \omega \;\&\; j < 2^i\}$ is defined by recursion on i. s_0^0, g_0^0 and w_0^0 are null. $U^\delta_{s_0^0, g_0^0, w_0^0}$ has an infinite branch because X_0 is a solution of (1). $\langle s_{i+1}^{2j}, g_{i+1}^{2j}, w_{i+1}^{2j} \rangle$ and $\langle s_{i+1}^{2j+1}, g_{i+1}^{2j+1}, w_{i+1}^{2j+1} \rangle$ are extensions of $\langle s_i^j, g_i^j, w_i^j \rangle$ provided by the splitting property. $\{s_i^j \mid i < \omega \;\&\; j < 2^i\}$ encodes a perfect subset of K.

To check the splitting property, assume that $U^\delta_{s,g,w}$ gives rise to a unique solution X of (1). The concluding arguments of Lemma 9.3 will show $X \in L(\omega_1^X)$ contrary to (1). Call a node $u \in U^\delta_{s,g,w}$ false if $(u)_0$ does not encode an initial segment of X. If u is false, then U^δ_u is wellfounded. Since $\omega_1^{X_0} \leq \omega_1^X$, it follows as in 9.3 that: U^δ_u can be construed as a tree hyperarithmetic in X (uniformly in u); the set of all false nodes is hyperarithmetic in X; the strict supremum of $|U^\delta_u|$ over all false u is some $\gamma < \omega_1^X$.

u is a false node iff there is a map f from U^δ_u onto the ordinal $|U^\delta_u|$ such that $f(x) = |U^\delta_x|$ for all $x \in U^\delta_u$. If f exists, then it is unique and defined by recursion on the wellfounded relation U^δ_u. The recursion can be performed at level $\delta + |U^\delta_u|$ of L. Thus f, if it exists, is first order definable over $L(\delta + |U^\delta_u|)$. It follows that the set of false nodes, hence X, is definable at level $\delta + \gamma$ of L. $\square$

The proof of Theorem 9.5 ((iii) → (i)) was modeled on the proof of Theorem 6.2 ((iii) → (i)). The tree U^δ corresponds to a tree T_R not made explicit in 6.2. The nodes of T_R are of the form $\langle s, t \rangle$, where s and t are sequence numbers of the same length and $R(s, t)$ holds. The Σ_1^1 set K of 6.2 is the set of all X such that $(\mathrm{E}g)(x)R(\bar{X}(x), \bar{g}(x))$. The proof of the splitting property for T_R was based on the fact that (A) if a Σ_1^1 predicate has a unique solution, then that solution is hyperarithmetic. The corresponding fact for 9.5 is: (B) if all the infinite branches of U^δ carry the same subset of ω, namely X, then $X \in L(\omega_1^X)$. According to Exercise 9.12, a subset of ω is hyperarithmetic iff it belongs to $L(\omega_1^{\mathrm{CK}})$. It is possible, although tedious, but certainly instructive, to give a proof of (A) in the style of the proof given of (B).

9.6 A Hierarchy of Π_1^1 Singletons. Proposition 9.1 suggests that the Π_1^1 singletons might be generated by iterating the hyperjump. To be precise, define

$$\begin{aligned} h_0 &= \text{hyperdegree of the empty set,} \\ h_{\delta+1} &= \text{hyperjump of } h_\delta \text{, and} \\ h_\lambda &= \text{least upper bound of } \{h_\delta \mid \delta < \lambda\}. \end{aligned}$$

It turns out that h_λ is undefined for a certain countable ordinal σ_0. σ_0 is fairly large. It is the least Σ_1 admissible ordinal that is a limit of Σ_1 admissible ordinals (Richter 1967, Sacks 1971).

For all $\delta < \sigma_0$, h_δ is the hyperdegree of a Π_1^1 singleton. But $\{h_\delta \mid \delta < \sigma_0\}$ does not exhaust the hyperdegrees of the Π_1^1 singletons.

Suzuki 1964 proved that the partial ordering of hyperdegrees, restricted to the Π_1^1 singletons, is a wellordering. He observed: if X and Y are Π_1^1 singletons, then

$$X \leq_h Y \leftrightarrow \omega_1^X \leq \omega_1^Y. \tag{1}$$

Assume the right side of (1) holds. Suppose X is the unique solution of

$$\{e\}^Z \text{ is wellordered.} \tag{2}$$

The ordertype of $\{e\}^X$ is less than ω_1^Y, hence (2) (with X in place of Z) is equivalent to $|\{e\}^X| < |b|_Y$ for some fixed $b \in O^Y$. Thus (2) is Δ_1^1 in Y. Hence $X \leq_h Y$ by Theorem 2.5.II relativized to Y.

9.7–9.13 Exercises

9.7. Show $X \in L(\omega_1^X)$ is a Π_1^1 predicate.

9.8. (Gaspari, Kechris, Sacks). Call a Π_1^1 subset of 2^ω scattered if it contains no perfect subset. Show the union of all scattered Π_1^1 sets is a scattered Π_1^1 set.

9.9. (Kondo, Addison). Suppose $P(X, Y)$ is Π_1^1. Show $P(X, Y)$ can be uniformized by some Π_1^1 $Q(X, Y)$.

9.10. (Shoenfield). The Δ_2^1 reals are a basis for the Σ_2^1 predicates of reals.

9.11. Show $X \in L(\omega_1^Z, Z)$ iff $X \leq_h Z$. (The definition of $L(\beta, Z)$ is the same as that of $L(\beta)$ in subsection 9.2 save for the initial step, which is replaced by "$L(0, Z) = \{Z\}$".)

9.12. Show X is hyperarithmetic iff $X \in L(\omega_1^{\mathrm{CK}})$.

9.13. Prove reduction, as defined in Section 3.6.II, for Π_1^1 sets of reals.