Chapter II
The Hyperarithmetic Hierarchy

The hyperarithmetic sets are defined by iterating the Turing jump through the recursive ordinals, and are shown to equal the Δ_1^1 sets. The equality is important for two reasons. First, it reveals that Δ_1^1 is more constructive than it appears to be. Second, it allows properties of Δ_1^1 sets to be proved by induction, since hyperarithmetic sets fall into a hierarchy and can be assigned ordinal ranks less than ω_1^{CK}.

Hyperarithmetic reducibility, hyperdegrees and the hyperjump are defined.

1. Hyperarithmetic Implies Δ_1^1

The H-sets are defined after some properties of the Turing jump are reviewed. A set is defined to be hyperarithmetic if it is recursive in some H-set. Then an effective transfinite recursion produces an effective method for passing from the index of an H-set X to a Δ_1^1 index for X.

1.1 *H*-Sets. Let c_X be the characteristic function of the set X. Y is said to be Turing reducible to (or recursive in) Z if

(1) $$(\mathrm{E}e)[c_Y=\{e\}^{c_Z}].$$

$\{e\}$ is sketched in Chapter I, subsection 1.1. Formula (1) is often rendered as $X\leq_T Y$.

The Turing jump of X is denoted by X_1' and is defined by

$$\{e|\{(e)_0\}^X((e)_1) \text{ is defined}\}.$$

X' can be regarded as the effective disjoint union of all sets recursively enumerable in X.

The following elementary facts about Turing reducibility and jump are proved in Rogers 1967.

(2) X' is not recursive in X.

(3) X is recursive in X' uniformly. (There is an e_0 such that for all X, $X = \{e_0\}^{X'}$.)

(4) There is a recursive function t such that

$$X = \{e\}^Y \rightarrow X' = \{t(e)\}^{Y'}$$

for all X, Y and e.

(5) There is a recursive function θ such that

$$X = \{d\}^Y \ \& \ Y = \{e\}^Z \rightarrow X = \{\theta(d, e)\}^Z$$

for all X, Y, Z, d and e.

The H-sets are defined by recursion on $<_O$:

$$H_1 = \phi, \tag{6}$$

$$H_{2^a} = (H_e)' \tag{7}$$

$$H_{3.5^e} = \{\langle x, n\rangle | x \in H_{\{e\}(n)}\} \quad (\langle x, n\rangle = 2^x \cdot 3^n). \tag{8}$$

Let $H\colon O \rightarrow 2^\omega$ be the unique function that satisfies (6)–(8). X is an H-set if $X = H_b$ for some $b \in O$. X is said to be *hyperarithmetic* if X is recursive in some H-set.

Let HYP be the set of all hyperarithmetic sets. HYP theory is largely the creation of Kleene, although pioneering work was done by M. Davis, and by A. Mostowski. Recall that X and Y are said to have the same Turing degree (in symbols $X \equiv_T Y$) if $X \leq_T Y$ and $Y \leq_T X$. It will be seen in Section 2 that the Turing degree of H_b is determined by $|b|$.

1.2 Lemma (Kleene). *There exists a recursive function k such that for all a, b*

$$a <_O b \rightarrow H_a = \{k(a, b)\}^{H_b}.$$

Proof. By effective transfinite recursion on b. Let e_0 and θ be as in subsection 1.1, (3) and (5). Fix a and assume $a <_O b$. The definition of k has three cases.

(i): $b = 2^a$. $k(a, b) = e_0$, since $H_b = H_a'$.

(ii): $b = 2^d \neq 2^a$. $k(a, b) = \theta(k(a, d), e_0)$.

(iii): $b = 3 \cdot 5^z$. $k(a, b)$ is defined thus. Let p be the recursive function of Theorem 3.5 of Chapter I. Simultaneously enumerate $W_{p(\{z\}(n))}$ $(n \geq 0)$ until an n is found such that $a \in W_{p(\{z\}(n))}$. Such an n must exist if $a <_O 3 \cdot 5^z$. $k(a, b)$ is

$$\theta(k(a, \{z\}(n)), h(n)),$$

where h is a recursive function such that

$$\{h(n)\}^X = \{v | \langle v, n\rangle \in X\}$$

for all n and X. Thus $\{h(n)\}^{H_{3\cdot 5^z}}$ is $H_{\{z\}(n)}$ when $3 \cdot 5^z \in O$.

Before the definition of k can be made precise, it is necessary to elaborate case (iii). A trick is needed to cover the possibility that $3 \cdot 5^z$ may not belong to O. In that event there may not be an n such that $\{z\}(n)$ is defined and $a \in W_{p(\{z\}(n))}$. The trick is to define $k(a, 3 \cdot 5^z)$ so as to embody the search for n without worrying about whether such an n exists. Let $\theta_*(e, a, z)$ be a recursive function with the following properties. For all e, a, and z, $\theta_*(e, a, z)$ is the index of a Turing reduction procedure. $\{\theta_*(e, a, z)\}^X$ makes sense for all X but may be partial for some X. The instructions for computing $\{\theta_*(e, a, z)\}^X$ are: simultaneously enumerate $W_{p(\{z\}(n))}$ $(n \geq 0)$ until an n is uncovered such that $\{z\}(n)$ is defined and $a \in W_{p(\{z\}(n))}$. Let n_O be the first such n uncovered. (If n_O does not exist, then $\{\theta_*(e, a, z)\}^X(v)$ is undefined for all v.) Then

$$\{\theta_*(e, a, z)\}^X \simeq \{\theta(\{e\}(a, \{z\}(n_O)), h(n_O))\}^X.$$

The recursive iterater I needed for the definition of k by effective transfinite recursive is given by:

$$\{I(e)\}(a,b) = \begin{cases} e_O & \text{if } b = 2^a, \\ \theta(\{e\}(a,d), e_O) & \text{if } b = 2^d \neq 2^a, \\ \theta_*(e, a, z) & \text{if } b = 3 \cdot 5^z \\ 0 & \text{otherwise.} \end{cases},$$

Let $\{c\} \simeq \{I(c)\}$. k is $\{c\}$. k is total because θ and θ_* are. $\square$

1.3 Theorem. *Each of the following predicates is* Π_1^1.

(i) $x \in O$ & $y \in H_x$.
(ii) $x \in O$ & $y \notin H_x$.

Proof. (i) Let $A(X)$ be the conjunction of:

$$(X)_1 = \phi,$$

$$(a)\ [a \in O \rightarrow (X)_{2^a} = (X)'_a], \text{ and}$$

$$(e)\ [3 \cdot 5^e \in O \rightarrow (X)_{3 \cdot 5^e} = \{\langle x, n \rangle \mid x \in (X)_{\{e\}(n)}\}$$

Recall that $(X)_m = \{n \mid \langle m, n \rangle \in X\}$ and that $\langle m, n \rangle = 2^m \cdot 3^n$. Define X^* to be the set of all $\langle x, y \rangle$ satisfying predicate 1.3.(i). Then $A(X^*)$ holds by induction on $<_O$.

Suppose $A(X)$ to show $X^* \subseteq X$. If $a \notin O$, then $(X^*)_a = \phi$ and so $(X^*)_a \subseteq (X)_a$. Assume $a \in O$. Then $(X^*)_a = (X)_a$ by induction on $<_O$.

Thus X^* is the intersection of all solutions of $A(X)$. By Theorem 1.6. I, X^* is Π_1^1, since O is Π_1^1.

(ii) Similar to (i). $\square$

1.4 Corollary (Kleene)

(i) *If* X *is hyperarithmetic, then* X *is* Δ_1^1.
(ii) *The predicate,* $X \in \mathrm{HYP}$, *is* Π_1^1.

Proof. Consider the predicate

$$(1) \qquad (Es)\,(Ez)\,[T(s, e, y, z)\ \&\ \ell h(s) = z\ \&\ U(z) = i$$
$$\&\,(j)_{j<z}((s)_j = 1 \to x \in O\ \&\ j \in H_x)$$
$$\&\,(j)_{j<z}((s)_j \neq 1 \to x \in O\ \&\ j \notin H_x)].$$

By Theorem 1.3, (1) is Π_1^1. Fix x and e, and assume $X = \{e\}^{H_x}$. Then

$$y \in X \leftrightarrow (1) \text{ holds with } i = 1,$$
$$y \notin X \leftrightarrow (1) \text{ holds with } i = 0.$$

Hence X is Δ_1^1.

The predicate, $X \in \mathrm{HYP}$, is equivalent to

$$(2) \qquad (Ex)\,(Ee)\,[x \in O\ \&\ X = \{e\}^{H_x}].$$

The Π_1^1-ness of (1) implies (2) is Π_1^1. $\square$

More information is to be had concerning Corollary 1.4(i). Suppose X is Δ_1^1. $2^c \cdot 3^d$ is said to be an index for X as a Δ_1^1 set (or simply a Δ_1^1-index for X) if c (d respectively) is a Π_1^1 index for X ($\omega - X$ respectively). Thus

$$y \in X \leftrightarrow (f)\,(Ex)\,[T(\bar{f}(x), c, y, x)\quad \&\quad U(x) = 1],$$
$$y \notin X \leftrightarrow (f)\,(Ex)[T(\bar{f}(x), d, y, x)\quad \&\quad U(x) = 0].$$

1.5 Theorem (Kleene). *There exists a recursive function f such that*

$$f(b) \text{ is a } \Delta_1^1\text{-index for } H_b$$

for all $b \in O$.

Proof. By effective transfinite recursion on $<_O$. First a recursive function j is defined such that

$$m \text{ is a } \Delta_1^1\text{-index for } X \to j(m)$$

is a Δ_1^1-index for X' for all m and X. The definition of X' yields

$$(1) \qquad y \in X' \leftrightarrow (Es)[T(s, (y)_0, (y)_1 \ell h(s))$$
$$\&\ \ (i)_{i<\ell h(s)}((s)_i = 1 \to i \in X)$$
$$\&\ \ (i)_{i<\ell h(s)}((s)_i \neq 1 \to i \notin X)].$$

Let m be a Δ_1^1-index for X:

(2) $$i \in X \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), (m)_0, i, x) \quad \& \quad U(x)=1],$$

(3) $$i \notin X \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), (m)_1, i, x) \quad \& \quad U(x)=0].$$

Substitute the right side of (2) for $i \in X$, and the right side of (3) for $i \notin X$, in (1). Let c be the index of the resulting Π_1^1 formula after normalization. Thus c is a Π_1^1-index for X'. In a similar fashion a Π_1^1-index d can be found for $\omega - X'$. Let $j(m) = 2^c \cdot 3^d$.

Next a recursive function r is defined so that for all z and all $\{Y_n | n < \omega\}$,

$$\text{if } (n)\ [\{z\}(n) \text{ is a } \Delta_1^1\text{-index for } Y_n],$$
$$\text{then } r(z) \text{ is a } \Delta_1^1\text{-index for } \{\langle y, n\rangle | y \in Y_n\}.$$

Define

(4) $$\langle y, n\rangle \in Z \leftrightarrow (f)(\mathrm{Ex})[T(\bar{f}(x), \{z\}(n), y, x) \quad \& \quad U(x) = 1].$$

(The right side of (4) is false if $\{z\}$ (n) is undefined.) Let u be the Π_1^1-index for the right side of (4) after normalization. Similarly a Π_1^1-index v for $\omega - Z$ can be found. Let $r(z) = 2^u \cdot 3^v$.

Now for the definition of f by transfinite recursion. Let c be a recursive function such that

$$\{c(e, z)\}(n) \simeq \{e\}(\{z\}(n))$$

for all n. There exists a recursive I such that

$$\{I(e)\}(b) \simeq \begin{array}{ll} e_0 & \text{if } \ b = 1 \\ j(\{e\}(m)) & \text{if } \ b = 2^m \\ r(c(e, z)) & \text{if } \ b = 3 \cdot 5^z \\ 0 & \text{otherwise.} \end{array}$$

Choose d so that $\{d\} \simeq \{I(d)\}$. Then $\{d\}$ is the sought-after f. □

Theorem 1.5 yields another proof of Theorem 1.4(i), and some information concerning *persistent* Δ_1^1 definitions. Fix $X \in \Delta_1^1$. Suppose

$$y \in X \leftrightarrow (\mathrm{Ef})A(f, y) \quad \text{and} \quad y \notin X \leftrightarrow (\mathrm{Ef})B(f, y)$$

for some arithmetic A and B. Clearly

$$(y)(\mathrm{Ef})[A(f, y) \vee B(f, y)].$$

($\vee$ is the exclusive "or".) For each y, choose an f_y that satisfies the matrix above, and let W be $\{f_y | y < \omega\}$. Then

$$y \in X \leftrightarrow (\mathrm{Ef})_{f \in V} A(f, y) \quad \text{and} \quad y \notin X \leftrightarrow (\mathrm{Ef})_{f \in V} B(f, y)$$

whenever $2^\omega \supseteq V \supseteq W$. The above state of affairs is described by saying: X has an upward persistent Δ_1^1 definition over W. It will be shown in the next section that each H_b has an upward persistent Δ_1^1 definition over $\{H_a | a <_O b\}$.

There is a persistence phenomenon hidden in the proof of Theorem 1.5. Define b^* by: $1^* = 1$, $(3 \cdot 5^e)^* = 3 \cdot 5^e$, and $(2^a)^* = 2^{2^{(a)^*}}$. Let V_b be $\{g | g' \leq_T H_b\}$ for each $b \in O$. If f is the recursive function defined in the proof of Theorem 1.5, then $f(b)$ is the index of a Δ_1^1 definition of H_b upward persistent over V_{b^*} (Exercise 1.9).

An early persistence result is due to Gödel: L has a Σ_1 definition upward persistent over L. To be more precise, there is a Δ_0 formula $P(x, y)$ of set theory such that

$$x \in L \leftrightarrow (\mathrm{Ey})_{y \in V} P(x, y)$$

for all models $V \supseteq L$.

1.6 Lemma. *If X has a Δ_1^1 definition upward persistent over $\{f | f \leq_T B\}$, then X' has a Δ_1^1 definition upward persistent over $\{f | f \leq_T B''\}$.*

Proof. Recall formulas (1)–(3) from the proof of Theorem 1.5. Assume the Δ_1^1 definition of X given by (2) and (3) is upward persistent over $\{f | f \leq_T B\}$. If the right side of (2) is substituted for $i \in X$, and the right side of (3) for $i \notin X$, in (1), then $y \in X'$ becomes

$$(4) \qquad (\mathrm{Es})_{s \in \mathrm{Seq}}(f)[f \leq_T B \rightarrow (\mathrm{Ex})R(\bar{f}(x), s, y)]$$

for some recursive R. Since f is restricted, the quantifier manipulations that transform (4) into a Δ_1^1 formula upward persistent over $\{f | f \leq_T B''\}$ have to be considered with care. $y \notin X'$ is

$$(5) \qquad (s)_{s \in \mathrm{Seq}}(\mathrm{Ef})[f \leq_T B \quad \& \quad \sim (\mathrm{Ex})R(\bar{f}(x), s, y)].$$

In order to move the universal quantifier on s in (5) past the existential quantifier of f, it is necessary to choose, for each s, an f that satisfies the matrix of (5). Since $f \leq_T B$, a choice of f amounts to a choice of n such that $\{n\}^B$ is total. The set of all such n is many-one reducible to B''. It follows that the set of all n such that $\{n\}^B$ is a total function f and satisfies $\sim (\mathrm{Ex})R(\bar{f}(x), s, y)$ is also many-one reducible to B''. Hence (5) is equivalent to

$$(6) \qquad (\mathrm{Ef})[f \leq_T B'' \quad \& \quad (s)_{s \in \mathrm{Seq}} \sim (\mathrm{Ex})R((\bar{f})_s(x), s, y)].$$

The f of (6) is such that $(f)_s = \{t(s)\}^{B''}$ for some $t \leq_T B''$. A similar argument begins with $y \notin X'$ and ends with a Σ_1^1 formula for $y \in X'$ upward persistent over $\{f | f \leq_T B''\}$. $\square$

1.7–1.9 Exercises

1.7. Recall $A(X)$ from the proof of Theorem 1.3. Is $A(X)$ a closure condition in the sense of the remarks following the proof of Theorem 1.6.I?

1.8. Obtain Theorem 1.5 as an immediate corollary of Theorem 1.3.

1.9. Let f be the recursive function developed in the proof of Theorem 1.5. Show, for each $b \in O$, $f(b)$ is a Δ_1^1-index for H_b upward persistent over V_{b*}. $1^* = 1$, $(3 \cdot 5^e)^* = 3 \cdot 5^e$, and $(2^a)^* = 2^{2^{(a*)}}$. $V_b = \{f \mid f \leq_T B\}$.

2. Δ_1^1 *Implies Hyperarithmetic*

The main result of this section is that every Δ_1^1 set is hyperarithmetic. Along the way some related boundedness, uniformization and selection principles are proved.

For each $b \in O$, define

$$O_b = \{a \mid a \in O \quad \& \quad |a| < |b|\}.$$

2.1 Lemma. *Each of the following predicates is* Π_1^1.

(i) $x \in O$ & $y \in O_x$.
(ii) $x \in O$ & $y \notin O_x$.

Proof. Similar to that of Theorem 1.3. This time $A(X)$ is the conjunction of

$$(X)_1 = \phi,$$

$$(e)\ [3 \cdot 5^e \in O \rightarrow (X)_{3 \cdot 5^e} = \bigcup_n (X)_{\{e\}(n)}], \text{ and}$$

$$(a)\ [2^a \in O \rightarrow (X)_{2^a} = \{1\} \tag{1}$$

$$\cup \{y \mid (\mathrm{E}m)(y = 2^m \quad \& \quad m \in (X)_a)\}$$

$$\cup \{y \mid (\mathrm{E}e)[y = 3 \cdot 5^e \quad \& \quad (n)(\{e\}(n) \in (X)_a)]\}].$$

The set of all $\langle x, y \rangle$ satisfying (1) is the intersection of all X satisfying $A(X)$. $\square$

2.2 Uniformization. $P(x, y)$ is said to uniformize $Q(x, y)$ if

$$(x)(y)[P(x, y) \rightarrow Q(x, y)], \text{ and}$$

$$(x)[(\mathrm{E}y)Q(x, y) \rightarrow (E_1 y)P(x, y)].$$

In short Q contains P, and P is the graph of a function whose projection on the x-axis is the same as that of Q. (The term, "uniformization", was coined by descriptive set theorists.) Proving a uniformization principle usually amounts to proving a selection, or choice, principle. If $Q(x, y)$ is recursively enumerable ($x, y \in \omega$), then it is trivial but instructive to uniformize it by a recursively enumerable P as follows. Let e be such that

$$Q(x, y) \leftrightarrow (\mathrm{E}z)\, T(e, x, y, z).$$

Define $P(x, y)$ by

$$(\mathrm{E}z)[T(e, x, y, z) \;\&\; (w)_{w<z} \sim (\mathrm{E}y)\, T(e, x, y, w) \;\&\; (v)_{v<y} \sim T(e, x, v, z)].$$

$P(x, y)$ is recursively enumerable because $T(e, x, y, w)$ implies $y < w$ (a useful convention satisfied by Kleene's T-predicate). Fix x and assume $(\mathrm{E}y)Q(x, y)$. $P(x, y)$ first minimizes the length z of computation needed to uncover some y that satisfies $Q(x, y)$, and then singles out the least y associated with the minimum computation. This selection procedure is adequate for the proof of the next theorem, additional evidence that a Π_1^1 set is some kind of recursively enumerable set.

2.3 Theorem (Kreisel). *Π_1^1 predicates (of numbers) can be uniformized by Π_1^1 predicates.*

Proof. By Theorem 5.4.I, there is a recursive function g such that

$$Q(x, y) \leftrightarrow g(x, y) \in O$$

for all $x, y \in \omega$. Let $P(x, y)$ be

$$g(x, y) \in O \;\&\; (z)_{z \neq y}[g(x, z) \notin O_{g(x, y)}] \;\&\; (z)_{z<y}[g(x, z) \notin O_{2^{g(x, y)}}].$$

$P(x, y)$ is Π_1^1 according to Lemma 2.1. Fix x and assume $(\mathrm{E}y)Q(x, y)$. $P(x, y)$ first minimizes $|g(x, y)|$ and then singles out the least y that gives rise to the least ordinal. $\square$

Theorem 2.3 remains true when x and y are set variables, but the proof is more complicated because there is no effective wellordering of 2^ω. Nonetheless the idea of minimizing certain ordinals is still pertinent, as will be seen below in Section 9 of Chapter III.

2.4 Theorem (Spector 1955). *There exists a recursive function g such that*

$$O_b = \{g(b)\}^{H_{2^b}}$$

for all $b \in O$.

Proof. By effective transfinite recursion. The most interesting step occurs in the limit case, since it is that case that requires $H_2 b$ rather than H_b in the statement of the theorem.

Limit case: $b = 3 \cdot 5^e$. Let k be the recursive function of Lemma 1.2. Define

$$\theta(e, n) \text{ by } k(2^{\{e\}(n)}, 3 \cdot 5^e).$$

If $3 \cdot 5^e \in O$, then for all n, $\theta(e, n)$ is defined and

$$H_{2^{\{e\}(n)}} = \{\theta(e, n)\}^{H_{3 \cdot 5^e}}.$$

Let $c(u, v)$ be a recursive function such that $\{c(u, v)\}^X \simeq \{u\}^{\{v\}^X}$ for all X. The predicate

$$\text{(1)} \qquad (\mathrm{En})[\{e\}(n) \ \ \& \ \ \{z\}(n) \ \text{ are defined} \ \ \& \ \ y \in \{c(\{z\}(n), \theta(e, n))\}^X]$$

is recursive in X'. Hence, for some recursive t, (1) is equivalent to $y \in \{t(e, z)\}^{X'}$. t has the power to extend any $\{z\}$ that satisfies the theorem below $3 \cdot 5^e$ to one that satisfies it at $3 \cdot 5^e$. Suppose $3 \cdot 5^e \in O$ and for each n, $\{z\}(n)$ is defined and

$$O_{\{e\}(n)} = \{\{z\}(n)\}^{H_{2^{\{e\}(n)}}}$$

Then $O_{3 \cdot 5^e} = \bigcup_n O_{\{e\}(n)} = \{t(e, z)\}^{H_{2^{3 \cdot 5^e}}}$.

Successor case: $b = 2^a$. Define $e \in E$ by $(n)\,[\{e\}\,(n)$ is defined & $\{e\}\,(n) \in W_{p(\{e\}(n+1))}]$, where p is the recursive function of Theorem 3.5.I. If $1 \neq a \in O$, then $y \in O_a$ iff

$$\text{(2)} \qquad y = 1 \lor (\mathrm{Em})_{m<y}[y = 2^m \ \ \& \ \ m \in O_a] \lor (\mathrm{Ee})_{e<y}[y = 3 \cdot 5^e \ \ \& \ \ e \in E \ \ \& \ \ (n)(\{e\}(n) \in O_a)].$$

If $\{s\}^{X'}$ is substituted for O_a in (2), then the resulting predicate is recursive in X'', since E is recursive in ϕ'', and since

$$(n)(\{e\}(n) \in \{s\}^{X'})$$

can be construed as

$$(n)(z) \sim [\{e\}(n) \text{ is defined} \ \ \& \ \ T(\overline{X'}(z), s, \{e\}(n), z) \ \ \& \ \ U(z) = 0].$$

Hence the result of substituting $\{s\}^{X'}$ for O_a in (2) is equivalent to $y \in \{j(s)\}^{X''}$ for some recursive function j. If $a \in O$ and

$$O_a = \{s\}^{H_{2^a}}, \quad \text{then} \quad O_{2^a} = \{j(s)\}^{H_{2^{2^a}}}.$$

Choose c_0 so that $\phi = \{c_0\}^X$ for all X. There exists a recursive I such that

$$\{I(z)\}(b) \simeq \begin{array}{ll} c_0 & \text{if } \ b = 1 \\ j(\{z\}(a)) & \text{if } \ b = 2^a \\ t(e,z) & \text{if } \ b = 3\cdot 5^e \\ 0 & \text{otherwise.} \end{array}$$

Let $\{d\} \simeq \{I(d)\}$, and g be $\{d\}$. $\square$

2.5 Theorem (Kleene). *If X is Δ_1^1, then X is hyperarithmetic.*

Proof. Assume $X \in \Delta_1^1$. Theorem 5.4.I implies there is a recursive g such that

$$y \in X \leftrightarrow g(y) \in O$$

for all y. Define A by

(1) $$z \in A \leftrightarrow (\mathrm{Ey})(y \in X \quad \& \quad z = \ g(y)).$$

A is Σ_1^1. Spector's boundedness theorem, 5.6.I, yields a $b \in O$ such that $A \subseteq O_b$. Thus

$$y \in X \leftrightarrow g(y) \in O_b.$$

O_b is hyperarithmetic by Theorem 2.4. $\square$

Theorems 1.4(i) and 2.5 combine to produce $\Delta_1^1 = \mathrm{HYP}$ for sets of numbers. The notion of Δ_1^1 is thought to be less clear, or less constructive, than the notion of HYP. This is a way of drawing attention to the fact that Δ_1^1 sets are defined from above by quantification over ω^ω, while hyperarithmetic sets are defined from below by iterating the Turing jump, or number quantifier, through the recursive ordinals. It is sometimes said that the HYP sets constitute a predicative analysis of the Δ_1^1 sets. A more precise statement of the situation is: the HYP sets provide a hierarchy for the Δ_1^1 sets. It is a general problem of considerable interest to develop a hierarchy for a family of sets defined en masse. The interest is more than philosophical, since a hierarchy makes it possible to prove theorems about the family by transfinite induction. This is the approach taken in the next section to show each Δ_1^1 set is implicitly arithmetically definable, that is the unique solution of an arithmetic predicate.

There is a uniformity lacking in the proof of Theorem 2.5, which will be supplied by Corollary 3.5. It consists of recursive functions f and g such that: if e is a Δ_1^1 index for X, then $f(e) \in O$ and $X = \{g(e)\}^{H_{f(e)}}$. The only information needed to make the proof of 2.5 yield f and g is an effective method of passing from a Σ_1^1 index of A, the set defined by 2.5.(1), to the bound b.

2.6 Lemma (Kreisel). *Suppose $Q(x,y)$ is Π_1^1. Then*

$$(x)(\mathrm{Ey})Q(x,y) \rightarrow (\mathrm{Ef})_{f \in \mathrm{HYP}}(x)Q(s,f(x)).$$

Proof. By Theorem 2.3 there is a $\Pi_1^1 P(x,y)$ that uniformizes $Q(x,y)$. Define $f(x)=y$ by $P(x,y)$. f is Δ_1^1 by Proposition 1.7.I, hence hyperarithmetic by Theorem 2.5. □

It is reasonable, but risky, to view Lemma 2.6 as analogous to the following result of elementary recursion theory: suppose $Q(x,y)$ is recursively enumerable, then

$$(x)(\mathrm{Ey})Q(x,y) \rightarrow (\mathrm{Ef})\ [f \text{ is recursive} \quad \& \quad (x)Q(x,f(x))].$$

($f(x)$ is the first y such that $Q(x,y)$ is enumerated.) Reasonable because it adds to the evidence that Π_1^1-ness is akin to recursive enumerability. Risky because it suggests that hyperarithmeticity is akin to recursiveness. In Part B of this book it will be seen that the analogy between Π_1^1 and recursively enumerable entails an analogy between hyperarithmetic and finite. This outcome is signaled by Theorem 2.4. If a recursive enumeration of A is cut short, then the result is a finite set. Analogously, if the natural enumeration of Kleene's O (so termed in the proof of Theorem 2.2.I) is cut short, then the result is some O_b, a hyperarithmetic set according to Theorem 2.4.

2.7–2.11 Exercises

2.7. Show each $\Sigma_n^0 Q(x,y)$ can be uniformized by some $\Sigma_n^0 P(x,y)$ $(n>0)$.

2.8. Show the range of a total hyperarithmetic function is hyperarithmetic.

2.9. Formulate precisely and prove: a hyperarithmetic union of hyperarithmetic sets is hyperarithmetic.

2.10. Show each hyperarithmetic set is many-one reducible to some H-set.

2.11. Suppose C is a Π_1^1 set of hyperarithmetic reals, $P(X,Y)$ is Π_1^1, and $(X)(\mathrm{EY})[Y \in C \quad \& \quad P(X,Y)]$. Show there exists a hyperarithmetic map h from 2^ω into C such that $(X)P(X,h(X))$. (Read Section 5 before solving this problem.)

3. *Selection and Reduction*

It is useful to extend the domain of $\|$ from O to ω by defining $|b|$, when $b \notin O$, to be ∞. ∞ is greater than every constructive ordinal. Thus $|a| < \infty$ means $a \in O$.

It is helpful to think of b as a code for some sort of infinite computation that either terminates at some constructive ordinal or fails to terminate. The main result of this section is a selection principle: there exists a recursive function t such that

$$\text{(1)} \qquad a \in O \vee b \in O \rightarrow t(a,b) \in O$$
$$\& \quad \min(|a|,|b|) \leq |t(a,b)|.$$

t selects in the following sense. If either a or b terminates, then $t(a,b)$ terminates at a constructive ordinal large enough to serve as a vantage point from which one can look down and select a terminating element of the pair (a, b). In classical recursion theory an element can be selected from a nonempty recursively enumerable set A by simply enumerating all computations until one is found that puts an element into A.

In the proof of Theorem 2.3 it was seen that an element of a nonempty Π_1^1 set B can be selected in essentially the same manner. Let f be a recursive function such that

$$y \in B \leftrightarrow f(y) \in O.$$

Select the unique y such that

$$\text{(2)} \qquad f(y) \in O \quad \& \quad (z)_{z \neq y}[f(z) \notin O_{f(y)}] \quad \& \quad (z)_{z < y}[f(z) \notin O_{2f(y)}].$$

The notion of selection in (1) is somewhat more subtle than that in (2). In (1) a pair of computations is given with an assurance that at least one terminates. In (2) a set $\{f(y) \mid y \in B\}$ of computations, all of which terminate is given. The counterpart of (1) in classical recursion theory is proved by alternating between a and b until one of them terminates. This is the so-called computing-in-tandem trick. It is needed to show the disjunction of two recursively enumerable predicates is recursively enumerable. It plays an important part in the proof of Gandy selection in Part D.

As in subsection 4.2.I, let R_e be the e-th binary, recursively enumerable relation, and let $|R|$ be the ordinal height of R when R is wellfounded. Define $|R|$ to be ∞ when R is not wellfounded (WF).

3.1. Lemma. *There exists a recursive function k such that:*

(i) $[R_c \in WF \vee R_d \in WF] \leftrightarrow R_{k(c,d)} \in WF$;
(ii) $[R_c \in WF \vee R_d \in WF] \rightarrow \min(|R_c|, |R_d|) \leq |R_{k(c,d)}|$.

Proof. $R_{k(c,d)}$ is $R_c \otimes R_d$, a certain kind of product. The field of $R_c \otimes R_d$ consists of all numbers that are codes for ordered pairs $\langle r, s\rangle$, where $r \in$ field R_c and $s \in$ field R_d.

$$\langle r_1, s_1\rangle R_c \otimes R_d \langle r_2, s_2\rangle \quad \text{iff} \quad r_1 R_c r_2 \quad \text{and} \quad s_1 R_d s_2.$$

(i) Let Z be a nonempty subset of the field of $R_c \otimes R_d$. Z_{R_c} is $\{r \mid (\mathrm{Es})(\langle r,s\rangle \in Z)\}$, and $Z_{R_d} = \{s \mid (\mathrm{Er})(\langle r, s\rangle \in Z)\}$. Z has a minimal element iff Z_{R_c} or Z_{R_d} does.

(ii) Suppose either R_c or R_d is wellfounded. By (i) $R_c \otimes R_d$ is wellfounded. $|R_c \otimes R_d|$ is

$$\mu\beta[\langle r,s\rangle \in \text{field } R_c \otimes R_d \rightarrow |\langle r, s\rangle| < \beta].$$

If $r \in \text{field } R_c$ and the restriction of R_c to $\{r_1 | r_1 R_c r\}$ is wellfounded, then

$$|r| = \mu\beta[r_1 R_c r \rightarrow |r_1| < \beta].$$

Otherwise let $|r|$ be ∞. Treat R_d similarly. An induction on $\min(|r|, |s|)$ shows

$$(1) \qquad \min(|r|, |s|) \leq |\langle r, s \rangle|$$

for all $\langle r, s \rangle \in \text{field } R_c \otimes R_d$, because

$$(2) \qquad \min(|r_2|, |s_2|) = \max\{\min(|r_1|, |s_1|) \,\big|\, \langle r_1, s_1 \rangle R_c \otimes R_d \langle r_2, s_2 \rangle\}.$$

(1) implies $\min(|R_c|, |R_d|) \leq |R_c \otimes R_d|$. (max is *strict* lub.) $\square$

The proof of Lemma 3.1 centers on the min-max trick of equation (2). It is worth pondering, because it will reappear in Section 1.XII.

3.2 Proposition. *There exists a recursive function h such that*

(*i*) $b \in O \leftrightarrow R_{h(b)} \in WF$;
(*ii*) $b \in O \rightarrow |b| \leq |R_{h(b)}|$.

Proof. Since $O \in \Pi_1^1$, there is a recursive R such that

$$b \in O \leftrightarrow (f)(\mathrm{E}x)\, R(\bar{f}(x), b).$$

Define $S_R(b)$ as in subsection 5.2.I. Let q be the recursive function of Theorem 3.5.I.

$$b \in O \leftrightarrow S_R(b) \in WF, \text{ and}$$
$$b \in O \rightarrow W_{q(b)} \in WF \quad \& \quad |b| = |W_{q(b)}|.$$

Consequently the effective disjoint union of $S_R(b)$ and $W_{q(b)}$ will serve as $R_{h(b)}$. The effective disjoint union of U and V is W:

$$(2u_1)\, W(2u_2) \leftrightarrow (u_1)\, U(u_2),$$
$$(2v_1 + 1)\, W(2v_2 + 1) \leftrightarrow (v_1)\, V(v_2). \qquad \square$$

3.3 Theorem. *There exists a recursive function t with the following properties*:

(*i*) $a \in O \vee b \in O \leftrightarrow t(a, b) \in O$;
(*ii*) $a \in O \vee b \in O \rightarrow \min(|a|, |b|) \leq |t(a, b)|$.

Proof. Let f be the recursive function of Lemma 4.3.I. $t(a,b)$ is $f(k(h(a), h(b)))$, where h and k are the recursive functions of Proposition 3.2. and Lemma 3.1. $\square$

Theorem 3.3. makes it possible to effectivize Spector's boundedness theorem (Corollary 5.6.I). (There is a substantially different approach based on the "completeness" and "creativity" of O; it avoids the use of t.)

3.4. Corollary. *There exists a recursive function f such that for all c: if c is a Σ_1^1-index of A, then*

(i) $A \subseteq O \leftrightarrow f(c) \in O$, *and*
(ii) $A \subseteq O \rightarrow A \subseteq O_{f(c)}$.

Proof. According to Theorem 5.4.I, there is a recursive function h such that

$$(1) \qquad (\mathrm{E}f)(x) \sim T(\bar{f}(x), c, y, x) \leftrightarrow h(c,y) \notin O$$

for all c and y. Let t be as in Theorem 3.3. Let k be a recursive function such that

$$W_{k(c)} = \{t(h(c,y),y) \mid y < \omega\}$$

Assume c is a Σ_1^1 index of A. If $A \subseteq O$, then $W_{k(c)} \subseteq O$. Let $f(c)$ be $2^{g(k(c))}$, where g is the bounding function of Lemma 4.1.I.

Suppose $A \subseteq O$. Then

$$y \in A \rightarrow h(c,y) \notin O \rightarrow |y| \leq |t(h(c,y),y)|,$$

and so $A \subseteq O_{f(c)}$.

Suppose $f(c) \in O$. By Lemma 4.1(i) of Chapter I, $W_{k(c)} \subseteq O$; hence by Theorem 3.3(i),

$$h(c,y) \in O \vee y \in O$$

for all y. Then (1) implies $A \subseteq O$. □

Corollary 3.4 yields an effective form of Theorem 2.5.

3.5 Corollary. *There exist recursive functions f and g such that for all e: if e is a Δ_1^1-index for X, then*

$$f(e) \in O \quad \text{and} \quad X = \{g(e)\}^{H_{f(e)}}.$$

Proof. The only noneffective step in the proof of Theorem 2.5 is the transition from a Σ_1^1 index for A to a bound b for A. But that can now be managed by Corollary 3.4. □

3.6 Reduction and Separation. Let Z be a set and F a family of subsets of Z. Reduction is said to hold for F if for each pair $A, B \in F$, there exists a pair $A_0, B_0 \in F$

such that

(i) $A_0 \subseteq A$ and $B_0 \subseteq B$,
(ii) $A_0 \cap B_0 = \phi$, and
(iii) $A_0 \cup B_0 = = A \cup B$.

The notion of reduction originated in descriptive set theory. In this section it is proved for Π_1^1 sets of numbers, and later for Π_1^1 sets of reals (Exercise 9.13.III). It is easily verified for recursively enumerable sets of numbers. Enumerate A and B simultaneously. If a number comes up in A that has not come up in B at an earlier stage, then put it in A_0. If a number comes up in B that has not come up in A at the same or an earlier stage, then put it in B_0.

Separation is said to hold for F if for each pair C, $D \in F$ such that $C \cap D = \phi$, there exists a pair C_1, $D_1 \in F$ such that

(iv) $C \subseteq C_1$ and $D \subseteq D_1$, and
(v) $C_1 = Z - D_1$.

C_1 is said to separate C and D.

Let $\sim F$ be $\{A | Z - A \in F\}$. Observe that reduction for $(\sim F)$ implies separation for F. Suppose C, $D \in F$ and $C \cap D = \varnothing$. Reduce $Z - C$ to A_0, and $Z - D$ to B_0. Then $C \subseteq Z - A_0$, $D \subseteq Z - B_0$, and $Z - A_0 = B_0$.

3.7 Theorem (Kleene). *Suppose A, $B \in \Pi_1^1$. Then there exist A_0, $B_0 \in \Pi_1^1$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cap B_0 = \varnothing$ and $A_0 \cup B_0 = A \cup B$.*

Proof. By Theorem 5.4.I, there are recursive functions h and j such that for all n,

$$n \in A \leftrightarrow h(n) \in O \quad \text{and} \quad n \in B \leftrightarrow j(n) \in O.$$

Let $n \in A_0 \leftrightarrow n \in A \;\; \& \;\; |h(n)| \leq |j(n)|$, and
$n \in B_0 \leftrightarrow n \in B \;\; \& \;\; |j(n)| < |h(n)|$.

Lemma 2.1 implies A_0, $B_0 \in \Pi_1^1$. □

It follows from Theorem 3.7 that any two disjoint Σ_1^1 sets can be separated by a Δ_1^1 set. By Corollary 3.5 the separating set is a hyperarithmetic set whose index can be obtained effectively from the indices of the Σ_1^1 sets.

3.8–3.10 Exercises

3.8. Extend $\|$ from O to ω as in the beginning of Section 3. Show $|a| < |b|$ is Π_1^1.

3.9. Suppose $A \in \Sigma_1^1$, $B \in \Pi_1^1$ and $A \subseteq B$. Find a Δ_1^1 C such that $A \subseteq C \subseteq B$.

3.10. Find two disjoint Π_1^1 sets which cannot be separated by a Δ_1^1 set.

4. Π_2^0 *Singletons*

Suppose $(E_1X)P(X)$. Let A be the unique X that satisfies $P(X)$. A is said to be implicitly defined by $P(X)$. If the form of $P(X)$ is F, then A is said to be an F singleton. Thus Theorem I.6.I asserts that each Σ_1^1 singleton is Δ_1^1. It is immediate that every Δ_1^1 set is a Σ_1^1 singleton. It follows from Corollary 1.4(i) that every hyperarithmetic set is a Σ_1^1 singleton. In this section it will be shown that every H-set is a Π_2^0 singleton. (Later it will be shown by forcing that some hyperarithmetic set is not an arithmetic singleton, cf. Exercise 3.18.IV.) It follows that the hierarchy of H-sets can be construed as a hierarchy for the Π_2^0 singletons. In order to make the last assertion more precise, Spector's uniqueness theorem will be proved: $[a, b \in O \ \&\ |a| = |b|] \to H_a \equiv_T H_b$.

The next proposition suggests that the Π_2^0 singletons can be generated by iterating the Turing jump.

4.1 Proposition. *If A is a Π_2^0 singleton, then so are A', and any $B \equiv_T A$.*

Proof. Let A be the unique solution of $(u)(\mathrm{E}v)R(u, v, X)$ for some recursive R. Choose e_0 so that $X = \{e_0\}^{X'}$ for all X, and so that $\{e_0\}^Y$ is total for all Y. Then A' is the unique solution of

$$(1) \qquad \begin{aligned} &(u)(\mathrm{E}v)R(u, v, \{e_0\}^Y) \\ &\&\ (n)[n \in Y \leftrightarrow (\mathrm{E}z)T(\overline{\{e_0\}^Y}(z), (n)_0, (n)_1, z)]. \end{aligned}$$

Formula (1) is Π_2^0 because the predicate $z \in \{e_0\}^Y$ is recursive. □

4.2 Theorem.

(*i*) *There exists a Π_2^0 predicate $H(a, X)$ such that*

$$(E_1X)H(a, X) \quad \& \quad H(a, H_a)$$

for all $a \in O$.

(*ii*) *There exists a recursive predicate $R(a, y)$ such that*

$$(\mathrm{E}f)(x)R(a, \bar{f}(x)) \quad \& \quad (f)[(x)R(a, \bar{f}(x)) \to f \equiv_T H_a]$$

for all $a \in O$.

Proof. By effective transfinite recursion on a according to $<_O$. Let $\Pi(e, a, X)$ denote $(u)(\mathrm{E}v)T(\bar{X}(v), e, a, u, v)$, the e-th Π_2^0 predicate. The proof of Proposition 4.1 yields a recursive function h such that: if A is the unique solution of $\Pi(e, m, X)$, then A' is the unique solution of $\Pi(h(e), 2^m, X)$.

Let j be a recursive function such that for all e and d: $\Pi(j(e, d), 3 \cdot 5^d, X)$ is:

$$(n)[\{e\}(\{d\}(n)) \text{ is defined} \rightarrow \Pi(\{e\}(\{d\}(n)), \{d\}(n), (X)_n)].$$

Let $\Pi(e_0, 1, X)$ be $(n)(n \notin X)$.

There exists a recursive I such that

$$\{I(e)\}(a) \simeq \begin{array}{ll} e_0 & \text{if } \ a = 1 \\ h(\{e\}(m)) & \text{if } \ a = 2^m \\ j(e, d) & \text{if } \ a = 3 \cdot 5^d \\ 0 & \text{otherwise.} \end{array}$$

Let c be a fixed point of I. Then $\{I(c)\} \simeq \{c\}$, and $H(a, X)$ is $\Pi(\{c\}(a), a, X)$. Note that $\{c\}$ is total as in 3.3.I.

To prove (ii) let $H(a, X)$ be $(x)(Ev)R_1(x, v, a, X)$ for some recursive R_1. Define $Q(a, x, g, X)$ by

$$R_1(s, g(x), a, X) \quad \& \quad g(x) = \mu v R_1(s, v, a, X).$$

Q is recursive and

$$H(a, X) \leftrightarrow (Eg)[g \leq_T X \quad \& \quad (x)Q(a, x, g, X)].$$

The normal form theorem for Σ_1^0 predicates implies there is a recursive R such that

(2) $$(x)R(a, \bar{f}(x)) \leftrightarrow (x)Q(a, x, (f)_0, (f)_1) \quad \& \quad (i)_{i>1}(x)((f)_i(x) = 0).$$

If $a \in O$, then $(f)_0 \leq_T (f)_1$ and $(f)_1$ is the characteristic function of the unique X that satisfies $H(a, X)$. $\square$

The Π_2^0 predicate $H(a, X)$ is less mysterious if viewed as follows. The Π_1^1 predicate

(3) $$a \in O \quad \& \quad X = H_a$$

has H_a as a unique solution when $a \in O$. (3) can be arithmetically approximated by dropping the requirement that a represent a wellordering. Instead a represents a recursive linear ordering that bears a superficial, arithmetic resemblance to $\{\langle u, v\rangle | u <_O v <_O a\}$. Thus a might be required to represent an ordering of ω such that each member of b of the field of the ordering has 2^b as its immediate successor save for the last member which is a. In addition the arithmetic approximation of (3) would refer to a hierarchy of sets attached to the elements of the field in much the same way that the H-sets are attached to the elements of O.

It turns out that the content of (3) can be expressed by a Π_2^0 formula save for the clause that requires the recursive linear ordering represented by a to be a wellordering. In short (3) has a Π_2^0 approximation equivalent to (3) when $a \in O$.

From the vantage point of model theory there is another view of Theorem 4.2. Let M be a nonstandard ω-model of a fragment of mathematics strong enough to develop the theory of hyperarithmetic sets. (An ω-model is one whose integers are the same as those of the real world.) Take nonstandard to mean there exist nonwellfounded, recursive linear orderings which are wellfounded as far as M is concerned; they belong to M, but none of their infinite descending sequences do. It follows that the hyperarithmetic hierarchy of M is a proper end extension of its counterpart in the real world.

Let a be a notation for a constructive ordinal in the sense of M but not in the real world. Thus a represents a recursive linear ordering whose maximal wellfounded initial segment is of height ω_1^{CK}. Let H_a^M be the H-set attached to a by M. a and H_a^M satisfy every reasonable Π_2^0 approximation of (3).

These matters are discussed further in Section 2.III.

4.3 Corollary. *Each H-set is a Π_2^0 singleton.*

4.4 Corollary. *For each $a \in O$, H_a has a Δ_1^1 definition upward persistent over $\{H_b | b <_O a\}$.*

Proof. Let $H(a, X)$ be the Π_2^0 formula of Theorem 4.2. Clearly

$$z \in H_1 \leftrightarrow z \neq z,$$

$$z \in H_{2^b} \leftrightarrow (\mathrm{E}X)[H(b, X) \;\&\; z \in X']$$

$$\leftrightarrow (X)[H(b, X) \rightarrow z \in X'], \text{ and}$$

$$z \in H_{3 \cdot 5^e} \leftrightarrow (\mathrm{E}X)(\mathrm{E}x)(\mathrm{E}n)[H(\{e\}(n), X) \;\;\&\;\; x \in X \;\;\&\;\; z = \langle x, n\rangle]$$

$$\leftrightarrow (X)(\mathrm{E}x)(\mathrm{E}n)[(H(\{e\}(n), X) \rightarrow x \in X) \;\;\&\;\; z = \langle x, n\rangle]. \quad \square$$

4.5 Theorem (Spector 1955). *There exists a recursive function $h(a, b)$ such that*

$$a, b \in O \;\;\&\;\; |a| \leq |b| \rightarrow H_a = \{h(a, b)\}^{H_b}$$

Proof. By effective transfinite recursion on O^2 wellordered as follows. $(a_1, b_1) < (a_2, b_2)$ iff

$$|a_1| < |a_2| \vee (|a_1| = |a_2| \;\;\&\;\; |b_1| < |b_2|).$$

The definition of the recursive iterater $I(e, a, b)$ has four cases. $h(a, b)$ will be $\{c\}(a, b)$, where c is a fixed point of I; that is $I(c, a, b) \simeq \{c\}(a, b)$.

Case 1: $a = 1$. $I(e, a, b) = c_1$ for some c_1 such that $\{c_1\}^X = H_1 = \phi$ for all X.

Case 2: $a = 2^m$ & $b = 2^n$. $I(e, a, b)$ is $j(\{e\}(m, n))$, where j is recursive and $\{j(d)\}^{X'} \simeq (\{d\}^X)'$.
Case 3: $a = 3 \cdot 5^z$. Let r and s be recursive functions such that

$$\{e\}(\{z\}(n), b) \simeq \{s(e, z, b)\}(n), \quad \text{and}$$

$$\{r(z)\}^X = \{\langle x, n\rangle | x \in \{\{z\}(n)\}^X\}.$$

Then

$$\{I(e, a, b)\}^X \simeq \{r(s(e, z, b))\}^X.$$

Case 4: $a = 2^m$ & $b = 3 \cdot 5^d$. Let g be as in Theorem 2.4 and k as in Lemma 1.2. If m, $3 \cdot 5^d \in O$, and $|2^m| \leq |3 \cdot 5^d|$, then there is an n such that $|2^m| < |\{d\}(n)|$. Such an n can be computed from $H_{3 \cdot 5^d}$ because

$$|2^m| < |\{d\}(n)| \leftrightarrow 2^m \in O_{\{d\}(n)}, \text{ and}$$

(1) $$O_{\{d\}(n)} = \{g(\{d\}(n))\}^{\{k(2^{\{d\}(n)}, 3 \cdot 5^d)\}^{H_{3 \cdot 5^d}}}$$

With the above pair of formulas in mind it is straightforward to find an index c_2 such that: if m, $3 \cdot 5^d \in O$ and $|2^m| \leq |3 \cdot 5^d|$, then

$$\{c_2\}^{H_{3 \cdot 5^d}}(m, d) \text{ is an } n \text{ such that } |2^m| < |\{d\}(n)|.$$

$\{c_2\}^X(m, d)$ is computed as follows. Let $X^*_{d,n}$ be the result of replacing $H_{3.5^d}$ by X in the right side of (1). Compute the truth-value of $2^m \in X^*_{d,n}$ for all n simultaneously. $\{c_2\}^X(m, d)$ is the first n (if there is any) for which the computation of $2^m \in X^*_{d,n}$ terminates affirmatively. Then

$$\{I(e, a, b)\}^X \simeq \{\{e\}(2^m, \{d\}(n)\}^{(X)_n},$$

where $n \simeq \{c_2\}^X(m, d)$.
If (a, b) fails to satisfy any of the above four cases, let $I(e, a, b)$ be 0. □

4.6 Corollary. *If $a, b \in O$, then*

$$|a| = |b| \rightarrow H_a \equiv_T H_b.$$

(*Spector's uniqueness theorem*)

It is tempting to think that Corollary 4.6 remains true when Turing degrees are replaced by many-one degrees. Moschovakis (1966) has shown that such an improvement is impossible.

For each $\delta < \omega_1^{\mathrm{CK}}$, let h_δ be the Turing degree of H_a for some a such that $|a| = \delta$. Spector's uniqueness theorem implies h_δ is well defined. $\{h_\delta | \delta \in \omega_1^{\mathrm{CK}}\}$ is a hierarchy for the Turing degrees of the H-sets. Its definition is based on constructive ordinal notations whose principal function was to clarify the notion of effective union needed to define h_λ when λ is a limit ordinal. Sacks 1971 shows

$$h_\lambda = \inf\{d'' | (\delta)_{\delta<\lambda}(h_\delta < d)\}.$$

It is known that h_λ is definable in the partial ordering of Turing degrees with jump. It is likely that h_λ is definable in the Turing degrees without jump. For successful attempts to define h_γ for $\gamma \geq \omega_1^{\mathrm{CK}}$, see Jockusch and Simpson 1976.

4.7 Hierarchy of Π_2^0 Singletons. Let $\Pi(e, X)$ be the e-th Π_2^0 formula, namely $(u)(\mathrm{E}v)T(\bar{X}(v), e, u, v)$. The set of Π_2^0 formulas is closed under recursive unions. If r is a recursive function, then the recursive union of $\Pi(r(e), X)(e \geq 0)$ is defined by

$$(1) \qquad \bigcup_e [\Pi(r(e), X)] \leftrightarrow (e)\Pi(r(e), (X)_e)$$

The right side of (1) is readily put in standard form $\Pi(e^*, X)$ for some e^* computable from a Gödel number for r.

The proof of Proposition 4.1 yields a recursive h such that: if A is the unique solution of $\Pi(e, X)$, then A' is the unique solution of $\Pi(h(e), X)$.

A hierarchy for the Π_2^0 singletons is defined by iteration of the Turing jump through the constructive ordinals. Let p be a recursive function that satisfies the following effective transfinite recursion on O.

$$\Pi(p(1), X) \leftrightarrow X = \phi.$$

$$\Pi(p(2^a), X) \leftrightarrow \Pi(h(p(a)), X)$$

$$\Pi(p(3 \cdot 5^d), X) \leftrightarrow \bigcup_e [\Pi(p(\{d\}(e)), X)].$$

For each $\delta < \omega_1^{\mathrm{CK}}$, let Π_δ be the Turing degree of the unique solution of $\Pi(p(a), X)$ for some $a \in O$ such that $|a| = \delta$. Π_δ is well defined thanks to Corollary 4.6 and the proof of Theorem 4.2.

$\{\Pi_\delta | \delta < \omega_1^{\mathrm{CK}}\}$ is aptly termed a hierarchy for the Turing degrees of the Π_2^0 singletons, since each Π_2^0 singleton is Δ_1^1, hence hyperarithmetic, and consequently one-one reducible to some Π_2^0 singleton in some Π_δ. (Recall: $X \leq_1 X'$; $X \leq_T Y \rightarrow X' \leq_1 Y'$.) Thus a cofinal subset of the Π_2^0 singletons are generated by starting with the null set and closing under Turing jump and recursive union.

4.8–4.10 Exercises

4.8. Let $H(a, X)$ be the Π_2^0 predicate of Theorem 4.2. Find b and Y such that $H(b, Y)$ holds and every H-set is recursive in Y.

4.9. Suppose A is Δ_1^1 definable over Δ_1^1 (that is, the function quantifiers range over the Δ_1^1 elements of ω^ω instead of ω^ω). Show A is Δ_1^1.

4.10. (Martin Davis). Suppose $a, b \in O$ and $|a| = |b| < \omega^\omega$. Show H_a and H_b belong to the same one-one degree.

5. Hyperarithmetic Reducibility

Let Y be an arbitrary subset of ω. All the results of Part A up to now relativize easily to Y. The relativization is initiated by replacing the recursive predicates of subsection 1.2.I by predicates recursive in Y. Note that $P(f, x)$ is recursive in Y iff there is a recursive predicate $R(X, f, x)$ such that $P(f, x)$ is $R(Y, f, x)$. In short a typical predicate recursive in Y is obtained by substituting the parameter Y for a free set variable in some recursive predicate. The process of *relativization* is straightforward because the presence of Y tends to have little effect on the proofs given so far in this book. Y simply goes along for the ride.

The consequences of relativization are not trifling. Results about number-theoretic predicates are lifted to predicates of reals by regarding the parameter Y as variable. A surprising aspect of relativization is that some recursive functions do not become recursive in Y. For example, $+_{O^Y}$ is recursive rather than recursive in Y.

5.1 Predicates Analytical in Y. A predicate is Σ_n^1 (or Π_n^1 or Δ_n^1) in Y if it is the result of substituting Y for a free set variable in some Σ_n^1 (or Π_n^1 or Δ_n^1) predicate. Thus $p(X, Y)$ is Σ_n^1 in Y (with free variable X) if $P(X, Y)$ is Σ_n^1 (with free variables X, Y). A predicate is said to be boldface Σ_n^1 (or Π_n^1 or Δ_n^1) if it is Σ_n^1 (or Π_n^1 or Δ_n^1) in some parameter Y. The collection of boldface Σ_n^1 predicates is denoted by $\utilde{\Sigma}_n^1$. The predicates of subsection 1.3.I are said to be lightface.

5.2 Proposition (Shoenfield). *For $n \geq 1$, the relation, X is Δ_n^1 in Y, is transitive.*

Proof. Suppose A is Δ_n^1 in B, $B \Delta_n^1$ in C, and

$$n \in A \leftrightarrow P_i(n, B), \; n \in B \leftrightarrow Q_i(n, C),$$

where $P_0, Q_0 \in \Sigma_n^1$ and $P_1, Q_1 \in \Pi_n^1$. Then

$$\begin{aligned} n \in A &\leftrightarrow (\mathrm{E}Y)[P_0(n, Y) \;\; \& \;\; Y = B] \\ &\leftrightarrow (\mathrm{E}Y)[P_0(n, Y) \;\; \& \;\; (n)(n \in Y \to Q_0(n, C)) \\ &\qquad\qquad\qquad\quad \& \;\; (n)(n \notin Y \to {\sim} Q_1(n, C))], \end{aligned}$$

and so A is Σ_n^1 in C. Similarly A is Π_n^1 in C. $\square$

It follow from Proposition 5.2 that "$A \Delta_n^1$ in B and $B \Delta_n^1$ in A" is an equivalence relation. Two sets are said to have the same Δ_n^1 degree if each is Δ_n^1 in the other. The Δ_1^1 degrees have been studied extensively. In Chapter IV a minimal Δ_1^1 degree will be constructed, that is a degree greater than 0 with nothing between it and 0. (0 is the degree of the empty set.)

5.3 Ordinals Constructive in Y. The definition of O^Y differs from that of O only in the limit case. $3 \cdot 5^e \in O^Y$ iff $\{e\}^Y$ is total and

$$\{e\}^Y(n) <_{O^Y} \{e\}^Y(n+1)$$

for all n. $|a|^Y$ is the ordinal constructive in Y represented by a when $a \in O^Y$. ω_1^Y is the least ordinal not constructive in Y. O^Y is Π_1^1 in Y uniformly: $x \in O^Y$ is a Π_1^1 predicate whose free variables are x and Y.

The function $u +_{O^Y} v$ is recursive rather than recursive in Y, because the function h of subsection 3.3(1).I can be replaced by a recursive function h^* such that

$$\{h^*(e, a, d)\}^Y(n) \simeq \{e\}^Y(a, \{d\}^Y(n))$$

for all Y.

The functions p and q of Theorem 3.5.I remain recursive, but their meanings change. $W_{p(b)}$ becomes $W^Y_{p(b)}$, the $p(b)$-th set of numbers recursively enumerable in Y.

The function g of Lemma 4.1.I remains recursive.

As in Theorem 4.4.I, the ordinals recursive in Y equal those constructive in Y.

5.4 Π_1^1 Predicates of Reals. Let $P(n)$ be a number-theoretic predicate Π_1^1 in Y. The relativization of Theorem 5.4.I provides a recursive function k such that for all n,

(1) $$P(n) \leftrightarrow k(n) \in O^Y,$$

uniformly in Y. k does not depend on Y. It is determined by the Π_1^1 predicate $Q(n, Z)$ (n and Z are free) such that $P(n)$ is $Q(n, Y)$.

A normal form for $P(n)$ is

(2) $$(f)(\mathrm{E}x)R(Y, \bar{f}(x), n)$$

for some recursive R. If the free variable n of (2) is suppressed by being set equal to 0, and if the parameter Y is regarded as a free variable, then (2) becomes a typical Π_1^1 predicate $N(Y)$ whose only free variable ranges over 2^ω. According to (1)

(3) $$N(Y) \leftrightarrow k(0) \in O^Y$$

for all Y. In this manner relativization to Y lifts the ordinal analysis of Π_1^1 predicates of numbers to Π_1^1 predicates of reals.

Every Π_1^1 predicate $Q(Y)$ of reals can be put in the form

$$\{e\}^Y \text{ is wellfounded,}$$

where e is such that $\{e\}^Y$ is total and a binary relation between numbers for all Y. Suppose $Q(Y)$ is $(f)(\mathrm{E}x)R(Y, \bar{f}(x), 0)$. Then $\{e\}^Y$ is $S_R^Y(0)$, the relativization of $S_R(0)$, from Proposition 5.3.I.

Fix n_0 and suppose $n_0 \in O^Y$ for all Y. Then n_0 can be thought of as defining a function f from 2^ω into ω_1: $f(Y) = |n_0|_Y$. The next result states that f obeys a sharp bounding principle.

5.5 Lemma. *Fix n and suppose $n \in O^Y$ for all Y. Then there exists a recursive ordinal δ such that*

$$|n|_Y < \delta$$

for all Y.

Proof. Suppose not. Then O is Σ_1^1:

$$b \in O \leftrightarrow (\mathrm{E}f)(\mathrm{E}Y)(x)(y)[x, y \in S_R(b) \ \& \ x > y$$
$$\rightarrow \langle f(y), f(x) \rangle \in W_{q(n)}^Y].$$

S_R is defined as in Proposition 5.3.I so that for all b, $b \in O$ iff $S_R(b)$ is wellfounded. $|f(x)|_Y$ is the height (or rank) of x in $S_R(b)$. q is the recursive function of Theorem 3.5.I. $\square$

It follows from Lemma 5.5 that if $P(Y)$ is Π_1^1 and $(Y)P(Y)$ holds, then the latter is "seen" to be true by some recursive ordinal.

5.6 Hyperarithmetic Predicates of Reals. The H^Y-sets are defined by recursion on O^Y.

$$H_1^Y = Y. \qquad H_{2^m}^Y = (H_m^Y)'.$$
$$H_{3\cdot 5^e}^Y = \{\langle x, n \rangle \mid x \in H_{\{e\}(n)}^Y\}.$$

X is said to be hyperarithmetic in Y (in symbols $X \leq_h Y$) if X is recursive in some H^Y-set. As in Corollary 1.4(ii), $X \leq_h Y$ is a Π_1^1 predicate (with X and Y as free variables). The relativization of Spector's uniqueness theorem (Corollary 4.6) to Y implies that the Turing degree of H_b^Y depends only on $|b|_Y$. Thus it makes sense to refer to $Y^{(\delta)}$, the δ-th iterate of the Turing jump of Y, when $\delta < \omega_1^Y$.

A predicate $P(Y)$ is said to be *hyperarithmetic* if there exist $b \in O$ and e such that for all Y

$$(1) \qquad P(Y) \leftrightarrow \{e\}^{H_b^Y}(0) \text{ is defined.}$$

$\langle b, e \rangle$ is said to be a code for $P(Y)$. The set of all codes for hyperarithmetic predicates is Π_1^1. Inserting a superscript Y in appropriate places in the proof of Theorem 1.3(i) shows

$$b \in O \quad \& \quad n \in H_b^Y$$

is Π_1^1 (with n and Y free). It follows, as in the proof of Corollary 1.4, that $P(Y)$ is Δ_1^1.

Suppose $Q(Y)$ is Δ_1^1 to show it hyperarithmetic. According to formula 5.4(3), there exist k_0 and k_1 such that for all Z,

$$Q(Z) \leftrightarrow k_0 \in O^Z \leftrightarrow k_1 \notin O^Z.$$

The relativization of Theorem 3.3 to Z implies:

$$\min(|k_0|_Z, |k_1|_Z) \leq |t(k_0, k_1)|_Z$$

for all Z. Let n be $2^{t(k_0,k_1)}$.Then

(2) $$Q(Z) \leftrightarrow k_0 \in O_n^Z.$$

It follows from the relativization of Theorem 2.4 to Z that for some e,

(3) $$O_n^Z = \{e\}^{H_{2^n}^Z}$$

for all Z. The function g of 2.4 remains recursive, rather than becoming recursive in Z, when 2.4 is relativized to Z. $n \in O^{\varnothing}$ since $n \in O^Z$ for all Z. ($\varnothing$ is the empty set.) There is very little difference between O and $O^{\varnothing}$, so n can be regarded as a member of O (cf. Lemma 7.5). Thus (2) and (3) imply $Q(Z)$ is hyperarithmetic.

5.7–5.12 Exercises

5.7. Clarify the assertion concerning O and $O^{\varnothing}$ made at the end of subsection 5.6.

5.8 Verify that $X \leq_h Y$ is Π_1^1.

5.9. Let $P(X)$ be Σ_1^1. Suppose

$$(X)[P(X) \rightarrow \{e\}^X \text{ is total and is a wellfounded relation}].$$

Show there is a $\delta < \omega_1^{\text{CK}}$ such that

$$(X)[P(X) \rightarrow |\{e\}^X| < \delta].$$

5.10. Suppose $P(X, y)$ is Π_1^1 and

$$(X)(\text{Ey})[y \in O^X \ \&\ P(X, y)].$$

Show there is a $\delta < \omega_1^{\text{CK}}$ such that

$$(X)(\text{Ey})[y \in O^X \ \&\ |y|_X < \delta \ \&\ P(X, y)].$$

5.11. (Kleene separation). Let $A, B \subseteq \omega^\omega$ be Σ_1^1 and disjoint. Find a $\Delta_1^1 C$ such that $A \subseteq C$ and $B \cap C = \varnothing$.

5.12. Suppose $A \subseteq B \subseteq \omega^\omega$, $A \in \Sigma_1^1$ and $B \in \Pi_1^1$. Find a Δ_1^1C such that $A \subseteq C \subseteq B$.

6. *Incomparable Hyperdegrees Via Measure*

X and Y belong to the same hyperdegree if $X \leq_h Y$ and $Y \leq_h X$. $\leq_h$ is transitive because "Δ_1^1 in" is by Proposition 5.2. Thus the hyperdegrees constitute a partition of 2^ω. The hyperdegree of X is denoted by $\underset{\sim}{X}$. $\underset{\sim}{X} \leq \underset{\sim}{Y}$ iff $X \leq_h Y$. $\underset{\sim}{\varnothing}$, the hyperdegree of the empty set, is the least hyperdegree. $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ have a least upper bound, the hyperdegree of

$$\{2n | n \in X\} \cup \{2n+1 | n \in Y\},$$

denoted by $\underset{\sim}{X} \cup \underset{\sim}{Y}$.

In this section the existence of two incomparable hyperdegrees is established by a measure-theoretic argument of extraordinary simplicity. In Chapter IV the same result will be obtained by a forcing argument analogous to the Kleene–Post construction of incomparable Turing degrees. The approach via measure is swift once the measurability of Π_1^1 sets is established.

6.1. Measurable Subsets of 2^ω. The subbasic open subsets of 2^ω are obtained by fixing single coordinates.

$$\{X | X \in 2^\omega \ \& \ m \in X\} \quad \text{and} \quad \{X | X \in 2^\omega \ \& \ n \notin X\}$$

are typical subbasic open sets. A basic open set is a finite intersection of subbasic sets. The measure μ of a basic open set b is 2^{-i}, where i is the number of coordinates fixed by b. Thus $\mu(b) = 2^{-i}$ if

$$b = \{X | m_1 \in X \ \& \ldots \& \ m_j \in X \ \& \ m_{j+1} \notin X \ \& \ldots \& \ m_i \notin X\}$$

and $m_1, m_2, \ldots, m_i$ are distinct. An open set is a union of basic open sets.

Let J be an arbitrary subset of 2^ω. An open cover of J is a family K of basic open sets such that $J \subseteq \cup K$. Define

$$I(K) = \Sigma\{\mu(b) | b \in K\}.$$

The outer measure of J, denoted by $\mu_0(J)$, is

$$\inf\{I(K) | K \text{ is an open cover of } J\}.$$

$\mu_i(J)$, the inner measure of J, is $1 - \mu_0(2^\omega - J)$. J is said to be *measurable* if $\mu_0(J) = \mu_i(J)$. If J is measurable, then its measure $\mu(J)$ is $\mu_0(J)$.

Every open set is measurable. The family of all measurable sets is a σ-algebra, that is a Boolean algebra closed under countable unions. The operations of meet, joint and complementation are the set-theoretic operations of intersection, union and complementation. μ is countably additive: if $\{J_i | i < \omega\}$ is a sequence of pairwise disjoint, measurable sets, then

$$\mu\left(\bigcup_i J_i\right) = \sum_i \mu(J_i).$$

The proof of countable additivity makes essential use of the countable axiom of choice. In order to find a suitable open set containing $\bigcup_i J_i$, it is necessary to choose a suitable open set containing J_i for each i.

A subset of 2^ω is *Borel* if it belongs to the least σ-algebra containing all the open sets. It follows that every Borel set is measurable, but not that every measurable set is Borel. A set is measurable iff it differs from a Borel set by a subset of a Borel set of measure 0. Thus J is measurable iff there exist Borel sets B_0 and B_1 such that

$$(B_0 - J) \cup (J - B_0) \subseteq B_1 \quad \text{and} \quad \mu(B_1) = 0.$$

Each Borel set can be fabricated from open sets in countably many steps. Let B_0 be the family of all subsets of 2^ω that are either open or closed. For each countable ordinal δ, let $B_{\delta+1}$ be the result of adding to B_δ all countable intersections of elements of B_δ, and the complements of such intersections. Let B_{ω_1} be $\cup\{B_\delta | \delta < \omega_1\}$. B_{ω_1} is a σ-algebra containing all the open sets, and clearly the least such. Hence a set is Borel iff it belongs to B_δ for some countable δ.

Each hyperarithmetic set is Borel since it can be fabricated in δ steps for some $\delta < \omega_1^{\mathrm{CK}}$. According to Exercise 6.4, the Borel sets are the same as the boldface $\boldsymbol{\Delta}_1^1$ sets.

6.2 Lemma (Lusin). *Π_1^1 subsets of 2^ω are measurable.*

Proof. Suppose $J \subseteq 2^\omega$ is Π_1^1. As in subsection 5.4 there is an integer k such that

$$X \in J \leftrightarrow k \in O^X$$

for all X. The measurability of J will follow easily from the measurability of Borel sets once a countable bound is found on the ordinals represented by elements of O^X as X ranges over J. First it must be checked that: for each integer j and countable ordinal δ, the set

$$(1) \qquad \{X \mid j \in O^X \;\&\; |j| = \delta\}$$

is Δ_1^1 in any Y such that Y is a wellordering of ω of height δ. Let $S^X(j)$ be a linear ordering of a set of sequence numbers, recursive uniformly in X, such that $j \in O^X$ iff $S^X(j)$ is wellfounded, as in Proposition 5.3.I relativized to X. $X \in (1)$ iff there is an f that maps $S^X(j)$ in a one-one, orderpreserving fashion onto Y. $X \notin (1)$ iff there is a one-one, orderpreserving f that maps one of $S^X(j)$ and Y onto a proper initial segment of the other.

Since (1) is boldface Δ^1_1, it is Borel by Exercise 6.4, hence measurable. Since measure is countably additive, there must be a δ_∞ such that (1) has measure 0 for all δ when $\delta \geq \delta_\infty$. (Corollary 1.6.IV: $\delta_\infty = \omega_1^{\mathrm{CK}}$.) Let

$$J_1 = \{X \mid k \in O^X \ \& \ |k|_X < \delta_\infty\}.$$

Clearly $J_1 \subseteq J$. Since J_1 is Borel, it remains only to show that $J - J_1$ is contained in a Borel set of measure O. Observe that

$$X \in (J - J_1) \rightarrow (\mathrm{Ej})[j \in O^X \ \& \ |j|_X = \delta_\infty]. \tag{2}$$

The set of all X that satisfy the right side of (2) is Borel and has measure zero thanks to the choice of δ_∞ and the countable additivity of measure. □

6.3 Theorem (Spector 1958). *There exist X and Y such that $X \nleq_h Y$ and $Y \nleq_h X$.*

Proof. As noted in subsection 5.6, $\leq_h$ is Π^1_1, hence measurable by Lemma 6.2. According to Fubini's theorem the measure of $\{(X, Y) \mid X \leq_h Y\}$ can be computed by integrating the measure of $\{X \mid X \leq_h Y\}$ along the Y-axis. But $\{X \mid X \leq_h Y\}$ is countable, hence of measure O. So $\{(X, Y) \mid X \leq_h Y\}$ has measure 0. In the same manner $\{(X, Y) \mid Y \leq_h X\}$ has measure 0. Thus almost every pair satisfies the conclusion of the theorem. □

One shortcoming of Spector's proof of Theorem 6.3 is the absence of a concrete pair of incomparables. In Chapter III Kleene's basis theorem will be applied to the conclusion of 6.3 to produce a pair of incomparables recursive in O.

In Chapter IV the measure-theoretic approach will be refined to show $\omega_1^Y = \omega_1^{\mathrm{CK}}$ for almost all Y.

6.4–6.7 Exercises

6.4. (Addison). Show a subset of 2^ω is Borel iff it is boldface Δ^1_1.

6.5. (Addison). Call a subset of 2^ω *analytic* if it is the projection of a Borel subset of $2^\omega \times 2^\omega$. Show a subset of 2^ω is analytic iff it is boldface Σ^1_1.

6.6. Show every analytic subset of 2^ω is measurable. (Gödel has shown the consistency of ZFC and the existence of an unmeasurable Δ^1_2 set, if ZF is consistent.)

6.7. Repeat 6.4–6.6 for ω^ω.

7. *The Hyperjump*

The hyperjump of X is O^X. According to subsection 5.3, the graph of the hyperjump function is Π^1_1. There is an imperfect analogy between the Turing- and hyperjumps. Each is a completion. For the hyperjump this means: O^X is Π_1^1 in X, and every set Π_1^1 in X is many-one reducible to O^X (the relativization of Theorem 5.4.I). However, the partial ordering of hyperdegrees of Π_1^1 sets differs radically from that of the Turing degrees of recursively enumerable sets.

Iteration of the Turing jump generated the hyperarithmetic sets, which turned out to be a hierarchy for the Turing degrees of the Π_2^0 singletons. Iteration of the hyperjump generates a hierarchy for some, but not all, of the hyperdegrees of the Π_1^1 singletons. To say a set is generated by the hyperjump is to say it is E-recursive in the hyperjump function. E-recursion is defined in part D.

The next proposition implies the hyperjump is well defined for hyperdegrees.

7.1 Proposition. $A \leq_h B \leftrightarrow O^A \leq_m O^B$.

Proof. Assume $O^A \leq_m O^B$. Since A, $\omega - A$ are Π_1^1 in A, they are many-one reducible to O^A by the relativization of Theorem 5.4.I to A. The transitivity of $\leq_m$ implies A, $\omega - A \leq_m O^B$. Thus A, $\omega - A$ are Π_1^1 in B, and so A is Δ_1^1 in B, hence hyperarithmetic in B by the relativization of Theorem 2.5 to B.

Assume $A \leq_h B$. Then A is Δ_1^1 in B. Since O^A is Π_1^1 in A, it follows that O^A is Π_1^1 in B as in the proof of Proposition 5.2. Therefore $O^A \leq_m O^B$ by the completeness of O^B with respect to sets Π_1^1 in B. $\square$

The next lemma implies that every Π_1^1 set has the same hyperdegree as the null set or Kleene's O. It is the first result to suggest that Π_1^1 sets are not analogous to recursively enumerable sets. Friedberg 1955 and Muchik 1955 independently found a pair of incomparable Turing degrees of recursively enumerable sets. Kreisel alone resisted the suggestion, on the grounds that Δ_1^1 sets were not analogous to recursive sets, but bore a relation to Π_1^1 sets much like that of finite sets to recursively enumerable sets. His insight led to the creation of metarecursion theory and a Friedberg–Muchnik-type theorem for the Π_1^1 sets, as detailed in Part B.

7.2 Proposition (Spector 1955). *If* X, $Y \in \Pi_1^1$ *and* $Y \notin \mathrm{HYP}$, *then* $X \leq_h Y$.

Proof. It suffices to show O is Δ_1^1 in Y. By Theorem 5.4.I there is a recursive f such that

$$x \in Y \leftrightarrow f(x) \in O.$$

If there were a $b \in O$ such that $f[Y] \subseteq O_b$, then $Y \leq_m O_b$ and $Y \in \mathrm{HYP}$ by Theorem 2.4. Thus $f[Y]$ is unbounded in O. Hence

$$a \in O \leftrightarrow (\mathrm{E}x)[x \in Y \ \& \ f(x) \in O \ \& \ a \in O_{f(x)}],$$

$$a \notin O \leftrightarrow (x)[x \in Y \rightarrow (f(x) \in O \ \& \ a \notin O_{f(x)})].$$

It follows from Lemma 2.1 that O is Δ_1^1 in Y. $\square$

Try for a moment to think of the proof of Proposition 7.2 in terms of generalized computations. If $a \in O$, then this is seen to be so by a computation of height $|f(x)|$

for some $x \in Y$. If $a \notin O$, then a set of computations makes it so. The supremum of the heights of computations in the set is at most ω_1^{CK}. Thus $a \in O$ is decided by a computation of height at most ω_1^{CK}. It would be more just to allow only one computation of height less than ω_1^{CK} to decide $a \in O$, since the computations that enumerate O all have height less than ω_1^{CK}. Allowing a computation of height ω_1^{CK} in the classification of Π_1^1 sets is no more fair than allowing a computation of height ω in the classification of recursively enumerable sets.

7.3 Theorem (Spector 1955). $A \leq_h B \rightarrow \omega_1^A \leq \omega_1^B$.

Proof. Suppose $A \leq_h B$ but $\omega_1^B < \omega_1^A$. Fix $b \in O^A$ such that $|b|_A = \omega_1^B$ with the intent of showing O^B is Σ_1^1 in B, an impossibility according to the relativization of Corollary 5.5.I. It suffices to show O^B is Σ_1^1 in A, since A is Δ_1^1 in B.

When Proposition 3.2 is relativized to B, the function h remains recursive. Thus

$$x \in O^B \leftrightarrow R_{h(x)}^B \text{ is wellfounded,}$$

$$x \in O^B \rightarrow |x|_B \leq |R_{h(x)}^B|,$$

where R_n^B is the n-th binary relation recursively enumerable in B. As noted in subsection 5.3, the function q of Theorem 3.5.I remains recursive when 3.5 is relativized. Thus

$$W_{q(a)}^A = \{\langle x, y\rangle \mid x <_{O^A} y <_{O^A} a\}$$

when $a \in O$. The following formula is Σ_1^1 and equivalent to $x \in O^B$.

(Ef) [f is an orderpreserving map of $R_{h(x)}^B$ into $W_{q(b)}^A$]. Remember that b was chosen so that $|W_{q(b)}^A| = \omega_1^B$. $\square$

7.4 Corollary (Spector 1955). $X \in \mathrm{HYP} \rightarrow \omega_1^X = \omega_1^{\mathrm{CK}}$.

The converse of Corollary 7.4 is false. In fact the set of all X such that $\omega_1^X = \omega_1^{\mathrm{CK}}$ has measure 1, as will be proved in Chapter IV.

The next lemma makes it possible to regard O^A as an initial segment of O^B when A is recursive in B.

7.5 Lemma. *Suppose A is recursive in B. Then there exists a recursive function f such that*

(*i*) $(x)(y)[x <_{O^A} y \leftrightarrow f(x) <_{O^B} f(y)]$, *and*
(*ii*) $(x)[x \in O^A \rightarrow |x|_A = |f(x)|_B]$.

Proof. Let $A = \{e_0\}^B$. There is a recursive I such that

$$\{I(c)\}(b) \simeq \begin{cases} 1 & \text{if } b = 1 \\ 2^{\{c\}(m)} & \text{if } b = 2^m \\ 3 \cdot 5^{h(c,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases}$$

h is a recursive function such that

$$\{h(c, d)\}^B(n) \simeq \{c\}(\{d\}^A(n)) \simeq \{c\}(\{d\}^{\{e_0\}^B}(n)).$$

Choose e so that $\{I(e)\} \simeq \{e\}$, and let f be $\{e\}$. f is total by induction on ω. An induction on $<_{O^A}$ establishes (ii) and the left-to-right direction of (i). The other direction of (i) requires an induction on $<_{O^B}$. □

7.6 Theorem (Spector 1955)

(i) $O^A \leq_h B \to \omega_1^A < \omega_1^B$
(ii) $\omega_1^A < \omega_1^B$ & $A \leq_h B \to O^A \leq_h B$.

Proof

(i) Suppose $O^A \leq_h B$. By Theorem 7.3, $\omega_1^{O^A} \leq \omega_1^B$, so it need only be shown that $\omega_1^A < \omega_1^{O^A}$. The proof of Lemma 4.3.I is unchanged by relativization to Y. Let Y be O^A. $<_{O^A}$ is a wellfounded relation recursively enumerable in O^A, and so $|<_{O^A}|$ is an ordinal constructive in O^A.

(ii) First consider the special case of $A \leq_T B$. Let f be the recursive function of Lemma 7.5. Suppose $\omega_1^A < \omega_1^B$. Then there is a $c \in O^B$ such that $\omega_1^A = |c|_B$ and

$$x \in O^A \leftrightarrow f(x) \in O_c^B.$$

Hence $O^A \leq_h B$, by the relativization of Theorem 2.4 to B. Now suppose $A \leq_h B$. Then $A \leq_T H_b^B$ for some $b \in O^B$. Theorem 7.3 implies

$$\omega_1^B = \omega_1^{H_b^B}.$$

By the special case, $O^A \leq_h H_b^B$. □

7.7 Corollary. $\omega_1^{\text{CK}} < \omega_1^X \leftrightarrow O \leq_h X$.

7.8–7.10 Exercises

7.8. Suppose $P(x, y)$ is Σ_1^1 and $\{\langle x, y\rangle \mid P(x, y)\}$ is a wellordering. Show its ordinal height is less than ω_1^{CK}.

7.9. Suppose $A <_h O$. Show $O^A \equiv_h O$.

7.10. (Platek). Call an ordinal $\gamma\, \Pi_1^1$ if there is a Π_1^1 binary relation $P(x, y)$ such that $P(x, y)$ is wellfounded and $\gamma = |P(x, y)|$. Show that ω_1^O is the least non-Π_1^1 ordinal.