Part A

Hyperarithmetic Sets

Hyperarithmetic theory is the first step beyond classical recursion theory. It is the primary source of ideas and examples in higher recursion theory. It is also a crossroads for several areas of mathematical logic. In set theory it is an initial segment of Gödel's L. In model theory, the least admissible set after ω. In descriptive set theory, the setting for effective arguments, many of which are developed below. It gives rise directly to metarecursion theory (Part B), and yields the simplest example of both α-recursion theory (Part C) and E-recursion theory (Part D).

Chapter I
Constructive Ordinals and Π^1_1 Sets

It is shown that a universal quantifier ranging over the real numbers is equivalent in certain circumstances to an existential quantifier ranging over the recursive ordinals, a countable set. Along the way notations for ordinals and the method of defining partial recursive functions by effective transfinite recursion are developed.

1. Analytical Predicates

The analytical predicates are obtained by applying function quantifiers to recursive predicates. Chapter I focuses on analytical predicates in which at most one function quantifier occurs, since in that case an analysis based on ordinals goes smoothly.

1.1 Partial Recursive Functions. Some conventions, occasionally violated, in this book are:

ω is $\{0, 1, 2, \ldots\}$, the set of natural numbers.
b, c, e, m, n are constants that denote natural numbers.
$x, y, z, \ldots$ are variables that range over ω.
$f, g, h, \ldots$ are total functions from ω into ω.
$X, Y, Z, \ldots$ are subsets of ω.
ϕ, ψ, θ are partial functions from ω into ω, that is functions whose graphs are subsets of ω^2.
$\phi(b) \simeq c$ is true iff (if and only if) $\phi(b)$ is defined and equal to c. $\phi(b) \simeq \psi(c)$ iff both $\phi(b)$ and $\psi(c)$ are defined and equal, or neither is defined.
$\{e\}^f$ is the e-th item in the standard enumeration of functions partial recursive in f. There exist a recursive predicate T and a recursive function U, both devised by Kleene, such that

(1) $$\{e\}^f(b) \simeq c \text{ iff } (\mathrm{Ey})[T(\bar{f}(y), e, b, y) \quad \& \quad U(y) = c].$$

$\bar{f}(y)$ encodes $\{\langle i, f(i)\rangle \mid i < y\}$:

$$\bar{f}(y) = \prod_{i<y} p_i^{1+f(i)}.$$

p_i is the i-th smallest prime; $p_0 = 2$. The right side of (1) says there is a computation y derived from the e-th set of equations, and the values of f restricted to $i < y$, whose final outcome is c. All of the above extends to

$$\{e\}^{f_0, \ldots, f_{m-1}}(x_0, \ldots, x_{n-1})$$

for all nonnegative m and n.

1.2 Function Quantifiers. A predicate $R(f, x)$ is recursive if there is an e such that:

(i) $(f)(x)[\{e\}^f(x)$ is defined]; and
(ii) $(f)(x)[R(f, x) \leftrightarrow \{e\}^f(x) = 0]$.

Thus the truth value of $R(f, x)$ is determined by a finite computation. As in subsection 1.1 the definition of recursive predicate extends routinely to predicates of the form $R(f_0, \ldots, f_{m-1}, x_0, \ldots, x_{n-1})$ for all $m, n \geq 0$. For simplicity $R(f, x)$ will be used somewhat ambiguously to denote a recursive predicate with an arbitrary number of function and number variables.

A predicate is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and function quantifiers. Thus

$$(\mathrm{Ex})(f)(\mathrm{Eg})R(x, y, f, g, h) \quad \text{and} \quad (\mathrm{Ef})(h)S(f, h, z) \tag{1}$$

are analytical if R and S are recursive. A predicate is *arithmetic* if it is analytical but includes no function quantifiers.

There is an aspect of the classification of predicates which will seem picayune now but which will matter a great deal later. A predicate may be classified by virtue of its explicit form, as were the predicates of (1), or by being proved equivalent to another predicate already classified. For example, the predicate, "f is constructible in the sense of Gödel", is seen to be analytical only after it is shown that every constructible number-theoretic function is constructible via a countable ordinal.

1.3 Theorem (Kleene 1955). *If $P(f, x)$ is analytical, then it can be put in one of the following forms:*

$$A(f,x) \qquad \begin{array}{ll} (\mathrm{Eg})(y)R(f, x, g, y), & (\mathrm{Eg})(h)(\mathrm{Ey})R(f, x, g, h, y) \ldots \\[1em] (g)(\mathrm{Ey})R(f, x, g, y), & (g)(\mathrm{Eh})(y)R(f, x, g, h, y) \ldots \end{array}$$

where A is arithmetic and R is recursive.

Proof. First $P(f, x)$ is put in prenex normal form with a recursive matrix by the usual quantifier manipulations associated with first order logic. Then the resulting prefix is put in one of the desired forms by applying the following rules. K is arbitrary.

$$(x)(\mathrm{Ef})K(f, x) \leftrightarrow (\mathrm{Ef})(x)K((f)_x, x). \tag{1}$$

$(f)_x$ is defined by $(f)_x(y) = f(2^x \cdot 3^y)$.

f is thus interpretable as a code for $\{f_n | n < \omega\}$. Rule (1) is a nontrivial consequence of the axiom of choice. If K is $(\mathrm{Eg})(h)A(f, x, g, h)$ for some arithmetic A, then (1) is provable in Zermelo–Fraenkel set theory (ZF). (See Chapter III, Section 9.) On the other hand there is a K of the form $(g)(\mathrm{Eh})B(f, x, g, h)$ with B arithmetic such that (1) is not provable in ZF.

The dual of (1) is

$$(1^*) \qquad (\mathrm{Ex})(f)K(f, x) \leftrightarrow (f)(\mathrm{Ex})K((f)_x, x).$$

Each of the remaining three rules has a dual.

$$(2) \qquad (\mathrm{Ex})K(x) \leftrightarrow (\mathrm{Ef})K(f(0)).$$

$$(3) \qquad (\mathrm{Ef})(\mathrm{Eg})K(f, g) \leftrightarrow (\mathrm{Ef})K((f)_0, (f)_1).$$

$$(4) \qquad (\mathrm{Ex})(\mathrm{Ey})K(x, y) \leftrightarrow (\mathrm{Ex})K((x)_0, (x)_1).$$

$(x)_i$ is the exponent of p_i, the i-th smallest prime, in the unique factorization of x. Note that the substitution of $(f)_x$ or $(x)_i$ for some of the variables of a recursive predicate leaves it recursive. The following illustrates the normalization of a prefix.

$(\mathrm{Ef})(x)(\mathrm{Ey})(h)(z)$	is given.
$(\mathrm{Ef})(x)(h)(\mathrm{Ey})(z)$	by (1*).
$(\mathrm{Ef})(g)(h)(\mathrm{Ey})(z)$	by (2*).
$(\mathrm{Ef})(g)(\mathrm{Ey})(z)$	by (3*).
$(\mathrm{Ef})(g)(\mathrm{Ey})(h)$	by (2*).
$(\mathrm{Ef})(g)(h)(\mathrm{Ey})$	by (1*).
$(\mathrm{Ef})(g)(\mathrm{Ey})$	by (3*).

Observe that a prefix can be normalized by deleting all number quantifiers, collapsing each block of function quantifiers of the same sort to a single one of that sort, and adding a single number quantifier on the right dual to the rightmost function quantifier. □

Each of the nonarithmetic normal forms of Theorem 1.3 has a Greek name. An analytical predicate in normal form is said to be Σ_n^1 (Π_n^1 respectively) if its prefix begins with an existential (universal respectively) function quantifier and encompasses $n-1$ alternations of function quantifiers. Thus the forms of Theorem 1.3 are

$$\text{Arithmetic,} \qquad \begin{matrix} \Sigma_1^1, \Sigma_2^1, \Sigma_3^1, \ldots \\ \Pi_1^1, \Pi_2^1, \Pi_3^1, \ldots \end{matrix}$$

A predicate is said to be Δ_n^1 if it is both Σ_n^1 and Π_n^1. The Δ_1^1 predicates will eventually be proved to be the same as the hyperarithmetic predicates.

1.4 Set Quantifiers. The effect of applying function quantifiers to predicates can also be realized by applying set quantifiers. X encodes a function if

$$(1) \qquad (x)[E_1 y][2^x \cdot 3^y \in X].$$

If (1) holds, then the function encoded by X is denoted by f_X. Thus

$$f_X(x) = y \leftrightarrow 2^x \cdot 3^y \in X.$$

A predicate $R(X, y)$ is said to be recursive if it is equivalent to some recursive predicate $R(c_X, y)$, where c_X is the characteristic function of X. Let $R_0(X, x, y)$ be a recursive predicate such that (1) is equivalent to $(x)(\mathrm{E}y)R_0(X, x, y)$. The rule for replacing function quantifiers by set quantifiers is:

$$(2) \qquad (\mathrm{E}f)K(f) \leftrightarrow (\mathrm{E}X)(x)(\mathrm{E}y)[R_0(X, x, y) \quad \& \quad K(f_X)].$$

Rule (2), and its dual, are all that is needed to transform Theorem 1.3 into Theorem 1.5. It is a fact that the single alternation of number quantifiers occurring in the normal forms of Theorem 1.5 cannot be reduced to a single number quantifier. There exists a recursive predicate $R(X, y, z, x)$ such that $(\mathrm{E}X)(y)(\mathrm{E}z)R(X, y, z, x)$ is not equivalent to $(\mathrm{E}X)(y)S(X, y, x)$ for any recursive S.

A predicate $P(Z, x)$ is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and set quantifiers; it is *arithmetic* if no set quantifiers are allowed.

1.5 Theorem (Kleene (1955).) *If $P(Z, x)$ is analytical, then it can be put in one of the following forms*:

$$A(Z, x) \qquad \begin{matrix} (\mathrm{E}X)(y)(\mathrm{E}z)R(X, y, z, Z, x), & (\mathrm{E}X)(Y)(\mathrm{E}y)(z)R \ldots \\ (X)(\mathrm{E}y)(z)R(X, y, z, Z, x), & (X)(\mathrm{E}Y)(y)(\mathrm{E}z)R \ldots \end{matrix}$$

where A is arithmetic and R is recursive.

There is no harm in mixing set and function variables. Thus a predicate is analytical if it is built up from recursive predicates by application of propositional connectives, number quantifiers, function quantifiers and set quantifiers. It is arithmetic if all quantifiers are number-theoretic. The resulting forms are again denoted by Σ_n^1 or Π_n^1 $(n \geq 1)$.

The most important of all Π_1^1 predicates is: X encodes a countable wellordering. It turns out to be universal Π_1^1, hence not Σ_1^1. It gives rise to a bounding principle with numerous applications. For example, it is used in Chapter IV to compute the Lebesgue measure of a Π_1^1 set of reals.

If $K \subseteq 2^\omega$, then K is said to be Π_n^1 (Σ_n^1 respectively) if $X \in K$ is Π_n^1 (Σ_n^1 respectively). Similar conventions are in force when $K \subseteq \omega$ or $K \subseteq \omega^2$ etc.

1.6 Theorem (Spector 1955). *Suppose* $A(X)$ *is* Σ_1^1.
(*i*) $\cap\{X|A(X)\}$ *is* Π_1^1.
(*ii*) *If* $(E_1X)A(X)$, *then the unique* X *that satisfies* $A(X)$ *is* Δ_1^1.

Proof.
(i) Let B be $\cap\{X|A(X)\}$. Then

$$x\in B\leftrightarrow(X)[A(X)\rightarrow x\in X].$$

(ii) Let C be the unique solution of $A(X)$. Then

$$\begin{aligned} x\in C &\leftrightarrow(\mathrm{E}X)[A(X)\quad\&\quad x\in X]\\ &\leftrightarrow(X)[A(X)\rightarrow x\in X]. \qquad \square\end{aligned}$$

In Chapter III, Section 6, it will be shown that every Σ_1^1 set with a non-Δ_1^1 member has a continuum of members. The proof will require more than trivial quantifier manipulations, namely an analysis of Σ_1^1 predicates by means of recursive trees with infinite branching.

Part (i) of Theorem 1.6 is often alluded to as follows: a set (of numbers) is Π_1^1 if it is the closure of a Π_1^1 set under a Σ_1^1 closure condition. A predicate $A(X)$ is a *closure condition* if the intersection of any non-empty collection of solutions of $A(X)$ is a solution of $A(X)$, and if every set (of numbers) is contained in some solution of $A(X)$. Let $A(X)$ be a closure condition. It follows that for each Y there is a least X, call it Y_0, such that $Y\subseteq X$ and $A(X)$:

$$Y_0=\cap\{X|Y\subseteq X\quad\&\quad A(X)\}.$$

Y_0 is called the closure of Y under A. By Theorem 1.6(i), $Y_0\in\Pi_1^1$ if $Y\in\Pi_1^1$ and $A(X)\in\Sigma_1^1$, because then

$$Y\subseteq X\quad\&\quad A(X)$$

is Σ_1^1.

1.7 Proposition. $f\in\Sigma_n^1\leftrightarrow f\in\Pi_n^1\leftrightarrow f\in\Delta_n^1$.

Proof. Since f is a function,

(1) $$f(x)=y\leftrightarrow(z)[y\neq z\rightarrow f(x)\neq z].$$

If the left side of (1) is Σ_n^1 (Π_n^1 respectively), then the right side is Π_n^1 (Σ_1^n respectively). $\square$

1.8 – 1.12 Exercises

1.8. Show there exists a universal Π_n^1 predicate, that is a Π_n^1 predicate $P(e,f,x)$ such that for each Π_n^1 predicate $Q(f,x)$ there is a c for which $P(c,f,x)$ and $Q(f,x)$ are equivalent for all f and x.

1.9. Show $\Pi_n^1 \subseteq \Sigma_{n+1}^1$, $\Sigma_n^1 \subseteq \Pi_{n+1}^1$, $\Pi_n^1 \nsubseteq \Sigma_n^1$ and $\Sigma_n^1 \nsubseteq \Pi_n^1$.

1.10. Let ω have the discrete topology and ω^ω the product topology. Basic closed subsets of ω^ω can be coded by finite sequences of natural numbers, hence by natural numbers. A closed subset of ω^ω, regarded as an intersection of basic closed sets, can be coded by a subset of ω. Show "X codes a closed subset of ω^ω" is arithmetic. Show "X codes a countable, closed subset of ω^ω" is Π_1^1.

1.11. Let L be a first order language whose set of primitive symbols is recursive. Let $S(X)$ be "X codes a countable set of sentences of L". Show "$S(X)$ and X is consistent (that is yields no contradiction via first order logic)" is arithmetic. Show "$S(X)$ and X has a model" is Σ_1^1.

1.12. Suppose $<_u$ and $<_v$ are order-isomorphic, recursive wellorderings of ω. Show there exists a $\Delta_1^1 f$ such that $x <_u y \leftrightarrow f(x) <_v f(y)$ for all $x, y \in \omega$.

2. Notations for Ordinals

Suppose $<$ is a wellordering of ω. $<$ is said to be recursive if the predicate $x < y$ is recursive. An ordinal is called recursive if it is finite or the ordertype of a recursive wellordering. The recursive ordinals form a countable, initial segment of the countable ordinals with strong closure properties. They constitute an effective analogue of the countable ordinals. For example, the Cantor–Bendixson analysis of a recursively encodable, countable closed set terminates at a recursive ordinal.

The definition of recursive ordinal is, in a manner of speaking, from above. The question of whether or not a recursive linear ordering ℓ is wellordered is complex. A straightforward resolution would require examination of every function f from ω into ω to see if f defines an infinite descending sequence in ℓ. A more constructive approach would be from below. It seems reasonable to expect that the successor of a constructive ordinal be constructive, and that the limit of a recursive sequence of constructive ordinals be constructive. The constructive approach is made precise with the help of notations. Afterwards it is shown the approaches from above and below yield the same result; the recursive and constructive ordinals coincide. The notion of ordinal notation is useful in proof theory as well as in recursion theory. It facilitates delicate recursions and inductions.

2.1 Kleene's *O*. The formula $x <_O y$ is to be read: x and y are notations for constructive ordinals and x is less than y according to the ordering of notations. The ordering $<_O$ is not linear, because the same ordinal may have two different notations.

The predicate $x <_O y$, regarded as a set of ordered pairs, is the closure of a finite set under a Σ_1^1 closure condition as described in the remarks following subsection 1.6.

The closure condition $A(X)$ has three clauses.

(1) $(u)(v)[\langle u, v\rangle \in X \rightarrow \langle v, 2^v\rangle \in X]$.

(2) $(n)\ [\{e\}(n)$ is defined & $\langle \{e\}(n), \{e\}(n+1)\rangle \in X] \rightarrow (n)[\langle \{e\}(n), 3\cdot 5^e\rangle \in X]$.

(3) $(u)\ (v)\ (w)\ [\langle u, v\rangle, \langle v, w\rangle \in X \rightarrow \langle u, w\rangle \in X]$.

(1) deals with successors, (2) with limits, and (3) with transitivity. All three are positive in nature: if some elements belong to X, then some other elements belong to X. The positiveness of $A(X)$ implies it is a closure condition. Hence there is a least X such that $\langle 1,2\rangle \in X$ and $A(X)$. $<_O$ is defined to be that least X.

Kleene's O, the set of notations for constructive ordinals, is the field of $<_O$.

A binary relation r is said to be *wellfounded* if there is no f such that $(x)r(f(x+1), f(x))$.

2.2 Theorem

(i) $<_O$ *and* O *are* Π_1^1.
(ii) $<_O$ *is a wellfounded partial ordering*
(iii) If $v \in O$, *then the restriction of* $<_O$ to $\{u|u <_O v\}$ *is linear.*

Proof.

(i) $<_O$ is Π_1^1 by Theorem 1.6(i), since $A(X)$ is arithmetic. Then O is Π_1^1 by rule (1*) of the proof of Theorem 1.3, since $u \in O \leftrightarrow (\mathrm{E}w)[w <_O u \vee u <_O w]$.

(ii) The following *natural enumeration* of $<_O$ is equivalent to a redefinition of $<_O$ by transfinite recursion on the ordinals.

Stage 0: enumerate $1 <_O 2$.

Stage $\delta+1$: enumerate $v <_O 2^v$ and $u <_O 2^v$ if $u <_O v$ was enumerated at stage δ.

Stage λ (limit): enumerate $\{e\}(n) <_O 3\cdot 5^e$ and $u <_O 3\cdot 5^e$, if not enumerated earlier, if for each n, $\{e\}(n) <_O \{e\}(n+1)$ has been enumerated earlier, and if for some n, $u <_O \{e\}(n)$ has been enumerated earlier.

By induction on γ, a pair enumerated at stage γ belongs to $<_O$. On the other hand the set of all pairs enumerated is a solution of $A(X)$ and hence contains $<_O$. Also by induction, if $u <_O v$ and $v <_O w$, then $u <_O v$ is enumerated at an earlier stage than $v <_O w$. It follows that $<_O$ is wellfounded, since otherwise there would be an infinite, descending sequence of ordinals.

The natural enumeration also makes clear there is no x such that

(1) $u <_O x \;\&\; x <_O 2^u$, or $x <_O 1$, or
(2) $(n)[\{e\}(n) <_O \{e\}(n+1) <_O x] \;\&\; x < 3\cdot 5^e$.

Consequently 2^u is said to be the successor of u, and $3\cdot 5^e$ the limit of $\{e\}(n)\,(n<\omega)$.

(iii) is proved by induction on $<_O$. Assume $u_1, u_2 <_O v$ to check $u_1 <_O u_2$ or $u_1 = u_2$ or $u_2 <_O u_1$. If $v = 2^u$, then (1) implies u_1, $u_2 \leq_O u$ and the desired result follows by induction. If $v = 3\cdot 5^e$, then apply (2). □

In Part B of this book a generalization of recursive enumerability is offered that allows $<_O$ to be viewed as a higher kind of recursively enumerable relation. The natural enumeration of $<_O$ becomes a proof that $<_O$ is metarecursively enumerable. Elements enter metarecursively enumerable sets by means of metafinite computations, which are infinite but have many of the properties of finite computations.

2.3 Constructive Ordinals. The function $|\ |: O \to$ Ordinals is defined by transfinite recursion on $<_O$.

$$|1| = 0.$$
$$|2^u| = |u| + 1;$$
$$|3 \cdot 5^e| = \lim_{n \to \infty} |\{e\}(n)|.$$

The definition is sound by 2.2(1)–(2). If $u \in O$, then u is said to be a notation for the ordinal $|u|$. An ordinal δ is constructive if $\delta = |u|$ for some $u \in O$. There are no gaps in the constructive ordinals. They form a countable, initial segment of the ordinals. The least non-constructive ordinal is called Church–Kleene omega-one and is written ω_1^{CK}.

The fact that each infinite constructive ordinal has many notations is a consequence of the "approach from below". The choice of a preferred notation for ω is no simple matter. A bad choice for ω might make choices difficult further on (cf. Exercise 2.4). Later it will be seen that there exists a Π_1^1 subset of O, linearly ordered by $<_O$, and of ordertype ω_1^{CK}. Such a subset is called a set of unique notations. It will be defined from above.

2.4–2.6 Exercises

2.4. A path in O is a set $Z \subseteq O$ such that Z is linearly ordered by $<_O$ and $(u)(v)[u <_O v \in Z \to u \in Z]$. A path can be continued if there is a $w \in O$ such that $(u)[u \in Z \to u <_O w]$. Find a path in O of ordertype less than ω_1^{CK} but which cannot be continued.

2.5. Spell out the details omitted from the proof of Theorem 2.2(ii).

2.6. Prove Theorem 2.2(ii) without any reference to ordinals. For example, prove $\sim(\mathrm{E}x)[x <_O 1]$ by showing $A(X) \to A(X - \{\langle x, 1\rangle\})$.

3. *Effective Transfinite Recursion*

Let f map ω into ω, and let $f \restriction n$ denote the restriction of f to $\{m \mid m < n\}$. To say f is defined by recursion on ω is to say there exists an iterater I such that

(1) $$f(n) = I(f \restriction n) \text{ for all } n.$$

(1) is called a recursion equation. For each I there is a unique f such that (1) holds. If I is computable, then f is computable by virtue of a straightforward, but limiting, intuition. $f(n)$ is computable from I and the set of previous values, $\{f(m) \mid m < n\}$. More precisely, $f(n)$ is computed by iterating I n times. The record of that n-fold iteration is the computation of $f(n)$. With this intuition in mind, it appears

farfetched to replace the standard wellordering of ω by some arbitrary wellordering of ω and still expect f to be computable when I is. For one thing, n may have infinitely many predecessors and the iteration of I infinitely many times is not a finite computation. For another, it is no longer clear what is meant by the effectiveness of I, since a typical argument of I in (1) may be an infinite object.

Church and Kleene made the remarkable discovery that (1) remains a valid scheme for defining recursive functions when the standard wellordering of ω is replaced by an arbitrary one, so long as I remains effective in an appropriate sense. It is tempting to think that the wellordering should be recursive, but that limitation is unnecessary, and fortunately so, since many of the applications are to $<_O$. Rogers was the first to use the phrase, "effective transfinite recursion", and to provide a general result similar to Theorem 3.2.

First a technical result of classical recursion theory, Kleene's fixed point theorem. The fact that (1) above has a unique solution f for each I is often described in a set theoretic setting by noting that f is a fixed point of I. The next result is an effective counterpart of the essential existence argument employed in the set theoretic treatment of definition by transfinite recursion.

3.1 Theorem (Kleene). *Suppose $I\colon \omega \to \omega$ is recursive. Then for some c, $\{I(c)\} \simeq \{c\}$.*

Proof. Let t be a recursive function such that

$$\{t(e)\} \simeq \{\{e\}(e)\}$$

for all e. Choose b so that

$$\{b\}(x) = I(t(x)) \tag{1}$$

for all x. Then

$$\{I(t(b))\} \simeq \{\{b\}(b)\} \simeq \{t(b)\}.$$

Thus $t(b)$ will serve for c. $\square$

Note that a fixed point c of I is computable in a uniform manner from a Gödel number of I, since t is independent of I, and since the composition that occurs in (1) is effective. It follows that if I depends effectively on some parameter p, then the fixed point c can be construed as a recursive function of p.

3.2 Theorem. *Let $<_R$ be a wellfounded relation whose field is a subset of ω, and I: $\omega \to \omega$ a recursive function. Suppose for all $e < \omega$ and x in the field of $<_R$, $\{e\}(y)$ defined for all $y <_R x$ implies $\{I(e)\}(x)$ defined. Then for some c, $\{c\}(x)$ is defined for all x in the field of $<_R$, and $\{c\} \simeq \{I(c)\}$.*

Proof. By theorem 3.1 there is a c such that $\{c\} \simeq \{I(c)\}$. Suppose x is a minimal (in the sense of $<_R$) element such that $\{c\}(x)$ is not defined. Thus $\{c\}(y)$ is defined for all $y <_R x$. But then $\{I(c)\}(y) \simeq \{c\}(y)$ is defined. □

Warning: definition by effective transfinite recursion (ETR) is more than an effective version of the set theoretic method of definition by transfinite recursion. There is an element of self-reference in ETR with no counterpart in set theory. There is also a use of indices to transform potentially infinite computations into finite ones. Thus $f(n)$ is computed not by iterating I n times, but by having I act on an index for $f \restriction n$. An excellent example of ETR is the definition of $+_O$.

3.3 Addition of Notations. The key property of $+_O$, the addition function for notations in O, is: if $a, b \in O$, then $a +_O b \in O$ and $|a +_O b| = |a| + |b|$. The definition of $+_O$ by effective transfinite recursion is aimed at realizing this key property.

Let h be a recursive function such that

$$\{h(e, a, d)\}(n) \simeq \{e\}(a, \{d\}(n)) \tag{1}$$

for all e, a, d and n. By the use of pleonasms (that is, each partial recursive function has infinitely many Gödel numbers), it is safe to insist h be one-one from ω^3 into ω.

Let I be a recursive function such that

$$\{I(e)\}(a, b) \simeq \begin{array}{ll} a & \text{if } b = 1 \\ 2^{\{e\}(a, m)} & \text{if } b = 2^m \\ 3 \cdot 5^{h(e, a, d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{array} \tag{2}$$

The first three clauses of (2) mimic the definition of $+$ for ordinals. Thus $\alpha + \beta = \alpha$ if $\beta = 0$, $\alpha + \beta = (\alpha + \gamma) + 1$ if $\beta = \gamma + 1$, and $\alpha + \lambda = \lim\limits_n (\alpha + \gamma_n)$ if $\lambda = \lim\limits_n \gamma_n$. I is recursive, despite the non-recursiveness of $<_O$, because the splitting of O into notations for zero, successors, and limits is effective. Also the instruction coded by $I(e)$ makes sense whether or not a and b belong to O.

By Theorem 3.1, I has a fixed point c. Define $a +_O b$ to be $\{c\}(a, b)$. Since $\{I(c)\} \simeq \{c\}$,

$$a +_O b \simeq \begin{array}{ll} a & \text{if } b = 1 \\ 2^{a +_O m} & \text{if } b = 2^m \\ 3 \cdot 5^{h(c, a, d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{array}$$

Note that $\{h(c, a, d)\}(n) \simeq a +_O \{d\}(n)$ by (1).

Nothing in Theorem 3.1 requires $+_O$ to be defined anywhere. The proof of Theorem 3.2 shows the domain of $+_O$ contains O. (2) has a quirk that compels $+_O$

to be total. Suppose $a+_O b$ is not defined. Then b must be 2^m, since h is total, and so $a+_O m$ is not defined for some $m<b$. Thus an induction on ω shows $+_O$ is total.

3.4 Theorem (Kleene). *The recursive function $+_O$ has the following properties for all a and b.*

(i) $a,\ b\in O \leftrightarrow a+_O b\in O$.
(ii) $a,\ b\in O \rightarrow |a+_O b|=|a|+|b|$.
(iii) $a,\ b\in O\ \ \&\ \ b\neq 1 \rightarrow a<_O a+_O b$.
(iv) $a\in O\ \ \&\ \ c<_O b \leftrightarrow a+_O c<_O a+_O b$.
(v) $a\in O\ \ \&\ \ b=c\in O \leftrightarrow a+_O b=a+_O c$.

Proof. First all inferences from left to right are proved simultaneously by induction on b, an induction that proceeds according to $<_O$.
Case 1: $b=1$. Then $a+_O b=a$, and (i) and (ii) from left to right are immediate. (iv) is vacuous by 2.2(1).
Case 2: $b=2^m$. Then $m\in O$. Hence by induction $a+_O m\in O$, $|a+_O m|=|a|+|m|$, and $a\leq_O a+_O m$. So $a+_O b\in O$, $|a+_O b|=|a|+|b|$, and $a<_O a+_O b$.
Suppose $c<_O b$. Then $c\leq_O m$ by 2.2(1). Hence by induction $a+_O c\leq_O a+_O m$, and so $a+_O c<_O a+_O b$.
Case 3: $b=3\cdot 5^e$. Then by 2.2(2), $\{e\}(n)$ is defined and $\{e\}(n)<_O\{e\}(n+1)<_O 3\cdot 5^e$ for all n. By induction $a\leq_O a+_O\{e\}(n)<_O a+_O\{e\}(n+1)$ and $|a+_O\{e\}(n)|=|a|+|\{e\}(n)|$ for all n. Hence $a+_O b=3\cdot 5^{h(c,a,e)}\in O$, $|a+_O b|=\lim\limits_n |a+_O\{e\}(n)|=|a|+|b|$, and $a<_O a+_O b$.
Suppose $c<_O b$. Then $c<_O\{e\}(n)$ for some n by Theorem 2.2(1) and 2.2(2). By induction $a+_O c<_O a+_O\{e\}(n)<_O a+_O b$.
Now the inferences from right to left are proved by induction on $a+_O b$ with respect to $<_O$.
Case 1: $a+_O b=a$ and $b=1$. Trivial.
Case 2: $a+_O b=2^{a+_O m}$ and $b=2^m$. Then $a+_O m\in O$, and by induction $a, m, b\in O$.
Subcase 2a: $a+_O c=a+_O b$. Then c is of the form 2^n, and $a+_O n=a+_O m$. By induction $n=m$, so $c=b$.
Subcase 2b: $a+_O c<a+_O b$. Then $a+_O c\leq_O a+_O m$, $c\leq_O m$ and $c<_O b$.
Case 3: $a+_O b=3\cdot 5^{h(c,a,e)}$ and $b=3\cdot 5^e$. Then $a+_O\{e\}(n)<_O a+_O\{e\}(n+1)$ $<_O 3\cdot 5^{h(c,a,e)}$ for all n. By induction $a\in O$ and $\{e\}(n)<_O\{e\}(n+1)$ for all n, hence $b\in O$.
Subcase 3a: $a+_O c=a+_O b$. Then $c=3.5^d$ for some d such that $h(c,a,d)=h(c,a,e)$. Recall that h is one-one. Thus $d=e$ and $c=b$.
Subcase 3b: $a+_O c<a+_O b$. Then $a+_O c<_O a+_O\{e\}(n)$ for some n. By induction $c<_O\{e\}(n)$, hence $c<_O b$. □

From now on W_e is the e-th recursively enumerable subset of ω, that is the domain of $\{e\}$, the e-th partial recursive function.

3.5 Theorem (Kleene). *There exist recursive functions p and q such that for all $b \in O$:*

(*i*) $W_{p(b)} = \{a \mid a <_O b\}$;
(*ii*) $W_{q(b)} = \{\langle u, v\rangle \mid u <_O v <_O b\}$.

Proof. The essential properties of p are:

$$\begin{aligned} & W_{p(1)} = \phi, \\ (1) \qquad & W_{p(2^a)} = \{a\} \cup W_{p(a)}, \\ & W_{p(3\cdot 5^d)} = \cup\{W_{p(\{d\}(n))} \mid \{d\}(n) \text{ is defined}\}. \end{aligned}$$

An induction on $<_O$ shows that any p that satisfies (1) also satisfies 3.5(i). The existence of a recursive such p is obtained by an effective transfinite recursion on $<_O$. Let e_o be a Gödel number and j and k recursive functions such that

$$\begin{aligned} & W_{e_o} = \phi, \\ (2) \qquad & W_{j(e,a)} = \{a\} \cup W_{\{e\}(a)}, \\ & W_{k(e,d)} = \cup\{W_{\{e\}(\{d\}(n))} \mid n < \omega\}. \end{aligned}$$

In (2) it is intended that $W_{\{e\}(a)} = \phi$ when $\{e\}(a)$ is undefined; similarly for $W_{\{e\}(\{d\}(n))}$.

There exists a recursive I such that

$$\{I(e)\}(b) \simeq \begin{cases} e_o & \text{if } b = 1 \\ j(e,a) & \text{if } b = 2^a \\ k(e,d) & \text{if } b = 3\cdot 5^d \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 3.1, I has a fixed point c: $\{I(c)\} \simeq \{c\}$. Define $p(b)$ to be $\{c\}(b)$. Then

$$p(b) = \begin{cases} e_o & \text{if } b = 1 \\ j(c,a) & \text{if } b = 2^a \\ k(c,d) & \text{if } b = 3\cdot 5^d \\ 0 & \text{otherwise.} \end{cases}$$

Note that p is total because j and k are. (2) implies that p satisfies (1).

The definition of q is similar to that of p. $\square$

Some applications of Theorem 3.5 need only the fact that $\{a \mid a <_O b\}$ is recursively enumerable whenever $b \in O$. But it does not seem possible to establish this fact without establishing it uniformly, that is without developing a uniform method p for enumerating the predecessors of an element of O. Not surprising, since the only general approach to showing every element of O has some constructive property is by effective transfinite recursion, an approach that can only be made by means of an effective recursion step.

3.6–3.7 Exercises

3.6. Prove there exists a recursive function q that satisfies Theorem 3.5(ii).

3.7. Fill in the details of the proof of Theorem 3.4.

4. Recursive Ordinals

An ordinal is said to be recursive if it is finite or the ordertype of some recursive wellordering of ω. In a moment it will be shown that the recursive ordinals coincide with the constructive ordinals. This theme will recur in Part D, where it will be seen that the ordinals E-recursive in x are cofinal with the ordinals constructive in x for every set x. It is sometimes said that the notion of recursive ordinal is more intrinsic than that of constructive ordinal. This merely means there is no element of chaos in the notion of recursive predicate, but there is great freedom in the development of notations for ordinals. In Part D it will be the case that every ordinal constructive in x is recursive in x, and every ordinal recursive in x is less than some ordinal constructive in x. In the present section every ordinal less than a constructive ordinal is constructive, but that will not be so in Part D.

There exist very small subclasses of the recursive predicates which give rise to the recursive ordinals. Thus each recursive wellordering is order isomorphic to some primitive recursive wellordering and even to some rudimentary (in the sense of Smullyan) wellordering. In fact O'Neill has observed that predicates computable in polynomial time suffice. In the other direction if $\{\langle x, y\rangle | x <_u y\}$ is Σ_1^1 and $<_u$ is a wellordering, then the ordertype of $<_u$ is a recursive ordinal.

The next lemma states that every recursively enumerable subset of O is bounded in a highly effective manner. Later it will be shown that every Σ_1^1 subset of O is bounded in a somewhat less effective manner.

4.1 Lemma. *There exists a recursive g such that for all e:*

(i) $g(e) \in O \leftrightarrow W_e \subseteq O$;
(ii) $g(e) \in O \rightarrow |a| < |g(e)|$ *for all* $a \in W_e$.

Proof. $g(e)$ is an "infinite sum" of W_e. Let r be a total recursive function such that for all e: $\{r(e)\}$ is total and the range of $\{r(e)\}$ is $W_e \cup \{1\}$. There exists a total recursive function s defined by recursion on n such that for all e:

$$\{s(e)\}(0) = \{r(e)\}(0) = 1,$$
$$\{s(e)\}(n+1) = \{s(e)\}(n) +_O 2^{\{r(e)\}(n+1)}.$$

$+_O$ was defined in subsection 3.3.

Let $g(e)$ be $3 \cdot 5^{s(e)}$.

First assume $g(e) \in O$ to see $W_e \subseteq O$. Clearly $\{s(e)\}(n) \in O$ for all n. Fix $n > O$. By Theorem 3.4(i), $2^{\{r(e)\}(n)} \in O$, and so $\{r(e)\}(n) \in O$.

Now assume $W_e \subseteq O$. Then for each n, $\{r(e)\}(n) \in O$, and by Theorem 3.4(iii) $\{s(e)\}(n) <_O \{s(e)\}(n+1)$. Hence $g(e) \in O$.

Lastly assume $g(e) \in O$ and $1 \neq a \in W_e$. Choose $n > 0$ so that $\{r(e)\}(n) = a$. Then

$$\{s(e)\}(n) = \{s(e)\}(n-1) +_O 2^a.$$

Hence $2^a <_O \{s(e)\}(n)$, and so $|a| < |g(e)|$. □

4.2 Enumeration of Wellfounded Relations. A binary relation $R(s, y)$ is said to be a wellordering if it is:

(i) connected ... $R(x, y) \vee R(y, x) \vee x = y$;
(ii) transitive ... $R(x, y) \ \& \ R(y, z) \rightarrow R(x, z)$; and
(iii) wellfounded ... if S is a nonempty subset of the field of R, then $(\mathrm{E}y)_{y \in S}(x)_{x \in S} \sim R(x, y)$.

Note that (iii) implies R is

(iv) irreflexive ... $\sim R(x, x)$, and
(v) antisymmetric ... $R(x, y) \rightarrow \sim R(y, x)$.

Certain aspects of the effective study of ordinals make it necessary to consider wellfounded relations rather than wellorderings. In addition a computation is a wellfounded tree rather than a wellordering.

If a binary relation R is wellfounded, then it has a height denoted by $|R|$ and measured by an ordinal. Let β be an ordinal variable. Read "$\mu\beta$" as "the least β such that". Define

$$|x| = \mu\beta[R(y, x) \rightarrow |y| < \beta],$$

$$|R| = \mu\beta(x)[x \in \text{field of } R \rightarrow |x| < \beta].$$

The notion of height is useful for proving theorems about wellfounded relations by transfinite induction.

Let R_e be the e-th recursively enumerable binary relation, that is

$$R_e(x, y) \leftrightarrow \{e\}(x, y) \text{ is defined.}$$

Thus $\{R_e | e < \omega\}$ is a simultaneous recursive enumeration of all recursively enumerable binary relations.

4.3 Lemma. *There exists a recursive f such that for all e:*

(i) R_e is wellfounded $\leftrightarrow f(e) \in O$; and
(ii) R_e is wellfounded $\rightarrow |R_e| \leq |f(e)|$.

Proof. The idea is to define a one-one, order-preserving map from the field of R_e into O by an effective transfinite recursion on R_e. One difficulty is the uncertain nature of the field of R_e. It is recursively enumerable, but may be empty or finite.

Let h be a total recursive function such that

$$R_{h(e,n)}(x, y) \leftrightarrow R_e(x, y) \;\&\; R_e(x, n) \;\&\; R_e(y, n)$$

for all e, n, x and y. $R_{h(e,n)}$ is the initial segment of R_e below n. $R_{h(e,n)}$ is empty if n is not in the field of R_e. There exists a total recursive t such that

$$(1) \qquad W_{t(b,e)} = \begin{matrix} \phi \quad \text{if} \quad R_e = \phi, \\ \{\{b\}(h(e,n)) \mid n < \omega\} \text{ otherwise.} \end{matrix}$$

Recall g from Lemma 4.1. Let k be a recursive function such that

$$\{k(b)\}(e) \simeq g(t(b, e));$$

let c_0 be a fixed point of k, that is $\{k(c_0)\} \simeq \{c_0\}$. Define

$$f(e) \simeq \{c_0\}(e), \quad \text{and}$$

$$t(e) = t(c_0, e).$$

Then

$$W_{t(e)} = \begin{matrix} \phi \quad \text{if} \quad R_e = \phi, \\ \{f(h(e,n)) \mid n < \omega\} \text{ otherwise,} \end{matrix}$$

and $f(e) = g(t(e))$.

Suppose R_e is wellfounded to show $f(e) \in O$ and $|R_e| \leq |f(e)|$. If the field of R_e is empty, then $W_{t(e)} = \phi$ and $f(e) \in O$ by Lemma 4.1. Assume $R_e \neq \phi$. Then $|R_{h(e,n)}| < |R_e|$ for all $n < \omega$, and the desired result follows by transfinite induction on $|R_e|$ with the aid of Lemma 4.1.

Now suppose $f(e) \in O$ to show R_e is wellfounded. For all n, $|f(h(e,n))| < |f(e)|$ by Lemma 4.1. So by transfinite induction on $<_O$, $R_{h(e,n)}$ is wellfounded for all n. Hence R_e is wellfounded. □

It can be shown that "$\leq$" in 4.3(ii) cannot be improved to "$=$".

Clause (i) of Theorem 4.3 implies that the predicate "R_e is wellfounded" is many-one reducible to Kleene's O. This is the first indication that O is some kind of complete or universal set. In Section 5 it will be shown that O is a complete Π_1^1 set, that is every Π_1^1 set is many-one reducible to O. With the aid of insight supplied by metarecursion theory in Part B, it will be seen that O is a complete Π_1^1 set in the same sense that

$$K = \{\langle e, n\rangle \mid \{e\}(n) \text{ is defined}\}$$

is a complete, recursively enumerable set. From the viewpoint of Part B, a number belongs to a Π_1^1 set if and only if some metafinite computation says it does. A metafinite computation will be a wellfounded relation of recursive ordinal height and of Δ_1^1 complexity.

4.4 Theorem (Kleene, Markwald). *The recursive ordinals equal the constructive ordinals.*

Proof. Suppose β is a recursive ordinal. Let R be a recursively enumerable, wellfounded binary relation of height β. By Theorem 4.3 there is a $b \in O$ such that $|R| \leq |b|$. Since every ordinal less than $|b|$ is constructive, β must be constructive.

Now suppose β is constructive. Then $\beta = |b|$ for some $b \in O$. By Theorem 3.5(ii) there is a recursively enumerable wellordering R such that $|R| = \beta$, namely $W_{q(b)}$. Assume β is infinite. Then there exists a one-one recursive f that maps ω onto the field of R. Define $x < y$ by $\langle f(x), f(y)\rangle \in W_{q(b)}$. $<$ is recursive and $|<| = \beta$. $\square$

4.5–4.6 Exercises

4.5. Show each recursive wellordering is order-isomorphic to some primitive recursive wellordering.

4.6. Recall q from Theorem 3.5. Suppose $a, b \in O$ and $|a| = |b|$. Let f be the unique, one-one, order-preserving map from $W_{q(a)}$ onto $W_{q(b)}$. Show f is Δ_1^1. Show f need not be partial recursive.

5. *Ordinal Analysis of* Π_1^1 *Sets*

In this section Π_1^1 sets are analyzed by means of recursive ordinals represented by recursive wellfounded relations on sequence numbers. The analysis is applied to show Kleene's O is a complete Π_1^1 set, and to obtain Spector's bounding principle for Σ_1^1 subsets of O.

5.1 The Partial Ordering of Sequence Numbers. Define $\bar{f}(x)$ as in subsection 1.1. y is said to be a *sequence number* if $y = \bar{f}(x)$ for some f and x. $\bar{f}(x)$ is thought of a code for the sequence $\langle f(0), f(1), \ldots, f(x-1)\rangle$. $\bar{f}(0) = 1$, and 1 encodes the null sequence. The length of $\bar{f}(x)$ is x, and is denoted by $\ell h(\bar{f}(x))$. If y is a sequence number, then y encodes

$$\langle (y)_0, (y)_1, \ldots, (y)_{\ell h(y)-1}\rangle.$$

If y and z are sequence numbers, then y is properly extended by z (symbolically $y > z$) if $\ell h(y) < \ell h(z)$ and $(y)_i = (z)_i$ for all $i < \ell h(y)$.

Let Seq be the set of all sequence numbers. Seq is a recursive set, and $>$ is a recursive, antisymmetric, transitive binary relation. Seq (with $>$) is useful in the study of Π_1^1 sets because it presents ω^ω effectively as a tree.

5.2 Normalization of Π_1^1 Predicates. Suppose $R_1(f, x, y)$ is recursive. Then there is an e such that for all f, x and y:

$$\{e\}^f(x, y) \text{ is defined, and}$$
$$R_1(f, x, y) \leftrightarrow \{e\}^f(x, y) = 0.$$

As in subsection 1.1, there exist recursive T and U such that

$$\{e\}^f(x, y) = 0$$
$$\leftrightarrow (\mathrm{Ez})[T(\bar{f}(z), e, x, y, z) \quad \& \quad U(z) = 0].$$

It follows that $(f)(\mathrm{Ex})R_1(f, x, y)$ is equivalent to

(1) $$(f)(\mathrm{Ex})R(\bar{f}(x), y)$$

for some recursive R. (1) is the preferred normal form for a Π_1^1 predicate. Since Kleene's T-predicate is "universal", an enumeration of the Π_1^1 predicates is provided by $(f)(\mathrm{Ex})T(\bar{f}(x), e, y, x)$ $(e = 0, 1, 2, \ldots)$.

For each y, let $S_R(y)$ be the restriction of Seq (and $>$) to those sequence numbers $\bar{f}(x)$ such that $(i)_{i<x} \sim R(\bar{f}(i), y)$. Clearly $S_R(y)$ is recursive uniformly in y.

5.3 Proposition. $(f)(\mathrm{Ex})R(\bar{f}(x), y)$ *iff* $S_R(y)$ *is wellfounded.*

Proof. Fix y. $\sim (f)(\mathrm{Ex})R(\bar{f}(x), y)$ iff there is an f such that $(x) \sim R(\bar{f}(x), y)$ iff there is an f such that $\bar{f}(0) > \bar{f}(1) > \bar{f}(2) \ldots$ is an infinite descending sequence in $S_R(y)$ iff $S_R(y)$ is not wellfounded. $\square$

Proposition 5.3 equates the problem of checking membership in Π_1^1 sets with the problem of checking wellfoundedness of recursive relations. Thus recursive ordinals suffice to analyze Π_1^1 sets.

5.4 Theorem (Kleene). *Each* Π_1^1 *set is many-one reducible to* O.

Proof. Suppose $B \in \Pi_1^1$. According to subsection 5.2 there is a recursive R such that

$$y \in B \leftrightarrow (f)(\mathrm{Ex})R(\bar{f}(x), y)$$

for all y. By proposition 5.3,

$$y \in B \leftrightarrow S_R(y) \text{ is wellfounded.}$$

Since $S_R(y)$ is recursive uniformly in y, there is a recursive function t such that $S_R(y) = R_{t(y)}$. (R_e is the e-th recursively enumerable, binary relation as defined in subsection 4.2.) Let f be the recursive function of Lemma 4.3. Then

(1) $$y \in B \leftrightarrow f(t(y)) \in O. \quad \square$$

There is a uniformity concealed in the proof of Theorem 5.4 worth making explicit. As in subsection 5.2, let

$$(f)(\mathrm{Ex})T(\bar{f}(x), e, y, x) \tag{2}$$

be the e-th Π_1^1 predicate. If $Q(y)$ is equivalent to (2) for all y, then e is said to be an index for the predicate $Q(y)$. An index for a Π_1^1 set B is an index for the predicate $y \in B$. The recursive function t of 5.4(1) is a function of two variables, e and y, where e is an index for B. Thus there is a uniform method for passing from an index for B to a recursive function that reduces B to O. In short the Π_1^1 sets are uniformly many-one reducible to O.

5.5 Corollary. $O \notin \Sigma_1^1$.

Proof. Analogous to a proof that a complete, recursively enumerable subset of ω is not recursive. Observe that any set many-one reducible to a Σ_1^1 set is also Σ_1^1. Hence by Theorem 5.3, if O were Σ_1^1, then every Π_1^1 set would be Σ_1^1. So it suffices to find a Π_1^1 set that is not Σ_1^1.

Define $Q(y)$ by $(f)(\mathrm{Ex})T(\bar{f}(x), y, y, x)$. Suppose $\sim Q(y)$ is Π_1^1. Then $\sim Q(y)$ is equivalent to $(f)(\mathrm{Ex})T(\bar{f}(x), e, y, x)$ for some e. But then $\sim Q(e)$ iff $Q(e)$. □

5.6 Corollary (Spector 1955). *Suppose $X \subseteq O$ and $X \in \Sigma_1^1$. Then there exists $b \in O$ such that $|x| \leq |b|$ for all $x \in X$ (Σ_1^1 boundedness).*

Proof. As in the proof of Theorem 5.4, with O in place of B, there is a recursive function t such that for all y,

$$y \in O \leftrightarrow R_{t(y)} \text{ is wellfounded},$$

where R_e is the e-th recursively enumerable relation. Let $Q(y)$ be

$$(\mathrm{Ez})[z \in X \ \&\ (\mathrm{Ef}) \\ (u)(v)(R_{t(y)}(u, v) \rightarrow \langle f(u), f(v) \rangle \in W_{q(z)})].$$

q is the recursive function of Theorem 3.5 $Q(y)$ is Σ_1^1. If $Q(y)$ holds, then $R_{t(y)}$ is wellfounded.

Suppose the hoped-for b does not exist. If $R_{t(y)}$ is wellfounded, then by Lemma 4.3 there is a $z \in X \subseteq O$ such that $|R_{t(y)}| \leq |z|$, and so $Q(y)$ holds. But then $y \in O$ is Σ_1^1 despite Corollary 5.5. □

There is a uniformity lacking in the statement of Corollary 5.6, whose existence will be established in Corollary 3.4.II. e is said to be an index for X as a Σ_1^1 set if

$$y \in X \leftrightarrow (\mathrm{Ef})(x) \sim T(\bar{f}(x), e, y, x)$$

for all y. The effective version of Corollary 5.6 is: there exists a recursive function f such that for all e and X, if e is a Σ_1^1 index for X, then

(i) $X \subseteq O \leftrightarrow f(e) \in O$, and
(ii) $X \subseteq O \rightarrow (z)[z \in X \rightarrow |z| \leq |f(e)|]$.

5.7–5.8 Exercises

5.7. Show each Π_1^1 set is one-one reducible to O.

5.8. Suppose A is a Π_1^1 set such that every Π_1^1 set is many-one reducible to A. Show every Π_1^1 set is one-one reducible to A.