Perspectives in Mathematical Logic

Gerald E. Sacks

Higher Recursion Theory

Springer-Verlag

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Ω-Group:
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In Memory Of My Father

Irwin “Pete” Sacks

Preface to the Series
Perspectives in Mathematic Logic

(Edited by the "Ω-group for Mathematical Logic" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps of guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of* results *and* techniques.

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of values, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R.O. Gandy, A. Levy, G.H. Müller, G. Sacks, D.S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F.K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and* *that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the overall plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the autor was keen to write and which would indeed contribute a perspective.*

Oberwolfach, September 1975

Acknowledgements. *In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag und Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970–1973) as an initial help which made our existence as a working group possible.*

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F.K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R.O. Gandy *H. Hermes*
A. Levy *G.H. Müller*
G. Sacks *D.S. Scott*

Author's Preface

Higher recursion theory (HRT) has been one of my two major obsessions for the last twenty years. Nonetheless my interest has not waned. Perhaps because, as Browning claimed:

"The best is yet to be."

I was talked into the subject, skittish all the way, by G. Kreisel. The old devil insisted, in several conversations beginning in 1961, on the existence of golden generalizations of recursion theory in which infinitely long computations converged. I listened for hours, without understanding a word, to his tales of the mother lode of recursion theory hidden far below the peaks of effective descriptive set theory.

My initial reaction to his yarns was naive. One could readily generalize the static (or syntactic) aspects of classical recursion theory (CRT) such as the enumeration theorem, but one could not hope to lift dynamic results such as the Friedberg-Muchnik solution of Post's problem to higher domains. Clearly the dynamic facts of CRT were inseparable from certain combinatoric properties of finite sets, and these properties, being "truly finite" in nature, could not be generalized fruitfully. In 1961 I was trying to prove the density of the recursively enumerable degrees, and that investigation had a no-nonsense flavor that rendered all exotic pursuits unpalatable.

In 1963 I began to understand what Kreisel was talking about. He had unearthed a compactness theorem for ω-logic in which *hyperarithmetic* played the part of *finite*. His result was: if A is a Π_1^1 set of axioms of ω-logic and every hyperarithmetic subset of A has a model, then A has a model. (This eye-opener was the forerunner of Barwise's compactness theorem for Σ_1 admissible sets.) Kreisel's proposal of 1961, as understood by me in 1963, was: replace the natural numbers by a Π_1^1 set I of indices for the hyperarithmetic sets, and recursively enumerable by Π_1^1; most important of all, let "finite" mean hyperarithmetic. The more one scrutinized his initially obscure proposal, the more lucid it became. Right away it was clear that a "finite" union of "finite" sets was "finite". A set A was "recursive" if A and I-A were Π_1^1; in Kreisel's terminology A was hyperarithmetic on I. It was immediate that a "recursive" function restricted to a "finite" set was "finite".

Now it was plausible that the dynamic results of CRT would lift up. The new notions were dubbed metarecursively enumerable, metafinite, and metarecursive. For smoothness the set I of indices of hyperarithmetic sets was replaced by a Π_1^1 set of unique notations for recursive ordinals, and finally by the set of recursive ordinals. Thus a metarecursively enumerable set was simply a set of recursive ordinals whose unique notations constituted a Π_1^1 set. The first test of metarecursion theory, Kreisel unerringly pronounced, was to prove the Friedberg-Muchnik theorem. A major technical obstacle stood in the way. It was possible for a metarecursively enumerable subset of a metafinite set not to be metafinite. (For me this obstacle was a source of meta-delight.) It was overcome in 1963 by a partial metarecursive map of ω onto ω_1^{CK}, a foretaste of Jensen's projectum techniques in L.

In 1966 I finished my work on metarecursion theory and conceived a plan for writing a book called *Higher Recursion Theory*. Part A would develop hyperarithmetic theory from scratch and would include connections with forcing and compactness. Part B would expound metarecursion, and Part C would deal with Σ_1 admissible ordinals. The principal flaw in the plan was my ignorance of Kleene's theory of recursion in normal objects of finite type. Platek's lecture on the superjump in Manchester in 1969 gave me my first toehold on the Kleene theory, and long talks with Gandy and Grilliot in 1970 brought me to the top of the subject. Thus Part D was born. In the 1970's and early 1980's considerable progress took place in admissible ordinals and in finite types, and so Parts C and D had to be started over and over again. Even hyperarithmetic theory, fairly stable for some years, saw new developments.

By 1980 I had formed a definite view of the contents of higher recursion theory (HRT). There is only one fork in the road upward. The natural numbers turn into ordinals or into sets. If ordinals, then recursively enumerable becomes Σ_1. If sets, then belonging to a recursively enumerable set means there exists a convergent computation presented as a wellfounded, possibly infinite tree. The ordinal approach was blazed by G. Takeuti, the setcomputational by S.C. Kleene, both in the 1950's. Happily each leads in a different way to a proof of the Friedberg-Muchnik theorem.

One of the early hopes for HRT was the discovery of new theorems in CRT via ideas from above. It is now obvious (as usual, after the case) that CRT is not "low" enough for applications of HRT. A theorem of HRT is proved by overcoming the lack of some combinatoric facts taken for granted in CRT. Moving upward means leaving behind some of the power of CRT. The same loss occurs when moving downward. It is not surprising, then, that HRT has been applied successfully "below" CRT. Consider the splitting theorem of CRT: each non-recursive, recursively enumerable degree is the join of two incomparable such degrees. T. Slaman and H. Woodin have shown that splitting is not provable from a fragment of Peano arithmetic known as Σ_1 bounding. Their argument draws on tricks from HRT. M. Mytilinaios has shown that splitting does follow from Σ_1 induction. His result is inspired by R. Shore's proof of splitting for every Σ_1 admissible ordinal; it uses Σ_2 blocking. J. Shinoda and T. Slaman have shown that the theory of polynomial-time degrees of recursive sets interprets first order arithmetic. They apply an idea from Moschovakis's work (on Kleene's theory of recursion in normal objects of finite type) that views divergence as Σ_1 in character.

The book, on an informal level, is almost self contained. Some of the arguments present, in an easygoing fashion, material that a typical reader has encountered elsewhere in a more formal setting. For example a course on Gödel's L is not assumed. The essential facts about L are given ab initio, but some readers may want more details. No previous acquaintance with forcing is necessary, but it would help clarify the various effective notions of forcing studied here. The more one knows of CRT, the better, but little more than the enumeration theorem is assumed, and at least one well known logician managed to learn priority arguments in the setting of Σ_1 admissible ordinals before applying them to CRT.

The book has four parts:

A. Hyperarithmetic Sets
B. Metarecursion
C. α-Recursion
D. E-Recursion

Part A is perhaps longer than some would wish. It lingers, as if life were not short, on effective transfinite recursion (ETR), the method invented by Church and Kleene in their study of notations for ordinals. My treatment of ETR follows that of H. Rogers, the first to present it intuitively. The classic theorems are proved, most of them in the original spirit, theorems such as recursive ordinals equal constructive ordinals, Kleene's O is Π^1_1 complete, Σ^1_1 bounding, and hyperarithmetic equals Δ^1_1. In addition measure and forcing are developed and applied in a hyperarithmetic context. The set of all real X such that the ordinals recursive in X are the recursive ordinals has measure 1. Forcing with hyperarithmetically encoded perfect sets yields a minimal hyperdegree. Forcing with Σ^1_1 sets (originated by Gandy) leads to Louveau's separation theorem.

Hyperarithmetic theory (HT) is often regarded, and rightly so, as the source of effective descriptive set theory. In this book it is the prologue to higher recursion theory. Many of the major developmets of HRT are foreshadowed in HT. Part B carries out of the program of metarecursion sketched above. It is easy to follow once one becomes accustomed to thinking of hyperarithmetic as "finite". Part B speedily verifies that the priority method of Friedberg and Muchnik can be executed in a higher domain. Simpson's dichotomy applies metarecursion to create new categories of Π^1_1 sets.

Part C tackles α, an arbitrary Σ_1 admissible ordinal. Post's problem is solved by combining fine structure of L with priority. The catch phrase here is "Σ_1 doing the work of Σ_2". The priority method, when applied in CRT, needs Σ_2 replacement. In α-recursion Σ_1 suffices with the assistance of effective approximations, downward projecta and Gödel-Jensen condensation. Shore's density theorem is an example of a Σ_3 (more precisely $\Sigma_{2.5}$) argument of CRT lifted to every Σ_1 admissible α. An early result that points to the flexibility of such α's is the regular sets theorem. A subset of α is said to be regular if its intersection with each ordinal less than α is α-finite (that is, belongs to $L(\alpha)$). An α-recursively enumerable set may fail to be regular, but it always has the same α-degree as some regular, α-recursively enumerable set.

Part D assigns a meaning to $\{e\}(x)$ for every set x via a notion of computation following schemes devised by Normann, and (subsequently) by

Moschovakis. A structure is E-closed if it is closed under application of the partial E-recursive function $\{e\}$ for all e. The biggest twist in E-closed structures is the existence of reflection phenomena and their application to priority and forcing thanks to a crucial connection between reflecting ordinals and Moschovakis divergence witnesses uncovered by Harrington and Kechris. On a more mundane level, the key to progress in E-recursion is usually a new selection theorem, such as those proved by Gandy, Grilliot, Moschovakis and Normann. This is true in priority arguments, and even more so in forcing arguments. Intuitively, a selection theorem provides an effective method of selecting a member of a nonempty, "recursively enumerable" set. van de Wiele's theorem explains why some Σ_1 functions are not E-recursive. The book ends, fittingly I think, with Slaman's density theorem for E-closed $L(\kappa)$'s.

I owe a great deal to those of my students who wrote theses on HRT: C. Bailey, G. Driscoll, S. Friedman, E. Griffor, L. Harrington, S. Homer, F. Lowenthal, M. Machtey, J. Macintyre, D. MacQueen, J. Owings, R. Shore, S. Simpson, T. Slaman, J. Sukonick and S. Thomason.

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Lastly there is the debt to J.B. Rosser (1907–1989), A. Nerode, H. Rogers and B. Dreben. The final four gave me my start and kept me going. If they are pleased with HRT, then so am I.

Gerald E. Sacks
June 1990
Cambridge, Chicago, Pasadena and Princeton

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