1. The Basics

1.1 Preliminaries and Notation

We assume that the reader is familiar with the basic definitions and results normally found in a first course in mathematical logic. Specifically, we will freely use the concepts of a *first-order language*, a *structure* or *model* in that language, and the *satisfaction relation* between models and formulas. We also assume that the reader knows the Compactness and Omitting Types Theorems, and can carry out an elimination of quantifiers argument for a specific theory such as dense linear orders without endpoints or divisible abelian groups. In this first section we will review some of these results as a way of setting our notation and viewpoint and jogging the student's memory.

Notation. (Model Theory)

- A first-order language is denoted by L, L', L_0, etc. The cardinality of a language L, $|L|$, is simply the cardinality of the set of nonlogical symbols of L.
- Formulas are denoted by lower case Greek letters. Writing $\varphi(v_0, \ldots, v_n)$ indicates that the free variables in φ are in $\{v_0, \ldots, v_n\}$. If $t_0, \ldots, t_n$ are terms in the language, $\varphi(t_0, \ldots, t_n)$ is the formula obtained by substituting t_i for v_i. A sentence is a formula with no free variables.
- We use $\mathcal{M}$ or $\mathcal{N}$, decorated with various subscripts and superscripts, to denote a model or structure in a first-order language. The universe of, e.g., $\mathcal{M}_0$, is M_0. Elements of the universe are denoted by lower case letters such as a, b, c, etc. If X is an element of the language in which $\mathcal{M}$ is a model $X^{\mathcal{M}}$ denotes the interpretation of X in $\mathcal{M}$.
- Given models $\mathcal{M}$ and $\mathcal{N}$ in a language L, a function $f : M \longrightarrow N$ is an *isomorphism* if f is a bijection and for all symbols $X \in L$, $f(X^{\mathcal{M}}) = X^{\mathcal{N}}$. When there is an isomorphism of $\mathcal{M}$ onto $\mathcal{N}$ we write $\mathcal{M} \cong \mathcal{N}$ and say $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. An *automorphism* of $\mathcal{M}$ is an isomorphism of $\mathcal{M}$ onto itself. For $\mathcal{M}$ a model, $\mathrm{Aut}(\mathcal{M})$ denotes the automorphism group of $\mathcal{M}$.
- A *theory* in the language L is a consistent set of sentences of L. A set of sentences need not be complete in order to be called a theory. (A set of sentences is *consistent* if it has a model. A theory is *complete* if all

models of the theory satisfy exactly the same sentences.) For T a theory, $Mod(T) = \{\mathcal{M} : \mathcal{M} \models T\}$. A class of models is *elementary* if it is $Mod(T)$ for some theory T. The complete theory of a model $\mathcal{M}$ is $Th(\mathcal{M}) = \{\sigma : \sigma$ is a sentence and $\mathcal{M} \models \sigma\}$. Models $\mathcal{M}$ and $\mathcal{N}$ are called *elementarily equivalent*, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$. Given a theory T and $n < \omega$ define an equivalence relation $\sim_n$ on the formulas in n free variables by: $\varphi(\bar{v}) \sim_n \psi(\bar{v})$ if for all models $\mathcal{M}$ of T, $\mathcal{M} \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. The *cardinality of* T, denoted $|T|$, is the supremum over n of the number of $\sim_n$ –classes, which is always infinite when T has an infinite model.
- If $\mathcal{M}$ is a structure in the language L and L_0 is a sublanguage of L, then $\mathcal{M} \upharpoonright L_0$ is the restriction of $\mathcal{M}$ to L_0. This restriction is defined to be the model in the language L_0 with the same universe and the same interpretation for the elements of L_0.

Notation. (Set Theory) The set-theoretic notation used here is quite standard. Less basic concepts will be defined later when they are needed.

When discussing the "logical" properties of a model there is little difference between a finite sequence $(a_1, \ldots, a_n)$ and the set $\{a_1, \ldots, a_n\}$. We will muddy the difference by writing $\bar{a} \subset M$ when $\bar{a}$ is a finite subset of M or a finite sequence from M ($\mathcal{M}$ is a model). Given a finite set of elements $\{a_1, \ldots, a_n\}$ we may juxtapose the elements and write $a_1 \ldots a_n$ for the sequence $(a_1, \ldots, a_n)$.

If L is a language and X is a set, $L(X)$ denotes the language obtained from L by adding a new constant symbol for each element of X. We will usually use a to denote both the element $a \in X$ and the corresponding constant symbol. Given a structure $\mathcal{M}$ for L, if $X \subset M$, then $\mathcal{M}_X$ denotes the expansion of $\mathcal{M}$ to $L(X)$ which interprets the constant $a \in X \cap L(X)$ by the element a. If $X = \{a_0, \ldots, a_n\}$ we may also write $(\mathcal{M}, a_0, \ldots, a_n)$ for $\mathcal{M}_X$.

The satisfaction relation is defined in most books as a relation on triples $\langle \mathcal{M}, \varphi, s \rangle$, where $\mathcal{M}$ and φ are as usual, and s is an *assignment*; i.e., a function from variables (including those free in φ) into M. In this book we adopt an approach, developed by Shoenfield in [Sho67], which is more streamlined and reflects the way we view elements of a model as parameters which can be used in formulas of the language. Briefly, for $\mathcal{M}$ a model we define satisfaction in $\mathcal{M}_M$ of *sentences* in $L(M)$ by a standard inductive argument. For $\varphi(v_0, \ldots, v_n)$ a formula of L and elements $a_0, \ldots, a_n \in M$, we say that $(a_0, \ldots, a_n)$ *satisfies* $\varphi(v_0, \ldots, v_n)$ *in* $\mathcal{M}$, and write $\mathcal{M} \models \varphi(a_0, \ldots, a_n)$, if $\mathcal{M}_M \models \varphi(a_0, \ldots, a_n)$ (where $\varphi(a_0, \ldots, a_n)$ is treated as a sentence in $L(M)$).

When studying the relations defined on a model by the formulas of the language it is common to fix a certain set of elements A and study the relations between A and other elements of the universe. For example, given an algebraically closed field k and polynomials over $A \subset k$, the set A is fixed throughout the study of the sets defined on k by the polynomials.

Definition 1.1.1. *If $\mathcal{M}$ is a model and $A \subset M$, we call φ a* formula over A *if φ is a formula in $L(A)$. For φ a formula over A we let $\varphi(\mathcal{M})$ denote $\{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}$, where φ has n free variables. We call $X \subset M^n$* definable over A in $\mathcal{M}$ *if $X = \psi(\mathcal{M})$ for some formula ψ over A having n free variables. We may say* A−definable in $\mathcal{M}$ *instead of definable over A in $\mathcal{M}$.*

Formulas with parameters from a model will be used frequently, and they will be introduced without formally changing the language.

All results in this book hold not just for a 1-sorted first-order language, but also for a many-sorted language (see, e.g., [End72]). To simplify the notation we will work in the context of a 1-sorted language (until Section 4.1 where we introduce a many-sorted expansion of a theory).

The starting point for model theory is the following result due to Gödel.

Theorem 1.1.1 (Compactness Theorem). *A theory T has a model if and only if every finite subset of T has a model.*

We assume that the reader has seen the Henkin construction of a model which proves the Compactness Theorem (see, e.g., [Hod93, 6.1.1]). The proof shows that when every finite subset of T has a model, T has a model of cardinality $\leq |T|$. As a first application of compactness we state

Corollary 1.1.1 (Löwenheim-Skolem Theorem). *A theory T in a language L which has an infinite model has a model in each infinite cardinality $\lambda \geq |T|$.*

Proof. Let $C = \{c_\alpha : \alpha < \lambda\}$ be a set of λ new distinct constant symbols, $L' = L \cup C$. Let $T' = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \lambda\}$. By the Compactness Theorem T' has a model. As a corollary to the proof of the Completeness Theorem we know that T' has a model $\mathcal{M}'$ of cardinality $|L'| + \aleph_0 = \lambda$. The restriction of $\mathcal{M}'$ to L is the desired model of T.

A similar proof shows that a theory which has arbitrarily large finite models has an infinite model. Other elementary applications of the Compactness Theorem are stated in the exercises.

Recall that $\mathcal{M}$ is a *submodel* of a model $\mathcal{N}$, written $\mathcal{M} \subset \mathcal{N}$, if

- $M \subset N$,
- $c^{\mathcal{M}} = c^{\mathcal{N}}$, for any constant symbol c in L,
- for $F \in L$ an n−ary function symbol, $F^{\mathcal{M}} = F^{\mathcal{N}} \restriction M^n$, and
- for $R \in L$ an n−ary relation symbol, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^n$.

Definition 1.1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be structures in the language L. We say that $\mathcal{M}$ is an* elementary submodel of $\mathcal{N}$*, and write $\mathcal{M} \prec \mathcal{N}$, if $M \subset N$ and for all formulas $\varphi(v_0, \ldots, v_n)$ of L, and $a_0, \ldots, a_n \in M$,*

$$\mathcal{M} \models \varphi(a_0, \ldots, a_n) \text{ if and only if } \mathcal{N} \models \varphi(a_0, \ldots, a_n)$$

(Notice that we could have stated the key condition in the definition as $\mathcal{M}_M \equiv \mathcal{N}_M$.) When $\mathcal{M} \prec \mathcal{N}$ we will also call $\mathcal{N}$ an *elementary extension of* $\mathcal{M}$. Our first question is: How can a submodel fail to be an elementary submodel? After doing Exercise 1.1.5 the reader will see that all failures are in the form: there are $a_0, \ldots, a_n \in M$ and $\varphi(v, v_0, \ldots, v_n)$ such that $\mathcal{N} \models \exists v \varphi(v, a_0, \ldots, a_n)$ and $\mathcal{M} \not\models \exists v \varphi(v, a_0, \ldots, a_n)$. This gives the elementary submodel relation the flavor of a closure condition; witnesses to existential quantifiers must be added to form an elementary submodel. This is exhibited in the next lemma, whose proof is left to the exercises.

Lemma 1.1.1 (Tarski-Vaught Test). *For models $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$ if and only if*

- *$\mathcal{M} \subset \mathcal{N}$ and*
- *for all formulas $\varphi(v, v_0, \ldots, v_n)$ and $a_0, \ldots, a_n \in M$, if $\mathcal{N} \models \exists v \varphi(v, a_0, \ldots, a_n)$, there is a $b \in M$ such that $\mathcal{N} \models \varphi(b, a_0, \ldots, a_n)$.*

By the Compactness theorem any infinite model $\mathcal{M}$ has elementary extensions of any cardinality $\geq |M| + |T|$. (Just apply the Löwenheim-Skolem Theorem to $Th(\mathcal{M}_M)$.) A natural companion to this is

Theorem 1.1.2 (Downward Löwenheim-Skolem-Tarski Theorem). *Let T be a theory, λ, κ cardinals with $\lambda \geq \kappa \geq |T|$ and $\mathcal{M}$ a model of T of cardinality λ. Then for any $X \subset M$ with $|X| \leq \kappa$, $\mathcal{M}$ has an elementary submodel of cardinality κ containing X.*

Proof. Form a chain of sets $X = X_0 \subset X_1 \subset X_2 \subset \ldots$ such that if $\varphi(v, v_0, \ldots, v_n)$ is a formula of the language of T, $a_0, \ldots, a_n \in X_i$ and $\mathcal{M} \models \exists v \varphi(v, a_0, \ldots, a_n)$, then there is a $b \in X_{i+1}$ such that $\mathcal{M} \models \varphi(b, a_0, \ldots, a_n)$. Since there are $|T|$ many formulas to consider and $|X_i| + \aleph_0$ many tuples $(a_0, \ldots, a_n)$ from X_i, we may require each X_i to have cardinality κ. Let $N = \bigcup_{i<\omega} X_i$. Considering suitable choices for φ shows that N is the universe of a submodel $\mathcal{N}$ of $\mathcal{M}$. Furthermore, since any n–tuple from N is from some X_i, the Tarski-Vaught test implies that $\mathcal{N}$ is an elementary submodel of $\mathcal{M}$, as desired.

For models $\mathcal{M}$ and $\mathcal{N}$ in the same language, a function f is an *elementary embedding of $\mathcal{M}$ into $\mathcal{N}$* if f is an isomorphism from $\mathcal{M}$ onto an elementary submodel of $\mathcal{N}$. We say $\mathcal{M}$ is *elementarily embeddable into $\mathcal{N}$* if there is such an elementary embedding.

Let T be a theory in L. An *n–type in $\bar{v} = (v_0, \ldots, v_{n-1})$* is a consistent set of formulas p in the variables $\bar{v}$. (Consistency is computed relative to the ambient theory T.) We may write $p(\bar{v})$ for p when it is helpful to exhibit the variables. Let p be an n–type in the theory T and $\mathcal{M} \models T$. Given $\bar{a} \in M^n$, we say that $\bar{a}$ *realizes p in $\mathcal{M}$* (or *satisfies p in $\mathcal{M}$*) if $\mathcal{M} \models \varphi(\bar{a})$ for each $\varphi \in p$. Extending the notation used for formulas, $p(\mathcal{M})$ denotes $\{\bar{a} \in M^n : \bar{a} \text{ realizes } p \text{ in } \mathcal{M}\}$. The model $\mathcal{M}$ *realizes* (or *satisfies*) p if some tuple from $\mathcal{M}$ realizes p; otherwise, $\mathcal{M}$ *omits* p. As usual, this notion can be relativized over a set of parameters. Given $A \subset M$, an *n–type over A in $T = Th(\mathcal{M})$* is a consistent set of formulas over A (where consistency is computed with respect to $Th(\mathcal{M}_A)$.) Given p a type over a set A, the *domain of p*, denoted *dom*(p), is the minimal set $B \subset A$ such that p is a type over B.

The type $p(\bar{v})$ is called *complete* if it is a maximally consistent set of formulas in $\bar{v}$ (i.e., there is no formula $\varphi(\bar{v})$ such that both $p \cup \{\varphi\}$ and $p \cup \{\neg\varphi\}$ are consistent). We will deal extensively with complete types in complete theories.

Notation. Fix a model $\mathcal{M}$, $A \subset M$ and $\bar{v}$ an n–tuple of variables. Let $S_n(A)$ denote the set of complete n–types over A in $\bar{v}$. (If $p = p(\bar{v})$ and $\bar{w}$ is another sequence of n variables we equate p and $p(\bar{w})$ in almost all model-theoretic settings.) Let $S(A) = \bigcup_{n<\omega} S_n(A)$. Given an n–tuple $\bar{a}$ from M there is a unique $p \in S(A)$ realized by $\bar{a}$ in $\mathcal{M}$, called the *type of $\bar{a}$ over A in $\mathcal{M}$* and denoted $tp_{\mathcal{M}}(\bar{a}/A)$. Explicitly,

$$tp_{\mathcal{M}}(\bar{a}/A) = \{\, \varphi : \ \varphi \text{ is a formula over } \ A \text{ and } \ \mathcal{M} \models \varphi(\bar{a}) \,\}.$$

We may compress the notation and write, e.g.,

$$tp_{\mathcal{M}}(a_0, \ldots, a_n) \text{ for } \ tp_{\mathcal{M}}(\,(a_0, \ldots, a_n)\,).$$

Definition 1.1.3. *Let T be a theory, $\mathcal{M} \models T$, $A \subset M$ and p, q n–types over A. We say that p* implies q in T, *written $p \models q$, if for any model $\mathcal{N} \succ \mathcal{M}$, $p(\mathcal{N}) \subset q(\mathcal{N})$. We say that p and q are* equivalent in T *if for every model $\mathcal{N} \succ \mathcal{M}$, $p(\mathcal{M}) = q(\mathcal{M})$; i.e., $p \models q$ and $q \models p$.*

Let T be a theory and p, q n–types (over $\emptyset$, for simplicity). The Compactness Theorem gives the equivalent:

– $p \models q$ if and only if
– for all formulas $\varphi(\bar{v}) \in q$ there are $\psi_0(\bar{v}), \ldots, \psi_n(\bar{v}) \in p$ such that $T \models \forall \bar{v}(\bigwedge_{i \leq n} \psi_i(\bar{v}) \rightarrow \varphi(\bar{v}))$.

(See the exercises). Additionally, $p \models q$ if and only if for every complete n–type r over A, $r \supset p \implies r \supset q$.

Definition 1.1.4. *Fix a theory T. A set of formulas p (in n variables) is said to be* isolated by φ *if φ is consistent with T and for all $\psi \in p$, $T \models \forall \bar{v}(\varphi(\bar{v}) \rightarrow \psi(\bar{v}))$. p is* isolated *if it is isolated by some formula. If p is not isolated it is called* nonisolated.

Notice that a nonisolated set of formulas need not be consistent. If p is isolated by φ and $\mathcal{M} \models T$ then $\varphi(\mathcal{M}) \subset p(\mathcal{M})$.

As the term "isolated" suggests there is a topology in the background. For $\varphi(\bar{v})$ a formula in n free variables, $\mathcal{O}_\varphi = \{\, p \in S_n(\emptyset) : \ \varphi \in p \,\}$. (Here, we equate the formulas $\varphi(\bar{v})$ and $\varphi(\bar{x})$, where $\bar{x}$ is another sequence of n variables: $\mathcal{O}_{\varphi(\bar{v})}$ is, by definition, the same as $\mathcal{O}_{\varphi(\bar{x})}$.) The sets of the form $\mathcal{O}_\varphi$ comprise the basic open sets of a topology on $S_n(\emptyset)$, called a *Stone space of* T. The topology is compact (by the Compactness Theorem) and Hausdorff. A type $p \in S_n(\emptyset)$ is isolated exactly when it is an isolated point in the Stone space topology on $S_n(\emptyset)$.

Given a model $\mathcal{M}$ and $A \subset M$, any $p \in S(A)$ is realized in *some* elementary extension $\mathcal{N}$ of $\mathcal{M}$. (This is a compactness argument, left to the reader in Exercise 1.1.8.) In general, though, there is no reason to think that p is realized in $\mathcal{M}$. For any formula $\varphi \in p$ (equivalently, any finite conjunction of formulas in p) the consistency of p requires that $\mathcal{M} \models \exists\bar{v}\varphi$; i.e., $\mathcal{M} \models \varphi(\bar{a})$ for some $\bar{a} \in M^n$. However, there may not be a single $\bar{a}$ which simultaneously satisfies all formulas in p. The obvious exception (which is immediate by the definition) is when p is isolated: If p is an isolated type over A then p is realized in any model of $Th(\mathcal{M}_A)$. That this is the single case when a type is realized in every model of a *countable* theory is proved in

Theorem 1.1.3 (Omitting Types Theorem). *If T is a countable theory and p is a nonisolated set of formulas in T, then T has a countable model which omits p.*

The proof is to use a Henkin construction to build a model which omits the nonisolated set of formulas. The restriction to a countable theory is necessary; there is a theory in which some nonisolated type is realized in every model. A related point is that there may not be an uncountable model omitting a nonisolated type even in a countable theory. (The reader is asked to find an example in the exercises.) The proof of the following slightly more complicated version of the Omitting Types Theorem is assigned as an exercise.

Corollary 1.1.2 (Extended Omitting Types Theorem). *Let T be a countable theory and for each i, let p_i be a nonisolated set of formulas in n_i variables. Then T has a countable model which omits each p_i.*

The Omitting Types Theorem is very useful when constructing nonisomorphic models of a theory. It enables us to show that a certain elementary class is "rich"; i.e., contains models with varying properties. The goal of the next chapter is to carry further this program of finding a wide variety of models of fixed theory.

1.1.1 Elimination of Quantifiers

The method of elimination of quantifiers provides model theorists with a powerful tool for understanding the definable subsets of a particular structure.

Definition 1.1.5. *Given a language L and a class $\mathcal{K}$ of structures in L, we say that a set Φ of formulas of L is an* elimination set *for $\mathcal{K}$ if for every formula $\varphi(\bar{v})$ of L there is a formula $\varphi'(\bar{v})$ such that*

– *φ' is a boolean combination of formulas in Φ and*
– *for every model $\mathcal{M} \in \mathcal{K}$, $\mathcal{M} \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \varphi'(\bar{v}))$.*

When T is a theory and $\mathcal{K}$ is the class of models of T we say elimination set for T.

The elimination of quantifiers program is: find an elimination set for a given class $\mathcal{K}$. Of course, the set of all formulas is always an elimination set for a class, but for many classes there is a simpler set which illuminates the model-theoretic properties of $\mathcal{K}$. Take, for example, the theory T in $L = \{<\}$ of dense linear orders without endpoints. (This is the theory axiomatized by the statements: $<$ defines a linear order, there is no least element or greatest element, and $\forall xy(x < y \rightarrow \exists z(x < z < y))$. $(\mathbb{Q}, <)$ is a model of T.) We leave it to the exercises to show that $\Phi = \{v_1 < v_2, v_1 = v_2\}$ is an elimination set for T. (Again, we do not distinguish between formulas which are obtained by a change of variables.)

In this example, the elimination set consists of atomic formulas — every formula is equivalent to a quantifier-free formula, so we really have eliminated the quantifiers. When the elimination set for T consists of atomic formulas we say that T *has elimination of quantifers*, T is *quantifier-eliminable*, or T is q.e. The term "elimination of quantifiers" has its origins, however, in the key step of a proof that Φ is an elimination set. The following is proved by induction on formulas (see the exercises). For Φ a set of formulas $\Phi^- = \{\neg\varphi : \varphi \in \Phi\}$.

Lemma 1.1.2. *Let $\mathcal{K}$ be a class of structures in L and Φ a set of formulas of L. Suppose that*

(1) *every atomic formula of L is in Φ, and*
(2) *for every formula $\theta(\bar{v})$ of the form $\exists w \bigwedge_{i<n} \psi_i(\bar{v}, w)$, where each $\psi_i \in \Phi \cup \Phi^-$, there is a formula $\theta'(\bar{v})$ which is a boolean combination of formulas from Φ and is equivalent to θ on every structure in $\mathcal{K}$.*

Then Φ is an elimination set for $\mathcal{K}$.

The following elimination of quantifier results will be used freely throughout the book. The reader is referred to [Hod93] for details.

– The class $\mathcal{K}$ of vector spaces over a division ring F has elimination of quantifiers. (The elements of $\mathcal{K}$ are structures in the language with a binary operation $+$, a unary function α for each $\alpha \in F$ and a constant symbol 0.)
– The class $\mathcal{K}$ of algebraically closed fields of a fixed characteristic has elimination of quantifiers.
– For $\mathcal{K}$ the class of real-closed fields $\{\exists y(y^2 = t(x)) : t(x)$ a term of L not containing $y\}$ is an elimination set for $\mathcal{K}$.
– The class of divisible abelian groups has elimination of quantifiers.

Historical Notes. The compactness theorem was proved for countable theories by Gödel in [Go30], for arbitrary propositional theories by Gödel in

[Go31] and in general by Mal'tsev [Mal36]. The Löwenheim-Skolem Theorem (as stated here) is found in [TV57], where the Tarski-Vaught test is also formulated. The Omitting Types Theorem can be attributed to several sources, Ehrenfeucht (in [Vau61]) and Grzegorczyk, Mostowski and Ryll-Nardzewski in [GMRN61]. The Downward Löwenheim-Skolem Theorem was proved first for some special languages in [Lo15] and in another form in [Sko20]. (They both consider the special case of showing that a countable theory with an infinite model has a countable one.)

Exercise 1.1.1. Prove that a theory which has arbitrarily large finite models has an infinite model.

Exercise 1.1.2. An ordered field $(F, +, \cdot, 0, 1, \leq)$ is called *Archimedean* if for any two positive elements a, $b \in F$ there is an n such that $na \geq b$. Show that $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ has an elementary extension which is not Archimedean.

Exercise 1.1.3. Let σ be a sentence which holds in any algebraically closed field of characteristic 0. Then there is a $p \neq 0$ such that σ holds in all algebraically closed fields of characteristic $\geq p$.

Exercise 1.1.4. Let T be a theory. Prove that $p \models q$ if and only if for all formulas $\varphi(\bar{v}) \in q$ there are $\psi_0(\bar{v}), \ldots, \psi_n(\bar{v}) \in p$ such that $T \models \forall \bar{v}(\bigwedge_{i \leq n} \psi_i(\bar{v}) \rightarrow \varphi(\bar{v}))$.

Exercise 1.1.5. Which of the following submodel relations are elementary?
(a) $(\mathbb{Q}, \leq) \subset (\mathbb{R}, \leq)$.
(b) $(2\mathbb{Z}, +, 0) \subset (\mathbb{Z}, +, 0)$.
(c) $(\mathbb{Q}, +, \cdot, 0, 1) \subset (\mathbb{C}, +, \cdot, 0, 1)$.

Exercise 1.1.6. Use induction on formulas to prove the Tarski-Vaught Test.

Exercise 1.1.7. Prove that $\mathcal{M}$ is elementarily embeddable into $\mathcal{N}$ (both structures in the same language) if and only if $\mathcal{N}$ can be expanded to a model of $Th(\mathcal{M}_M)$.

Exercise 1.1.8. Show that if $\mathcal{M}$ is a model, $A \subset M$ and $p \in S(A)$, then p is realized in some elementary extension $\mathcal{N}$ of $\mathcal{M}$. Also show that if p is an isolated type over A then p is realized in any model of $Th(\mathcal{M}_A)$.

Exercise 1.1.9. Prove the Extended Omitting Types Theorem.

Exercise 1.1.10. Give an example of a countable theory T and a nonisolated type p which is realized in every uncountable model of T.

Exercise 1.1.11. Let $\mathcal{M}$ be a finite model in a language L. Show that

$$\mathcal{N} \equiv \mathcal{M} \implies \mathcal{N} \cong \mathcal{M}.$$

Exercise 1.1.12. Prove Lemma 1.1.2.

Exercise 1.1.13. Prove that the theory of dense linear orders without endpoints has elimination of quantifiers.

Exercise 1.1.14. Let $L = \{E\}$ where E is a binary relation and let T be the theory in L saying that E is an equivalence relation with infinitely many infinite classes and no finite classes. Prove that T has elimination of quantifiers.