XI. Changing Cofinalities; Equi-Consistency Results

§0. Introduction

We formulate a condition which is (strongly) preserved by revised countable support iteration, implies $\aleph_1$ is not collapsed, no real is added and is satisfied e.g. by Namba forcing, and any $\aleph_1$-complete forcing. So we can iterate forcing notions collapsing $\aleph_2$ but preserving $\aleph_1$ up to some large cardinal.

Our aim is to improve the results of chapter of X to equi-consistency results. If you want to add reals, look at Chapter XV. To prove the preservation we use partition theorems and Δ-system theorems on tagged trees (3.5, 3.5A, 3.7 (and 4.3A)). Some of them are from Rubin and Shelah [RuSh:117], see detailed history there on pages 47, 48 and more on mathematics see [RuSh:117], [Sh:136] 2.4, 2.5 (pages 111 – 113).

§1. The Theorems

1.1 Discussion. In this chapter we list the demands that we would like our condition to satisfy, and show how, having a condition satisfying these demands we can prove our theorems. Then, in the following sections we will formulate the condition and prove it satisfies all our demands. Lastly we shall prove some more complicated theorems applying the condition.

The Demands. We will have a condition for forcing notions such that:

(i) If P satisfies the condition *then* forcing with P does not collapse $\aleph_1$ and, moreover, (when CH holds) it does not add reals.

(ii) If $P = \operatorname{Rlim} \bar{Q}$, where $\bar{Q}$ is an RCS iteration of forcing notions such that each of them satisfies the condition *then* P satisfies it as well.

(RCS iteration was defined in X §1. In 1.9 we will recall its basic properties).

Really we do not get (ii) but a slightly different version (ii)$'$, which is as good for our purpose:

(ii)$'$ Assume V satisfies: if $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta \rangle$ is an RCS iteration, $\underset{\sim}{Q}_{2i+1}$ is Levy collapse of $2^{|P_{2i+1}|+|i|}$ to $\aleph_1$ (by countable conditions), each $\underset{\sim}{Q}_{2i}$ satisfies the condition. *Then* P_δ, the revised limit of $\bar{Q}$, satisfies the condition (see also 6.2A).

(iii) If $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ is an RCS iteration as in (ii)$'$, κ is a strongly inaccessible cardinal and $|P_i| < \kappa$ for $i < \kappa$ *then* P_κ, the revised limit of $\bar{Q}$, satisfies the κ-c.c.

(iv) The condition is satisfied by the following forcing notions:

a. Namba forcing. (See 4.1, it adds a cofinal countable subset to ω_2 without collapsing ω_1.) We denote this forcing notion by Nm.

b. Any $\aleph_1$-closed forcing notion.

c. $P[S]$, where S is a stationary subset of ω_2 such that $\alpha \in S \Rightarrow \operatorname{cf}(\alpha) = \omega$, and $P[S] = \{f : f$ is an increasing and continuous function from $\alpha + 1$ into S, for some $\alpha < \omega_1\}$. Note that $P[S]$ shoots a closed copy of ω_1 into S hence collapses $\aleph_2$.

Remark. The condition on P is, by the terminology we shall use, essentially the $\{\lambda \le |P| : \lambda \text{ regular} > \aleph_1\}$-condition; more exactly, the definition of such a notion is in 6.7, where (ii)$'$ is proved. Now (iv)$_a$ holds by 4.4, (iv)$_b$ by 4.5, (iv)$_c$ holds by 4.6, (i) by 3.2 and (iii) automatically follows from 6.3A(1) as in X 5.3, see 1.13.

Remark. The preservation theorem in this chapter is in a sense orthogonal to the one of Chapter X, since here we are not interested in semiproperness of forcing notions (e.g. Namba forcing may fail to be semiproper, but it always satisfies the condition in this chapter). In chapter XV we will present a generalization of the S-condition which also generalizes semiproperness.

Assume we have a condition satisfying all of these demands and let us get to the proofs of our theorems.

1.2 Theorem. If "ZFC + G.C.H. + there is a measurable cardinal" is consistent then so is "ZFC + G.C.H. + there is a normal precipitous filter D on ω_2 such that $S_0^2 \in D$".

Remark.

(1) S_0^2 is $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \aleph_0\}$.

(2) By [JMMP] the converse of this theorem is also true, so we have an equiconsistency result.

(3) In fact if "ZFC + there is a measurable cardinal" is consistent then so is "ZFC + G.C.H. + there is a measurable cardinal", so we can delete "G.C.H." from the hypothesis of our theorem.

Proof. We start with a model of ZFC + G.C.H. with a measurable cardinal κ. We iterate, by the RCS iteration, forcing with Nm κ many times. More exactly let $\bar{Q} = \langle P_i, Q_i : i < \kappa\rangle$ be an RCS iteration, $\underset{\sim}{Q}_{2i}$ is Nm (see $(\text{iv})_a$ above), $\underset{\sim}{Q}_{2i+1} = \mathrm{Levy}(\aleph_1, 2^{|P_{2i+1}|})$. Let V denote our ground model, and P denote $\mathrm{Rlim}\,\bar{Q}$. We can prove by induction $|P_i| < \kappa$; moreover if $\lambda \leq \kappa$ is strongly inaccessible then $i < \lambda \Rightarrow |P_i| < \lambda$, and P_λ has power λ.

By 1.1 (ii)$'$, P satisfies the condition (remembering $(\text{iv})_\text{a}$+$(\text{iv})_\text{b}$) hence by 1.1(i), forcing by P does not collapse $\aleph_1$ nor add reals and so $V^P \models CH$. On the other hand clearly $|P_i| \geq i$, hence $\underset{\sim}{Q}_{2i+1}$ collapses $|i|$ to $\aleph_1$, hence all λ, $\aleph_1 < \lambda < \kappa$ are collapsed by P. By 1.1(iii) (or X 5.3(1)) P satisfies the κ-chain condition hence κ is not collapsed. So clearly $\aleph_1^{V^P} = \aleph_1^V$, $\aleph_2^{V^P} = \kappa$ and V^P satisfies G.C.H.

Let F be a normal κ-complete ultrafilter over κ (in V), then by (iii) and X.6.5 (see references there), F generates a normal precipitous filter on κ in V^P. Let A be $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ (in V) then $A \in F$ so we are done with the proof once we show that $\lambda \in A$ implies λ has cofinality ω in $V[P]$. As Nm satisfies our condition (by demand (iv)$_a$) and λ is inaccessible in V we know that the iteration up to stage λ satisfies the λ-c.c. (by demand (iii), or by using X5.3(1) provided that we restrict A to Mahlo cardinals). Hence after forcing with P_λ we have $\lambda = \aleph_2$ and at the next step in the iteration Nm shoots a cofinal ω-sequence into λ, a sequence that exemplifies $\mathrm{cf}(\lambda) = \aleph_0$ in V^P, see 4.1A. $\square_{1.2}$

1.3 Theorem. If "ZFC + G.C.H. + there is a Mahlo cardinal" is consistent *then* so is "ZFC + G.C.H. + for every stationary $S \subseteq S^2_0$ there is a closed copy of ω_1 included in it".

Remark. Earlier Van-Liere has shown the converse and is a variant of a problem of Friedman, see on this X 7.0.

For the clarity of the exposition we prove here a weaker theorem and postpone the proof of the theorem as stated above to Sect. 7 of this chapter.

1.4 Theorem. If "ZFC + G.C.H. + there is a weakly compact cardinal" is consistent *then* so is "ZFC + G.C.H. + for every stationary $S \subseteq S^2_0$ there is a closed copy of ω_1 included in it".

Proof. The proof is very much like the proof of Theorem 7.3 of X; the only difference is that now we do not have to demand that there will be measurable cardinals below the weakly compact cardinal. We give here only an outline of the proof. Let κ be weakly compact, w.l.o.g. $V = L$, so by Jensen's work there is $\langle \bar{A}_\alpha : \alpha < \kappa,\ \alpha \text{ inaccessible}\rangle$, $\bar{A}_\alpha = \langle A_{\alpha,e} : e < n_\alpha\rangle$, a diamond sequence satisfying: $A_{\alpha,e} \subseteq H(\alpha)$, and for every finite sequence $\bar{A}$ of subsets of $H(\kappa)$, and Π^1_1 sentence ψ such that $(H(\kappa), \in, \bar{A}) \vDash \psi$ there is some inaccessible λ such that $\bar{A} \restriction H(\lambda) = \bar{A}_\lambda$ (i.e. $n_\lambda = \ell g(\bar{A})$ and $A_{\lambda,e} = A_e \cap H(\lambda)$) and

$(H(\lambda), \in, \bar{A}_\lambda) \vDash \psi$. Now we define an RCS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$. Let $\underset{\sim}{Q}_\alpha = P[\underset{\sim}{S}_\alpha]$ (as it was defined in 1.1 (iv)$_c$) whenever $\bar{A}_\alpha = \langle P_\alpha, p_\alpha, \underset{\sim}{S}_\alpha \rangle$, α strongly inaccessible and $p_\alpha \in P_\alpha$ and $p_\alpha \Vdash_{P_\alpha}$ "$\underset{\sim}{S}_\alpha$ is a stationary subset of S_0^λ $(= \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\})$", and in all other cases we force with the usual Levy $\aleph_1$-closed conditions for collapsing $2^{|P_\alpha| + |\alpha|}$.

In the model we get after the forcing $\kappa = \aleph_2$ and every stationary subset of S_0^2 includes a closed copy of ω_1. (For checking the details note that our forcing notion, and any initial segment of it, satisfies our condition thus no reals are added, $\aleph_1$ is not collapsed and in any λ-stage for inaccessible λ, the initial segment of the forcing satisfies the λ-c.c. so at that stage $\lambda = \omega_2$, when we use $P[S_\lambda]$ we are forcing with $P[S]$ for S which is a stationary subset of S_0^2). $\square_{1.3}$

1.4A Remark. If κ is only a Mahlo cardinal then this proof suffices if we just want "for every $S \subseteq S_0^2$, S or $S_0^2 \setminus S$ contains a closed copy of ω_1 " or even if we want "if h is a pressing-down function on S_0^2, then for some α, $h^{-1}(\{\alpha\})$ contains a closed copy of ω_1". See more in 7.2.

1.5 Theorem. If "ZFC + G.C.H. + there is an inaccessible cardinal" is consistent *then* so is "ZFC + G.C.H. + there is no subset of $\aleph_1$ such that all ω-sequences of $\aleph_2$ are constructible from it".

Remark.

(1) This theorem answers a question of Uri Abraham who has also proved its converse.

(2) Again we can omit G.C.H. from the hypothesis.

Proof. Let κ be inaccessible and let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ be an RCS iteration, $\underset{\sim}{Q}_{2i} = (\mathrm{Nm})^{V^{P_{2i}}}$, $\underset{\sim}{Q}_{2i+1} = \mathrm{Levy}(\aleph_2, 2^{|P_{2i}| + |i|})$. In the resulting model $\kappa = \aleph_2$ and as the forcing satisfies the κ-c.c. any subset of $\aleph_1$ is a member of a model obtained by some proper initial segment of our iteration, but the ω-sequence added to ω_2 by the next Nm forcing does not belong to this model so it is not constructible from this subset of $\aleph_1$. $\square_{1.5}$

1.6 Theorem. If the existence of a Mahlo cardinal is consistent with ZFC *then* so is "G.C.H. + for every subset A of $\aleph_2$ there is some ordinal δ such that $\mathrm{cf}(\delta) = \aleph_0$ but δ is a regular cardinal in $L[A \cap \delta]$".

Remark. Again this is an answer to a question of Uri Abraham and again he has shown that the converse of the theorem is true as well by using the square on λ.

Proof. Let κ be Mahlo (in a model V of ZFC + G.C.H.), w.l.o.g. $V = L$ and iterate as in the proof of 1.5. Let A be a subset of $\aleph_2$ in the resulting model. As the forcing notion satisfies the κ-c.c., we can find a closed and unbounded $C \subseteq \aleph_2$ such that for $\delta \in C$ we have $A \cap \delta \in V[P_\delta]$ where P_δ is the Rlim of δ'th initial segment of our iteration. As κ is Mahlo, $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in it so there is some inaccessible λ in C. Such λ exemplifies our claim. P_λ satisfies the λ-c.c., so in $V[P_\lambda]$ we have $\lambda = \aleph_2$ hence Nm at the λ-step of the iteration adds a cofinal ω sequence into λ, so in $V[P_\kappa]$, which is our model, $\mathrm{cf}(\lambda) = \aleph_0$. But $L[A] \subseteq V[P_\lambda]$ as $\lambda \in C$ (and $P_\delta \lessdot P = P_\kappa$). $\square_{1.6}$

One more answer to a question from Uri Abraham's dissertation is to get V such that if $A \subseteq \omega_2$ and $\aleph_2^{L[A]} = \aleph_2$ then $L[A]$ has $\geq \aleph_2$ reals. We had noted that for the statement to hold in V, it is enough to have: $L[\{\delta < \aleph_2^V : \mathrm{cf}^V \delta = \aleph_0\}]$ has at least $\aleph_2^V$ reals; more explicitly, it is enough to produce a model in which there are $\aleph_2$ distinct reals r, such that for some $\lambda \in \mathrm{Car}^L$ we have $r(\ell) = 0$ iff $\mathrm{cf}((\lambda^{+\ell})^L) = \aleph_0$ (i.e., the cofinality is in V, $\lambda^{+\ell}$ is computed in L). Then the answer below was obtained by Shai Ben-David using the same method as of 1.5, 1.6:

1.7 Theorem. [Ben David] The consistency of "ZFC + there exists an inaccessible cardinal" is equivalent with the consistency with ZFC of the statement: "There is no cardinal preserving extension of the universe in which there is a set $A \subseteq \aleph_2$ such that $L[A]$ satisfies C.H. and $\aleph_2^{L[A]} = \aleph_2$".

However, this proof relies on a preliminary version of this chapter in which forcing notions adding reals were permitted, which unfortunately seems doubtful and was abandoned. The framework given in 1.1, is not enough since in (iv) no forcing notions adding reals appear, but we can use XV §3 instead (i.e. for unboundedly many $i, \underset{\sim}{Q}_i$ is Cohen forcing)

Remark. In fact there is no class of V which is a model of ZFC, having the same $\aleph_1$ and $\aleph_2$ and satisfying C.H.

1.8 Remark. The partition theorems presented later can be slightly generalized to monotone families (instead of ideals) as done in the first version of this book. But this is irrelevant to our main purpose.

We now recall the main properties of RCS iterations. Whenever it is convenient, we will assume that all partial orders under consideration are complete Boolean algebras i.e. are $(B \setminus \{0\}, \geq)$.

1.9 Definition. We say that a sequence $\langle P_\alpha, \underset{\sim}{Q}_\beta, \mathbf{i}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is an RCS iteration of length δ (δ not necessarily limit ordinal), if:

(1) For all β, $\mathbf{i}_\beta$ is a dense embedding from $P_\beta * \underset{\sim}{Q}_\beta$ into $P_{\beta+1}$, or into the complete Boolean algebra generated by $P_{\beta+1}$. [We usually do not mention $\mathbf{i}_\beta$ and identify $P_\beta * \underset{\sim}{Q}_\beta$ with $P_{\beta+1}$].

(2) For all $\alpha < \beta \leq \delta$, $P_\alpha \lessdot P_\beta$. [We assume that for all $p \in P_\beta$, the projection of p to P_α exists, and we write it as $p \restriction \alpha$. We write $P_{\alpha,\beta}$ for the quotient forcing P_β / P_α or P_β / G_α in V^{P_α} or $V[G_\alpha]$, and let $p \mapsto p \restriction [\alpha, \beta)$ be the obvious map in V^{P_α}.]

(3) Whenever $\alpha \leq \delta$ is a limit ordinal, then $P_\alpha = \mathrm{Rlim}\langle P_\beta : \beta < \alpha \rangle$.

Also, if we write $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta \rangle$, we automatically define $P_\delta \stackrel{\text{def}}{=} \mathrm{Rlim}\, \bar{Q}$ (if δ is a limit) or $P_\delta \stackrel{\text{def}}{=} P_{\delta-1} * \underset{\sim}{Q}_{\delta-1}$ (if δ is a successor), respectively.

We say that $\langle P_\alpha : \alpha < \delta \rangle$ is an RCS iteration iff $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta \rangle$ is one, with $\underset{\sim}{Q}_\alpha \stackrel{\text{def}}{=} P_{\alpha+1} / P_\alpha$.

We will not define here what Rlim $\bar{Q}$ actually is. A possible definition is in chapter X. Here we will only collect some properties of RCS iterations which we will use. First, we need a definition:

1.10 Definition. If $\underset{\sim}{\alpha}$ is a P_δ-name we say that $\underset{\sim}{\alpha}$ is *prompt*, if $\Vdash$ "$\underset{\sim}{\alpha} \le \delta$", and for all ordinals $\xi \le \delta$, all conditions $q \in P_\delta$:

$$\text{whenever } \ q \Vdash \text{“}\underset{\sim}{\alpha} = \xi\text{”}, \ \text{ then already } \ q{\restriction}(\xi+1) \Vdash \text{“}\underset{\sim}{\alpha} = \xi\text{”}.$$

(where $q{\restriction}(\delta+1) = q$)

Note that

1.10A Observation. 1) For a P_δ-name $\underset{\sim}{\alpha}$ we have:

$\underset{\sim}{\alpha}$ is prompt iff $\Vdash$ "$\underset{\sim}{\alpha} \le \delta$", and for all $\xi \le \delta$ and all $p \in P_\delta$:

$$\text{if } p \Vdash \text{“}\underset{\sim}{\alpha} \le \xi\text{”}, \text{ then } p{\restriction}(\xi+1) \Vdash \text{“}\underset{\sim}{\alpha} \le \xi\text{”}.$$

Also the inverse implication holds, of course.

2) If S is a set of prompt P_δ-names, *then* also $\mathrm{Sup}(S)$ is a prompt P_δ-name and $\min(S)$ is a prompt P_δ-name.

Proof. Easy.

1.11 Definition. If $\underset{\sim}{\alpha}$ is a prompt P_δ-name, *then*

(a) $P_{\underset{\sim}{\alpha}} \stackrel{\text{def}}{=} \{p \in P_\delta : (\forall q \ge p)\ [\text{if } q \Vdash \text{“}\underset{\sim}{\alpha} = \xi\text{”, then } q{\restriction}(\xi+1) \Vdash_{P_\delta} \text{“}p \in \underset{\sim}{G}_\delta\text{”}]\}$

(b) for an atomic $\bar{Q}$-condition p, $p{\restriction}\underset{\sim}{\alpha}$ is naturally defined: for $G_\delta \subseteq P_\delta$ generic over V, $(p{\restriction}\underset{\sim}{\alpha})[G_\delta]$ is $p[G_\delta]$ if $\underset{\sim}{\zeta}_p[G_\delta] \le \underset{\sim}{\alpha}[G_\delta]$ and $\emptyset$ otherwise

(c) for $p \in P_\delta$, let $p{\restriction}\underset{\sim}{\alpha} = \{r{\restriction}\underset{\sim}{\alpha} : r \in p\}$

It may be more illustrative to consider the following dense subset:

$$P^*_{\underset{\sim}{\alpha}} \stackrel{\text{def}}{=} \bigcup_{\xi \leq \delta} \{p \in P_\xi : p \Vdash_{P_\delta} \text{“}\underset{\sim}{\alpha} = \xi\text{”}\}$$

1.11A Remark.

(a) For any prompt P_δ-name $\underset{\sim}{\alpha}$ we have $P_{\underset{\sim}{\alpha}} \lessdot P_\delta$. Moreover, if $\Vdash_{P_\delta}$ “$\underset{\sim}{\alpha}_1 \leq \underset{\sim}{\alpha}_2$”, then $P_{\underset{\sim}{\alpha}_1} \lessdot P_{\underset{\sim}{\alpha}_2}$.

(b) If $\dot{\alpha}$ is the canonical name of α, then $P_\alpha = P^*_{\dot{\alpha}}$.

1.12 Properties of RCS iterations. Let $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta \rangle$ be an RCS iteration with RCS limit P_δ. *Then*

(0) Assume $\alpha(*) < \delta$, $\underset{\sim}{\alpha}$ is a prompt $P_{\alpha(*)+1}$-name of an ordinal $\geq \alpha(*)$, $\mathcal{I}$ is an antichain of $P_{\alpha(*)+1}$ such that $p \in \mathcal{I} \Rightarrow p \Vdash_{P_\delta}$ “$\underset{\sim}{\alpha} = \alpha(*)$” and for each $p \in \mathcal{I}$, $\underset{\sim}{\beta}_p$ is a prompt P_δ-name of an ordinal $\geq \alpha(*)$. *Then* for some prompt P_δ-name $\underset{\sim}{\gamma}$ we have $\Vdash_{P_\delta}$ “ if $p \in \mathcal{I} \cap \underset{\sim}{G}_{P_\delta}$ then $\underset{\sim}{\gamma} = \underset{\sim}{\beta}_p$ and if $\mathcal{I} \cap \underset{\sim}{G}_{P_\delta} = \emptyset$ then $\underset{\sim}{\gamma} = \underset{\sim}{\alpha}$”.

(1) Whenever $\langle \underset{\sim}{\alpha}_n : n \leq \omega \rangle$ is a sequence of prompt P_δ-names, satisfying $\Vdash_{P_\delta}$ “$\underset{\sim}{\alpha}_n < \underset{\sim}{\alpha}_{n+1}$” for all n, and $\Vdash_{P_\delta}$ “$\underset{\sim}{\alpha}_\omega = \sup_n \underset{\sim}{\alpha}_n$”, then $P_{\underset{\sim}{\alpha}_\omega}$ is the inverse limit of $\langle P_{\underset{\sim}{\alpha}_n} : n < \omega \rangle$. So in particular: whenever $\langle p_n : n < \omega \rangle$ is a sequence of conditions in P_δ, and $p_n \in P_{\underset{\sim}{\alpha}_n}$, and $p_{n+1} \restriction \underset{\sim}{\alpha}_n = p_n$ for all n, then there is $p \in P_{\underset{\sim}{\alpha}_\omega}$ such that for all n, $p \restriction \underset{\sim}{\alpha}_n = p_n$. Moreover if $p_0 \in P_{\underset{\sim}{\alpha}_0}$ and $\underset{\sim}{p}_{n+1}$ is a $P_{\underset{\sim}{\alpha}_n}$-name of a member of $P_{\underset{\sim}{\alpha}_{n+1}}$ such that $\underset{\sim}{p}_{n+1} \restriction \underset{\sim}{\alpha}_n = \underset{\sim}{p}_n$ then there is p as above.

(2) Let $\alpha(*) < \delta$ be non-limit, $G_{\alpha(*)} \subseteq P_{\alpha(*)}$ generic over V, and $\langle \alpha_\zeta : \zeta \leq \beta \rangle$ an increasing continuous sequence of ordinals in $V[G_{\alpha(*)}]$, $\alpha_0 = \alpha(*)$, $\alpha_\beta = \delta$, each $\alpha_{\zeta+1}$ a successor ordinal.
In $V[G_{\alpha(*)}]$, we define $P'_\zeta = P_{\alpha_\zeta}/G_{\alpha(*)}$, $\underset{\sim}{Q}'_\zeta = P_{\alpha_{\zeta+1}}/\underset{\sim}{G}_{\alpha_\zeta}$ (where $\underset{\sim}{G}_{\alpha_\zeta}$ is the P_{α_ζ}-name of the generic subset of P'_ζ, which essentially means a generic subset of P_{α_ζ} over V extending $G_{\alpha(*)}$ ($\underset{\sim}{Q}'_\zeta$ is still a P'_ζ-name) $\bar{Q}' = \langle P'_\zeta, \underset{\sim}{Q}'_\zeta : \zeta < \beta \rangle$. *Then* in $V[G_{\alpha(*)}]$, $\bar{Q}'$ is an RCS iteration with limit P'_β.

(3) If δ is limit, *then* for all $p \in P_\delta$,

$$\Vdash_{P_\delta} \text{“}p \in \underset{\sim}{G}_\delta \text{ iff } \forall \alpha < \delta\ [p{\restriction}\alpha \in \underset{\sim}{G}_\delta]\text{”}$$

(4) If δ is a limit, *then* for all $p \in P_\delta$ we have a countable set $\{\underset{\sim}{\zeta}^k(p) : k < \omega\}$ of prompt names with $\Vdash$ “$\underset{\sim}{\zeta}^k(p) < \delta$” for all k such that letting $\underset{\sim}{\zeta}^* = \sup\{\underset{\sim}{\zeta}^k(p) : k < \omega\}$, we have $\underset{\sim}{\zeta}^*$ is an almost prompt P_δ-name and $p \in P_{\underset{\sim}{\zeta}^*}$.

(5) If δ is a strongly inaccessible cardinal and for every $\alpha < \delta$ we have $|P_\alpha| < \delta$ *then* P_δ is the direct limit of $\bar{Q}$ that is $\bigcup\limits_{\alpha<\delta} P_\alpha$ is a dense sunset of P_δ.

1.13 More properties of RCS iterations. As corollaries of (4) and (5) above we get:

(1) Let $\bar{Q}$ be an RCS iteration as above, and assume

$$\Vdash_{P_\delta} \text{“cf}(\delta) > \aleph_0\text{”}$$

Then

(a) $\bigcup_{\alpha<\delta} P_\alpha$ is (essentially) a dense subset of P_δ that is for every $p \in P_\delta$ for some $q \in \bigcup\limits_{\alpha<\delta} P_\alpha$ we have $q \Vdash$ “$p \in \underset{\sim}{G}_{P_\delta}$” (so P_δ is the direct limit of $\langle P_\alpha : \alpha < \delta\rangle$)

(b) No new ω-sequences of ordinals are added in stage δ, i.e., whenever $p \in P_\delta$, $\underset{\sim}{\tau}$ a P_δ-name and $p \Vdash_{P_\delta}$ “$\underset{\sim}{\tau} : \omega \to \text{Ord}$”, *then* there is an $\alpha < \delta$, a P_α-name $\underset{\sim}{\tau}^*$ and a condition $q \geq p$ such that $q \Vdash$ “$\underset{\sim}{\tau} = \underset{\sim}{\tau}^*$”.

(2) *If* $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \kappa\rangle$ is an RCS iteration, κ a strongly inaccessible cardinal, P_κ does not collapse $\aleph_1$, and for all $\alpha < \kappa$ we have $|P_\alpha| < \kappa$, *then* P_κ satisfies the κ-chain condition.

Proof. (1a) Let $p \in P_\delta$, $\underset{\sim}{\zeta}^*$ as in 1.12(4), and let $q \geq p$ decide the value of $\underset{\sim}{\zeta}^*$, say $q \Vdash_{P_\delta}$ “$\underset{\sim}{\zeta}^* = \xi$”. Then $q{\restriction}(\xi + 1) \in P_{\xi+1}$ is essentially stronger than p.

(1b) Not hard.

(2) Easy, since we take direct limits on the stationary set $S_1^\kappa = \{\delta < \lambda : \text{cf}(\delta) = \aleph_1\}$, by (1)(a). $\square_{1.13}$

§2. The Condition

In this section we get to the heart of this chapter, the definition of our condition for forcing notions. We need some preliminary definitions.

2.1 Definition. A tagged tree (or an ideal tagged tree) is a pair $(T, \mathbf{I})$ such that:

(A) T is a tree i.e., a nonempty set of finite sequences of ordinals such that if $\eta \in T$ then any initial segment of η belongs to T; here with no maximal nodes if not said otherwise. T is partially ordered by initial segments, i.e., $\eta \lhd \nu$ iff η is an initial segment of ν.

(B) $\mathbf{I}$ is a function with domain including T such that for every $\eta \in T : \mathbf{I}(\eta)$ $(\stackrel{def}{=} \mathbf{I}_\eta)$ is an ideal of subsets of some set called the domain of $\mathbf{I}_\eta$, and $\mathrm{Suc}_T(\eta) \stackrel{def}{=} \{\nu : \nu \text{ is an immediate successor of } \eta \text{ in } T\} \subseteq \mathrm{Dom}(\mathbf{I}_\eta)$.

(C) For every $\eta \in T$ we have $\mathrm{Suc}_T(\eta) \neq \emptyset$ and above each $\eta \in T$ there is some $\nu \in T$ such that $\mathrm{Suc}_T(\nu) \notin \mathbf{I}_\nu$.

2.1A Convention. For any tagged tree $(T, \mathbf{I})$ we can define $\mathbf{I}^\dagger$, $\mathbf{I}^\dagger_\eta = \{\{\alpha : \eta\hat{\ }\langle\alpha\rangle \in A\} : A \in \mathbf{I}_\eta\}$; we sometimes, in an abuse of notation, do not distinguish between $\mathbf{I}$ and $\mathbf{I}^\dagger$; e.g. if $\mathbf{I}^\dagger_\eta$ is constantly I^*, we write I^* instead of $\mathbf{I}$. Sometimes we also write $\mathrm{Suc}_T(\eta)$ for $\{\alpha : \eta\hat{\ }\langle\alpha\rangle \in T\}$.

2.2 Definition. η will be called a splitting point of $(T, \mathbf{I})$ if $\mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta$ (just like ν in (3) above). Let $\mathrm{sp}(T, \mathbf{I})$ be the set of splitting points of $(T, \mathbf{I})$.

We call $(T, \mathbf{I})$ normal if $\eta \in T \setminus \mathrm{sp}(T, \mathbf{I}) \Rightarrow |\mathrm{Suc}_T(\eta)| = 1$ (we may forget to demand this).

2.3 Definition. We now define orders between tagged trees:

(a) $(T_2, \mathbf{I}_2) \leq (T_1, \mathbf{I}_1)$ if $T_1 \subseteq T_2$ and whenever $\eta \in T_1$ is a splitting point of T_1 then $\mathrm{Suc}_{T_1}(\eta) \notin \mathbf{I}_2(\eta)$ and $\mathbf{I}_1(\eta){\restriction}\mathrm{Suc}_{T_1}(\eta) = \mathbf{I}_2(\eta){\restriction}\mathrm{Suc}_{T_1}(\eta)$ (where $I{\restriction}A = \{B : B \subseteq A \text{ and } B \in I\}$) and $\mathrm{Dom}(I{\restriction}A) = A$.

(b) $(T_2, \mathbf{I}_2) \leq^* (T_1, \mathbf{I}_1)$ iff $(T_2, \mathbf{I}_2) \leq (T_1, \mathbf{I}_1)$ and every $\eta \in T_1$ which is a splitting point of T_2 is a splitting point of T_1 as well.

2.3A Notation. We omit $\mathbf{I}_1$ and denote a tagged tree by T_1 whenever $\mathbf{I}_\eta = \{A \subseteq \mathrm{Suc}_T(\eta) : |A| < |\mathrm{Suc}_T(\eta)| \text{ if } |\mathrm{Suc}_T(\eta)| > \aleph_0, \text{ and } A = \emptyset \text{ if } |\mathrm{Suc}_T(\eta)| \leq \aleph_0\}$ for every $\eta \in T$.

2.4 Definition.

(1) For a set $\mathbb{I}$ of ideals, a tagged tree $(T, \mathbf{I})$ is an $\mathbb{I}$-tree if for every $\eta \in T, \mathbf{I}_\eta \in \mathbb{I}$ (up to an isomorphism) or $|\mathrm{Suc}_T(\eta)| = 1$.

(2) For a set S of regular cardinals, T is called an S-tree if for some $\mathbf{I}$, $(T, \mathbf{I})$ is an $\mathbb{I}_S$-tagged tree where $\mathbb{I}_S = \{\{A \subseteq \lambda : |A| < \lambda\} : \lambda \in S\}$

2.5 Definition.

(1) For a tree T, lim$T$ is the set of all ω-sequences of ordinals, such that every finite initial segment of them is a member of T. The set $\lim T$ is also called the set of "branches" of T.

(2) A subset J of a tree T is a front if $\eta, \nu \in J$ implies none of them is an initial segment of the other, and every $\eta \in$ lim$T$ has an initial segment which is a member of J.

2.6 Main Definition. Let S be a set of regular cardinals; we say that a forcing notion P satisfies the S-condition if there is a function F with values of the "right" forms, so that for every S-tree T:

if f is a function $f : T \to P$ satisfying

(a) $\nu \lhd \eta$ implies $f(\nu) \leq_P f(\eta)$ and

(b) there are fronts $J_n (n < \omega)$ (of T) such that $\bigcup_{n<\omega} J_n = \mathrm{sp}(T, I)$, every member of J_{n+1} has a proper initial segment belonging to J_n and $\eta \in J_n$ implies

$$\langle \mathrm{Suc}_T(\eta), \langle f(\nu) : \nu \in \mathrm{Suc}_T(\eta) \rangle \rangle = F(\eta, w[\eta], \langle f(\nu) : \nu \trianglelefteq \eta \rangle)$$

($w[\eta]$ is defined below) and $\mathrm{Suc}_T(\eta) = \{\eta\hat{}\langle\alpha\rangle : \alpha < \lambda\}$ for some $\lambda \in S$ (for simplicity).

then for every $T^\dagger$, $T \leq^* T^\dagger$ there is some $p \in P$ such that $p \Vdash_P$ "$\exists\eta \in \lim T^\dagger$ such that $\forall k < \omega$ we have $f(\eta{\restriction}k) \in \underset{\sim}{G}_P$" where $\underset{\sim}{G}_P$ is the P-name of the generic subset of P; note that in general η is not from V, i.e. it may be a branch which forcing by P adds.

2.6A Explanation. First for the notation: $w[\eta] = \{k < \ell g(\eta) : \eta{\restriction}k \in \bigcup_{\ell<\omega} J_\ell\}$.

Now for the meaning: One can regard the situation as a kind of a *game.* There are two players. In ω many steps they define a tree T and an increasing function $f : T \to P$. In the n'th move, player I defines an initial segment T_n of the tree T (so T_n will be the set of nodes up to some member of the front J_n) and a function $f_n : T_n \to P$ which is increasing such that $m < n \Rightarrow f'_m \subseteq f_n$ (see below). Player II end-extends the tree T_n to a tree T'_n by adding successors to each leaf (=node without successor) in T_n and extends f_n to a function f'_n on T'_n. Then player I plays T_{n+1} (an end extension of T'_n with no infinite branches), and a function f_{n+1} ($\supseteq f'_n$), etc. Finally, $T = \bigcup_n T_n$, $f = \bigcup_n f_n$. Player II wins a play if for all $T^\dagger$: if $T \leq^* T^\dagger$, then there is $p \in P$ such that $p \Vdash_P$ "$(\exists\eta \in \lim T^\dagger)(\forall\kappa < \omega) f(\eta{\restriction}k) \in \underset{\sim}{G}_P$". P satisfies the S-condition if there is a winning strategy F which at each point η depends only on what happened so far on the nodes below η.

However F, the "winning strategy" of player II, has only partial memory.

Remark. It does not matter if we require $\bigcup_n J_n = \mathrm{sp}(T, I)$ or $\bigcup_n J_n \subseteq \mathrm{sp}(T, I)$, or equivalently whether we allow player I to play any end extension T_n of T'_{n-1} or only end extensions with no new splitting points.

Remark. (1) If P is a dense subset of Q, then P has the $\mathbb{I}$-condition (see 2.7 below) iff Q has it.

(2) If $P \lessdot Q$, and Q has the $\mathbb{I}$-condition, then also P has the $\mathbb{I}$-condition.

The proof of (2) uses the fact that if: $f : T \to P$, then the existence of a branch in $\{\eta \in T : f(\eta) \in G_p\}$ is absolute between the universes V^P and V^Q.

2.6B Convention.

(1) In Definition 2.6, the value F gives to $\mathrm{Suc}_T(\eta)$ is w.l.o.g. $\{\eta\hat{\ }\langle\alpha\rangle : \alpha < \lambda\}$ for some λ, and we do not strictly distinguish between λ and $\mathrm{Suc}_T(\eta)$.

(2) The domain of F consists of triples of the form $\langle\eta, w, f\rangle$, where η is a finite sequence of ordinals, $w \subseteq \mathrm{Dom}(\eta)$, and f is an increasing function from $\{\eta\restriction k : k \leq \ell g(\eta)\}$ into P. The value $F(\eta, w, f)$ has two components: The first is of the form $\{\eta\hat{\ }\langle i\rangle : i \in A\}$ for some set A of ordinals (by (1), without loss of generality $A = |A|$) and the second component is a family of elements of P above $f(\eta)$, indexed by the first component.

When we define such a function F, we usually call the first component "$\mathrm{Suc}_T(\eta)$" (here "T" is just a label, not an actual variable), and we write the second component as $f\restriction\mathrm{Suc}_T(\eta)$ or $\langle f(\nu) : \nu \in \mathrm{Suc}_T(\eta)\rangle$ (i.e. we use the same variable "f" that appears in the input of F).

2.7 Definition. For a set $\mathbb{I}$ of ideals we define similarly when does a forcing notion P satisfies the $\mathbb{I}$-condition (the only difference is dealing with $\mathbb{I}$-trees instead of S-trees), so now

$$\langle \mathrm{Suc}_T(\eta), \mathbf{I}_\eta, \langle f(\nu) : \nu \in \mathrm{Suc}_T(\eta)\rangle\rangle = F(\eta, w[n], f\restriction\{\nu : \nu \trianglelefteq \eta\})$$

and $\mathrm{Suc}_T(\eta) = \mathrm{Dom}(\mathbf{I}_\eta)$. We allow ourselves to omit $\mathrm{Suc}_T(\eta)$ when it is well understood. (We can let the function depend on $\mathbf{I}_\nu(\nu \triangleleft \eta)$ too).

2.7A Remark. If I is restriction closed (i.e. $I \in \mathbb{I}$, $A \subseteq \mathrm{Dom}(I)$, $A \notin I$ then $I\restriction A \in \mathbb{I}$ at least for some $B \subseteq A$, $B \notin I^+$ and $J \in \mathbb{I}$ we have $I\restriction B \cong J$) *then* we can weaken the demand to

$$\mathrm{Suc}_T(\eta) \subseteq \mathrm{Dom}(\mathbf{I}_\eta), \quad \mathrm{Suc}_T(\eta) \notin I_\eta).$$

§3. The Preservation Properties Guaranteed by the S-Condition

3.1 Definition.

(1) An ideal I is λ-complete if any union of less than λ members of I is still a member of I.

(2) A tagged tree $(T, \mathbf{I})$ is λ-complete if for each $\eta \in T$ the ideal $\mathbf{I}_\eta$ is λ-complete.

(3) A family $\mathbb{I}$ of ideals is λ-complete if each $I \in \mathbb{I}$ is λ-complete.

3.2 Theorem. (CH) If P is a forcing notion satisfying the $\mathbb{I}$-condition for an $\aleph_2$-complete $\mathbb{I}$ *then* forcing with P does not add reals.

As an immediate conclusion we get:

3.3 Theorem. (CH) If P is a forcing notion satisfying the S-condition for a set S of regular cardinals greater than $\aleph_1$ *then* forcing with P does not add reals.

The main tool for the proof of the theorem is the combinatorial Lemma 3.5 from [RuSh:117], for which we need a preliminary definition. More on such theorems and history see Rubin and Shelah [RuSh:117].

3.4 Definition. We define a topology on $\lim T$ (for any tree T) by defining for each $\eta \in T$ the set $T_{[\eta]} = \{\nu : \eta \trianglelefteq \nu \text{ or } \nu \trianglelefteq \eta\}$ and letting $\{\lim T_{[\eta]} : \eta \in T\}$ generate the family of open subsets of $\lim T$ (so each such set $\lim(T_{[\eta]})$ is also closed and is called basic open, and an open subset is an arbitrary union of basic open sets). The family of Borel sets is the σ-algebra generated by the open sets.

3.5 Lemma. 1) If $(T, \mathbf{I})$ is a λ^+-complete tree and H is a function from $\lim T$ to λ such that for every $\alpha < \lambda$ the set $H^{-1}(\{\alpha\})$ is a Borel subset of $\lim T$ (in the topology that was defined in Definition 3.4) *then* there is a tagged subtree $(T^\dagger, \mathbf{I})$, $(T, \mathbf{I}) \leq^* (T^\dagger, \mathbf{I})$ (see 2.3(b)) such that H is constant on $\lim T^\dagger$.

2) In part (1) we can let H be multivalued, i.e. assume $\lim(T)$ is $\bigcup_{\alpha<\lambda} H_\alpha$, each H_α is a Borel subset of $\lim(T)$. If $(T, \mathbf{I})$ is λ^+-complete *then* there is $(T^\dagger, \mathbf{I})$ such that $(T, \mathbf{I}) \leq^* (T^\dagger, \mathcal{I})$ and for some α we have $\lim(T^\dagger) \subseteq H_\alpha$.

Proof. 1) First note that *if* $T_1 \subseteq T$ is such that: $\langle\rangle \in T_1$; for every $\eta \in T_1$ if η is a splitting point of $(T, \mathbf{I})$ then $\mathrm{Suc}_{T_1}(\eta) = \mathrm{Suc}_T(\eta)$ and if η is not a splitting point of T then $|\mathrm{Suc}_{T_1}(\eta)| = 1$, *then* $(T, \mathbf{I}) \leq^* (T_1, \mathbf{I}\restriction T_1)$, so w.l.o.g. we can assume that in T every point is either a splitting point or it has only one immediate extension.

For each $\alpha < \lambda$ let us define a game $\eth_\alpha$: in the first move player I chooses the node η_0 in the tree such that $\ell g(\eta_0) = 0$, player II responds by choosing a proper subset A_0 of $\mathrm{Suc}_T(\eta_0)$ such that $A_0 \in \mathbf{I}_{\eta_0}$, in the n-th move player I chooses an immediate extension of η_{n-1}, η_n such that $\eta_n \notin A_{n-1}$ or η_{n-1} is not a splitting point of $(T, \mathbf{I})$, and player II responds by choosing $A_n \in \mathbf{I}_{\eta_n}$.

Player I wins if for the infinite branch η defined by $\eta_0, \eta_1, \eta_2, \dots$ we have $H(\eta) = \alpha$. By the assumption of the lemma this is a Borel game so by Martin's Theorem, [Mr75] one of the players has a winning strategy. We claim that there is some $\alpha < \lambda$ for which player I has a winning strategy in the game $\eth_\alpha$. Assume otherwise, i.e., for every $\alpha < \lambda$ player II has a winning strategy F_α. We construct an infinite branch inductively: let $\eta_0 = \langle\rangle$, $\eta_0 \in T$. At stage n let A_n be $\bigcup_{\alpha<\lambda} F_\alpha(\eta_0, \eta_1, \dots, \eta_{n-1})$; now if η_{n-1} is a splitting point (of $(T, \mathbf{I})$) then $\mathbf{I}_{\eta_{n-1}}$ is λ^+-complete and each $F_\alpha(\eta_0, \dots, \eta_{n-1})$ is a member of it, hence $A_n \in \mathbf{I}_{\eta_{n-1}}$, so clearly $\mathrm{Suc}_T(\eta_{n-1}) \nsubseteq A_n$.

If η_{n-1} is not a splitting point it has only one immediate successor and let it by η_n, otherwise since $\mathrm{Suc}(\eta_{n-1}) \notin I_{\eta_{n-1}}$, $A_n \in I_{\eta_{n-1}}$, we have $(\mathrm{Suc}(\eta_{n-1}) \setminus A_n) \neq \emptyset$ so we choose $\eta_n \in (\mathrm{Suc}_T(\eta_{n-1}) \setminus A_n)$. Let $\eta = \bigcup_{n<\omega} \eta_n$ be the infinite branch that we define by our construction and let $\alpha(*) = H(\eta)$. Now in the game $\eth_{\alpha(*)}$ if player I will choose η_n at stage n (for all n) and player II will play by $F_{\alpha(*)}$, player I will win although player II has used his winning strategy $F_{\alpha(*)}$, contradiction.

So there must be $\alpha(*)$ such that player I has a winning strategy $H_{\alpha(*)}$ for $\eth_{\alpha(*)}$ and let $T^\dagger$ be the subtree of T defined by $\{\eta : \langle \eta\restriction 0, \dots, \eta\restriction(n-1)\rangle$ are the

first n moves of player I in a play in which he plays according to $H_{\alpha(*)}\}$. Now for $\eta \in T^\dagger \cap \mathrm{sp}(T)$, let $A = \mathrm{Suc}_{T^\dagger}(\eta)$. Then $A \notin I_\eta$, otherwise player II could have played it as A_n. So $T \leq^* T^\dagger$, and $T^\dagger$ is as required.
2) Same proof replacing $H^{-1}(\{\alpha\})$ by H_α so $H(\eta) = \alpha$ by $\eta \in H_\alpha$. $\square_{3.5}$

3.5A Corollary. If $(T, \mathbf{I})$ is a λ^+-complete tree, and g is a function from T into λ, and $\lambda^{\aleph_0} = \lambda$, *then* there is a tagged subtree $(T^\dagger, I)$, $(T, I) \leq^* (T^\dagger, I)$ such that $g \restriction T^\dagger$ depends only on the length of its argument, i.e. for some function $g^\dagger : \omega \to \lambda$, for all $\eta \in T^\dagger$, $g(\eta) = g^\dagger(\ell g(\eta))$. $\square_{3.5A}$

Proof of Theorem 3.2. Let $\underset{\sim}{\tau}$ be a name of a real in $V[P]$ and $p_0 \in P$ and we will find a condition $p \in P$ forcing $\underset{\sim}{\tau}$ to be equal to a real from V and $p_0 \leq p$. Let f, $\langle T, \mathbf{I} \rangle$ be such that $\mathrm{Rang}(\mathbf{I}) \subseteq \mathbb{I}$, $f : T \to P$ and be defined as follows: we define by induction on k, for a sequence η of ordinals of length k, the truth value of $\eta \in T$, $f(\eta)$, and then $\mathbf{I}_\eta$. We let $\langle\rangle \in T, f(\langle\rangle) = p_0$. For $\eta \in T$ of even length $2k$, we use F from the definition of the $\mathbb{I}$-condition, to define $\mathrm{Suc}_T(\eta), \mathbf{I}_\eta, f \restriction \mathrm{Suc}_T(\eta)$. For $\eta \in T$ of length $2k+1$, we let $\mathrm{Suc}_\eta(T) = \{\eta\hat{\ }\langle 0 \rangle\}$, and we define $f(\eta\hat{\ }\langle 0 \rangle)$ such that it will be an extension of the value of f on its predecessor and such that $f(\eta)$ forces a value for $\underset{\sim}{\tau}(k)$ (the k'th place of the real that $\underset{\sim}{\tau}$ names).

We continue by defining $H : \lim T \to \mathbb{R}^V$ (as we assume C.H. clearly $|\mathbb{R}^V| = \aleph_1$, so it is just like a function from T to ω_1) by letting $H(\eta)(k) =$ the value forced by $f(\eta \restriction (2k+1))$ for $\underset{\sim}{\tau}(k)$. By Lemma 3.5 there is $(T^\dagger, \mathbf{I})$, $(T, \mathbf{I}) \leq^* (T^\dagger, \mathbf{I})$ on which H is constant, now let p be the forcing condition that by Definition 2.6 forces "$\exists \eta \in \lim T^\dagger$ such that $\forall k [f(\eta \restriction k) \in \underset{\sim}{G}_P]$". This p forces $\underset{\sim}{\tau}$ to equal the constant value of H on $T^\dagger$ which is a member of V, and $p \Vdash$ "$p_0 \in \underset{\sim}{G}_P$". $\square_{3.2}$

3.6 Theorem. If P is a forcing notion satisfying the S-condition for a set of regular cardinals S and $\aleph_1 \notin S$ *then* forcing by P does not collapse $\aleph_1$.

Remark. 1) Note that this is stronger than 3.3, as we do not assume C.H. and that we allow $\aleph_0 \in S$. The proof is quite similar to the proof of Theorem 3.2 but here we use a somewhat different combinatorial lemma. Note also that we shall not use this theorem;

2) We can generalize 3.2 and 3.6 to $\mathbb{I}$-condition when $\mathbb{I}$ is $\aleph_2$-complete, striaght-forwardly, see 3.8.

3.7 Lemma. Let $(T, \mathbf{I})$ be such that for some regular uncountable λ, for every $\eta \in T$ either $\mathbf{I}_\eta$ is λ^+-complete or $|\mathrm{Suc}_T(\eta)| < \lambda$, *then* for every $H : T \to \lambda$ satisfying $\{\eta \in \lim T : H(\eta) < \alpha\}$ is a Borel subset of $\lim T$ for any successor $\alpha < \lambda$, there is $\alpha < \lambda$ and $(T', \mathbf{I})$, $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ such that for all $\eta \in T'$ we have $H(\eta) < \alpha$, and for all η in T', if $|\mathrm{Suc}_T(\eta)| < \lambda$, then $\mathrm{Suc}_{T'}(\eta) = \mathrm{Suc}_T(\eta)$.

Proof of the lemma. We define for each successor $\alpha < \lambda$ a game $\eth_\alpha$ very much like the way we did it for proving Lemma 3.5, the only difference being that if $|\mathrm{Suc}_T(\eta_n)| < \lambda$ player II chooses A_n such that $|\mathrm{Suc}_T(\eta_n) \setminus A_n| = 1$ (otherwise player II chooses $A_n \in \mathbf{I}_{\eta_n}$ just like in 3.3); player I wins if for every $n < \omega$ $H(\eta_n) < \alpha$. Here again the game $\eth_\alpha$ is determined for every α (here simply because if player II wins a play he does so at some finite stage). Again we claim that there should be at least one α for which player I has a winning strategy. Assume the contrary and let F_α be player's II winning strategy for each $\alpha < \lambda$. We construct a subtree T^* deciding by induction on the height of the members of T which of them are the members of T^*. For η that is already in T^*, if $|\mathrm{Suc}_T(\eta)| < \lambda$ we include all the members of $\mathrm{Suc}_T(\eta)$ in T^*; otherwise $\mathbf{I}_\eta$ is λ^+-complete so $\mathrm{Suc}_T(\eta) \setminus \bigcup_{\alpha<\lambda} F_\alpha(\eta{\restriction}0, \eta{\restriction}1, \ldots, \eta)$ is not empty, so we pick one extension of η from this set and the rest of $\mathrm{Suc}_T(\eta)$ will not be in T^*. Now T^* is a tree of height ω branching to less than λ successors at each point, so as λ is regular uncountable $|T^*| < \lambda$ and there is some $\alpha^* < \lambda$ such that $\eta \in T^*$ implies $H(\eta) < \alpha^*$. Regarding the game $\eth_{\alpha^*}$, there is a play of it in which player I chooses all along the way members of T^* and player II plays according to F_{α^*}, of course player I wins this game contradicting the assumption that F_{α^*} is a winning strategy for player II.

We define T' just like we did in the proof of Lemma 3.5, collecting all the initial segments of plays of player I in the game ∂_{α^*} when he plays according to his winning strategy H_{α^*}. $\square_{3.7}$

Proof of Theorem 3.6. Just like in the proof of 3.2, having a name $\underset{\sim}{\tau}$ of a function in $V[P]$ mapping ω into ω_1 we take an S-tree T, define a function $h : T \to P$ using the F in odd stages and in even stages forcing more and more values for $\underset{\sim}{\tau}$. Using 3.7 with $\lambda = \aleph_1$, we get a condition $p \in P$ forcing the function that $\underset{\sim}{\tau}$ names to be bounded below ω_1, so we are done. $\square_{3.6}$

Similarly we can prove:

3.8 Theorem. If P satisfies the $\mathbb{I}$-condition, and λ is regular uncountable and $(\forall I \in \mathbb{I})$ $[|\bigcup I| < \lambda$ or I is λ^+-complete$]$ *then* $\Vdash_P$ "$\mathrm{cf}(\lambda) > \aleph_0$". If $(\forall I \in \mathbb{I})$ $[I$ is λ^+-complete$]$ and $\lambda = \lambda^{\aleph_0}$ *then* P adds no new ω-sequence from λ.

3.8 Warning. The statement "in $V^P, \underset{\sim}{Q}$ satisfies the S-condition" may be interpreted as "in $V^P, \underset{\sim}{Q}$ satisfies the $\mathbb{I}$-condition" in two ways:

(a) $\mathbb{I} = \{\{A \in V^P : A \subseteq \lambda, V^P \models |A| < \lambda\} : \lambda \in S\}$

(b) $\mathbb{I} = \{\{A \in V : A \subseteq \lambda, V \models |A| < \lambda\} : \lambda \in S\}$

Note that $I \in \mathbb{I}$ is identified with the ideal it generates.

However the two interpretations are equivalent if P satisfies the λ-chain condition (or is λ^+-complete) for each $\lambda \in S$ (and even weaker conditions) [and this will be the case in all our applications.]

§4. Forcing Notions Satisfying the S-Condition

4.1 Definition. Namba forcing Nm is the set $\{T : T$ is a tagged $\{\aleph_2\}$-tree, such that for every $\eta \in T$, for some ν, $\eta \trianglelefteq \nu \in T$ and $|\mathrm{Suc}_T(\nu)| = \aleph_2\}$ with the order $T \leq T'$ iff $T \supseteq T'$ (see 2.4); so smaller trees carry more information and we identify T and $(T, I^{bd}_{\omega_2})$, $I^{bd}_{\omega_2}$ is the ideal of bounded subsets of ω_2. We will write $\underset{\sim}{h}$ or $\underset{\sim}{h}_{Nm}$ for the generic branch added by Nm, i.e. $\Vdash$"$\underset{\sim}{h} : \omega \rightarrow \omega_2^V$", and for every p in the generic filter, $\mathrm{tr}(p) \overset{\mathrm{def}}{=} \mathrm{trunk}(p) \subseteq h$", where the trunk of T is $\eta \in T$ of maximal length such that $\nu \in T$ & $\ell g(\nu) \leq \ell g(\eta) \Rightarrow \nu \trianglelefteq \eta$. We can restrict ourselves to normal members: T such that $\eta \in T \Rightarrow |\mathrm{Suc}_T(\eta)| \in \{1, \aleph_2\}$. For I an ideal of ω_2, Nm(I) is defined similarly (not used in this chapter).

4.1A Claim. Nm changes the cofinality of $\aleph_2$ to $\aleph_0$ ($\underset{\sim}{h}_{Nm}$ exemplifies this).

Remark. In X 4.4 (4) the variant of Namba forcing Nm′ is the set of all trees of height ω such that each tree has a node, the trunk such that below its level the tree-order is linear and above it each point has $\aleph_2$ many immediate successors, the order is inversed inclusion. Namba introduces Nm in [Nm]. Both forcing notions add a cofinal ω-sequence to ω_2 (Nm by 4.1A, Nm′ by X 4.7(2)) without collapsing $\aleph_1$, and (if CH holds) neither of them adds reals (Nm by 4.4, Nm′ by 4.7(1), (3)), but they are not the same.

4.2 Claim. (Magidor and Shelah). Assume CH. If $h[h']$ is a Namba sequence [Namba′-sequence] *then* in $V[h]$ we cannot find a Namba′-sequence over V, nor can we in $V[h']$ find a Namba-sequence over V.

Proof. Trivially we can in Nm and Nm′ restrict ourselves to conditions which are trees consisting of strictly increasing finite sequences of ordinals. First we look in $V[h']$, let $\underset{\sim}{h}'$ be the Nm′-name of the Namba′ sequence, and let $\underset{\sim}{f}$ be a Nm′-name of an increasing function from ω to ω_2. Let $T^0 \in \mathrm{Nm}'$ and suppose $T^0 \Vdash_{Nm'}$ "Sup Rang$(\underset{\sim}{f}) = \omega_2^V$". Now it is easy to find T^1 in Nm′, $T^1 \geq T^0$, such that for each $m, n < \omega$ the truth values of "$\underset{\sim}{f}(n) = \underset{\sim}{h}'(m)$", and "$\underset{\sim}{f}(n) < \underset{\sim}{h}'(m)$" are determined by T^1 (i.e., forced), (possible by X 4.7(1), as forcing by Nm′ does not add reals.)

Let $A = \{k < \omega$: for some m, T^1 forces that for every $i < k$ we have $\underset{\sim}{f}(i) < \underset{\sim}{h}'(m) \leq \underset{\sim}{f}(k)\}$, so A is an infinite subset of ω, in V, and let $A = \{k_\ell : \ell < \omega\}$, $k_0 < k_1 < k_2 < \dots$ and there are $\langle m_\ell : \ell < \omega\rangle$ such that $m_\ell < m_{\ell+1} < \omega$ and $T^1 \Vdash$ "$\underset{\sim}{f}(i) < \underset{\sim}{h}'(m_\ell) \leq \underset{\sim}{f}(k_\ell)$ for $i < k_\ell$ and $\ell < \omega$". Now

$(*)$ $T^1 \Vdash_{\mathrm{Nm}'}$ " for every $F \in V$, an increasing function from ω_2 to ω_2, there is $\ell_0 < \omega$ such that for every $\ell > \ell_0 + 3$, $\underset{\sim}{f}(k_\ell) > F(\underset{\sim}{f}(k_{\ell-2}))$".

Why? This is because for every $F \in V$ (as above, without loss of generality strictly increasing) and $T^2 \geq T^1$ (in Nm') if ℓ_0 is the length of the trunk of T^2 then for some $T^3 \geq T^2$;

$(**)$ $T^3 \Vdash_{\mathrm{Nm}'}$ " if $\ell > \ell_0 + 3$ then $\underset{\sim}{f}(k_\ell) > F(\underset{\sim}{f}(k_{\ell-2}))$"

Why? Simply choose $T^3 = \{\eta \in T^2 : \text{if } \ell g(\eta) \geq \ell + 1 > \ell_0 \text{ then } \eta(\ell) > F(\eta(\ell - 1))\}$, clearly $T^3 \in \mathrm{Nm}', T^3 \geq T^2$ and T^3 satisfies $(**)$.

So $(*)$ holds, but it exemplifies $\underset{\sim}{f}$ is not a Nm-sequence, i.e., $T^1 \Vdash_{\mathrm{Nm}'}$ "$\underset{\sim}{f}$ is not a Namba-sequence". [Why? Because $\Vdash_{\mathrm{Nm}}$ " for some function $F \in V$ from ω_2 to ω_2 for arbitrarily large $\ell < \omega$ we have $\underset{\sim}{h}(k_\ell) < F(\underset{\sim}{h}(k_{\ell-2}))$" as if $T \in \mathrm{Nm}$ and for simplicity each $\eta \in T$ is strictly increasing we let $F : \omega_2 \to \omega_2$ be such that $F(\alpha) = \min\{\delta : \text{if } \eta \in T \cap {}^{\omega>}\delta \text{ then for some } \nu,\ \eta \triangleleft \nu \in T \cap {}^{\omega>}\delta\}$, and let $T' = \{\eta \in T$: if $\ell < \ell g(\eta)$ and $\eta{\restriction}\ell \in \mathrm{sp}(T)$ and $|\{m < \ell : \eta{\restriction}m \in \mathrm{sp}(T)\}| \in \bigcup\{[k_{10i}, k_{10i+5}) : i < \omega\}$ then $\eta(\ell) < F(\eta(\ell - 1))\}$, and T' forces the failure.] So we have proved one half of 4.2.

Now let us prove the second assertion in the claim, i.e., let $\underset{\sim}{f}$ be a Nm-name of an increasing function from ω to ω_2, and we shall prove that it is forced, not to be generic for Nm' so assume $\Vdash_{\mathrm{Nm}}$ " $\bigcup\limits_{m<\omega} \underset{\sim}{f}(n) = \omega_2$". Clearly this is enough.

Let $T \in \mathrm{Nm}$, then we can find $T^0 \geq T, T^0 \in \mathrm{Nm}$ (normal) such that for every splitting point η of T_0 and $\nu = \eta\hat{\ }\langle\alpha\rangle \in T^0$:

1) for some n_ν, $T^0_{[\nu]} \Vdash_{Nm}$ "$n_\nu = \mathrm{Min}\{\ell : \underset{\sim}{f}(\ell) > \mathrm{Max\,Rang}(\nu)\}$,

2) for some γ_ν, $T^0_{[\nu]} \Vdash$ "$\underset{\sim}{f}(n_\nu) = \gamma_\nu$"

3) if ρ_ν is the trunk of $T^0_{[\nu]}$ then $(\forall\beta)[\rho_\nu\hat{\ }\langle\beta\rangle \in T^0 \to \beta > \gamma_\nu]$.

If n_ν is not defined let $n_\nu = \omega$ (this occurs if $\nu \notin \mathrm{sp}(T)$.

Now by 3.5A there is T^1, $T^0 \leq^* T^1$ (in Nm) and $n_\ell (\ell < \omega)$ such that $n_\eta = n_{\iota_g(\eta)}$ for every $\eta \in T^1$. Let $\{\ell_i : i < \omega\}$ be a list of $\{\ell < \omega : n_{\ell+1} \neq \omega\}$, such that $\ell_i < \ell_{i+1}$, so $\eta \in T^1$, $\ell g(\eta) = \ell_i$ implies η is a splitting point of T^1. Note that if $\eta \in T^1$, $\ell_{i+1} \in \mathrm{Dom}(\eta)$, then $T^1_{[\eta]} \Vdash$ "$\eta(\ell_i) < f(n_{\ell_i}) < \eta(\ell_{i+1})$". Let $T^2 = \{\eta \in T^1 : \text{if } \ell_{2i} < \ell g(\eta), \text{ then } \eta(\ell_{2i}) = \mathrm{Min}\{\alpha : (\eta{\restriction}\ell_{2i})\hat{\ }\langle\alpha\rangle \in T^1\}$, $F(\alpha) = \mathrm{Min}\{\gamma : (\forall k < \omega)\ (\forall \nu \in T^2 \cap {}^{\omega>}\alpha)(\exists \rho \in {}^k\gamma)(\nu\hat{\ }\rho \in T^2)\}$. So F is

nondecreasing and $T^2_{[\eta]} \Vdash$ "$\underset{\sim}{f}(n_{\ell_{2i}}) \leq \eta(\ell_{2i+1}) < F(\eta(\ell_{2i-1})) \leq F(\underset{\sim}{f}(n_{\ell_{2i-1}}))$" for $i \in (0, \omega)$.
Let $A \overset{\text{def}}{=} \{n_{\ell_{2i}} : 0 < i < \omega\}$.

Then $T^2 \in \mathrm{Nm}, T^2 \geq T^1$, and $F \in V$ is a function from ω_2 to $\omega_2, A \in V$ an infinite subset of ω and

$T^2 \Vdash_{Nm}$ "for every $n \in A, \underset{\sim}{f}(n) < F(\underset{\sim}{f}(n-1))$".

This shows that (T^2 forces) $\underset{\sim}{f}$ is not a Nm'-sequence. $\square_{4.2}$

4.3 Claim.

(1) Nm', Nm do not satisfy the $2^{\aleph_0}$-chain condition.

(2) It is consistent with ZFC that $2^{\aleph_0} = \aleph_1, 2^{\aleph_1}$ arbitrarily large and Nm, Nm' satisfies the $\aleph_4$-c.c.

4.3A Remark. The proof of (2) is inspired by the proof of Baumgartner of the consistency of: ZFC + $2^{\aleph_0}$ arbitrarily large + "there is no set of $\aleph_3$ subsets of $\aleph_1$ with pairwise countable intersection". Thinking a minute the close connection between the problems should be apparent. The other ingredient is the Δ-system theorem on trees from Rubin and Shelah (again see [RuSh:117]).
Note that Nm, Nm' necessarily colapse $\aleph_3$ (see [Sh:g, VII 4.9]) so 4.3(2) is best possible.

Proof. (1) For every real η (i.e. $\eta \in {}^{\omega}2$), let $T_\eta = \{\nu : \nu$ a finite sequence of ordinals $< \omega_2$, and $n < \ell g(\nu) \Rightarrow \nu(n) + \eta(n)$ is an even ordinal$\}$.

Clearly $T_\eta \in \mathrm{Nm}$ and $T_\eta \in \mathrm{Nm}'$, and the T_η's are pairwise incompatible (in Nm and in Nm') and there are $2^{\aleph_0}$ such T_η's.

(2) Let V satisfy G.C.H. $\kappa > \aleph_2$ and $P = \{f : f$ a countable function from κ to $\{0,1\}\}$ ordered by inclusion. Suppose in $V^P, \underset{\sim}{Q}$ is Nm or Nm', and it does not satisfy the $\aleph_4$-chain condition. So there is $p_0 \in P$ and P-names $\underset{\sim}{T}_i (i < \aleph_4)$ such that $p_0 \Vdash_P$ "each $\underset{\sim}{T}_i$ belongs to $\underset{\sim}{Q}$ (for $i < \aleph_4$) and they are pairwise incompatible in $\underset{\sim}{Q}$, $\underset{\sim}{Q}$ is Nm or Nm'". Without loss of generality $p_0 \Vdash$ " if $\underset{\sim}{Q} = \mathrm{Nm}'$, then every T_i has trunk $\langle\rangle$".

For each i we can now find a tree of conditions p^i_η deciding higher and higher splitting points of $\underset{\sim}{T}_i$. Specifically, we will define $A^i \subseteq {}^{\omega>}(\omega_2)$, p^i_η, ν^i_η for $\eta \in A^i$ such that

(a) $\langle\rangle \in A^i, p^i_{\langle\rangle} \geq p_0$

(b) $p^i_\eta \Vdash_P$ "$\eta \in \underset{\sim}{T}_i$" (and $p^i_\eta \in P$ of course).

(c) $p^i_\eta \Vdash_P$ "ν^i_η is a splitting point of $\underset{\sim}{T}_i, \eta \trianglelefteq \nu^i_\eta$, and: if $\rho \triangleleft \nu^i_\eta$ is a splitting point of T_i then for some $\ell < \ell g(\eta)$ we have $\rho = \nu^i_{\eta\restriction\ell}$".

(d) $\eta \in A^i, \rho \in A^i, \eta \triangleleft \rho$ implies $\nu^i_\eta \triangleleft \nu^i_\rho$

(e) $\nu^i_\eta{}^\frown\langle\alpha\rangle \in A^i$ iff for some $q \in P, p^i_\eta \leq q$ and $q \Vdash_P$ "$\nu^i_\eta{}^\frown\langle\alpha\rangle \in \underset{\sim}{T}_i$".

(f) if $\rho = \nu^i_\eta{}^\frown\langle\alpha\rangle \in A^i$ then $p^i_\eta \leq p^i_\rho$, and $p^i_\rho \Vdash_P$ "$\rho \in \underset{\sim}{T}_i$" [this actually follows from (b) and (d) and $\eta \trianglelefteq \nu^i_\eta$].

(g) if $\underset{\sim}{Q}$ is Nm$'$ then for every i and $\eta \in A^i$, $\nu^i_\eta = \eta$.

This is easily done, and let $T^0_i = \{\eta\restriction\ell : \ell \leq \ell g(\eta), \eta \in A^i\}$, and let $p^i_\eta (\eta \in T^0_i)$ be $p^i_\nu, \nu \in A_i$, where $\eta \trianglelefteq \nu$, and $(\forall\rho)[\eta \trianglelefteq \rho \triangleleft \nu \to \rho \notin A^i]$. By the Δ-system theorem on trees from [RuSh:117, Th.4.12, p.76] there is T^1_i satisfying $T^0_i \leq^* T^1_i$, and $q^i_\eta (\eta \in T^1_i)$ such that:

(α) $p^i_\eta \leq q^i_\eta$ hence $p_0 \leq q^i_\eta$ (and $q^i_\eta \in P$).

(β) if η is a splitting point of T^1_i, then $\eta{}^\frown\langle\alpha\rangle, \eta{}^\frown\langle\beta\rangle \in T^1_i$ & $\alpha \neq \beta$ implies $\mathrm{Dom}(q^i_{\eta^\frown\langle\alpha\rangle}) \cap \mathrm{Dom}(q^i_{\eta^\frown\langle\beta\rangle}) = \mathrm{Dom}(q^i_\eta)$ and $q^i_\eta \leq q^i_{\eta^\frown\langle\alpha\rangle}$, $q^i_\eta \leq q^i_{\eta^\frown\langle\beta\rangle}$

(γ) if η is not a splitting point of T^1_i, ($\eta \in T^1_i$) then for the unique α such that $\eta{}^\frown\langle\alpha\rangle \in T^1_i$, we have $q^i_{\eta^\frown\langle\alpha\rangle} = q^i_\eta$.

Now by the usual Δ-system theorem there are $i < j < \aleph_4$ such that $T^1_i = T^1_j$ and for every $\eta \in T^1_i, q^i_\eta, q^j_\eta$ are compatible. Let

$$q^* = q^i_{\langle\rangle} \cup q^j_{\langle\rangle} \in P$$

$$\underset{\sim}{T} = \{\eta \in T^1_i : q^i_\eta \in \underset{\sim}{G}_P \ \&\ q^j_\eta \in \underset{\sim}{G}_P\}$$

Clearly $\underset{\sim}{T}$ is a P-name of a subset of ${}^{\omega>}(\omega_2)$, closed under initial segments, $\underset{\sim}{T} \subseteq \underset{\sim}{T}_i, \underset{\sim}{T}_j$, so it suffices to prove

$$q^* \Vdash \text{“}\underset{\sim}{T}_i \leq^* \underset{\sim}{T} \;\&\; \underset{\sim}{T}_j \leq^* \underset{\sim}{T} \;\&\; \underset{\sim}{T} \in \underset{\sim}{Q}\text{”}$$

which is easy. $\square_{4.3}$

4.4 Lemma. Nm satisfies the S-condition for any S such that $\aleph_2 \in S$.

Proof. To show our claim holds we have to describe F and then show that F does its work. At a point η where we use F, F has to determine $\mathrm{Suc}_T(\eta)$, and $f(\eta')$ for any immediate successor η' of η (see 2.6B for the notation). At such a point $f(\eta)$ is already known and is a condition in Nm. Let ν_η be a point of minimal height in $f(\eta)$ such that ν_η has $\aleph_2$ many immediate successors (in $f(\eta)$). Let $\mathrm{Suc}_T(\eta)$ be $\{\eta\hat{\ }\langle\alpha\rangle : \nu_\eta\hat{\ }\langle\alpha\rangle \in f(\eta)\}$ and for each $\eta\hat{\ }\langle\alpha\rangle$ in $\mathrm{Suc}_T(\eta)$ let $f(\eta\hat{\ }\langle\alpha\rangle)$ be the subtree of $f(\eta)$ consisting of members of $f(\eta)$ which are comparable with $\nu_\eta\hat{\ }\langle\alpha\rangle$ (in the tree order of $f(\eta)$). When we want to check that our F does the work; we are given an S-tree T, fronts J_n and a function $f : T \to \mathrm{Nm}$ as above in 2.6 and we are given a subtree T', $T \leq^* T'$. We have to find a condition $r \in \mathrm{Nm}$ so that $r \Vdash$ “there exists an infinite η such that for every $n < \omega, \eta{\restriction}n \in T'$ and $f(\eta{\restriction}n) \in \underset{\sim}{G}$”. We produce r by passing from T' to a subtree $T'' \geq T'$ such that every point in T'' either belongs to some front J_n (and thus fits the demands of F and in particular has $\aleph_2$ many successors) and is a splitting point, or it has exactly one immediate successor. Now r is the tree of all the initial segments of trunks of $f(\eta)$ for some $\eta \in T'' \cap (\bigcup_{n<\omega} J_n)$; that is:

$$r = \{\rho : \exists\eta \in (\bigcup_{n<\omega} J_n) \cap T'' \text{ such that } \rho \trianglelefteq \nu_\eta\}$$

where ν_η is from the definition of $f(\eta)$ according to F. By the construction, if η_1, η_2 are $\triangleleft$-incomparable, then so are $\nu_{\eta_1}, \nu_{\eta_2}$, hence by the definition of Nm, r is a member of Nm. As any $p \in \mathrm{Nm}$ forces that “$\exists\eta \in \lim(p)$ such that for all n the subtree defined by $\eta{\restriction}n$ belongs to $\underset{\sim}{G}$”, it is not hard to see that r is as required. $\square_{4.4}$

4.4A Claim. If I^* is an $\aleph_2$-complete ideal on ω_2 to which every singleton belongs and $I^* \in \mathbb{I}$ then Nm satisfies the $\mathbb{I}$-condition.

Proof. Same, and really follows (see more generally in XV §4). $\square_{4.4A}$

4.5 Lemma. Any ω_1-closed forcing notion satisfies any S-condition.

Proof. This is trivial with no "real" demands on F, when we are in the relevant situation with an S-tree T', $T \leq^* T'$ and $f : T \to P$ we just pick $r \in P$ such that $r \geq \bigcup_{n<\omega} f(\eta \restriction n)$ for some $\eta \in \lim T'$, such r exists by the completeness of P and it forces that any smaller condition is a member of $\underset{\sim}{G}_P$, so we are done. $\square_{4.5}$

4.5A Remark. The same is true for strategically $\aleph_1$-closed forcing notions (games of length $\omega + 1$ suffice).

4.6 Lemma. Let W be a stationary subset of $S_0^2 = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ and let $P[W] = \{h : h$ is an increasing and continuous function from $\alpha + 1$ into W for some $\alpha < \omega_1\}$ ordered by inclusion, *then* $P[W]$ satisfies the S-condition for any S such that $\aleph_2 \in S$.

Proof. We define the F and then show why it works. Each $F(\eta)$ will determine $\mathrm{Suc}_T(\eta)$ to be $\{\eta\char`^\langle\alpha\rangle : \alpha < \omega_2\}$ and $f(\eta\char`^\langle\alpha\rangle)$ a condition above $f(\eta)$ such that $\mathrm{Max}(\mathrm{Rang}(f(\eta\char`^\langle\alpha\rangle))) > \alpha$ (note that by the definition of $P[W]$ each function h which belongs to $P[W]$ attains its maximum: $\max(\mathrm{Rang}(h)) = h(\max(\mathrm{Dom}(h)))$. Let us denote $\mathrm{Max}(\mathrm{Rang}(f(\eta)))$ by α_η. For proving that F works, assume T is an S-tree, J_n fronts, $f : T \to P[W]$ meets our requirements for F (see 2.6) and $T \leq^* T'$. Let C_1 be a closed unbounded subset of ω_2 such that if $\delta \in C_1$ and $\eta \in T'$ and $\eta \in {}^{\omega>}\delta$ then $\alpha_\eta < \delta$. Let C_2 be a closed unbounded subset of ω_2 such that for $\delta \in C_2, \eta \in T' \cap (\bigcup_{n<\omega} J_n)$ satisfying $\eta \in {}^{\omega>}\delta$ and $\alpha < \delta$ there is always some β such that $\alpha < \beta < \delta$ and $\eta\char`^\langle\beta\rangle \in T'$. Now for some $\eta \in T'$ we pick $\delta \in C_1 \cap C_2 \cap W$ such that $\alpha_\eta < \delta$ and construct an $\triangleleft$-increasing sequence $\langle \eta_n : n < \omega \rangle$ in T' such that $\lim_{n\to\omega} \alpha_{\eta_n} = \delta$ (this is possible as $\mathrm{cf}(\delta) = \omega$ using the definitions of C_1 and

C_2). Let $r = \bigcup_{n<\omega} f(\eta_n) \cup \{\langle \mathrm{Sup}_{n<\omega}\mathrm{Dom}(f(\eta_n)),\ \delta\ \rangle\}$, clearly $r \in P[W]$ and forces each $f(\eta_n)$ to belong to the generic G so we are done. $\square_{4.6}$

4.6A Lemma. Let $\bar{W} = \langle W_i : i < \omega_1 \rangle$ be a sequence of stationary subsets of $S^2_0 = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ and let the forcing notion $P[\bar{W}]$ be defined by

$$\begin{aligned} P[\bar{W}] \stackrel{\text{def}}{=} \{f : & f \text{ is an increasing and continuous function from} \\ & \alpha + 1 \text{ into } W_0 \text{ for some } \alpha < \omega_1, \text{and } h \text{ satisfies} \\ & \forall i \leq \alpha : h(i) \in W_i\} \end{aligned}$$

(ordered by inclusion), *then* $P[\bar{W}]$ satisfies the S-condition for any S such that $\aleph_2 \in S$.

Proof. We define the F as in the previous lemma: Each $F(\eta)$ will determine $\mathrm{Suc}_T(\eta)$ to be $\{\eta\hat{\ }\langle\alpha\rangle : \alpha < \omega_2\}$ and $f(\eta\hat{\ }\langle\alpha\rangle)$ a condition above $f(\eta)$ such that $\mathrm{Max}(\mathrm{Rang}(f(\eta\hat{\ }\langle\alpha\rangle)) > \alpha$. Let us write α_η for $\mathrm{Max}(\mathrm{Dom} f(\eta))$ and δ_η for $\mathrm{Max}(\mathrm{Rang}(f(\eta)))$.

Now assume T is an S-tree, and $f : T \to P[\bar{W}]$ obeys F, and let $T \leq^* T'$. By Lemma 3.5 with $\lambda = \aleph_1$ (not $\lambda = 2^{\aleph_0}$) we can find a subtree T'', $T' \leq^* T''$ and an $\alpha < \omega_1$ such that whenever $\eta_0 \lhd \eta_1 \ldots$ are elements of T'', then $\lim_{n\to\omega} \alpha_{\eta_n} = \alpha$. Now as in the proof of 4.6 let $\delta \in W_\alpha$ be such that

$$(\forall \eta \in {}^{<\omega}\delta \cap T'')(\forall i < \delta)(\exists j < \delta)[\alpha_\eta < \delta,\ \eta\hat{\ }\langle j\rangle \in T'' \text{ and } i < \alpha_{\eta\hat{\ }\langle j\rangle} < \delta].$$

Again we can construct a sequence $\eta_0 \lhd \eta_1 \lhd \ldots$ in T'' such that

$$\lim \mathrm{Sup}\mathrm{Rang}(\eta_n) = \delta.$$

Let $r = \bigcup_{n<\omega} f(\eta_n) \cup \{\langle \alpha, \delta \rangle\}$, then $r \in P[\bar{W}]$ and r forces each $f(\eta_n)$ to belong to the generic G. $\square_{4.6A}$

4.7 Lemma. Suppose

(a) P satisfies the $\mathbb{I}_0$-condition.

(b) For every $I_0 \in \mathbb{I}_0$ there is $I_1 \in \mathbb{I}_1$ such that $\bigcup_{A \in I_1} A \subseteq \bigcup_{A \in I_0} A$ and $(\forall B \in I_0)[B \subseteq \bigcup_{A \in I_1} A \rightarrow B \in I_1]$

Then P satisfies the $\mathbb{I}_1$-condition.

Proof. Trivial. $\square_{4.7}$

§5. Finite Composition

5.1 Theorem. Let Q_0 satisfies the $\mathbb{I}_0$-condition and let $\underset{\sim}{Q}_1$ be a Q_0-name of a forcing notion such that the weakest condition of Q_0 forces it to satisfy the $\mathbb{I}_1$-condition. Let

(a) μ be the first regular cardinal strictly greater than the cardinality of the domain of each member of $\mathbb{I}_0$

(b) λ be such that $\lambda = \lambda^{<\mu} \geq |Q_0|$

(c) assume $\Vdash_{Q_0}$ "$\mathbb{I}_1$ is λ^+-complete"

(d) let $\mathbb{I}$ be $\mathbb{I}_0 \cup \mathbb{I}_1$

Then $P = Q_0 * \underset{\sim}{Q}_1$ satisfies the $\mathbb{I}$-condition.

Remark. Note that $\mathbb{I}_1 \in V$ (we will not gain much by letting $\mathbb{I}_1 \in V^{Q_0}$.)

Proof. Once again we have to define the function F and then prove it does its work. We will need a combinatorial lemma and its proof will conclude the proof of the theorem. For $f(\eta) \in P$ we denote by $f^0(\eta)$ its Q_0-part and by $f^1(\eta)$ the $\underset{\sim}{Q}_1$-part (it is a Q_0-name of a condition in $\underset{\sim}{Q}_1$), let F^0 be the function exemplifying Q_0 satisfies the $\mathbb{I}_0$-condition and $\underset{\sim}{F}^1$ be the Q_0-name of the function exemplifying $\underset{\sim}{Q}_1$ satisfies the $\mathbb{I}_1$-condition.

We divide the definition of the F to even and odd stages. In even stages i.e., when $|w|$ is even, we will refer to the Q_0 part of P and use F^0. More precisely, let $\langle B, \mathbf{I}_\eta, \langle r_\nu : \nu \in B\rangle\rangle = F^0_{w_1}(\eta, \langle f^0(\eta \restriction \ell) : \ell \leq \ell g(\eta)\rangle)$ where $w_1 = \{\ell \in w : |\ell \cap w|$ even$\}$. Now let $\mathrm{Suc}_T(\eta) = B$ and for $\nu \in B$, $f^1(\nu) = f^1(\eta)$, $f^0(\nu) = r_\nu$. In odd stages we essentially do the same for the $\underset{\sim}{Q}_1$-part but we need a little modification; $\underset{\sim}{F}^1$ is just a $\underset{\sim}{Q}_0$-name of a function and we may not even know the domain of the ideal $\mathbf{I}_\eta$ it give ($=$ the $\mathrm{Suc}_T(\eta)$), so first we extend $f^0(\eta)$ to a condition q_0' satisfying $f^0(\eta) \leq q_0' \in Q_0$ and forcing a specific value for $\mathbf{I}_\eta$ (hence for $\mathrm{Suc}_T(\eta)$) as defined by $\underset{\sim}{F}^1$, and then proceed like in the Q_0 part (of course we change each $f^0(\eta)$ there to $f^1(\eta)$ and so on) and we let the Q_0 part $f^0(\eta')$ for each $\eta' \in \mathrm{Suc}_T(\eta)$ be the q_0' we have picked (the $\underset{\sim}{Q}_1$ part will be defined by $\underset{\sim}{F}^1(\eta, \langle f^1(\eta \restriction \ell : \ell < \ell g(\eta))\rangle)$ (if we want to allow F to have just $\mathrm{Suc}_T(\eta) \neq \emptyset \bmod \mathbf{I}_\eta$, act as in the proof 6.2).

Before we can show that this definition works we need a definition and a combinatorial lemma.

5.2 Definition. For a subset A of T we define by induction on the length of η, $\mathrm{res}_T(\eta, A)$ for each $\eta \in T$. Let $\mathrm{res}_T(\langle\rangle, A) = \langle\rangle$. Assume $\mathrm{res}_T(\eta, A)$ is already defined and we define $\mathrm{res}_T(\eta\char`\^\langle\alpha\rangle, A)$ for all members $\eta\char`\^\langle\alpha\rangle$ of $\mathrm{Suc}_T(\eta)$. If $\eta \in A$ then $\mathrm{res}_T(\eta\char`\^\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\char`\^\langle\alpha\rangle$ and if $\eta \notin A$ then $\mathrm{res}_T(\eta\char`\^\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\char`\^\langle 0\rangle$. Thus $\mathrm{res}(T, A) \stackrel{\mathrm{def}}{=} \{\mathrm{res}_T(\eta, A) : \eta \in T\}$ is a tree obtained by projecting, i.e., gluing together all members of $\mathrm{Suc}_T(\eta)$ whenever $\eta \notin A$.

5.3 Lemma. Let λ, μ be uncountable cardinals satisfying $\lambda^{<\mu} = \lambda$ and let $(T, \mathbf{I})$ be a tree in which for each $\eta \in T$ either $|\mathrm{Suc}_T(\eta)| < \mu$ or $\mathbf{I}(\eta)$ is λ^+-complete. *Then* for every function $H : T \to \lambda$ there exist $T', (T, \mathbf{I}) \leq^* (T', \mathbf{I})$ such that (letting $A = \{\eta \in T : |\mathrm{Suc}_T(\eta)| < \mu\}$) for $\eta, \eta' \in T'$: $\mathrm{res}_T(\eta, A) = \mathrm{res}_T(\eta', A)$ implies: $H(\eta) = H(\eta')$ and $\eta \in A$ iff $\eta' \in A$, and if $\eta \in T' \cap A$, then $\mathrm{Suc}_T(\eta) = \mathrm{Suc}_{T'}(\eta)$. (Note that the lemma is also true for $\lambda = \mu = \aleph_0$).

5.4 Continuation of the proof of 5.1. Using the lemma let us prove the theorem. So we are given $(T, \mathbf{I}), f, J_n$ for $n < \omega$ as in 2.6 for our F, and consider $f^0 : T \to Q_0$ as a function to λ, (remember $|Q_0| \leq \lambda$).

We let $A = \{\eta : |\mathrm{Suc}_T(\eta)| < \mu\})$. By the lemma for every $(T', \mathbf{I})$ satisfying $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ there is a subtree $T'', (T', \mathbf{I}) \leq^* (T'', \mathbf{I})$ such that for every $\eta, \eta' \in T''$ we have: $f^0(\eta) = f^0(\eta')$, and $\eta \in A$ iff $\eta' \in A$ whenever $\mathrm{res}_T(\eta, A) =$

$\mathrm{res}_T(\eta', A)$. Let $T^* = \{\mathrm{res}_T(\eta, A) : \eta \in T''\}$, it is an $\mathbb{I}_0$-tree (since the "even" fronts of the original tree now become splitting points) and f^0 induces a function $\hat{f}^0$ from it to Q_0 i.e., $\nu = \mathrm{res}_T(\eta, A)$ implies $\hat{f}^0(\nu) = f^0(\eta)$ (by the conclusion of 5.3, $\hat{f}^0$ is well defined).

By the definition of F for even $|w|$'s and the assumption that F^0 exemplify the $\mathbb{I}_0$-condition we can find an $r_0 \in Q_0$ and a Q_0-name $\underset{\sim}{\eta}$ of a member of $\lim T^*$ such that $r_0 \Vdash_{Q_0}$ "for every $k < \omega$ we have $\hat{f}^0(\underset{\sim}{\eta}\restriction k) \in \underset{\sim}{G}_0$" where $\underset{\sim}{G}_0$ is the Q_0-name for the generic subset of Q_0". Let $\underset{\sim}{T}^+ \overset{\text{def}}{=} \{\rho \in T'' : \mathrm{res}(\rho, A) = \mathrm{res}(\underset{\sim}{\eta}, A)\}$; this is a Q_0-name of an $\mathbb{I}_1$-tree and by the definition of F in the odd stages (i.e. F_w when $|w|$ is odd) there are a Q_0-name $\underset{\sim}{r}_1$ of a member of Q_1 and a $Q_0 * \underset{\sim}{Q}_1$-name $\underset{\sim}{\nu}$ of an ω-branch of $\underset{\sim}{T}^+$ such that $r_0 \Vdash_{Q_0} [\underset{\sim}{r}_1 \Vdash_{\underset{\sim}{Q}_1}$ " $\underset{\sim}{\nu} \in \lim \underset{\sim}{T}^+$ is such that $\underset{\sim}{f}^1(\underset{\sim}{\nu}\restriction k) \in \underset{\sim}{G}_1$ for every $k < \omega$"] where $\underset{\sim}{G}_1$ is the name of the generic set for $\underset{\sim}{Q}_1$ and $\underset{\sim}{\nu}$ is forced to be a name in $\underset{\sim}{Q}_1$ of a member of $\lim \underset{\sim}{T}^+$. The condition in $P = Q_0 * \underset{\sim}{Q}_1$ which witnesses that the $\mathbb{I}$-condition holds is of course $\langle r_0, \underset{\sim}{r}_1\rangle$, since $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash$ "$\underset{\sim}{\nu} \in \lim T$, and for all $k \in \omega$, $\underset{\sim}{f}^0(\underset{\sim}{\nu}\restriction k) = \underset{\sim}{\hat{f}}^0(\mathrm{res}(\nu\restriction k), A) \in \underset{\sim}{G}_0$, and $f(\nu\restriction k) \in G$. $\square_{5.1}$

We now pay our debt and prove Lemma 5.3; the proof is in the spirit of the proofs of the previous combinatorial Lemmas 3.3 and 3.5.

5.5. *Proof of Lemma 5.3* Without loss of generality $\eta\hat{\ }\langle\alpha\rangle \in T \Rightarrow \alpha < |\mathrm{Suc}_T(\eta)|$. We will prove the lemma by induction on μ. We start with a successor μ, in such cases there is a cardinal κ such that $\mu = \kappa^+$ and for each $\eta \in T$ we have $res_T(\eta, A) \in {}^{\omega>}\kappa$. Let $\{\langle h_\alpha, g_\alpha\rangle : \alpha < \lambda\}$ be a list of all the pairs (h, g) such that g is a function from ${}^{\omega>}\kappa$ to $\{0,1\}$ and h is in a function from ${}^{\omega>}\kappa$ to λ (by the assumption $\lambda^{<\mu} = \lambda$, hence there are at most λ many such pairs). For each $\alpha < \lambda$ we define a game $\eth_\alpha$ just like in the proof of 3.7, except that now player I wins if for the $\eta \in \lim T$ that they constructed along the play we have: $\eta\restriction k \in A$ iff $g_\alpha(\mathrm{res}_T(\eta\restriction k, A)) = 0$ and $H(\eta\restriction k) = h_\alpha(\mathrm{res}_T(\eta\restriction k, A))$ for every $k < \omega$.

If for every $\alpha < \lambda$ player II had a winning strategy we could build a subtree T^* by induction on the height of $\eta \in T$ taking into T^* all members of $\mathrm{Suc}_T(\eta)$

when $\eta \in A$, and otherwise picking as the only member of $\mathrm{Suc}_{T^*}(\eta)$ an element of $\mathrm{Suc}_T(\eta)$ that is not in any of the A_α's that are defined for player II at that stage by his winning strategy for $\eth_\alpha$ (this is possible as we assume that $\eta \notin A$ implies I_η is λ^+-complete).

The map $\eta \mapsto \mathrm{res}(\eta, A)$ is 1-1 on T^*, so there is a pair $\langle h_{\alpha_0}, g_{\alpha_0}\rangle$ in our list such that for each $\eta \in T^*$ we have $H(\eta) = h_{\alpha_0}(\mathrm{res}_T(\eta, A))$ and $g_{\alpha_0}(\eta) = 0$ iff $\eta \in A$. Now we define a play in the game $\eth_{\alpha_0}$: player I plays choosing only members of T^* while II plays according to his winning strategy for $\eth_{\alpha_0}$, but in such a play, player I surely wins and we get the desired contradiction.

So there exists some $\beta < \lambda$ for which player II has no winning strategy in $\eth_\beta$, but the game is determined hence player I has a winning strategy for $\eth_\beta$. Now let T' be the tree of all sequences η that can appear in a play where player I used this strategy. T' satisfies the requirements (similar to 3.5). This finishes the case where μ is a successor.

If μ is singular, then $\lambda \le \lambda^{<\mu^+} = \lambda^\mu \le (\lambda^{<\mu})^{\mathrm{cf}\mu} = \lambda^{\mathrm{cf}(\mu)} \le \lambda^{<\mu} = \lambda$, so we can without loss of generality replace μ by μ^+. If μ is a regular limit cardinal (or just $\aleph_0 < \mathrm{cf}(\mu) < \mu$), then we first use Lemma 3.7 to find $T', (T, \mathbf{I}) \le^* (T', \mathbf{I})$, and $\mu' < \mu$ such that for every $\eta \in T'$: $\mathbf{I}_\eta$ is λ^+-complete or $|\mathrm{Suc}_T(\eta)| \le \mu'$, and then use the induction hypothesis on μ'. $\square_{5.3}$

5.6 Corollary to 5.3. Assume that $1 = \lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \dots$ are cardinals satisfying $\lambda_{k+1}^{<\mu_k} = \lambda_{k+1}$ for all k. Let $(T, \mathbf{I})$ be a tagged tree, and assume $T = \bigcup_k A_k$ where for all $\eta \in A_k$:

$$|\bigcup \mathbf{I}_\eta| < \mu_k \text{ and } I_\eta \text{ is } \lambda_k^+ - \text{complete.}$$

Let $f_k : T \to \lambda_k$, for $k < \omega$.

Then there is a tree T^* such that $(T, \mathbf{I}) \le^* (T^*, \mathbf{I})$ and for all k and all η, ν in T^*:

$$(*) \qquad \text{if } \mathrm{res}(\eta, \bigcup_{i<k} A_i) = \mathrm{res}(\nu, \bigcup_{i<k} A_i), \text{ then} f_k(\eta) = f_k(\nu)$$

Proof. As $\lambda_0 = 1$ clearly $T_0 = T$ will satisfy the condition for $k = 0$. We apply Lemma 5.3 to T_0 (with $\lambda = \lambda_1$, $\mu = \mu_0$, $f = f_1$, $A = A_0$) to get a subtree T_1 satisfying the condition also for $k = 1$. We continue by induction. In the k-th step, we apply Lemma 5.3 to T_k with $\lambda = \lambda_{k+1}$, $\mu = \mu_k$, $f = f_{k+1}$, $A = \bigcup_{\ell<k+1} A_\ell$.

Finally, let $T^* = \cap_k T_k$. Clearly T^* satisfies $(*)$. Note that $\langle\rangle \in T^*$, and if $\eta \in T^* \cap A_k$, then by the conclusion of lemma 5.3,

$$\mathrm{Suc}_{T_k}(\eta) = \mathrm{Suc}_{T_{k+1}}(\eta) = \dots$$

so $\mathrm{Suc}_{T^*}(\eta) = \mathrm{Suc}_{T_k}(\eta)$. Hence T^* is a tree and $(T_0, \mathbf{I}) \le^* (T^*, \mathbf{I})$. $\square_{5.6}$

§6. Preservation of the $\mathbb{I}$-Condition by Iteration

6.1 Definition. We say that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is *suitable* for $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j} : \langle i,j\rangle \in W^*\rangle$ provided that the following hold:

(0) $W^* \subseteq \{\langle i,j\rangle : i < j \le \alpha, i$ is not strongly inaccessible $\}$ and $\{\langle i+1, j\rangle : i+1 < j < \bigcup_{\beta\le\alpha} \beta+1\} \subseteq W^*$ (we can use some variants, but there is no need)

(1) $\bar{Q}$ is a RCS iteration.

(2) $P_{i,j} = P_j/P_i$ satisfies the $\mathbb{I}_{i,j}$-condition for $\langle i,j\rangle \in W^*$.

(3) for every $I \in \mathbb{I}_{i,j}$ the set $\bigcup I$ is a uncountable cardinal, I is $\lambda^+_{i,j}$-complete, $\lambda_{i,j} < \bigcup I < \mu_{i,j}$, $\mu_{i,j}$ regular, and $|P_i| \le \lambda_{i,j}$, and $\lambda^+_{i,j} \ge \aleph_2$ (note that $\mathbb{I}_{i,j}$ is from V and not V^{P_i}, and $i \le \lambda_{i,j} < \mu_{i,j}$).

(4) if $i(0) < i(1) < i(2) \le \alpha$, $\langle i(0), i(1)\rangle \in W^*, \langle i(1), i(2)\rangle \in W^*$ then $\lambda_{i(1),i(2)}{}^{<\mu_{i(0),i(1)}} = \lambda_{i(1),i(2)}$.

(5) for every $I \in \mathbb{I}_{i(2),i(3)}$ and $i(0) < i(1) \le i(2) < i(3), I$ is $\lambda^+_{i(0),i(1)}$-complete.

6.2 Lemma. If $\bar{Q} = \langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$ is *suitable* for $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j} : i < j < \omega\rangle$, and $\mathbb{I} = \bigcup_{n<\omega} \mathbb{I}_{n,n+1}$ *then* $P_\omega = \mathrm{Rlim}\bar{Q}$ satisfies the $\mathbb{I}$-condition.

Proof. Let $\mathbb{I}_i = \mathbb{I}_{i,i+1}$, $\lambda_i = \lambda_{i,i+1}$, $\mu_i = \mu_{i,i+1}$, note that $P_{i,i+1} = \underset{\sim}{Q}_i$, so $\underset{\sim}{Q}_i$ satisfies the $\mathbb{I}_i$-condition, $|P_i| \le \lambda_i, \mu_i \le \lambda_{i+1} = \lambda_{i+1}{}^{<\mu_i} < \mu_{i+1}$.

For each $i < \omega$, let $\underset{\sim}{F}_i$ be a P_i-name of a function witnessing that $\underset{\sim}{Q}_i$ satisfies the $\mathbb{I}_i$-condition. We will act as in the proof of Theorem 5.1, but now we have countably many $\underset{\sim}{F}_i$'s rather than two. We can a priori partition the tasks, so let $\omega = \bigcup_{i<\omega} B_i$, the B_i pairwise disjoint, each B_i infinite.

Now we shall define the function F which exemplifies "P_ω satisfies the $\mathbb{I}$-condition". So we have to define $F(\eta, w, f\restriction\{\nu : \nu \trianglelefteq \eta\})$, (see Definition 2.6). Let i be the unique $i < \omega$ such that $|w| \in A_i$, let $w^* = \{\ell \in w : |\ell \cap w| \in A_i\}$, and let $\langle \underset{\sim}{B}, \underset{\sim}{I}, \langle \underset{\sim}{r}_\nu : \nu \in \underset{\sim}{B}\rangle\rangle = \underset{\sim}{F}_i(\eta, w^*, \langle f(\eta\restriction\ell)(i) : \ell \leq \ell g(\eta)\rangle)$, (so $\Vdash_{P_i}$ "$\underset{\sim}{B} = \mathrm{Dom}(\underset{\sim}{I})$").
We choose $q_\eta \in P_i$, such that $(f(\eta)\restriction i) \leq q_\eta$ and for some λ_η, $q_\eta \Vdash_{P_i}$ "$|\underset{\sim}{B}| = \underset{\sim}{B} = \lambda_\eta$" and $q_\eta \Vdash_{P_i}$ "$\underset{\sim}{I}$ is $\mathbf{I}_\eta$ which belongs to $\mathbb{I}$, in fact to $\mathbb{I}_i = \mathbb{I}_{i,i+1}$ (by the natural isomorphism $\underset{\sim}{f}_\eta$)", (see 2.6B). Let $p_\eta = f(\eta)\restriction[i+1, \omega)$. We choose $Suc_T(\eta) = \{\eta\hat{\ }\langle\alpha\rangle : \alpha < \bigcup \mathbf{I}_\eta\}$, and define: $F(\eta, w, \langle f(\nu\restriction\ell) : \ell \leq \ell g(\eta)\rangle) = \langle \lambda_\eta, \mathbf{I}_\eta, \langle \underset{\sim}{r}_\nu \cup p_\eta \cup q_\eta : \nu \in Suc(\eta)\rangle\rangle$, [really we should replace $\underset{\sim}{r}_\nu$ by the function $\{\langle i, \underset{\sim}{r}_\nu\rangle\}$, and λ_η by $\{\eta\hat{\ }\langle i\rangle : i < \lambda_\eta\}$ but we shall ignore such problems].

We now have to prove that P_ω, $\mathbb{I}$, and F satisfy Definition 2.6. So let $(T, \mathbf{I})$, $J_k (k < \omega)$ and $f : T \to P$ be as in Definition 2.6 and $(T, \mathbf{I}) \leq^* (T^0, \mathbf{I})$ and we have to find a $p \in P$, such that $p \Vdash$ "$\exists \eta \in \lim T^0$ such that $(\forall k < \omega) f(\eta\restriction k) \in \underset{\sim}{G}_{P_\omega}$".

First define $f_k : T^0 \to P_k$ by $f_k(\eta) = f(\eta)\restriction k$. Let $A_k = \{\eta \in T^0 : \mathbf{I}_\eta \in \mathbb{I}_k\}$. By 5.6 we can find a tree T^*, $(T^0, \mathbf{I}) \leq^* (T^*, \mathbf{I})$, such that whenever $\eta, \nu \in T^*$ and $\mathrm{res}(\eta, \bigcup_{i<k} A_i) = \mathrm{res}(\nu, \bigcup_{i<k} A_i)$, then $f_k(\eta) = f_k(\nu)$. Let $T^*_k = \mathrm{res}(T^*, \bigcup_{i<k+1} A_i)$. Define $\underset{\sim}{f}^*_k : T^*_k \to \underset{\sim}{Q}^*_k$ by $f^*_k(\mathrm{res}(\eta, \bigcup_{i<k+1} A_i)) = f_k(\eta)(k)$.

By induction on $i = 0, 1, 2\ldots$ we can now define P_{i+1}-names $\underset{\sim}{\eta}_i$ and conditions $p(i) \in \underset{\sim}{Q}_i$ such that $\langle p(0), \ldots, p(i)\rangle \Vdash_{P_{i+1}}$ "$\underset{\sim}{\eta}_i \in \lim \underset{\sim}{T}^*_i$ and $(\forall \ell < \omega)\ f^*_i(\underset{\sim}{\eta}_i\restriction\ell) \in \underset{\sim}{G}_{Q_i}$" and for all $i < j$,

$$\Vdash_{P_{j+1}} \text{"}(\forall \ell)\ [\underset{\sim}{\eta}_i\restriction\ell = \mathrm{res}(\underset{\sim}{\eta}_j\restriction\ell, \bigcup_{k<i+1} A_k)]\text{"}.$$

Finally we can find a P_ω-name $\underset{\sim}{\eta}$ such that for all $\ell \Vdash_{P_\omega}$ "for all large enough i $\underset{\sim}{\eta}\restriction\ell = \underset{\sim}{\eta}_i\restriction\ell$". It is now clear that $\Vdash_{P_\omega}$ "$\underset{\sim}{\eta} \in \lim T^*$, and $\forall \ell\ f(\underset{\sim}{\eta}\restriction\ell) \in G_{P_\omega}$". $\square_{6.2}$

6.2A Remark. Note that here as well in the next theorem we need that the $\mathbb{I}_{i,j}$'s are well separated (compare 6.1(4), (5)) i.e. some have small underlying sets, others have large completeness coefficients (e.g. in the previous theorem we required that ideals $\mathbb{I}_i$ are on sets $\subseteq \mu_i$, and ideals in $\mathbb{I}_{i+1}$ had to be λ_{i+1}-complete, $\lambda_{i+1} \geq \mu_i$). To satisfy these requirements we will in applications only work with iterations in which in every odd step some large enough cardinal is collapsed, see 1.1(ii)$'$.

6.3 Lemma.

(1) If $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_1 \rangle$ is suitable for $\langle \mathbb{I}_{\alpha,\beta}, \lambda_{\alpha,\beta}, \mu_{\alpha,\beta} : \alpha < \beta < \omega_1, \alpha$ non-limit$\rangle$, and $\mathbb{I} = \bigcup\{\mathbb{I}_{\alpha,\beta} : \alpha < \beta < \omega_1$ and α non-limit$\}$ *then* $P = P_{\omega_1} = \lim \bar{Q}$ satisfies the $\mathbb{I}$-condition.

(2) We can replace ω_1 by any δ such that $\aleph_0 < \mathrm{cf}^V \delta < \mathrm{Min}\{\lambda_{\alpha,\beta} : \langle \alpha, \beta \rangle \in W^*\}$.

Proof. 1) We will first prove this assuming CH (which is enough for all applications in this chapter), and then indicate how we can get rid of this extra assumption. The proof consists of two parts: In part A we define the function F, and in part B we show that it satisfies the requirements from definition 2.6.

Part A: To each $p \in P$ we have associated a countable set $\{\underset{\sim}{\zeta}^k(p) : k < \omega\}$ of prompt names, such that letting $\underset{\sim}{\zeta}^*(p) = \sup\{\underset{\sim}{\zeta}^k(p) : k < \omega\}$, we have $p \in P_{\underset{\sim}{\zeta}^*}$ (see 1.12(4)). Let $\bigcup_{i<\omega} B_i$ be the set of odd natural numbers > 2, the B_i infinite pairwise disjoint, be such that $(\forall \ell \in B_i)(i + 1 < \ell)$.

Let $\underset{\sim}{F}_{\alpha,\beta}$ be P_α-name of a function exemplifying "$P_{\alpha,\beta}$ satisfies the $\mathbb{I}_{\alpha,\beta}$-condition."

Let us explain our strategy; we cannot deal with all pairs (α, β) along a branch as the branch is countable, and α, β range over an uncountable set. So along each branch η we try to determine the $\bar{Q}$-named ordinals, $\underset{\sim}{\zeta}^m(f(\eta \restriction \ell))$, so we get a potential bound α^* to larger and larger parts of each $f(\eta \restriction \ell)$ and we shall use the functions $F_{\alpha^*_n, \alpha^*_{n+1}}$, where $\alpha^* = \bigcup \alpha^*_n$.

We shall define now the function F which exemplifies "P satisfies the $\mathbb{I}$-condition," so we have to define $F(\eta, w, \langle f(\eta \restriction \ell) : \ell \leq \ell g(\eta) \rangle)$. If $|w| \notin \bigcup_i B_i$

define it as any $\langle I, \langle f(\eta\hat{\ }\langle\xi\rangle) : \xi \in B\rangle\rangle$ such that $f(\eta\hat{\ }\langle\xi\rangle)$ forces a value to $\underset{\sim}{\zeta}^m(f(\eta\restriction\ell))$ for $m, \ell \le \ell g(\eta)$.

Now let $|w| \in \bigcup_i B_i$.

Naturally we shall use one of the $\underset{\sim}{F}_{\alpha,\beta}$, but we have to determine which one. By the way we are defining F, we can assume that for $k < \ell g(\eta)$, $f(\eta\restriction(k+1))$ determines (i.e. forces a value to) $\underset{\sim}{\zeta}^m(f(\eta\restriction\ell))$ for $\ell, m \le k$, so we can define the following:

Let $\alpha_0(\eta) = 0$, and for $0 < k < \ell g(\eta)$ let

$$\alpha_k(\eta) = \mathrm{Max}\{\underset{\sim}{\zeta}^m(f(\eta\restriction\ell)) + k : \ell, m \le k\}$$

Note that for any finite or infinite sequence ν: if $\eta \triangleleft \nu$, $k < \ell g(\eta)$, then $\alpha_k(\eta) = \alpha_k(\nu)$.

Let i be such that $|w| \in B_i$. Then $i + 1 < |w| \le \ell g(\eta)$, so

$$(*) \qquad \alpha \overset{\text{def}}{=} \alpha_i(\eta) \qquad \beta \overset{\text{def}}{=} \alpha_{i+1}(\eta)$$

are well defined.

Let $w^*_\eta = \{k \in w : |w \cap k| \in B_i\}$, and let

$$\langle \underset{\sim}{B}, \underset{\sim}{\mathbf{I}}, \langle \underset{\sim}{r}_\nu : \nu \in \underset{\sim}{B}\rangle\rangle = F_{\alpha,\beta}\left(\eta, w^*_\eta, \langle f(\eta\restriction\ell)\restriction[\alpha,\beta) : \ell \in w^*_\eta \cup \{\ell g(\eta)\}\rangle\right)$$

(recall α and β should have subscripts η and w, which we suppress for notational simplicity).

Now choose $q_\eta \ge f(\eta)\restriction\alpha$ such that $q_\eta \in P_\alpha$ and such that $q_\eta \Vdash_{P_\alpha}$ "$\underset{\sim}{\mathbf{I}}$ is isomorphic to $\mathbf{I}_\eta$, $\mathbf{I}_\eta \in \mathbb{I}_{\alpha,\beta}$", and let $F(\eta, w, \langle f(\nu\restriction\ell) : \ell \le \ell g(\eta)\rangle) = \langle \mathbf{I}_\eta, \langle q_\eta \bigcup \underset{\sim}{r}_\nu : \nu = \eta\hat{\ }\langle\alpha\rangle \text{ and } \alpha < \mathrm{Dom}(\mathbf{I}_\eta)\rangle\rangle$.

Part B. Now we have to prove that $P, \mathbb{I}$ and F satisfy Definition 2.6. So let $(T, \mathbf{I})$, J_k $(k < \omega)$ and f be as in Definition 2.6 for the F chosen above, and $(T, \mathbf{I}) \le^* (T^\dagger, \mathbf{I})$ and we have to find the required p. To each branch η of $T^\dagger$ we have associated a sequence $\langle \alpha_k(\eta) : k < \omega\rangle$ of countable ordinals. Since we assume CH we also have $\aleph_1^{\aleph_0} = \aleph_1$, so by Lemma 3.5A we can find T'' such that $(T^\dagger, \mathbf{I}) \leq^* (T'', \mathbf{I})$, and for some fixed sequence $\langle \alpha^*(k) : k < \omega \rangle$ we have $\alpha^*(k) = \alpha_k(\eta)$ for all $\eta \in \lim(T'')$.

Now we continue as in the proof of 6.2. We let $A_\ell = \{\eta : I_\eta \in \mathbb{I}_{\alpha^*(\ell),\alpha^*(\ell+1)}\}$ and $f_\ell(\eta) = f(\eta){\restriction}\alpha^*(\ell)$. By 5.6 we can find a tree T^*, $(T'', \mathbf{I}) \leq^* (T^*, \mathbf{I})$ such that for all η, ν in T^*:

If $\mathrm{res}(\eta, \bigcup_{\ell<k} A_\ell) = \mathrm{res}(\nu, \bigcup_{\ell<k} A_\ell)$, then $f(\eta){\restriction}\alpha^*(k) = f(\nu){\restriction}\alpha^*(k)$.

We let T^*_k be $\{\mathrm{res}_{T^*}(\eta, \bigcup_{\ell\leq k} A_\ell) : \eta \in T^*\}$ and $f^*_k : T^*_k \to P_{\alpha^*(k),\alpha^*(k+1)}$ is defined by

$$f^*_k(\mathrm{res}_{T^*}(\eta, \bigcup_{\ell\leq k} A_\ell)) = p \text{ iff } f(\eta){\restriction}[\alpha^*(k), \alpha^*(k+1)) = p$$

Now note that $\langle J_n : n \in B_0 \rangle$ is a system of fronts as in 2.6, and at each $\eta \in J_n$, if $n \in B_0$, then the function $F(\eta, \{k : \eta{\restriction}k \in \bigcup_m J_m\}, \langle f(\nu) : \nu \trianglelefteq \eta \rangle)$ used the function $\underset{\sim}{F}_{\alpha^*(0),\alpha^*(1)}(\eta, \{k : \eta{\restriction}k \in \bigcup_{m\in B_0} J_m\}, \langle f(\nu){\restriction}[\alpha^*(0), \alpha^*(1))\rangle)$ (but $\underset{\sim}{F}_{\alpha^*(0),\alpha^*(1)}$ is $F_{\alpha^*(0),\alpha^*(1)}$ as $\alpha^*(0) = 0$), so we can find $p_1 \in P_{\alpha^*(1)}$, such that for some $P_{\alpha^*(1)}$-name $\underset{\sim}{\eta}_0$

$$p_1 \Vdash_{P_{\alpha^*(1)}} \text{“}\underset{\sim}{\eta}_0 \in \lim T^*_0 \text{ and } (\forall \ell < \omega)\ f^*_0(\underset{\sim}{\eta}_0{\restriction}\ell) \in \underset{\sim}{G}_{P_{\alpha^*(1)}}\text{”}.$$

Continuing by induction, we define $p_n \in P_{\alpha^*(n+1)}$ satisfying $p_n{\restriction}\alpha^*(m+1) = p_m$ for $m < n$, such that for some $P_{\alpha^*(n+1)}$-name $\underset{\sim}{\eta}_n$

$p_n \Vdash_{P_{\alpha^*(n+1)}}$ “$\underset{\sim}{\eta}_n \in \lim T^*_n$ and $f^*_n(\underset{\sim}{\eta}_n{\restriction}\ell) \in \underset{\sim}{G}_{P_{\alpha^*(n+1)}}$ for every $\ell < \omega$ and for $m < n, \ell < \omega$, $\eta_m{\restriction}\ell = \mathrm{res}_{T^*}\left(\underset{\sim}{\eta}_n{\restriction}\ell, \bigcup_{\ell\leq m} A_\ell\right)$”.

So $p = \bigcup_{n<\omega} p_n \in P_{\alpha^*}$, where $\alpha^* = \bigcup_n \alpha^*(n)$, and there is a P_{α^*}-name $\underset{\sim}{\eta}$ such that $p \Vdash_{P_{\alpha^*}}$ “$\underset{\sim}{\eta} \in \lim T^* \subseteq \lim T''$, and for every $n, m < \omega$, $\mathrm{res}_{T^*}(\underset{\sim}{\eta}{\restriction}n, \bigcup_{\ell\leq n} A_\ell) = \underset{\sim}{\eta}_m{\restriction}m$” (this determines $\underset{\sim}{\eta}$ uniquely as $T^* \subseteq T''$). So $p \Vdash_{P_{\alpha^*}}$ “for every $m, \ell < \omega$, $f(\underset{\sim}{\eta}{\restriction}\ell){\restriction}\alpha^*(m) \in \underset{\sim}{G}_{P_{\alpha^*}}$” hence, by the definition of RCS, as α^* is limit: $p \Vdash_{P_{\alpha^*}}$ “for every $\ell < \omega$, $f(\underset{\sim}{\eta}{\restriction}\ell){\restriction}\alpha^* \in \underset{\sim}{G}_{P_{\alpha^*}}$”.

We have here a problem: A priori, $f(\underset{\sim}{\eta}{\restriction}\ell)$ is not necessarily in P_{α^*}, (only in P_{ω_1}) so $f(\underset{\sim}{\eta}{\restriction}\ell){\restriction}\alpha^* \in \underset{\sim}{G}_{P_{\alpha^*}}$ seems to be weaker than the required “$f(\underset{\sim}{\eta}{\restriction}\ell) \in$

$G_{P_{\omega_1}}$". However, we have $f\left(\underset{\sim}{\eta}\restriction(m+\ell+1)\right)\restriction\gamma \Vdash_{P_\gamma}$ "$\underset{\sim}{\zeta}^m(f(\underset{\sim}{\eta}\restriction\ell)) = \gamma$" for some γ, so $\gamma < \alpha_{m+\ell+2}(\eta) = \alpha^*(m+\ell+2) < \alpha^*$

Since also $p \Vdash_{P_{\alpha^*}}$ "$f(\underset{\sim}{\eta}\restriction(m+\ell+2))\restriction\gamma \in \underset{\sim}{G}_{P_{\alpha^*}}$", we conclude that

$$p \Vdash \text{“}\underset{\sim}{\zeta}^m(f(\underset{\sim}{\eta}\restriction\ell)) < \alpha^*$$

for every $\ell, m < \omega$.

So p essentially forces $\forall\ell\, f(\underset{\sim}{\eta}\restriction\ell) \in P_{\alpha^*}$. Hence clearly $p \Vdash_{P_{\omega_1}}$ "$f(\underset{\sim}{\eta}\restriction\ell) \in \underset{\sim}{G}_{P_{\omega_1}}$".

If we do not have CH, we modify the proof as follows: let $g : \omega \to \omega$ be such that $g(m) \le m$ and $(\forall n)(\exists^\infty m)[g(m) = n]$. Next, for each $\alpha < \omega_1$ let $\langle \rho^\alpha_\ell : \ell < \omega\rangle$ list all finite sequences of the form $\langle (i_k, \beta_k) : k \le k^*\rangle$, $\beta_0 = 0$, $\beta_k < \beta_{k+1} \le \alpha$, $i_k < i_{k+1} < \omega$ such that if $\rho^\alpha_{\ell_1} \triangleleft \rho^\alpha_{\ell_2}$ then $\ell_1 < \ell_2$. Let $\rho^\alpha_\ell = \langle (i_k(\alpha,\ell), \beta_k(\alpha,\ell)) : k \le k^*(\alpha,\ell)\rangle$, second, we write the odd natural numbers > 2 as a doubly indexed union $\bigcup_{i,m} B_{i,m}$ of infinite disjoint sets (instead of $\bigcup_i B_i$). Then, instead of $(*)$, we define $\rho[w,\eta]$ as $\rho^{\alpha_i(\eta)}_\ell$ when $|w| \in B_{i,\ell}$ where $\alpha_i(\eta)$ was defined in Part A and so define $i = i_w$, $\ell = \ell_w$. Next, we define by induction on $|w|$ when (w,η) is nice: it is nice when $k^*(\alpha_i(\eta),\ell) = 0$ or for $k < k^*(\alpha_i(\eta),\ell)$ we have $i_k(\alpha_i(\eta),\ell) \in w$, and $\rho[w \cap i_k(\alpha,\ell), \eta\restriction i_k(\alpha,\ell)] = \rho^{\alpha_i(\eta)}_\ell\restriction(k+1)$.

Now if (w,η) is not nice we do nothing, if it is nice, we let $k = k[w,\eta]$ be the $k = g(k^*(\alpha_i(\eta),\ell))$. We let

$$\alpha = \alpha[w,\eta] = \beta_{i_k(\alpha_i(\eta),\ell)}(\alpha_i(\eta),\ell)$$

$$\beta = \beta[w,\eta] = \beta_{i_{k+1}[w,\eta]}(\alpha_i(\eta),\ell)$$

$w^* = w^*[w,\eta] = \{i : (\alpha[w\cap i, \eta\restriction i], \beta[w \cap i, \eta\restriction i]) = (\alpha,\beta)$ and for some $m < k^*$, $i = i_m(\alpha_i(\eta),\ell)\}$. Then we define the function F as before.

In part B, when we check that this construction works, we can only find a tree T'' with the property that for some α^*, for all branches η in T'', $\lim_{k\to\omega}\alpha_k(\eta) = \alpha^*$(using 3.5). Let $\langle \alpha^*(k) : k < \omega\rangle$ be a sequence of ordinals converging to α^*. Now we can shrink T'', so as to use only $F_{\alpha^*(n),\alpha^*(n+1)}$ $(n < \omega)$, i.e. let us define by induction on n (stipulating $J'_{-1} = \{\langle\rangle\}$ $J'_n =$

$\{\eta : \eta \in \bigcup_{n<\omega} J_m$, and letting $w = \{\ell < \ell g(\eta) : \eta \restriction \ell \in \bigcup_{m<\omega} J_m\}$ for some k^* and $i_0 < \ldots i_{k^*-1}$ from w, $\bigwedge_{\ell<k^*} i_\ell \in J'_{\ell-1}$ and $\rho[w,\eta] = \langle (i_\ell, \alpha^*(\ell) : \ell < k^*\rangle\char`^\langle(\ell g\eta, \alpha^*(k^*))\rangle\}$ (it is a system of fronts, i.e., every branch of T'' meets each J_n infinitely often) and let T^0, $(T'', \mathbf{I}) \le (T^0, \mathbf{I})$ be such that:
if $\eta \in T^0, \eta \in \bigcup_n J_n$ then $(\forall \alpha)(\eta\char`^\langle\alpha\rangle \in T'' \Rightarrow \eta\char`^\langle\alpha\rangle \in T^0)$
if $\eta \in T^0, \eta \notin \bigcup_n J_n$ then $(\exists! \alpha)(\eta\char`^\langle\alpha\rangle \in T^0)$.

Now we continue as before.

(2) Left to the reader (essentially the same proof). $\square_{6.3}$

6.3A Corollary. (1) For $P = \operatorname{Rlim} \bar{Q}$ as in the previous lemma, $\bigcup_{\alpha<\omega_1} P_\alpha$ is (essentially) a dense subset of P i.e. for every $p \in P_{\omega_1}$ there are q and α such that $p \restriction \alpha \le q \in P_\alpha$, $q \Vdash$ "$p \in \underset{\sim}{G}_{P_{\omega_1}}$" (in fact $r \in p \Rightarrow q \Vdash$ "$\underset{\sim}{\zeta}(r) < \alpha$").

(2) For $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta\rangle$ as in the previous lemma (so $\delta = \omega_1$ or just $\mathrm{cf}(\delta) = \omega_1$), *if* $\underset{\sim}{\bar{\alpha}}$ is a P_δ-name of an ω-sequence of ordinals ($P_\delta = \operatorname{Rlim} \bar{Q}$, of course) $p \in P_\delta$ *then* for some $i < \delta, q \in P_i$, and $\underset{\sim}{\bar{\beta}}$ a P_i-name of an ω-sequence of ordinals, $P_\delta \vDash$ "$p \le q$", and $q \Vdash_{P_\delta}$ "$\underset{\sim}{\bar{\alpha}} = \underset{\sim}{\bar{\beta}}$".

Proof: By 1.13 (or directly from the proof of 6.3). $\square_{6.3A}$

6.4 Lemma. Suppose $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \kappa\rangle$ is suitable for $\langle \mathbb{I}_{\alpha,\beta}, \lambda_{\alpha,\beta}, \mu_{\alpha,\beta} : \langle i, j\rangle \in W^*, \rangle$, κ is strongly inaccessible $|P_i| + \lambda_{i,j} + \mu_{i,j} + |\bigcup I| < \kappa$ for every $\langle \alpha, \beta\rangle \in W^*$, $I \in \mathbb{I}_{\alpha,\beta}$ and let $\mathbb{I} = \bigcup_{\alpha,\beta} \mathbb{I}_{\alpha,\beta}$. *Then* $P_\kappa = \operatorname{Rlim}(\bar{Q})$ satisfies the $\mathbb{I}$-condition.

Proof. This is quite easy, because $P_\kappa = \bigcup_{\alpha<\kappa} P_\alpha$. So let $\underset{\sim}{F}_{\alpha,\beta}$ be a P_α-name of a witness to "$P_{\alpha,\beta} \overset{\text{def}}{=} P_\beta / P_\alpha$ satisfies the $\mathbb{I}_{\alpha,\beta}$-condition", for $\alpha < \beta < \kappa$, α non-limit, and let $\omega = \bigcup_{\alpha<\omega} A_i$, the A_i's are infinite, pairwise disjoint and $n \in A_{i+1} \Rightarrow n > 1 + i$ (so $\beta(k)$ is an ordinal $< \kappa$, not just a name). Now we shall define the function F, so we should define
$F(\eta, w, \langle f(\eta \restriction \ell) : \ell \le \ell g(\eta)\rangle)$ (See Definition 2.6). Let i be such that $|w| \in A_i$, $w^* = \{\ell \in w : |w \cap \ell| \in A_i\}$ and let $\beta(0) = 0$, $\beta(1) = 1$ and for $k > 1$, $k \le \ell g(\eta)$, let $\beta(k) = \operatorname{Min}\{\gamma + k : \ell \le k \Rightarrow f(\eta \restriction \ell) \in P_\gamma\}$. Now we shall

use $\underset{\sim}{F}_{\beta(i),\beta(i+1)}$, so let $\underset{\sim}{F}_{\beta(i),\beta(i+1)}(\eta, w^*, \langle f(\eta{\restriction}\ell){\restriction}[\beta(i), \beta(i+1) : \ell \le \ell g(\eta)\rangle)$ be $\langle \underset{\sim}{I}, \langle \underset{\sim}{r}_\nu : \nu \in \bigcup \underset{\sim}{I}\rangle\rangle$, and choose $q^0_\eta \in P_{\beta(i)}$ such that $q^0_\eta \Vdash_{P_{\beta(1)}}$ "$\underset{\sim}{I} = \mathbf{I}_\eta$", and $P_{\beta(i)} \Vdash$"$f(\eta){\restriction}\beta(i) \le q^0_\eta$". Now we can define $F(\eta, w, \langle f(\eta{\restriction}\ell) : \ell \le \ell g(\eta)\rangle) = \langle \mathbf{I}_\eta, \langle f(\nu) : \nu \in Suc_T(\eta)\rangle\rangle$ and $Suc_T(\eta) = \{\eta\hat{\ }\langle\alpha\rangle : \alpha < \bigcup \mathbf{I}_\eta\}$, and $f(\nu) = q^0_\eta \cup \underset{\sim}{r}_\nu$. Note that $\beta(k)$ depends on $\eta{\restriction}k$, so we should have written $\beta(\eta{\restriction}k)$, see below.

Now suppose we are given $(T, \mathbf{I}), J_n, f$ as in Definition 2.6. (for $P = \operatorname{Rlim}\bar{Q}$ and $\mathbb{I} = \bigcup_{i<j}\mathbb{I}_{i,j}$) and $(T, \mathbf{I}) \le^* (T', \mathbf{I})$ and we have to find p as required. Let for every $\eta \in T$, $\beta(\eta)$ be 0 if $\ell g(\eta) = 0$, 1 if $\ell g(\eta) = 1$ and $\operatorname{Min}\{\gamma + \ell g(\eta) : f(\eta) \in P_\gamma$ otherwise; so $\nu \lhd \eta \Rightarrow \beta(\nu) < \beta(\eta)$ and $\beta(\eta)$ is never a limit ordinal. So by a repeated use of Lemma 5.3 we can get T^*, $(T^\dagger, \mathbf{I}) \le^* (T^*, \mathbf{I})$ such that:

(∗) for every $\eta \in T^*$ and $\eta \lhd \nu_\ell \in T^*$, $\operatorname{res}(\nu_1, A_\eta) = \operatorname{res}(\nu_2, A_\eta)$ then $\langle f_{\nu_1{\restriction}\ell}{\restriction}\beta(\eta)\rangle : \ell \le \ell g\nu_1\rangle = \langle \rho_{\nu_2{\restriction}\ell}(\beta(\eta)) : \ell \le \ell g\nu_2\rangle$ where $A_\eta = \{\nu \in T : |\operatorname{Suc}_T(\nu)| < \mu_{\theta,\beta(\eta)}\}$

By induction on n we will now define prompt names $\underset{\sim}{\beta}_n$ of ordinals, conditions $p_n \in P_{\underset{\sim}{\beta}_n}$ and $P_{\underset{\sim}{\beta}_n}$-names $\underset{\sim}{\eta}_n$ and $\underset{\sim}{\nu}_n$ such that p_n forces the following

(1) $\underset{\sim}{\eta}_n \in T^*, \ell g(\underset{\sim}{\eta}_n) = n$

(2) $\underset{\sim}{\eta}_n \lhd \underset{\sim}{\nu}_n \in \lim(\operatorname{res}(T^*, A_{\eta_n}))$

(3) $\forall \rho \in T^* \forall l < \omega$: if $\nu_n{\restriction}\ell = \operatorname{res}(\rho, A_{\underset{\sim}{\eta}_n})$, then $f(\rho){\restriction}\underset{\sim}{\beta}_n \in G_{\underset{\sim}{\beta}_n}$ and $\underset{\sim}{\beta}_n = \beta(\underset{\sim}{\eta}_n)$.

For $n = 0$ there is nothing to do: ($\eta_0 = \langle\rangle$, $\beta_0 = 0$).
In stage $n+1$ we will work in $V[G_{P_{\underset{\sim}{\beta}_n}}]$, where $G_{P_{\underset{\sim}{\beta}_n}}$ is a generic filter on $P_{\underset{\sim}{\beta}_n}$ containing p_n, more formally, we have $\beta_n < \kappa$ and $\underset{\sim}{G}_{P_{\beta_n}} \subseteq P_{\beta_n}$ generic over V such that $\Vdash_{P_\kappa/G_{P_{\underset{\sim}{\beta}_n}}}$ "$\underset{\sim}{\beta}_n = \beta_n$". We let $\eta_{n+1} = \nu_n{\restriction}n+1$, $\beta_{n+1} = \beta(\eta_{n+1})$. Since we have used $F_{\beta(\eta_n),\beta(\eta_{n+1})}$ we can find a condition $p_{n,n+1}$ in $P_{\beta_n,\beta_{n+1}}$ and a $P_{\beta_n,\beta_{n+1}}$-name $\underset{\sim}{\nu}_{n+1}$ such that

$$\Vdash_{P_{\beta_{n+1}}/P_{\beta_n}} \text{“}\underset{\sim}{\nu}_{n+1}{\restriction}\ell = \operatorname{res}_{T^*}(\rho, A_{\eta_{n+1}}) \Rightarrow f(\rho){\restriction}[\beta_n, \beta_{n+1}) \in \underset{\sim}{G}_{P_{\underset{\sim}{\beta}_n,\underset{\sim}{\beta}_{n+1}}}\text{”}$$

Now we can return to V and translate everything back to $P_{\underset{\sim}{\beta}_n}$-names, and get a condition p_{n+1} from p_n and $p_{n,n+1}$.

Since we are using RCS iteration see (1.12)(1), letting $\beta^* = \sup_n \underset{\sim}{\beta}_n$, we can after ω many steps find a condition $p \in P_{\beta^*}$ which is stronger than every p_n, and a P_{β^*}-name $\underset{\sim}{\nu}$ of a branch in T^*, defined by $\underset{\sim}{\nu} = \bigcup_n \underset{\sim}{\eta}_n$. This is as required. $\square_{6.4}$

6.5 Lemma. Suppose $\bar{Q} = \langle P_\alpha, Q_\alpha : \alpha < \kappa\rangle$ is suitable for $\langle \mathbb{I}_{\alpha,\beta}, \lambda_{\alpha,\beta}, \mu_{\alpha,\beta} : \langle\alpha,\beta\rangle \in W^*\rangle$, κ is strongly inaccessible, $|\bigcup I| + \lambda_{\alpha,\beta} + \mu_{\alpha,\beta} < \kappa$ for any $\langle\alpha,\beta\rangle \in W^*$, and $I \in \mathbb{I}_{\alpha,\beta}$, $\underset{\sim}{Q}_\kappa$ is a P_κ-name of a forcing notion satisfying the $\mathbb{I}_\kappa$-condition and let $\mathbb{I}_0 = \bigcup_{\alpha,\beta} \mathbb{I}_{\alpha,\beta}, \mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_\kappa$. Let

$$\begin{aligned} A^* = \{\alpha < \kappa : \text{ for every } i < \alpha, \Vdash_{P_i} \text{“cf}(\alpha) > \aleph_0\text{” and for every } I \in \mathbb{I}_0,\\ V \models \text{“}|\bigcup I| \geq \alpha \Rightarrow I \text{ is } |\alpha|^+\text{-complete” and}\\ V \models \text{“}|\bigcup I| \geq \mathrm{cf}\alpha \Rightarrow I \text{ is } |\mathrm{cf}\alpha^+|\text{-complete}\} \end{aligned}$$

and assume:

(a) for every $I \in \mathbb{I}_\kappa$: either I is κ^+-complete or I is κ-complete and normal and $\kappa \setminus A^* \in I$.

(b) for some $I^* \in \mathbb{I}_\kappa$ and $\bigcup I^* = \kappa$ and all singletons are in I^*.

Then $P_\kappa * \underset{\sim}{Q}_\kappa$ satisfies the $\mathbb{I}$-condition.

Remark: In Gitik Shelah [GiSh:191], $(a) + (b)$ were weakened to: each $I \in \mathbb{I}$ is κ-complete (or see XV §3).

Proof. Let $\underset{\sim}{F}_{\alpha,\beta}$ witness "$P_{\alpha,\beta} = P_\beta / P_\alpha$ satisfies the $\mathbb{I}_{\alpha,\beta}$-condition" and $\underset{\sim}{F}_\kappa$ (a P_κ-name) witness "$\underset{\sim}{Q}_\kappa$ satisfies the $\mathbb{I}_\kappa$-condition" and $\bigcup_{i,j,k,m<\omega} A_{i,j,k,m} = \{3n+2 : n < \omega\}$, the $A_{i,j,k,m}$'s infinite pairwise disjoint and $n \in A_{i,j,k,m} \Rightarrow i,j,k,m < n$.

Now we shall define the function F, so we should define $\langle \mathbf{I}_\eta, \langle f(\nu) : \nu \in \mathrm{Suc}_T(\eta)\rangle\rangle = F(\eta, w, \langle f(\eta{\restriction}\ell) : \ell < \ell g(\eta)\rangle)$ (see Definition 2.6). If $p \in P_\kappa * \underset{\sim}{Q}_k$ we will write $p{\restriction}\kappa$ for the P_κ-component of p and $p(\kappa)$ for the Q_κ-component.

Case i. $|w|$ is divisible by 3.

We let $\mathbf{I}_\eta = I^*$ and for $\nu \in \mathrm{Suc}_T(\eta)$ we have $f(\nu) \geq f(\eta)$ be such that: if $\ell < \ell g(\eta)$, $\eta(\ell) \in A^*$, then for some $\alpha < \eta(\ell), f(\nu)\restriction\eta(\ell) \in P_\alpha$ (possible by 6.3(2)). We denote the minimal such α by $\alpha_{\nu,\ell}$. Thus, $\alpha_{\nu,\ell} < \nu(\ell)$ and $f(\nu)\restriction\nu(\ell) \in P_{\alpha_{\nu,\ell}}$.

Case ii. $|w| + 2$ is divisible by 3. We act as in the proof of 5.1, i.e., we use our winning strategy for the game on $\underset{\sim}{Q}_\kappa$: let $w^* = \{\ell \in w : |w \cap \ell| + 2$ is divisible by 3$\}$, and we let $\langle \underset{\sim}{I}, \langle \underset{\sim}{r}_\nu : \nu \in \underset{\sim}{B} \rangle\rangle = \underset{\sim}{F}_\kappa(\eta, w^*, \langle f(\eta\restriction\ell)(\kappa) : \ell \leq \ell g(\eta)\rangle)$.

Choose $q_\eta \in P_\kappa$, $q_\eta \geq f(\eta)\restriction\kappa$, $q_\eta \Vdash_{P_\kappa}$ "$\underset{\sim}{I} = \mathbf{I}_\eta$" for some $\mathbf{I}_\eta$, and let $F(\eta, w, \langle f(\eta\restriction\ell) : \ell \leq \ell g(\eta)\rangle) = \langle \mathbf{I}_\eta, (q_\eta, r_\nu) : \nu = \eta\hat{\ }\langle\alpha\rangle, \alpha < \bigcup \mathbf{I}_\eta\rangle\rangle$

Case iii. $|w| + 1$ is divisible by 3.

So for a unique quadruple $\langle i, j, k, m\rangle, |w|$ belongs to $A_{i,j,k,m}$, (hence $i, j, k, m < |w| \leq \ell g(\eta)$) and let $w^* = \{\ell \in w : |w \cap \ell| \in A_{i,j,k,m}\}$.

Now we shall use $\underset{\sim}{F}_{\xi,\zeta}(\eta, w^*, \langle f(\eta\restriction\ell)\restriction[\xi,\zeta) : \ell < \ell g(\eta)\rangle)$ were $\xi < \zeta < \kappa$ are chosen as follows

if $\eta(i) < \eta(j), \eta(i) \in A^*, \eta(j) \in A^*, i < k, j < k, k < m$, $|w \cap k|$ and $|w \cap m|$ are divisible by 3, (so $\alpha_{\eta\restriction k,i}, \alpha_{\eta\restriction m,i}$ are well defined) *then* let $\xi = \alpha_{\eta\restriction k,i} + k + 1, \zeta = \alpha_{\eta\restriction m,i} + m + 1$

if $\eta(i) < \eta(j), \eta(i) \in A^*$, $\eta(j) \in A^*, i < k, j < k, k = m$, $|w \cap k| - 1$ is divisible by 3 and $\eta(i) < \alpha_{\eta\restriction k,i}$ *then* let $\xi = \eta(i), \zeta = \alpha_{\eta\restriction k,i} + k + 1$

if $\eta(i) \geq \eta(j), \eta(i) \in A^*$, $i < k < m$, $|w \cap k|$ and $|w \cap m|$ are divisible by 3 *then* let $\xi = \alpha_{\eta\restriction k,i} + k + 1$, $\zeta = \alpha_{\eta\restriction m,i} + m + 1$

if $\eta(i) \geq \eta(j), \eta(i) \in A^*$, $i < k = m$, $|w \cap k|$ is divisible by 3 *then* let $\xi = 0, \zeta = \alpha_{\eta\restriction k,i} + k + 1$

if none of the above occurs *then* let $\xi = 0, \zeta = 1$.

So let $(T, \mathbf{I}), J_n, f$ be as in Definition 2.6 and $(T, \mathbf{I})^* \leq^* (T', \mathbf{I})$, w.l.o.g. $\bigcup_{n<\omega} J_n$ is the set of splitting points of $(T, \mathbf{I})$, (shrink T considering T'), and for notational simplicity we assume $J_n = \{\eta \in T : \ell g(\eta) = n\}$. So for η we have used $\omega = \{\ell : \ell < \ell g(\eta)\}$. Let $\sigma_\eta \stackrel{\text{def}}{=} \eta(\ell g(\eta) - 1)$.

We have to find p as required there.

Assume for simplicity CH. So by Lemma 3.5A, we can find $T^1, (T^\dagger, \mathbf{I}) \leq^* (T^1, \mathbf{I})$ such that

(a) For some $B_0 \subseteq \omega \times \omega$, for every $\eta \in \lim T^1$, $\eta(i) < \eta(j)$ *iff* $\langle i, j \rangle \in B_0$

(b) For $\eta \in T^1 : \ell g(\eta) \in B_1$ *iff* $\mathbf{I}_\eta$ is κ^+-complete; also $\ell g(\eta) \in B_1$ implies and $\ell g(\eta) + 2$ is divisible by 3

(c) If $\ell g(\eta) \in B_2 \stackrel{\text{def}}{=} \{n < \omega : n + 2 \text{ is divisible by } 3,\ n \notin B_1\}$ then $\kappa \setminus A^* \in \mathbf{I}_\eta$ and $\mathbf{I}_\eta$ is a normal ideal on κ

(d) If $\ell g(\eta) \in B_3 \stackrel{\text{def}}{=} \{n : n + 1 \text{ is divisible by } 3\ \}$ then $\bigcup \mathbf{I}_\eta$ has cardinality $< \kappa$.

Let $A^{**} = \{\alpha \in A^* : \alpha$ is strong limit and for every $\langle i, j \rangle \in W^*$, $I \in \mathbb{I}_{i,j}$, if $i < j < \alpha$ then $|\bigcup I| + \lambda_{i,j} + \mu_{i,j} + |P_i| < \alpha\}$

Clearly if $I \in \mathbb{I}_\kappa$ is not κ^+-complete then $\kappa \setminus A^{**} \in I$ (since I is normal)

(e) if $(\ell g(\eta) - 1) \in B_4 \stackrel{\text{def}}{=} \{n : n \text{ divisible by } 3\}$ and $\sigma_{\eta\restriction(m+1)} \in A^{**}$ then (being normal) $f(\eta)\restriction\sigma_{\eta\restriction(m+1)}$ is (equivalent to) a member of $\bigcup\{P_\gamma : \gamma < \sigma_{\eta\restriction(m+1)}\}$

say to some member of $P_{\alpha_{\eta,m}}$, where $\alpha_{\eta,m} < \eta(m)$ (see case (i)).

We can conclude (by (c) and (e) above) that without loss of generality

(f) If $\eta \in T^1, \ell g(\eta) - 1 \in B_2$ then $\sigma_\eta \in A^{**}$. Also $\eta(\ell) < \kappa \Rightarrow \eta(\ell) < \sigma_\eta$, and $[|\bigcup \mathbf{I}_{\eta\restriction m}| < \kappa \Rightarrow |\bigcup \mathbf{I}_{\eta\restriction m}| < \sigma_\eta]$ for every $\ell < \ell g(\eta) - 1$, $m < \ell g(\eta)$; and if $\eta\char`^\langle\alpha\rangle \in T^1, \mathbf{I}_\eta$ is κ^+-complete then $\alpha > \kappa$.

For $\eta \in T^1$ if $\ell g(\eta) - 1 \in B_2$ let

$A_\eta = \{\nu \in T^1 : \nu \trianglelefteq \eta \text{ or: } \eta \triangleleft \nu \text{ and } |\bigcup \mathbf{I}_\eta|^V < \sigma_\eta\}$

$A_\eta^\dagger = \{\nu \in T^1 : \nu \trianglelefteq \eta \text{ or: } \eta \triangleleft \nu \text{ and } |\bigcup \mathbf{I}_\eta|^V < \mathrm{cf}^V(\sigma_\eta)\}$

By a repeated use of 5.3 (starting at $\langle\rangle$ and going up in T^1) we can find T^2, $(T^1, \mathbf{I}) \leq^* (T^2, \mathbf{I})$ such that

(g) If $\eta \in T^2, \ell g(\eta) - 1 \in B_2$, $\ell g(\eta) \leq m < \omega$, m divisible by 3 and for $\ell = 1, 2$, $\nu_\ell \in T^2$, $\ell g(\nu_\ell) = m + 1$, $\mathrm{res}(\nu_1, A_\eta) = \mathrm{res}(\nu_2, A_\eta)$ then $\alpha_{\nu_1, lg(\eta)-1} = \alpha_{\nu_2, lg(\eta)-1}$ (notice that $\alpha_{\nu, lg(\eta)-1} < \sigma_\eta < \kappa$, so κ-completeness suffices).

(h) If $\eta \in T^2, \ell g(\eta) - 1 \in B_2$, $\ell g(\eta) \leq m < \omega$, m divisible by 3, then there is $\gamma_{\eta,m} < \sigma_\eta$ such that if $\nu \in T^2$, $\ell g(\nu) = m + 1$ then $\alpha_{\nu, \ell g(\eta)-1} \leq \gamma_{\eta,m}$.

Note

(i) If $\eta \in T^2, \ell g(\eta) - 1 \in B_2$, then σ_η has cofinality $> \aleph_0$ hence $\gamma_\eta = \bigcup\{\gamma_{\eta,m} : m \geq \ell g(\eta), m$ divisible by $3\}$ is $< \sigma_\eta$.

Now if $\eta \in T^2, \ell g(\eta) \in B_2$ then the function $\alpha \mapsto \gamma_{\eta\frown\langle\alpha\rangle}$ is a regressive function on a subset of κ which do not belong to $\mathbf{I}_\eta$. Hence for some γ, $\{\nu : \nu = \eta\frown\langle\alpha\rangle, \gamma_\nu = \gamma\}$ is not in $\mathbf{I}_\eta$.

So without loss of generality

(j) If $\eta \in T^2, \ell g(\eta) \in B_2$ then for some $\gamma_\eta < \kappa$:

$(\forall\alpha)[\eta\frown\langle\alpha\rangle \in T^2 \Rightarrow \gamma_{\eta\frown\langle\alpha\rangle} = \gamma_\eta]$

So w.l.o.g.

(k) If $\eta \in T^2, \ell g(\eta) \in B_2$, $\eta \lhd \nu \in T^2$ then $|\mathbf{I}_\nu| \leq \gamma_\eta$ or $\mathbf{I}_\nu$ is $|\sigma_{\nu\restriction(\ell g(\eta)+1)}|^+$-complete.

If $\eta \in T^2, \ell g(\eta) \in B_2$ let

$$T^2_\eta = \{\mathrm{res}(\nu, A_\eta) : \nu \in T^2\}.$$

So w.l.o.g.

(l) If $\eta \in T^2, \ell g(\eta) \in B_2$, and for $\ell = 1,2$ $\nu_\ell \in T^2$, $\eta \lhd \nu_\ell$, $\mathrm{res}(\nu_1, A_\eta) = \mathrm{res}(\nu_2, A_\eta)$ then $p_{\nu_1}\restriction\gamma_\eta = p_{\nu_2}\restriction\gamma_\eta$ and $|\cup \mathbf{I}_{\nu_1}| < \gamma_\eta \Leftrightarrow |\bigcup \mathbf{I}_{\nu_2}| < \gamma_\eta \Rightarrow \mathbf{I}_{\nu_1} = \mathbf{I}_{\nu_2},]$.

Let $\ell_0 = 0$ and $\{\ell_m : 1 \leq m < \omega\}$ be an enumeration in increasing order of $\{\ell < \omega : \ell - 1 \in B_2\}$. For any $\eta \in T^2, \ell g(\eta) = \ell_m$ we define β_η as $\mathrm{Sup}\{\gamma_\nu : \eta \lhd \nu, \gamma_\nu$ defined and $\ell g(\nu) < \ell_{m+1}\}$. Remembering that if $\ell g(\nu) \notin B_2$ then $|\bigcup \mathbf{I}_\nu| < \kappa$ or $\mathbf{I}_\nu$ is κ^+-complete it is clear that w.l.o.g.

(m) If $\eta \in T^2, \ell g(\eta) = \ell_m$ then $\beta_\eta < \kappa$; and $\eta \lhd \nu \in T^2$, $\ell g(\nu) = \ell_{m+1}$ implies $\beta_\eta < \sigma_\nu$. Let, for $\eta \in T^2, \ell g(\eta) = \ell_m$,

$$A^*_\eta \stackrel{\mathrm{def}}{=} \{\nu \in T^2 : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu, \ |\bigcup \mathbf{I}_\nu| < \gamma_\eta\}.$$

For every $\eta \in T^2, \ell g(\eta) = \ell_m$ we define $T^2_\eta \stackrel{\mathrm{def}}{=} \{\mathrm{res}(\nu, A^*_\eta) : \nu \trianglelefteq \eta \text{ or } \eta \lhd \nu\}$ and we define f_η and $\mathbf{I}^\eta$ (functions with domain $\subseteq T^2_\eta$) by: $f_\eta(\rho) = f(\nu)\restriction\beta_\eta$ and $\mathbf{I}^\eta_\rho = \mathbf{I}_\eta$ if $\eta \trianglelefteq \nu \in T_2$, $\rho = \mathrm{res}(\nu, A^*_\eta)$, is except that $\mathbf{I}^\eta_\rho$ is defined only if $|\bigcup \mathbf{I}_\eta| < \gamma_\eta$.

Now look at $T^2_{\langle\rangle}, f_{\langle\rangle}$ and $p\restriction\alpha_{\langle\rangle}$ (w.l.o.g. $p\restriction\alpha_{\langle\rangle} = p\restriction\kappa$). As in case (iii) of the definition of F we work hard enough (repeating previous proofs) there is a $P_{\beta_{\langle\rangle}}$-name $\underset{\sim}{\eta}_{\langle\rangle}$ and $q_{\langle\rangle} \in P_{\beta_{\langle\rangle}}$ such that :

$q_{\langle\rangle} \Vdash_{P_{\beta_{\langle\rangle}}}$ "$f_{\langle\rangle}(\underset{\sim}{\eta}_{\langle\rangle}\restriction\ell) \in \underset{\sim}{G}_{\beta_{\langle\rangle}}$ for every ℓ" as our strategy will often have used $F_{\alpha_{\langle\rangle},\beta_{\langle\rangle}}$.

Now comes a crucial observation: if $G_{\beta_{\langle\rangle}} \subseteq P_{\beta_{\langle\rangle}}$ is generic (over V) $\eta \in T_2, \ell g(\eta) = \ell_1$ and $\mathrm{res}(\eta, A^*_0) \lhd \eta_{\langle\rangle}[G_{\beta_{\langle\rangle}}]$, $\nu = \eta\hat{\ } < \alpha > \in T^2$ then we can (as in 5.1 and above) find $q_\nu \in P_{\beta_\nu}$, and P_{β_ν} -name $\underset{\sim}{\eta}_\nu$ such that: q_ν is compatible with every member of $G_{\beta_{\langle\rangle}}$ and

$q_\nu \Vdash_{\beta_{\langle\rangle}}$ "$f_\nu(\underset{\sim}{\eta}_\nu\restriction\ell) \in \underset{\sim}{G}_{\beta_\nu}$ for every ℓ, and for every $\eta \in T^2$, $[\mathrm{res}(\eta, A^*_\nu) \lhd \underset{\sim}{\eta}_\nu \Rightarrow \mathrm{res}(\eta_{\langle\rangle}, A^*_{\langle\rangle}) \trianglelefteq \eta_{\langle\rangle}]$". Note that $q_{\langle\rangle} \Vdash$ "$\{\alpha : q_{\eta\hat{\ }\langle\alpha\rangle} \in G_{P_\kappa}\} \neq \emptyset \bmod \mathbf{I}_\eta$". Hence each q_η can play the role of $q_{\langle\rangle}$ in the next step:

We can continue and define $q_\nu, \underset{\sim}{\eta}_\nu$ for every $\nu \in T^2, \bigvee_{m<\omega} \ell g(\nu) = \ell_m$ with the obvious properties:

(1) $q_\nu \in P_{\gamma_\nu}$,

(2) $q_\nu\restriction[\sigma_\nu, \gamma_\nu) = q_\nu$ when $\ell g(\nu) > 0$,

(3) $q_\nu \Vdash_{P_{\beta_\nu}}$ "$\underset{\sim}{\eta}_\nu \in \lim T^2_\nu$"

(4) $q_\nu \Vdash_{P_{\beta_\nu}}$ "if for every $\ell_m < \ell g(\nu)$, $\mathrm{res}(\nu, A^*_{\nu\restriction\ell_m}) \lhd \underset{\sim}{\eta}_{\nu\restriction\ell_m}$ and

$$f_{\nu\restriction\ell_m}(\underset{\sim}{\eta}_\nu\restriction k) \in \underset{\sim}{G}_{\nu\restriction\ell_m} \text{ for every } k$$

then for every $k < \omega$ and $\ell_m < \ell g(\nu)$,

$$\mathrm{res}(\underset{\sim}{\eta}_\nu\restriction k,\ A^*_{\nu\restriction\ell_m}) \lhd \underset{\sim}{\eta}_{\nu\restriction\ell_m} \text{ and } f_\nu(\underset{\sim}{\eta}_\nu\restriction k) \in \underset{\sim}{G}_{\beta_\nu} \text{ and } \nu \trianglelefteq \underset{\sim}{\eta}_\nu\text{''}.$$

Now we define a P_κ-name of a subtree of T^2 : $\underset{\sim}{T}^3 = \{\nu\restriction n : n < \ell g(\nu)$, $\ell g(\nu) = \ell_m$ for some $m, q_\nu \in \underset{\sim}{G}_{P_\kappa}$ and for $i < m$ $q_{\nu\restriction\ell_i} \in \underset{\sim}{G}_{P_\kappa}$ and $\mathrm{res}(\nu, A^*_{\nu\restriction\ell_i}) \lhd \underset{\sim}{\eta}_{\nu\restriction\ell_i}\}$. Clearly,

$$\begin{aligned} q_{\langle\rangle} \Vdash_{P_\kappa} \text{``}&\underset{\sim}{T}^3 \subseteq T^2, T_3 \text{ is closed under initial segments,}\\ &\mathrm{Suc}_{\underset{\sim}{T}^3}(\eta) \neq \emptyset \bmod \mathbf{I}_\eta \text{ for } \eta \in \underset{\sim}{T}^3 \text{ and}\\ &\text{if } \eta \in T^3, \ell g(\eta) \in B_1 \cup B_2 \text{ then } \mathrm{Suc}_{\underset{\sim}{T}^3}(\eta) = Suc_{T^2}(\eta)\text{''}\end{aligned}$$

$$q_{\langle\rangle} \Vdash_{P_\kappa} \text{“ for every } \eta \in \underset{\sim}{T}^3, f(\eta)\restriction[0,\kappa) \text{ belongs to } \underset{\sim}{G}_\kappa\text{”}.$$

Now we can use the hypothesis “$\underset{\sim}{F}_\kappa$ exemplifies that $\underset{\sim}{Q}_\kappa$ satisfies the $\mathbb{I}_\kappa$-condition” and case (ii) in the definition of F to finish. $\square_{6.5}$

6.6 Lemma. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \le \alpha, j < \alpha\rangle$ is suitable for $\langle \mathbb{I}_{i,j}\ , \lambda_{i,j}\ , \mu_{i,j}\ : i < j \le \alpha\ , i$ is non-limit$\rangle$, $i(*) < \alpha$ is non-limit, $G_{i(*)} \subseteq P_{i(*)}$ generic over V, and $\langle i_\zeta : \zeta \le \beta\rangle$ is an increasing continuous sequence of ordinals in $V[G_{i(*)}]$, $i_0 = i(*)$, $i_\beta = \alpha$, each $i_{\zeta+1}$ a successor ordinal .
In $V[G_{i(*)}]$ we define $P'_\zeta = P_{i_\zeta}/G_{i(*)}$, $\underset{\sim}{Q}'_\zeta = \underset{\sim}{Q}_{i_\zeta}/[G_{i(*)}]$, (still a P_ζ-name) $\bar{Q}' = \langle P'_\zeta, \underset{\sim}{Q}'_\xi : \zeta \le \beta, \xi < \beta\rangle$, *then* in $V[G_{i(*)}]$, $\bar{Q}'$ is suitable for $\langle \mathbb{I}_{i_\zeta, i_\xi}, \lambda_{i_\zeta,i_\xi}, \mu_{i_\zeta,i_\xi} : \zeta < \xi \le \beta,\ \zeta$ non-limit $\rangle$.

Remark. Had we allowed $\mathbb{I}_{i,j}$, $\lambda_{i,j}$, $\mu_{i,j}$ to be suitable names we would have obtained here a stronger theorem.

Proof. Straightforward. $\square_{6.6}$

6.7 Conclusion. Suppose

(a) $\bar{Q} = \langle P_i, Q_i : i < \alpha\rangle$ is an RCS iteration.

(b) $\underset{\sim}{Q}_i$-satisfies the $\mathbb{I}_i$-condition, and $\mathbb{I}_i$ is $\aleph_2$-complete (in V^{P_i}, but $\mathbb{I}_i \in V$)

(c) if $\mathrm{cf}(i) < i \vee (\exists j < i)|P_j| \ge i$, then for some λ, μ we have $\bigcup_{j\ge i} \mathbb{I}_j$ is λ^+-complete and $(\forall I \in \bigcup_{j<i} \mathbb{I}_j)(|\bigcup I| < \mu)$ and $|P_i| \le \lambda = \lambda^{<\mu}$

(d) if $\mathrm{cf}(i) = i\ \&\ (\forall j < i)|P_j| < i$ then every $I \in \mathbb{I}_i$ is i^+-complete *or* normal, and e.g. $A^*_i = \{\alpha < i : \mathrm{cf}(\alpha) \ne \aleph_1\} \in I$.

Then $\mathrm{Rlim}\,\bar{Q}$ satisfies the $(\bigcup_{i<\alpha}\mathbb{I}_i)$-condition; if in addition κ is strongly inaccessible and $\bigwedge_{i<\kappa}|P_i| < \kappa$ then $\mathrm{Rlim}\,\bar{Q}$ satisfies the κ-c.c.

Proof. We should prove by induction on $i \le \alpha$ that for every $j < i$, P_i/P_j satisfies the $\bigcup\{\mathbb{I}_\gamma : j \le \gamma < i\}$-condition using 5.1, 6.2, 6.3, 6.4, 6.5 (and 5.6). $\square_{6.7}$

6.8 Conclusion. We can satisfy the demands of 1.1 by "P satisfies $\{\lambda : \aleph_2 \leq \lambda \leq |P|, \lambda$ regular $\}$"-condition.

Concluding Remark. We could have strengthened somewhat the result 6.6, but with no apparent application by using a larger A_i^*.

§7. Further Independence Results

In this section we complete some independence results.

7.1 Theorem. The following are equiconsistent.

(a) ZFC + there is a Mahlo cardinal.

(b) ZFC + G.C.H.+ $Fr^+(\aleph_2)$ (where $Fr^+(\aleph_2)$ means that every stationary $S \subseteq S_0^2 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\}$ contains a closed copy of ω_1).

Remark. Our proof will use, in addition to the ideas of the proof of Theorem 1.4 also ideas of the proof of Harrington and Shelah [HrSh:99], but, for making the iteration work, we build a quite generic object rather than force it (as in [Sh:82]).

2) In b) we can also contradict G.C.H. (using XV §3 for a) $\Rightarrow$ b)).

Proof. The implication b) $\Rightarrow$ a) was proved by Van Lere, using the well known fact that if in L there is no Mahlo cardinal, then the square principle holds for $\aleph_2$. So the point is to prove a) $\Rightarrow$ b). As any Mahlo cardinal in V is a Mahlo cardinal in L, we can assume $V = L$, κ a strongly inaccessible Mahlo cardinal.

We shall define a revised countable support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$, $|P_i| \leq \aleph_{i+1}$. If i is not a strongly inaccessible cardinal $\underset{\sim}{Q}_i$ is the Levy collapse of $2^{\aleph_1}$ to $\aleph_1$ by countable conditions (in V^{P_i}). If i is strongly inaccessible then $\underset{\sim}{Q}_i$ is $P[\underset{\sim}{S}_i]$ (see 4.6), $\underset{\sim}{S}_i$ is a P_i-name of a stationary subset of $S_0^i = \{\delta < i : \mathrm{cf}(\delta) = \aleph_0$ in $V\}$ (note $\Vdash_{P_i}$ "$i = \aleph_2$" by 1.1(3)), where $\underset{\sim}{S}_i$ will be carefully chosen as described below. Note that $\aleph_1^V = \aleph_1^{V^{P_\kappa}}$, $\kappa = \aleph_2^{V^{P_\kappa}}$ (again we use 1.1(3)) so P_κ collapse no cardinal $\geq \kappa$ and in V^{P_κ} the GCH holds.

We let $P_\kappa = \mathrm{Rlim}\,\bar{Q}$. In V^{P_κ} we define an iterated forcing $\bar{Q}^* = \langle P_i^*, \underset{\sim}{Q}_i^* : i < \kappa^+ \rangle$, with support of power $\aleph_1$, such that (in V^{P_κ}) for each i, $\underset{\sim}{S}_i^*$ is a P_i^*-name of a subset of S_0^2 which does not contain a closed copy of ω_1, and:

$\underset{\sim}{Q}_i^* = \{f \in (V^{P_\kappa})^{P_i^*} : f$ an increasing continuous function from some $\alpha < \omega_2$ into $\omega_2 \setminus \underset{\sim}{S}_i^*$ and if α is a limit ordinal, $\bigcup_{i<\alpha} f(i) \notin \underset{\sim}{S}_i^*\}$ (so $\underset{\sim}{Q}_i$ makes S_i^* nonstationary).

We shall prove that (if the $\underset{\sim}{S}_i$'s were chosen suitably then):

$(*)$ For every $\alpha < \kappa^+$, forcing by P_α^* does not add new ω_1-sequences (to V^{P_κ}) and P_α^* contains a dense subset of power $\leq \aleph_2$ (everything in V^{P_κ}).

This implies that $P_{\kappa^+}^*$ satisfies the κ^+-chain condition, so by a suitable bookkeeping every $P_{\kappa^+}^*$-name of $S \subseteq S_0^2$ which does not contain a closed copy of ω_1 is $\underset{\sim}{S}_i^*$ for some i. So easily we can conclude that it is enough to prove $(*)$.

So let $\alpha^* < \kappa^+, p \in P_{\alpha^*}^*$, and $\underset{\sim}{\tau}$ be a $P_{\alpha^*}^*$-name of a function from ω_1 to ordinals.

For all those things we have P_κ-names, (but α^* is an actual ordinal in V, as P_κ satisfies the κ-c.c.). Now in V we can define an increasing continuous sequence $N_i^* (i < \kappa)$ of elementary submodels of $H(\kappa^{+++})$ of cardinality $< \kappa$ such that P_κ, $\bar{Q}$ and all the names involved belong to N_0^*, $\langle N_i^* : i \leq j \rangle \in N_{j+1}^*, N_{j+1}^*$ is closed under sequence of length $\leq |N_j^*|$.

Now in V, as $V = L$, $\diamondsuit_{\{\lambda < \kappa : \lambda \text{ is strongly inaccessible}\}}$ holds, so we have guessed $\langle N_i^* : i \leq \lambda \rangle$, $\underset{\sim}{\tau}, p, \ldots$ in some stage λ, $(\bigcup_{i<\lambda} N_i^*) \cap \kappa = N_\lambda^* \cap \kappa = \lambda$. Really we are only guessing subsets of κ, so we can only guess the isomorphism type of N_λ, etc., or equivalently, its Mostowski collapse. I.e. let f be a one to one function from κ onto $\bigcup\limits_{i<\kappa} N_i^*$, let h be a one to one function from κ^3 onto κ, and $h_\ell : \kappa \to \kappa$ for $\ell < 3$ be such that $\varepsilon = h(\alpha, \beta, \gamma) \Leftrightarrow \alpha = h_0(\alpha)$ & $\beta = h_1(\beta)$ & $\gamma = h_2(\gamma)$. Let

$$A = \{h(0, \alpha, \beta) : f(\alpha) \in f(\beta) \text{ and } \alpha, \beta < \kappa\} \cup \{h(1, i, \alpha) : f(\alpha) \in N_i\}$$

$$\cup\{h(2,0,\alpha) : f(\alpha) = \langle \bar{Q}, \bar{Q}^*, \underset{\sim}{T}, p, \alpha^* \rangle\}.$$

Let $\langle A_\lambda : \lambda < \kappa \text{ inaccessible}\rangle$ be a diamond sequence. Let N^λ be the unique transitive subset of $H(\kappa)$ isomorphic to $(\lambda, \in^\lambda)$ where $\in^\lambda = \{(\alpha,\beta) : \alpha < \lambda, \beta < \lambda, h(0,\alpha,\beta) \in A_\lambda\}$, if there is one. Let $g^\lambda : \lambda \to N^\lambda$ be the isomorphism. Let for $i < \lambda$, $N_i^\lambda = N^\lambda \restriction \{g^\lambda(\alpha) : h(1,i,\alpha) \in A_\lambda\}$ and $x^\lambda = (\bar{Q}^\lambda, \bar{Q}^{*,\lambda}, \underset{\sim}{T}^\lambda, p^\lambda, \alpha^{*,\lambda})$ be such that $x^\lambda = g^\lambda(\alpha)$, $\alpha = \min\{\beta : h(2,0,\beta) \in A_\lambda\}$ (if there are such α). Now necessarily $W = \{\lambda < \kappa : \lambda \text{ inaccessible, and } A \cap \lambda = A_\lambda, \text{ and } f \text{ maps } \lambda$ onto $N_\lambda^*\}$ is stationary and for $\lambda \in W$, N^λ is well defined and isomorphic to N_λ^* say by $g^* : N^\lambda \to N_\lambda^*$ and $g^*(N_i^\lambda) = N_i^*$ (for $i < \lambda$), $\bar{Q}^\lambda = \bar{Q} \restriction \lambda$, $g^*(\bar{Q}^{*,\lambda}) = \bar{Q}^*$, $g^*(\underset{\sim}{T}^\lambda) = \underset{\sim}{T}$, $g^*(p^\lambda) = p$, $g(\alpha^{*,\lambda}) = \alpha^*$.

So now we will explain what we did in stage λ to take care of this situation.

First we will give an overview of how to get the sets S_λ; In stage λ (an inaccessible below κ, so work in V^{P_α}) we use $\diamondsuit$, i.e. A_λ to obtain a continuous increasing sequence $\langle N_i^\lambda : i \leq \lambda \rangle$ of quite close models (which guesses (the isomorphic type of a) a sequence $\langle N_i^* : i \leq \kappa \rangle$ as above). We also guess an ordinal $\alpha = \alpha^{*,\lambda} \in N^\lambda$ (so actually we are only guessing $\mathrm{otp}(\alpha^* \cap N_\lambda^*)$) and x^λ "guess" $\langle \bar{Q}, \bar{Q}^*, \underset{\sim}{T}, p, \alpha^\lambda \rangle, \ldots$ Let $G_\lambda \subseteq P_\lambda$ be generic over V, $p^\lambda \in G_\lambda$. We now try to construct a sequence $\langle p_i : i < \lambda \rangle$ of conditions in $P^{*,\lambda}_{\alpha^{*,\lambda}} \cap N^\lambda[G_\lambda]$ which will induce an $N^\lambda[G_\lambda]$-generic set. If we succeed, letting $p_i' = g^*(p_i)$ in $V[G_\lambda]$ we have $p_i' \in N_\lambda^* \cap P$ is increasing and $p_\lambda = \lim_{i<\lambda}(p_i')$ will decide all names in $N_\lambda^*[G_\lambda]$ (p_λ has domain $N_\lambda^*[G_\lambda] \cap \alpha^*$, $p_\lambda(\beta) = \bigcup_{i<\lambda} p_i'(\beta) \cup \{\langle \lambda, \lambda \rangle\}$). Moreover, $p_\lambda(\beta)$ will be an actual function (at least above $p_\lambda \restriction \beta$) rather than just a P_β^*-name, for all $\beta \in \mathrm{Dom}(p_\lambda) \subseteq N_\lambda^*[G_\lambda]$. This will show that in V^{P_κ}, the set

$$D \overset{\text{def}}{=} \{p \in P_{\alpha^*}^* : \forall \beta \in \mathrm{Dom}(p) \exists f : p \restriction \beta \Vdash \text{“}p(\beta) = f\text{''}\}$$

is dense, $P_{\alpha^*}^*$ contains a dense subset of cardinality $\leq \kappa (= \aleph_2)$, in V^{P_κ}, which is one of the demands in $(*)$, [which implies that $P_{\alpha^*}^*$ satisfies the κ^+-c.c. The usual Δ-system argument (recalling that P_κ^* used $\aleph_1$-support) then shows that also $P_{\kappa^+}^*$ satisfies the κ^+-c.c.)].

We will try to build these p_i in $\mathrm{otp}(N^\lambda[G_\lambda] \cap (\alpha^{*,\lambda} + 1))$ many steps, by constructing initial parts $\langle p_i \restriction \gamma : i \leq \lambda \rangle$ for $\gamma \in N^\lambda \cap (\alpha^{*,\lambda} + 1)$, by induction on γ.

However, it is possible that our construction will get stuck in some stage $\gamma \in N^\lambda[G_\lambda] \cap (\alpha^{*,\lambda} + 1)$. In this case we show that we will have constructed a stationary subset of λ (which guesses $\underset{\sim}{S}^*_\gamma \cap \lambda$). We will use this set as S_λ (and hence contradict the guess, since no superset of one of the sets S_λ can appear in the second iteration as some S^*_γ). However, since our guess must be correct on a stationary set of λ's, the construction will be completed stationarily often.

Before we start the construction of the p_i's, we will try to guess its outcome. In our ground model $V = L$ we have $\diamondsuit^*_\lambda$, so as $|P_\lambda| = \lambda$, we still have in V^{P_λ} i.e. $V[G_\lambda]$, that $\aleph_2$ is λ, and $\diamondsuit_{\{i<\lambda:\mathrm{cf}(i)=\aleph_1\}}$ holds. Note $N^\lambda_i[G_\lambda]$ is well defined: $G_\lambda \cap N^\lambda_i$ is a generic subset of $P_\kappa^{N^\lambda_i}$. So we can choose for each $i < \lambda$ a sequence $\langle q_{i,\xi} : \xi < i \rangle$ and a name $\underset{\sim}{\tau}_i$ such that:

(α) every initial segment of the sequence $\langle (\underset{\sim}{\tau}_i, \langle q_{i,\xi} : \xi < i \rangle) : i < \lambda \rangle$ belongs to $N^\lambda_i[G_\lambda]$.

(β) $q_{i,\xi} \in N^\lambda_i[G_\lambda] \cap P^*_\alpha$, is increasing with ξ.

(γ) if $\langle q_\xi : \xi < \lambda \rangle$ satisfies (α) + (β) [i.e., it is increasing with ξ and $q_\xi \in (N^\lambda \cap P^*_{\alpha^{*,\lambda}})$], and $\underset{\sim}{\tau} \in N^\lambda[G_\lambda]$ is a P^*_γ-name of an ordinal then $\{i : \langle q_\xi : \xi < i \rangle = \langle q_{i,\xi} : \xi < i \rangle$ and $\underset{\sim}{\tau} = \underset{\sim}{\tau}_i\}$ is stationary.

Note that $N^\lambda[G_\lambda]$ is closed under taking i-sequences (in V^{P_λ}) for $i < \lambda$ so clause (α) is not necessary.

Now at least for ordinals i such that $\mathrm{cf}(i) = \omega$ it is not clear whether $\langle q_{i,\xi} : \xi < i \rangle$ has an upper bound in $P^{*,\lambda}_{\alpha^{*,\lambda}}$, however we can find $\alpha(i) \in N^\lambda[G_\lambda] \cap (\alpha^{*,\lambda} + 1)$ and $q^*_i \in N^\lambda[G_\lambda] \cap P^{*,\lambda}_{\alpha(i)}$, $q^*_i \geq q_{i,\xi} \restriction \alpha(i)$ for $\xi < i$, and if $\alpha(i) < \alpha^{*,\lambda}$ then $q^*_i \Vdash_{P^*_{\alpha(i)}}$ "$\langle q_{i,\xi}(\alpha(i)) : \xi < i \rangle$ has no upper bound (in $N^\lambda[G_\lambda]$!)".

[Just let $\alpha(i) \in N_i \cap (\alpha^{*,\lambda} + 1)$ be maximal such that $q^\dagger \overset{\mathrm{def}}{=} (\bigcup_{\xi<i} q_{i,\xi}) \restriction \alpha(i)$ belongs to $P_{\alpha(i)}$; hence $q^\dagger \nVdash_{P^{*,\lambda}_{\alpha(i)}}$ "$(\bigcup_{\xi<i} q_{i,\xi})(\alpha(i)) \in \underset{\sim}{Q}^{*,\lambda}_{\alpha(i)}$" hence some $r, q^\dagger \leq r \in P^{*,\lambda}_{\alpha(i)}$ is as required]. Moreover, we may also assume that q^*_i decides the value of $\underset{\sim}{\tau}_i$ if $\underset{\sim}{\tau}_i$ is a $P^{*,\lambda}_{\alpha(i)}$-name.

Now we define by induction on $\gamma \in N^\lambda[G_\lambda] \cap (\alpha^{*,\lambda}+1)$ a set C_γ, a sequence $\langle C_{\beta,\gamma} : \beta < \gamma, \beta \in N^\lambda[G_\lambda] \cap (\alpha^{*,\lambda}+1)\rangle$ and sequences $\langle p_{\gamma,i} : i \in C_\gamma\rangle$ satisfying the following:

(i) Each C_γ, and each $C_{\beta,\gamma}$ is a closed unbounded subset of λ, $C_{\beta,\gamma} \subseteq C_\beta \cap C_\gamma$.

(ii) $p_{\gamma,i} \in N^\lambda[G_\lambda] \cap P^{*,\lambda}_\gamma$, and is increasing with i.

(iii) $p_{\gamma,i}\restriction\beta = p_{\beta,i}$, for $\beta < \gamma$, $i \in C_{\beta,\gamma}$:

(iv) if $\langle p_{\gamma,j} : j < i\rangle$, $\langle q_{i,j} : j < i\rangle$ (from the diamond above) satisfy $(\forall j < i)(\exists \zeta < i)(p_{\gamma,j} \leq q_{i,\zeta}\restriction\gamma)$, $(\forall \zeta < i)(\exists j < i)(q_{i,\gamma}\restriction\gamma \leq p_{\gamma_j})$ and $\alpha(i) \geq \gamma$ *then* $q^*_i \leq p_{\gamma,i}$.

We will define this sequence by induction on γ. For $\gamma = 0$ there is nothing to do. If γ is a limit of cofinality $< \lambda$, let $\gamma = \bigcup_{\xi<\mathrm{cf}(\gamma)} \gamma_\xi$ with $\zeta < \xi < \mathrm{cf}(\gamma) \Rightarrow \gamma_\zeta < \gamma_\xi < \gamma$. We let

$$C_\gamma \stackrel{\mathrm{def}}{=} \bigcap_{\zeta<\xi<\mathrm{cf}(\gamma)} C_{\gamma_\zeta,\gamma_\xi}.$$

For $i \in C_\gamma$, we let $p_{\gamma,i} = \bigcup_{\xi<\mathrm{cf}(\gamma)} p_{\gamma_\xi,i}$. (This is a union of a sequence of at most $\aleph_1$ conditions which are end extensions of each other (in the sense that $p_{\gamma_\xi,i} = p_{\gamma_\zeta,i}\restriction\gamma_\xi$ for $\zeta < \xi < \mathrm{cf}(\gamma)$), hence this limit exists.) Finally, we let $C_{\beta,\gamma} = C_\beta \cap C_\gamma \cap \bigcap_{\xi<\mathrm{cf}(\gamma)} C_{\beta,\gamma_\xi}$ [where we let $C_{\alpha,\beta} = \lambda$ for $\alpha \geq \beta$]. If $\mathrm{cf}(\gamma) \geq \lambda$, then we can find an increasing unbounded sequence $\langle \gamma_\xi : \xi < \lambda\rangle$ in $\gamma \cap N^\lambda[G_\lambda]$ such that for all $\zeta < \lambda$ we have $\langle \gamma_\xi : \xi < \zeta\rangle \in N^\lambda[G_\lambda]$. We let C_γ be a diagonal intersection:

$$C_\gamma \stackrel{\mathrm{def}}{=} \{i < \lambda : i \bigcap_{\zeta<\xi<i} C_{\gamma_\zeta,\gamma_\xi}\}$$

and for $i \in C_\gamma$ we let $p_{\gamma,i} \stackrel{\mathrm{def}}{=} \bigcup\{p_{\xi,i} : \xi < i\}$. We let

$$C_{\beta,\gamma} \stackrel{\mathrm{def}}{=} \{i < \lambda : (\forall j < i)\ i \in C_{\beta,\gamma_j}\}.$$

An easy calculation shows that (ii) will be satisfied.

Successor step: Let $\gamma = \beta + 1$. If the set $\underset{\sim}{S}^*_\beta$ as computed by $\langle p_{\beta,i} : i \in C_\beta\rangle$ (i.e. the set $\{\varepsilon < \lambda :$ for some $i \in C_\beta$ we have $p_{\beta,i} \Vdash$ "$\varepsilon \in \underset{\sim}{S}^*_\beta$"$\})$ does not include a stationary subset of λ, let $C_\gamma \subseteq C_\beta$ be a club set disjoint from this set. We let

$C_{\beta,\gamma} \stackrel{\text{def}}{=} C_\gamma$. For $\beta' < \beta$ we let $C_{\beta',\gamma} = C_{\beta',\beta} \cap C_\gamma$. So for $i \in C_\gamma$, we will have $p_{\gamma,i} \restriction \beta = p_{\beta,i}$, so we only have to define $p_{\gamma,i}(\beta)$. We will do this by induction on i: If i is (in C_γ) the successor of j, then we let $p_{\gamma,i}(\beta)$ be a condition extending $p_{\gamma,j}(\beta)$ such that there is $\varepsilon \in C_\gamma$ with $\mathrm{ht}(p_{\gamma,j}(\beta)) < \varepsilon <\mathrm{ht}(p_{\gamma,i}(\beta))$ (where $\mathrm{ht}(r) \stackrel{\text{def}}{=} \sup(\mathrm{range}(r))$. This will ensure that in limit steps the supremum of the conditions constructed so far always exists. For limit i, we first take the supremum of the conditions constructed so far, and, if possible, increase the condition again to make it stronger than $q_i^*(\beta)$.
Finally if $\underset{\sim}{S}^*_\beta$ as computed by $\langle p_{\beta_i} : i \in C_\beta \rangle$ does contain a stationary set, we will choose this as S_λ when defining the first iteration $\bar{Q}$. Note that this choice of S_λ does not depend on $N^*_\lambda, \ldots$ but only on $N^\lambda, \ldots$ so all is O.K. As remarked above, this will not happen if $\diamondsuit$ has guessed correctly. $\square_{7.1}$

7.2 Theorem. The following are equi-consistent
(a) ZFC + there is a weakly compact cardinal.
(b) ZFC + G.C.H. + if S', $S'' \subseteq S^2_0$ are stationary, then for some $\delta \in S^2_1$, $S' \cap \delta$, $S'' \cap \delta$ are stationary.
(c)ZFC + G.C.H.+"if $S_i \subseteq S^2_0$ are stationary sets for $i < \omega_1$ *then* there is an increasing continuous sequence of ordinals $< \omega_2$, $\langle \alpha_i : i < \omega_1 \rangle$ such that $\alpha_i \in S_i$.
Remark. In (b), (c) we can also contradict G.C.H. (use XV §3 for Con(a) $\Rightarrow$ Con(c)).

Proof. The implication Con(a) $\Rightarrow$ Con(b) was proved by Baumgartner [Ba], the inverse by [Mg5]. Now Con(c)$\Rightarrow$Con(b) is trivial, and so the point is to prove Con(a)$\Rightarrow$Con(c) which is done just like the proof of theorem 1.4, using the forcing notion $P[\langle S_i : i < \omega_1 \rangle]$ from 4.6A. $\square_{7.2}$

Before we prove the next theorem, we recall the forcing notion for "shooting a club through S":

A known forcing is

7.3 Definition. For any set $S \subseteq \omega_2$, define the forcing notion Club(S) by

$$\text{Club}(S) \stackrel{\text{def}}{=} \{h : \text{for some } \alpha < \omega_2 \text{ we have:}$$
$$\text{Dom}(h) = \alpha + 1, \text{Rang}(h) \subseteq S, h \text{ continuous increasing}\}$$

This forcing notion "shoots a club through S". See on it in [BHK] and more in [AbSh:146].

For $h \in \text{Club}(S)$ let

$$\alpha(h) \stackrel{\text{def}}{=} \max \text{Dom}(h) \qquad \delta(h) \stackrel{\text{def}}{=} h(\alpha(h))$$

7.4 Lemma.

(a) If $S \cap S^2_0$ is stationary, then Club(S) does not add new ω-sequences of ordinals.

(b) If the set

$$\hat{S} \stackrel{\text{def}}{=} \{\delta \in S^2_1 \cap S : S \cap \delta \text{ contains a club subset of } \delta\}$$

is stationary, and CH holds, *then* Club(S) does not add ω_1-sequences of ordinals to V.

7.4A Remark. Instead CH it suffice to have for some list $\{a_\alpha : \alpha < \omega_2\}$ of subsetes of ω_2 that $\{\delta \in S^2_1$: there is a club C of δ such that $C \subseteq S$ and $\alpha < \delta \Rightarrow C \cap \alpha \in \{a_\beta : \beta < \alpha\}$.

Proof: We leave (a) to the reader as it is easier and we will need only (b).
To prove (b), let $p \in \text{Club}(S)$, $\underset{\sim}{\tau}$ a Club(S)-name such that $p \Vdash$ "$\underset{\sim}{\tau}$ is a function from ω_1 to the ordinals". Let $N \prec (H(\aleph_3), \in)$ be a model of size $\aleph_1$ which contains all relevant information (i.e., $\{S, p, \underset{\sim}{\tau}\} \subseteq N$), is closed under ω-sequences and satisfies $N \cap \omega_2 \in \hat{S}$. We can find such a model because we have CH and $\hat{S}$ is stationary. Let $C \subseteq \delta \cap S$ be a club set.

Now we can find a continuous increasing sequence $\langle N_\alpha : \alpha < \omega_1 \rangle$ satisfying the following for all $\alpha < \omega_1$:

(a) $N_\alpha \in N$, $N_\alpha \prec N$, N_α countable.

(b) $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$.

(c) $\sup(N_\alpha \cap \omega_2) \in C$.

(d) $\{S, p, \underset{\sim}{\tau}\} \subseteq N_0$.

W.l.o.g. $\bigcup_{\alpha<\omega_1} N_\alpha = N$. Let $p_0 = p$, and define a sequence $\langle p_i : i < \omega_1 \rangle$ satisfying

(0) $p_i \in N$

(1) p_{i+1} decides the value of $\underset{\sim}{\tau}(i)$

(2) Letting $\alpha_i \stackrel{\text{def}}{=} \min\{\alpha : p_i \in N_\alpha \;\&\; \alpha > \alpha(p_i)\}$, we demand $\alpha(p_{i+1}) > \sup(N_{\alpha_i} \cap \omega_2)$ and $\delta(p_{i+1}) > \sup(N_{\alpha_i} \cap \omega_2)$ (see 7.3).

(3) If $j < i$, then $p_j \leq p_i$.

Given p_i, it is no problem to find p_{i+1}. If i is a limit, then letting $\alpha^* \stackrel{\text{def}}{=} \sup_{j<i}\alpha(p_j)$, $\delta^* \stackrel{\text{def}}{=} \sup_{j<i}\delta(p_j)$, we have $\delta^* = N_{\alpha^*} \cap \omega_2 \in S$, so we can let $p_i = \bigcup_{j<i} p_j \cup \{(\alpha^*, \delta^*)\}$. Note that $p_i \in N$, because N was closed under ω-sequences.

Finally, $\bigcup_{i<\omega_1} p_i$ can be extended to a condition p_{ω_1} because $\delta \in S$. Now $p_{\omega_1} \Vdash \underset{\sim}{\tau} \in V$. $\square_{7.4}$

The following solves a problem of Abraham.

7.5 Theorem. The following are equi-consistent

a)ZFC + there is a 2-Mahlo cardinal.

b)ZFC + G.C.H. + $\{\delta < \aleph_2 : \delta$ inaccessible in L$\}$ contains a closed unbounded subset of ω_2.

Proof. Con(b) $\Rightarrow$ Con(a):

Let C be a closed unbounded subset of ω_2 consisting of regular cardinals of L. So each $\delta \in C \cap S^2_0$ is inaccessible in L, hence each $\delta \in C \cap S^2_1$ is Mahlo in L, hence $\aleph_2^V$ is 2-Mahlo in L, i.e., $\{\lambda : \lambda < \aleph_2^V, \lambda$ Mahlo in $L\}$ is stationary.

Con(a)$\Rightarrow$Con(b):

So without loss of generality $V = L$, κ is 2-Mahlo.

We define an RCS-iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$, where:

1)if i is not strongly inaccessible, $\underset{\sim}{Q}_i$ is Levy collapse of $2^{\aleph_1}$ to $\aleph_1$ by countable conditions.

2)if i is strongly inaccessible but not Mahlo, $\underset{\sim}{Q}_i = \mathrm{Nm}'$.

3)if i is a strongly inaccessible Mahlo cardinal, $\underset{\sim}{Q}_i = P[S_i]$ where $S_i = \{\lambda < i : \lambda$ strongly inaccessible $\}$.

It should be easy for the reader to prove that $P_\kappa = \mathrm{Rlim}\,\bar{Q}$ satisfies the κ-chain condition, and the S-condition, $S = \{\lambda : \aleph_1 < \lambda < \kappa, \lambda$ inaccessible (in L)$\}$.

Lastly in V^{P_κ} let P^* =Club($\{\lambda < \kappa : \lambda$ inaccessible in L$\}$). So our forcing is $P_\kappa * \underset{\sim}{P}^* \in V$.

Now P^* is not even $\aleph_1$-complete, but still P was constructed so that P^* does not add $\aleph_1$-sequences by 7.4(b), and $V^{P_\kappa * P^*}$ is as required. $\square_{7.5}$

7.6 Theorem. Assume κ^* is supercompact. *Then* for some forcing notion P, in V^P, for every regular $\lambda > \aleph_1$, $\mathrm{Fr}^\dagger(\lambda)$ holds (and we can ask also GCH).

Proof. W.l.o.g. $V \models$ GCH. Let κ will be the first strongly inaccessible cardinal κ which is κ^{+7}-supercompact. Let $\mathbf{j} : \kappa \to H(\kappa)$, $j(\alpha) \in H(|\alpha|^{+6})$ be a Laver diamond under this restriction. Let $\langle P_i^-, Q_j^- : i \le \kappa, j < \kappa\rangle$ be an Easton support iteration. $\underset{\sim}{Q}_j^-$ is $\mathbf{j}_0(j)$ if $\mathbf{j}(j) = \langle \mathbf{j}_\ell(j) : \ell < 2\rangle$ and $\mathbf{j}_0(j)$ is a P_i^--name of a j-directed complete forcing notion, j strong inaccessible, $(\forall \zeta < j)(|P_\zeta| < j)$, and the trivial forcing otherwise. Let $V_0 = V$, $V_1 = V_0^{P_\kappa^-}$. Clearly $V_1 \models \diamondsuit_{\{\mu<\kappa:\mu \text{ is strongly inaccessible }\}}$. Let $R = \mathrm{Levy}(\lambda, <\kappa^*)^{V_1}$, so in V^R, for every regular $\theta \ge \kappa^*$ we have:

$(*)_\theta$ if $S \subseteq \{\delta < \theta : \mathrm{cf}(\delta) = \aleph_0\}$ is stationary then for some $\delta^* < \lambda$ we have: $\mathrm{cf}(\delta^*) = \kappa$, and $S \cap \delta^*$ is a stationary subset of δ^*. (see Fact in X 7.4)

In $V_2 = V^R$ we have $\kappa^* = \kappa^+$ and define $\bar{Q}$, P_κ, $\bar{Q}^* = \langle P_i^*, \underset{\sim}{Q}_i^* : i < \kappa^+\rangle$, $P_{\kappa^+}^*$ as in the proof of 7.1 except that for $\delta < \kappa$ strong inaccessible, $\underset{\sim}{Q}_\delta$ is suggested by $\mathbf{j}_1(\delta)$ when $\mathbf{j}(\delta) = \langle \mathbf{j}_\ell(\delta) : \ell < 2\rangle$ as above, $\mathbf{j}_1(\delta)$ a $(P_{\delta+1}^- * P_\delta)$-

name of a forcing notion satisfying the S-condition (in the universe $V^{P_{\delta+1}\times P_\delta}$ of course). So we force with $R' \stackrel{\text{def}}{=} R * \underset{\sim}{P}_\kappa * P^*_{\kappa^+}$ (really we can arrange that $P_\kappa \in V_1$, $\underset{\sim}{P}^*_\kappa$ is an $R \times P_\kappa$-name). Looking at the proof of 7.1, the only point left is to prove $(*)_\theta$ for $\theta = \text{cf}(\theta) \geq \kappa^*$. If $\theta > \kappa^*$, as the density of $P_\kappa * \underset{\sim}{P}^*_{\kappa^+}$ ($\in V_2$) is κ^*, any stationary $S \subseteq \{\delta < \theta : \text{cf}(\delta) = \aleph_0\}$ from $V^{R'}$ contains a stationary subset from V_2. We can use $V_2 \models (*)_\theta$, so we are left with the case $\theta = \kappa^*$.

If in V_2, $\underset{\sim}{S}$ is a $P_\kappa * \underset{\sim}{P}^*_{\kappa^+}$ name, $p \in P_\kappa * \underset{\sim}{P}^*_{\kappa^+}$, $p \Vdash$ "$\underset{\sim}{S}_0 \subseteq \{\delta < \kappa^+ : \text{cf}(\delta) = \aleph_0\}$ stationary", let

$$S_1 = \{\delta < \kappa^+ : \text{cf}(\delta) = \aleph_0 \text{ and } p \nVdash \text{“}\delta \notin \underset{\sim}{S}_0\text{”}\}.$$

For $\delta \in S_1$ choose $p_\delta \in P_\kappa * \underset{\sim}{P}^*_{\kappa^+}$, $p \leq p_\delta$, $p_\delta \Vdash$ "$\delta \in S$". Let $S_2 = \{\delta \in S_1 : p_\delta \in \underset{\sim}{G}_{P_\kappa * P^*_{\kappa^+}}\}$, so $p \Vdash$ "$\underset{\sim}{S}_2 \subseteq \underset{\sim}{S}_0$".

Let $E = \{\alpha < \kappa^+ : \alpha \text{ limit and } \delta \in S_1 \cap \alpha \Rightarrow p_\delta \in P_\kappa * P^*_\alpha\}$ is a club of κ^+ in V_2. It is enough to show that

$\otimes$ $W \stackrel{\text{def}}{=} \{\delta < \kappa^+ : \text{cf}(\delta) = \kappa, \delta \in E$ and $p \Vdash_{P_\kappa * \underset{\sim}{P}^*_\delta}$ "$\underset{\sim}{S}_2 \cap \delta$ (which is a $(P_\kappa * \underset{\sim}{P}^*_{\delta^*})$-name) is a stationary subset of δ"$\}$ is a stationary subset of κ^+.

[Why? As then instead of guaranting $\underset{\sim}{S}_2 \cap \delta$ will continue to be stationary, we guess such name and related elementary submodel in some $\alpha < \kappa$ and in Q_α take care of $\underset{\sim}{S}_2 \cap \delta$ having a closed subset of order type ω_1.]

Let $G_R \subseteq R$ be the generic subset of R.

In V_1 we can find $\delta < \kappa^*$ such that $V_1 \models$ "$\kappa < \delta < \kappa^*$, δ is strongly inaccessible" and letting $R_\delta = \text{Levy}(\kappa, < \delta) \lessdot R$, $G_{R_\delta} = G_R \cap R_\delta$, in $V[G_{R_\delta}]$ we have $\bar{Q}$, $\bar{Q}^* \restriction \delta$ hence $P_\kappa * P^*_\delta$, $\langle p_i : i \in S_1 \cap \delta\rangle$ and $S_1 \cap \delta$ is stationary (and of course $V[G_{R_\delta}] \models \delta = \kappa^+$). Also in $V_1[G_{R_\delta}]$, the forcing notion $P_\kappa * P^*_\delta$ satisfies the δ-c.c. (just as in $V_1[G_R]$, $P_\kappa * P^*_\kappa$ satisfies the κ^*-c.c.). So for a club of $i < \delta$, $i \in S_1 \cap \delta$ implies $p_i \Vdash$ "$\underset{\sim}{S}^* \stackrel{\text{def}}{=} \{j \in S_1 \cap \delta : p_j \in \underset{\sim}{G}_{P_\kappa * \underset{\sim}{P}^*_\delta}\}$ is a stationary subset of δ^*" (as in Gitik, Shelah [GiSh:310]). Choose such $i(*)$. If this holds in $V_1[G_R]$ too, then we are done so assume towards contradiction that this fails, so moving back to V_0 for some $q \in P^-_\kappa$, $\underset{\sim}{r} \in R$ and $\underset{\sim}{p}$ we have $(q, \underset{\sim}{r}) \Vdash_{P_\kappa * \underset{\sim}{R}}$ "$p_{i(*)} \leq \underset{\sim}{p} \in P_\kappa * \underset{\sim}{P}^*_\delta$" and for some $P^-_\kappa * (\underset{\sim}{R} * (\underset{\sim}{P}_\kappa * \underset{\sim}{P}^*_\delta))$-name $\underset{\sim}{E}$

we have

$$(q, \underset{\sim}{r}, \underset{\sim}{p}) \Vdash_{P_\kappa^- * \underset{\sim}{R} * (\underset{\sim}{P}_\kappa * \underset{\sim}{P}^*_\delta)} \text{“}\underset{\sim}{E} \text{ is a club of } \delta \text{ disjoint to } \underset{\sim}{S}^*\text{”}.$$

Let $M \prec (H((\kappa^*)^{++}), \in)$ be such that $\|M\| = \kappa^+$, $H(\kappa^+) \subseteq M$, $x = \{\kappa, \kappa^*, \delta, \underset{\sim}{E}, q, (\underset{\sim}{r}, \underset{\sim}{p}), \underset{\sim}{R}/G_{R_\delta}, P_\kappa * \underset{\sim}{P}^*_\delta\} \subseteq M$ and ${}^\kappa M \subseteq M$.

Let (M', x') be isomorphic say by g to (M, x), M transitive, so $g : M \to M'$. We can find $\mathfrak{B} \prec (H(\kappa^{+6}), \in)$ to which x' and M' belong, such that letting $\theta = \mathfrak{B} \cap \kappa$ we have: θ is strongly inaccessible θ^{++}-supercompact, $\mathfrak{B} \approxeq (H(\theta^{+6}), \in)$ and $\mathbf{j}(\theta) = (\mathbf{j}_0(\theta), \mathbf{j}_1(\theta)) \in H(\theta^{++})$ is such that: $\mathbf{j}_0(\theta) = g(R_\delta)$ and $\mathbf{j}_1(\theta)$ is the $g(P_\kappa^- * \underset{\sim}{R}_\delta * \underset{\sim}{P}_\kappa)$-name i.e. $(P_\delta^- * Q_\delta^- * \underset{\sim}{P}_\delta)$-name of $g(P^*_\delta) * \text{club}(\underset{\sim}{S}^*)$. We can finish easily. $\square_{7.5}$

§8. Relativising to a Stationary Set

8.1 Definition. For a set $\mathbb{I}$ of ideals and stationary $\mathbf{W} \subseteq \omega_1$ we define when does a forcing notion P satisfies the ($\mathbb{I}$, $\mathbf{W}$)-condition (compare with 2.6). It means that there is a function F such that (letting $J^{\text{bd}}_{\omega_1}$ =the bounded subsets of ω_1):

if $(T, \mathbf{I})$, f satisfies the following properties:

(∗) (a) $(T, \mathbf{I})$ is an $(\mathbb{I} \cup \{J^{\text{bd}}_{\omega_1}\})$-tree.

(b) $f : T \to P$.

(c) $\nu \le \eta$ implies $P \models f(\nu) \le f(\eta)$.

(d) There are fronts $J_n (n < \omega)$ of T such that every member of J_{n+1} has a proper initial segment of J_n and:

(α) If $\eta \in J_n$ then $\langle \text{Suc}(\eta), \mathbf{I}_\eta, \langle f(\nu) : \nu \in \text{Suc}_T(\eta)\rangle\rangle = F(\eta, w[\eta], \langle f(\nu) : \nu \lhd \eta\rangle)$ (where $w[\eta] = \{\kappa : \eta \restriction \kappa \in \cup_n J_n\}$).

(β) $\cup J_n$ is the set of splitting points of $(T, \mathbf{I})$

(γ) If n is odd, $\eta \in J_n$ then $\mathbf{I}_\eta = J^{\text{bd}}_{\omega_1}$.

(δ) If n is even, $\eta \in J_n$ then $\mathbf{I}_\eta \in \mathbb{I}$.

then

(**) *if* $(T, \mathbf{I}) \leq (T^*, \mathbf{I})$, $[\eta \in T^* \cap (\bigcup_{n<\omega} J_{2n}) \Rightarrow \mathrm{Suc}_{T^*}(\eta) \notin \mathbf{I}_\eta]$ and for some limit $\delta \in \mathbf{W}$ for every $\eta \in \lim T^*$, $\delta = \sup\{\eta(k) : \eta{\restriction}k \in \bigcup_{n<\omega} J_{2n+1}\}$ *then* for some $q \in P$ we have $q \Vdash$ "$(\exists \eta)[\eta \in \lim T^* \& \bigwedge_{k<\omega} f(\eta{\restriction}k) \in \underset{\sim}{G}_P]$"

8.2 Claim. 1)If P satisfies the $\mathbb{I}$-condition, e.g. P is $\aleph_2$-complete *then* P satisfies the $(\mathbb{I}, \omega_1)$ -condition.

2)If $\mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \omega_1$, P satisfies the $(\mathbb{I}, \mathbf{W}_2)$-condition *then* p satisfies the $(\mathbb{I}, \mathbf{W}_1)$-condition.

3) If $\mathbf{W} \subseteq \omega_1$ is stationary, $\mathbb{I}$ is a family of $\aleph_2$-complete ideals, and the forcing notion P satisfies the $(\mathbb{I}, \mathbf{W})$-condition *then* forcing with P does not collapse $\aleph_1$ and preserves the stationarity of $\mathbf{W}$.

Proof. 1) If F witnesses "P satisfies the $\mathbb{I}$ -condition", define F' such that if $(T, \mathbf{I})$, f, $\langle J_n : n < \omega\rangle$ are as in Definition 8.1, then for $n = 2m$, $\eta \in J_n$:

$$\langle \mathrm{Suc}_T(\eta), \langle f(\nu) : \nu \in \mathrm{Suc}_T(\eta)\rangle\rangle = F'(\langle \eta, w[\eta], f(\nu) : \nu \trianglelefteq \eta\rangle\rangle)$$
$$= F(\langle \eta, \{\ell : \eta{\restriction}\ell \in \bigcup_{k<\omega)} J_{2k}\}, \quad \langle f(\nu) : \nu \trianglelefteq \eta\rangle\rangle)$$

This finishes the proof.

2) Trivial.

3) Suppose $p \in P$, $p \Vdash$ "$\underset{\sim}{C}$ is a club of ω_1". Choose $(T, \mathbf{I})$, f, $\langle J_n : n < \omega\rangle$ as in Definition 8.1 such that:

(i) $J_n = \{\eta \in T : \ell g(\eta) = 2n\}$

(ii)If $\eta \in T$, $\ell g(\eta)$ odd then $\mathrm{Suc}_T(\eta) = \{\eta\hat{\ }\langle 0\rangle\}$ and for some α_η

(α) min $(\omega_1 \cap \mathrm{Rang}(\eta)) < \alpha_\eta < \omega_1$

(β) $f(\eta\hat{\ }\langle 0\rangle) \Vdash_P$ "$\alpha_\eta \in \underset{\sim}{C}$"

(iii)$f(\langle\rangle) = p$

There is no problem in this. Let $T_0 = \{\eta \in T : \textit{if } k < \ell g(\eta),\ k = 4m+2,\ \ell < k,\ \ell = 2n+1 \textit{ then } \alpha_{\eta{\restriction}\ell} < \eta(k))\}$. Clearly $(T, \mathbf{I}) \leq (T^0, \mathbf{I})$ and the requirement in (**) of Definition 8.1 holds. By XV 2.6 (no vicious circle! as it does not use any intermidiate material) there is a club C^* of ω_1 and for each $\delta \in C^*$, a tree T_δ such that:

(a)$(T^0, \mathbf{I}) \leq (T_\delta, \mathbf{I})$

(b)$\eta \in T_\delta,\ \eta \in \bigcup_n J_{2n} \Rightarrow \eta \in \mathrm{sp}(T_\delta, \mathbf{I})$.

(c)$\eta \in \lim T_\delta \Rightarrow \delta = \sup(\mathrm{Rang}(\eta) \cap \omega_1) \& \delta \notin \mathrm{Rang}(\eta)$.

This last statement holds also for branches of T_δ in extensions of the universe, being absolute. Choose $\delta \in W \cap C^*$, and apply Definition 8.1 to $(T_\delta, \mathbf{I})$ (standing for $T^\dagger$ there), and get q as there. Now $q \Vdash$ "$\{f(\underset{\sim}{\eta}\restriction\ell) : \ell < \omega\} \subseteq \underset{\sim}{G}_P,\ \underset{\sim}{\eta} \in \lim T$" (for some P-name $\underset{\sim}{\eta}$). In particular $q \Vdash$ "$p \in \underset{\sim}{G}_P$" (as $p_{\langle\rangle} = p$) so w.l.o.g. $p \leq q$. Also by (ii) (α), (iii) above $q \Vdash_P$ "$\sup(\mathrm{Rang}(\eta) \cap \omega_1) = \sup\{\alpha_{\eta\restriction(2n+1)} : n < \omega\}$". And so $q \Vdash$ "$\delta = \sup\{\alpha_{\eta\restriction(2n+1)} : n < \omega\}$". As $q \Vdash$ "$f(\underset{\sim}{\eta}\restriction(2n+2)) \in \underset{\sim}{G}_P$" also (see (β) of (ii)) we have $q \Vdash$ "$\alpha_{\underset{\sim}{\eta}\restriction(2n+1)} \in \underset{\sim}{C}$" hence $q \Vdash$ "$\delta \in \underset{\sim}{C}$".

As $\delta \in \mathbf{W}$ we finish. $\square_{8.2}$

8.3 Lemma. Let $\mathbf{W} \subseteq \omega_1$, be stationary. All the theorems on preservations of the $\mathbb{I}$-condition (in §5, §6) for $\aleph_2$-complete $\mathbb{I}$ hold for the $(\mathbb{I}, \mathbf{W})$- condition.

Proof. Same proof, sometimes using XV 2.6.