Chapter VIII
Morasses and the Cardinal Transfer Theorem

By now it should be quite clear how it is that $V = L$ is of use when it is necessary to carry out intricate constructions by recursion. The uniform structure of L, and in particular the Condensation Lemma, enables us to take care of many "future" possibilities in a relatively small number of steps. For instance, in the construction of a Souslin tree, we need to take care of a collection of ω_2 potential uncountable antichains in ω_1 steps. As we shall see in this chapter, the uniformity of L enables us to do much more than this. In certain circumstances it is possible to construct structures of cardinality ω_2 in ω_1 steps. The idea is to use a sort of two-staged condensation principle, simultaneously approximating the final structure of size ω_2 by means of structures of size ω_1, and approximating each of these approximations by countable structures. In order to make this work, what is necessary is to investigate the way in which the two parts of such an approximation procedure must (and can) fit together. The essential combinatorial structure of L which is required is called a "morass". There is no need to stop there. We can go on to develop three-stage "morasses" which enable us to get up to ω_3 using only countable structures, and so on. In fact the subject of morasses is a vast area on its own, and would require an entire book of its own for a complete coverage. What we shall do in this chapter is look at the very simplest kind of morass, the one that gets us up to ω_2, in full detail, and then give little more than a glance at what comes after. In order both to motivate and illustrate the definition and use of a morass we take the problem which itself led to the development of morass theory, the Cardinal Transfer Problem of Model Theory.

1. Cardinal Transfer Theorems

Cardinals Transfer Theorems are generalised Löwenheim-Skolem Theorems. In its simplest form, the Löwenheim-Skolem Theorem says that if $\mathscr{A}$ is a model of a countable, first-order language, K, then there is a countable K-structure $\mathscr{B}$ such that $\mathscr{B} \equiv \mathscr{A}$. (More generally, given any infinite cardinal κ, there is a K-structure $\mathscr{B}$ of cardinality κ such that $\mathscr{B} \equiv \mathscr{A}$. Here and throughout it will be assumed that all structures considered are infinite, and that all first-order theories involved admit infinite models. This will exclude trivial special cases.) Now suppose that the language K contains a distinguished unary predicate symbol U. If κ, λ are cardinals (both infinite) we shall say that a K-structure $\mathscr{A}$ has *type* (κ, λ) iff $\mathscr{A}$ has the form

$$\mathscr{A} = \langle A, U^{\mathscr{A}}, \dots \rangle$$

where $|A| = \kappa$ and $|U^{\mathscr{A}}| = \lambda$. The idea of a cardinal transfer theorem is to obtain a Löwenheim-Skolem Theorem which preserves the relationship between the cardinality of the domain and that of the distinguished subset. The simplest case is the so-called "Gap-1 Cardinal Transfer Property", which says that every K-structure of type (κ^+, κ) (for some infinite κ) is elementarily equivalent to a K-structure of type (ω_1, ω). As we shall see presently, this result is provable in ZFC. Assuming GCH, we may replace (ω_1, ω) by any type (λ^+, λ) where λ is regular. Assuming $V = L$ we may drop the requirement that λ be regular here. (These results are considered in the Exercises.) The "Gap-2 Property" says (in the simplest case) that every K-structure of type (κ^{++}, κ) is elementarily equivalent to one of type (ω_2, ω). More generally there is a "Gap-n Property" for every positive integer n. There are also more general types of Cardinal Transfer Property, which we shall not consider here. The reader may consult *Chang-Keisler* (1973) for further details (including applications) of Cardinals Transfer Theorems.

As mentioned above, the Gap-1 Property is (in its simplest form) provable in ZFC. The Gap-2 Property, and indeed the Gap-n Property for any $n \geqslant 2$, is provable in ZFC + $(V = L)$. Our aim here is to use the simple version of the Gap-2 Property to motivate and illustrate the notion of a morass. In order to do this it is convenient to begin with a brief account of the proof of the Gap-1 Theorem. (In particular we shall need all of the model theoretic notions developed for the Gap-1 Theorem in order to prove the Gap-2 Theorem.)

We fix, once and for all, a countable, first-order language, K, with a distinguished unary predicate symbol, U. We shall show that if $\mathscr{A}$ is a K-structure of type (κ^+, κ) for some infinite cardinal κ, there is a K-structure $\mathscr{B}$ of type (ω_1, ω) such that $\mathscr{B} \equiv \mathscr{A}$. We recall some basic notions of model theory. For further details the reader should consult, for example, *Chang-Keisler* (1973).

K' will denote an arbitrary, countable expansion of K. A particular example of an expansion of K is obtained by adjoining to K an individual constant $\mathring{x}$ for each x in a given set X. This expansion will be denoted by K_X. In this case, X may be uncountable: this is the only case where uncountable languages may be considered. If $\mathscr{A}$ is any K-structure, then $\langle \mathscr{A}, (a)_{a \in A} \rangle$ is a K_A-structure. (We adopt the usual convention that A is the domain of $\mathscr{A}$, B the domain of $\mathscr{B}$, etc.)

The first-order theory of a structure $\mathscr{A}$ is denoted by $Th(\mathscr{A})$. Thus if $\mathscr{A}$ is a K-structure,

$$Th(\mathscr{A}) = \{\varphi \mid \varphi \text{ is a sentence of } K \text{ such that } \mathscr{A} \models \varphi\}.$$

Let T be a K_X-theory. An *element-type* of T is a set $\Sigma(x)$ of K_X-formulas with free variable at most x, such that for some model $\mathscr{A}$ of T and some element a of $\mathscr{A}$, $\mathscr{A} \models \Sigma(a)$. (The notation is self-explanatory.) In this case we say that a *realises* $\Sigma(x)$ *in* $\mathscr{A}$.

Let κ be an infinite cardinal. A structure $\mathscr{A}$ is said to be κ-saturated iff for every set $B \subseteq A$ such that $|B| < \kappa$, every element-type of $Th(\langle \mathscr{A}, (b)_{b \in B} \rangle)$ is realised in the structure $\langle \mathscr{A}, (b)_{b \in B} \rangle$. A structure $\mathscr{A}$ is said to be *saturated* iff it is $|\mathscr{A}|$-saturated. The following theorem is standard.

1.1 Theorem. (i) *Let $\mathscr{A}, \mathscr{B}$ be saturated K'-structures of cardinality κ, $\mathscr{A} \equiv \mathscr{B}$. Then $\mathscr{A} \cong \mathscr{B}$.*

Moreover, if $A' \subseteq A$, $B' \subseteq B$, $|A'| = |B'| < \kappa$, and h: $A' \leftrightarrow B'$ are such that

$$\langle \mathscr{A}, (a)_{a \in A'} \rangle \equiv \langle \mathscr{B}, (ha)_{a \in A'} \rangle,$$

then there is an isomorphism $\tilde{h}$: $\mathscr{A} \cong \mathscr{B}$ such that $\tilde{h} \upharpoonright A' = h$.

(ii) *Assume* GCH. *Let κ be an uncountable regular cardinal. Then any K'-theory has a saturated model of cardinality κ.* □

A structure $\mathscr{A}$ is said to be *homogeneous* iff, whenever $B \subseteq A$, $|B| < |A|$, and h: $B \to A$ is such that

$$\langle \mathscr{A}, (b)_{b \in B} \rangle \equiv \langle \mathscr{A}, (hb)_{b \in B} \rangle,$$

there is an automorphism $\tilde{h}$: $\mathscr{A} \cong \mathscr{A}$ such that $\tilde{h} \upharpoonright B = h$.

It is immediate from 1.1 (i) that any saturated structure is homogeneous. By virtue of 1.1 (ii), this provides an existence proof for uncountable homogeneous structures of regular cardinality, assuming GCH. As far as countable homogeneous structures are concerned, the existence is provable in ZFC alone.

1.2 Theorem. *Let T be a K'-theory. Then T has a countable, homogeneous model.* □

We shall make use of countable, homogeneous structures in our proof of the Gap-1 Theorem. The following result will also be required.

1.3 Theorem. *If $\mathscr{A}_0 \prec \mathscr{A}_1 \prec \ldots \prec \mathscr{A}_n \prec \ldots$ $(n < \omega)$ is an elementary chain of countable, homogeneous structures, then $\mathscr{A} = \bigcup_{n<\omega} \mathscr{A}_n$ is a countable, homogeneous structure.* □

A structure $\mathscr{A}$ is said to be *special* if there is an elementary chain

$$\mathscr{A}_0 \prec \mathscr{A}_1 \prec \ldots \prec \mathscr{A}_n \prec \ldots \prec \mathscr{A}$$

such that:

(i) $\mathscr{A} = \bigcup_{n<\omega} \mathscr{A}_n$;

(ii) $|A_0| < |A_1| < \ldots < |A_n| < \ldots < |A|$;

(iii) $\mathscr{A}_{n+1}$ is $|A_n|$-saturated for every n.

The following result tells us all we need to know about special structures.

1.4 Theorem. (i) *Every K'-theory has a special model (of some cardinality).*

(ii) *If $\mathscr{A}, \mathscr{B}$ are special structures of the same cardinality such that $\mathscr{A} \equiv \mathscr{B}$, then $\mathscr{A} \cong \mathscr{B}$.* □

The key step in the proof of the Gap-1 Theorem is the following lemma.

1.5 Lemma. *Let* $\mathscr{A} = \langle A, U, \dots \rangle$ *be a K-structure of type* (κ^+, κ) *for some* κ. *Then there are countable, homogeneous K-structures* $\mathscr{B}$, $\mathscr{C}$ *such that:*

(i) $\mathscr{B} = \langle B, V, \dots \rangle$, $\mathscr{C} = \langle C, V, \dots \rangle$ (*i.e.* $U^{\mathscr{B}} = U^{\mathscr{C}}$);
(ii) $\mathscr{B} \equiv \mathscr{C} \equiv \mathscr{A}$;
(iii) $\mathscr{B} \prec \mathscr{C}$ *and* $\mathscr{B} \neq \mathscr{C}$;
(iv) $\mathscr{B} \cong \mathscr{C}$.

Proof. Let $\mathscr{A}_0 \prec \mathscr{A}$ be such that $U \subseteq A_0$ and $|A_0| = \kappa$. Pick $a \in A - A_0$, and let $\mathscr{A}_1 \prec \mathscr{A}$ be such that $A_0 \cup \{a\} \subseteq A_1$ and $|A_1| = \kappa$. Let h: $A_0 \leftrightarrow A_1$, and form the structure

$$\mathscr{A}^* = \langle \mathscr{A}_1, A_0, h \rangle .$$

By 1.4(i), let $\mathscr{D}^*$ be a special structure such that $\mathscr{D}^* \equiv \mathscr{A}^*$, say

$$\mathscr{D}^* = \langle \mathscr{D}_1, D_0, k \rangle .$$

Let $\mathscr{D}_0$ be the restriction of $\mathscr{D}_1$ to domain D_0. Since $\mathscr{D}^* \equiv \mathscr{A}^*$, it is easily seen that $\mathscr{D}_0 \prec \mathscr{D}_1$. Since $\mathscr{D}^*$ is special, it is straightforward to check that both $\mathscr{D}_0$ and $\mathscr{D}_1$ are special. But k: $D_1 \leftrightarrow D_0$. Hence by 1.4(ii), $\mathscr{D}_0 \cong \mathscr{D}_1$. Note also that $U^{\mathscr{D}_0} = U^{\mathscr{D}_1}$. Let f: $\mathscr{D}_1 \cong \mathscr{D}_0$ and consider the structure

$$\mathscr{D}^{**} = \langle \mathscr{D}_1, D_0, f \rangle .$$

By 1.2, let $\mathscr{C}^{**} \equiv \mathscr{D}^{**}$ be countable homogeneous, say

$$\mathscr{C}^{**} = \langle \mathscr{C}_1, C_0, g \rangle .$$

Let $\mathscr{C}_0$ be the restriction of $\mathscr{C}_1$ to domain C_0. It is routine to check that $\mathscr{C}_0, \mathscr{C}_1$ are both countable, homogeneous structures, that $\mathscr{C}_0 \equiv \mathscr{C}_1 \equiv \mathscr{A}$, $U^{\mathscr{C}_0} = U^{\mathscr{C}_1}$, $\mathscr{C}_0 \prec \mathscr{C}_1$, $\mathscr{C}_0 \neq \mathscr{C}_1$, and that g: $\mathscr{C}_1 \cong \mathscr{C}_0$. Thus $\mathscr{B} = \mathscr{C}_0$ and $\mathscr{C} = \mathscr{C}_1$ satisfy the lemma. □

The above lemma shows that it is quite possible to have structures $\mathscr{A}$, $\mathscr{B}$ such that $\mathscr{A} \prec \mathscr{B}$ and $\mathscr{A} \cong \mathscr{B}$. The following lemma also involves this situation.

1.6 Lemma. *Let* $\mathscr{B}_0 \prec \mathscr{B}_1 \prec \dots \prec \mathscr{B}_n \prec \dots$ $(n < \omega)$ *be an elementary chain of isomorphic, countable, homogeneous models. Then* $\mathscr{B}_\omega = \bigcup_{n<\omega} \mathscr{B}_n$ *is countable and homogeneous and* $\mathscr{B}_\omega \cong \mathscr{B}_n$ *for all* $n < \omega$.

Proof. By 1.3 we know that $\mathscr{B}_\omega$ is countable and homogeneous. We prove that $\mathscr{B}_\omega \cong \mathscr{B}_0$. The idea is to construct enumerations $(b_n^0 \mid n < \omega)$, $(b_n^\omega \mid n < \omega)$ of B_0, B_ω, respectively, so that

$$\langle \mathscr{B}_0, (b_n^0)_{n<\omega} \rangle \equiv \langle \mathscr{B}_\omega, (b_n^\omega)_{n<\omega} \rangle ,$$

which at once implies that h: $\mathscr{B}_0 \cong \mathscr{B}_\omega$, where we define $h(b_n^0) = b_n^\omega$ for all $n < \omega$.

Suppose that $b_0^0, \ldots, b_n^0, b_0^\omega, \ldots, b_n^\omega$ are defined and satisfy

(i) $$\langle \mathscr{B}_0, b_0^0, \ldots, b_n^0 \rangle \equiv \langle \mathscr{B}_\omega, b_0^\omega, \ldots, b_n^\omega \rangle.$$

(The definition of b_0^0, b_0^ω is a degenerate case of this definition, so we omit it.) Let $b_{n+1}^0 \in B_0$. We show that there is an element b_{n+1}^ω of B_ω such that

(ii) $$\langle \mathscr{B}_0, b_0^0, \ldots, b_n^0, b_{n+1}^0 \rangle \equiv \langle \mathscr{B}_\omega, b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \rangle.$$

Since $\mathscr{B}_0 \prec \mathscr{B}_\omega$, we have

$$\langle \mathscr{B}_0, b_0^0, \ldots, b_n^0 \rangle \equiv \langle \mathscr{B}_\omega, b_0^0, \ldots, b_n^0 \rangle,$$

so by (i),

$$\langle \mathscr{B}_\omega, b_0^0, \ldots, b_n^0 \rangle \equiv \langle \mathscr{B}_\omega, b_0^\omega, \ldots, b_n^\omega \rangle.$$

Thus as $\mathscr{B}_\omega$ is homogeneous and $b_{n+1}^0 \in B_\omega$ there is a $b_{n+1}^\omega \in B_\omega$ such that

$$\langle \mathscr{B}_\omega, b_0^0, \ldots, b_n^0, b_{n+1}^0 \rangle \equiv \langle \mathscr{B}_\omega, b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \rangle.$$

Since $\mathscr{B}_0 \prec \mathscr{B}_\omega$, this at once yields (ii).

To complete the proof that $\mathscr{B}_\omega \cong \mathscr{B}_0$ we show that if we are given $b_0^0, \ldots, b_n^0$, $b_0^\omega, \ldots, b_n^\omega$ as in (i) and $b_{n+1}^\omega \in B_\omega$ is given, we can find an element b_{n+1}^0 of B_0 to satisfy (ii). (The required enumerations $(b_n^0 \mid n < \omega)$, $(b_n^\omega \mid n < \omega)$ can then be defined by recursion using a "back and forth" procedure to ensure that all elements of B_0, B_ω are included in these sequences.) Well, we can pick an integer $m < \omega$ such that $b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \in B_m$. Since $\mathscr{B}_m \prec \mathscr{B}_\omega$, we have

(iii) $$\langle \mathscr{B}_m, b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \rangle \equiv \langle \mathscr{B}_\omega, b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \rangle.$$

Since $\mathscr{B}_m \cong \mathscr{B}_0$ there are elements $b_0, \ldots, b_n, b_{n+1}$ of B_0 such that

(iv) $$\langle \mathscr{B}_m, b_0^\omega, \ldots, b_n^\omega, b_{n+1}^\omega \rangle \equiv \langle \mathscr{B}_0, b_0, \ldots, b_n, b_{n+1} \rangle.$$

By (i), (iii) and (iv),

$$\langle \mathscr{B}_0, b_0, \ldots, b_n \rangle \equiv \langle \mathscr{B}_0, b_0^0, \ldots, b_n^0 \rangle.$$

So as $b_{n+1} \in B_0$ and $\mathscr{B}_0$ is homogeneous, there is an element b_{n+1}^0 of B_0 such that

$$\langle \mathscr{B}_0, b_0, \ldots, b_n, b_{n+1} \rangle \equiv \langle \mathscr{B}_0, b_0^0, \ldots, b_n^0, b_{n+1}^0 \rangle.$$

Combining this with (iii) and (iv) we get (ii), as required. This completes the proof that $\mathscr{B}_\omega \cong \mathscr{B}_0$, and with it the proof of the lemma. □

We are now able to prove the Gap-1 Theorem.

1.7 Theorem. *Let $\mathscr{A} = \langle A, U, \ldots \rangle$ be a K-structure of type (κ^+, κ). Then there is a K-structure $\mathscr{B}$ of type (ω_1, ω) such that $\mathscr{B} \equiv \mathscr{A}$.*

Proof. By 1.5 there are countable, homogeneous structures $\mathcal{B}_0, \mathcal{B}_1$ such that $\mathcal{B}_0 \equiv \mathcal{B}_1 \equiv \mathcal{A}$, $\mathcal{B}_0 \prec \mathcal{B}_1$, $U^{\mathcal{B}_0} = U^{\mathcal{B}_1}$, $\mathcal{B}_0 \neq \mathcal{B}_1$, $\mathcal{B}_0 \cong \mathcal{B}_1$. The idea of the proof is to define, by recursion, a strictly increasing elementary chain

$$\mathcal{B}_0 \prec \mathcal{B}_1 \prec \dots \prec \mathcal{B}_\nu \prec \dots \quad (\nu < \omega_1)$$

of countable, homogeneous structures such that for all $\nu < \omega_1$, $\mathcal{B}_\nu \cong \mathcal{B}_0$ and $U^{\mathcal{B}_\nu} = U^{\mathcal{B}_0}$, so that $\mathcal{B} = \bigcup_{\nu < \omega_1} \mathcal{B}_\nu$ is as required by the theorem.

$\mathcal{B}_0, \mathcal{B}_1$ are already defined. Suppose we have defined $\mathcal{B}_\nu$. Since $\mathcal{B}_\nu \cong \mathcal{B}_0$, we may let $\mathcal{B}_{\nu+1}$ be related to $\mathcal{B}_\nu$ as $\mathcal{B}_1$ is related to $\mathcal{B}_0$. This leaves us with the case where $\delta < \omega_1$ is a limit ordinal and $\mathcal{B}_\nu$, $\nu < \delta$, are all defined. In this case we let $\mathcal{B}_\delta = \bigcup_{\nu < \delta} \mathcal{B}_\nu$. By 1.6, $\mathcal{B}_\delta$ is as required. The proof is complete. □

The above proof depended upon the countability of the structures $\mathcal{B}_\nu$ in a significant way. Consequently, there seems to be no possibility of extending the chain $(\mathcal{B}_\nu | \nu < \omega_1)$ to an ω_2-chain and thereby produce a model of type (ω_2, ω). In fact it is easily seen that it is not possible to increase the size of a "gap" in a Cardinal Transfer Theorem. But when it comes to trying to prove the Gap-2 Cardinal Transfer Theorem we have some extra initial information: we start with a structure of type (κ^{++}, κ). How can we make use of this fact to obtain an elementarily equivalent structure of type (ω_2, ω)? Ideally we would like to utilise the methods developed in order to prove the Gap-1 Theorem. Thus, the idea is to construct the desired (ω_2, ω)-model as a limit of some system of countable *approximations* to it, in the sense that each of the structures $\mathcal{B}_\nu$, $\nu < \omega_1$, of 1.7 is an *approximation* to the sought-after (ω_1, ω)-model $\mathcal{B}$. But if we are to obtain a model of type (ω_2, ω) as a limit of countable models, there is no point in trying to use an elementary chain of structures. Rather we require some kind of elementary directed system of models. We commence by considering a "naive" approach to this problem.

We wish to construct a model $\mathcal{B}$ of type (ω_2, ω). We may regard this model as a union of a chain

$$\mathcal{B}_0 \prec \mathcal{B}_1 \prec \dots \prec \mathcal{B}_\nu \prec \dots \prec \mathcal{B} \quad (\nu < \omega_2)$$

of structures of type (ω_1, ω), all having the same distinguished subset, U. Each of these (ω_1, ω)-structures $\mathcal{B}_\nu$ can itself be represented as the union of a chain

$$\mathcal{B}_{\nu 0} \prec \mathcal{B}_{\nu 1} \prec \dots \prec \mathcal{B}_{\nu\tau} \prec \dots \prec \mathcal{B}_\nu \quad (\tau < \omega_1)$$

of countable structures, all with the same U. Thus the structure $\mathcal{B}$ is a sort of limit of the system of countable structures $\mathcal{B}_{\nu\tau}$, $\nu < \omega_2$, $\tau < \omega_1$. The question is, can we construct such a system *from below* in order to determine the limit structure $\mathcal{B}$? It turns out that if we assume $V = L$, this can be done, though it is by no means an easy matter, and relies heavily upon the Fine Structure Theory. The central point is the construction of a framework upon which a suitable directed elementary system can be built. This framework is known as a *morass*. In the next section we shall give a precise definition of a morass and show how such a structure can be built in L. In a sense, when we write down the axioms for a morass we are simply stating some properties of the Fine Structure Theory of a certain hierarchy of structures of the form $\langle J_\varrho, A\rangle$. It is thus not altogether surprising to discover that the structure so defined is somewhat "richer" than is required to prove the Gap-2 Theorem. In order to prove this (and many other applications of morasses) a simpler structure suffices. This "simplified morass" will be described in section 4. In section 3 we shall give a proof of the Gap-2 Theorem using the "standard" morass constructed in L. Section 4 contains an alternative proof of the Gap-2 Theorem using the simplified morass structure. The reason for this duplication is that the proof in section 3, using the standard morass, illustrates just how the Fine Structure Theory enables this theorem to be proved (which is, of course, the main aim of this book), whereas the (simpler) proof in section 4 serves as a prototype for other applications of morasses. Thus the reader who symply wants to learn how to use a morass may go straight on to section 4 from this point. (Though some acquaintance with section 2 is necessary if the reader wishes to find out just where the simplified morass comes from.)

2. Gap-1 Morasses

We can obtain a structure of cardinality κ^+ as a limit of a κ^+-chain of structures of cardinality κ. In order to determine a structure of cardinality κ^{++} as a limit of κ^+ many structures each of cardinality κ, a chain of structures will not work, and we must define instead some sort of directed system of structures. The underlying set-theoretic problem then is to establish some sort of framework upon which such a system can be built, corresponding to the well-ordered set κ^+ used as the domain of κ^+-chains. Such a framework (or indexing system) is called a *morass*. or more precisely a $(\kappa^+, 1)$-morass. For definiteness, we shall present our development for the case $\kappa = \omega$. The general case is entirely similar. So what we shall describe is a $(\omega_1, 1)$-morass. (We shall then say a few words about (κ, n)-morasses for $n > 1$.)

In order to formulate the notion of a morass, let us fix some sort of schematic representation of what we require. We want to determine, by means of countable structures, all the structures of cardinality ω_1 which lie in an increasing ω_2-chain determining a structure of cardinality ω_2. Let $\mathscr{A}$ denote the structure of cardinality ω_2 we are aiming at, and let $\mathscr{A}_\nu$, $\omega_1 < \nu < \omega_2$ be the increasing chain of length ω_2, where $|\mathscr{A}_\nu| = \omega_1$ for each ν. We can represent this as in Fig. 1.

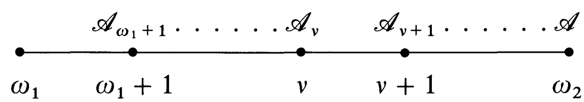


Fig. 1

For each $\nu, \omega_1 < \nu < \omega_2$, we have a chain of models with limit $\mathscr{A}_\nu$. Each member of this chain will be a countable structure. We shall index each such by a countable ordinal τ, so $\mathscr{A}_\nu$ will be the limit of the structures $\mathscr{A}_{\nu\tau}$ for certain τ. We shall not use all countable ordinals τ here, just a certain collection associated with ν. It will turn out to be both natural and convenient to specify the ordinals τ associated with ν by defining a well-founded relation $\dashv$ on ω_2 so that $\{\tau \mid \tau \dashv \nu\}$ is the set of ordinals for which $\mathscr{A}_{\nu\tau}$ is defined, with $\{\tau \mid \tau \dashv \nu\}$ being totally ordered by $\dashv$ and $\bar{\tau} \dashv \tau \dashv \nu$ implying that, in some sense $\mathscr{A}_{\nu\tau}$ "extends" $\mathscr{A}_{\nu\bar{\tau}}$. However, just being able to determine the structures $\mathscr{A}_\nu$ piecemeal will not be enough. We need to determine the *sequence* $(\mathscr{A}_\nu \mid \omega_1 < \nu < \omega_2)$. In order to do this, we do not simply approximate the models $\mathscr{A}_\nu$, but rather the initial segments $(\mathscr{A}_{\bar{\nu}} \mid \omega_1 < \bar{\nu} \leqslant \nu)$ of our final chain.

Dropping our reference to the models $\mathscr{A}_\nu$, $\mathscr{A}_{\nu\tau}$ now, let us concentrate on the indexing system upon which we shall define the model system: this will be our "morass". We have seen that we need to be able to obtain each interval $[\omega_1, \nu]$, $\omega_1 < \nu < \omega_2$, as a limit of intervals $[\alpha, \tau]$, $\alpha < \tau < \omega_1$, in order that we shall never have to consider uncountable models $\mathscr{A}_{\nu\tau}$ during the course of our eventual construction. Just what the ordinals α, τ here are will clearly be unimportant: what counts is how these intervals fit together to form the indexing system. Hence we may assume that all of the small approximating intervals are disjoint: i.e. if $[\alpha, \tau]$, $[\alpha', \tau']$ are part of our morass, and if $\alpha < \alpha'$, then $\tau < \alpha'$. (This is not a misprint!) In point of fact, when we come to give the formal definition of a morass, we shall not use the entire interval $[\alpha, \tau]$ but rather a certain closed subset of it. This does not effect the combinatorial properties of the morass at all, but will make matters a little simpler when we come to construct a morass in L.

If $[\bar{\alpha}, \bar{\tau}]$, $[\alpha, \tau]$ are intervals in the morass with $\bar{\tau} \dashv \tau$, there will be an embedding $\pi_{\bar{\tau}\tau}$ of $[\bar{\alpha}, \bar{\tau}]$ into $[\alpha, \tau]$. And if $\omega_1 < \tau < \omega_2$, $[\omega_1, \tau]$ will be a direct limit of all the intervals $[\bar{\alpha}, \bar{\tau}]$ in the morass with $\bar{\tau} \dashv \tau$, under the π-embeddings. All of this is indicated in Fig. 2, where we adopt the usual convention that the relationship $\bar{\tau} \dashv \tau$ is indicated by a line drawn from τ downwards to $\bar{\tau}$. In connection with Fig. 2, notice that we draw each of the morass intervals $[\alpha, \tau]$ horizontally, to emphasise how they all fit together. In reality, by the disjointness of the intervals, the whole thing could be drawn as a single straight line, and indeed that is what it really is. But it is clearer to draw each morass interval horizontally as shown, so we shall continue to do so.

The problem is how do we set this up so that it works? In particular, the ω_1 many countable intervals must all fit together neatly so that the limits on the top level do indeed give us the chain of intervals $\langle [\omega_1, \nu] \mid \omega_1 < \nu < \omega_2 \rangle$. In order to arrange this, we shall have to make matters somewhat more complicated than we have indicated so far. One aspect of this is that some morass intervals will be initial parts of other morass intervals, so that the disjointness of intervals that we spoke of a few moments ago will not be true for all pairs of intervals (though it will be the case that the only two possibilities are disjointness or initial segments).

We shall say that an ordinal α is *adequate* iff it is either admissible or else is a limit of admissible ordinals. The adequate ordinals thus form a proper class of ordinals which is closed and unbounded in every uncountable cardinal. Moreover, each adequate ordinal has strong closure properties under definability.

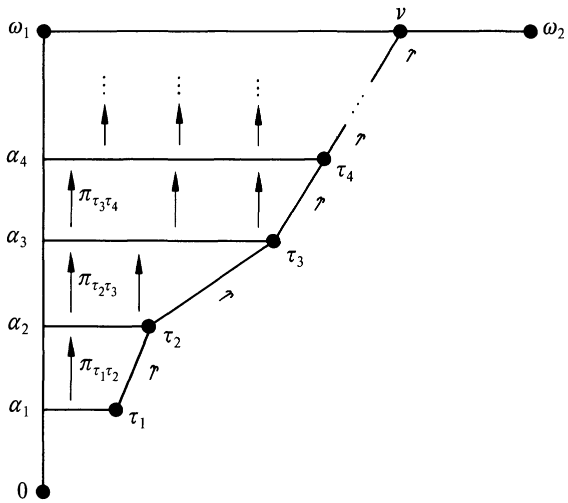


Fig. 2

Let $\mathscr{S}$ be a set of ordered pairs (α, ν) of adequate ordinals such that $\alpha < \nu < \omega_2$, $\alpha \leqslant \omega_1$, and whenever $(\alpha, \nu), (\alpha', \nu') \in \mathscr{S}$, then

$$\alpha < \alpha' \rightarrow \nu < \alpha'.$$

Define:

$$S^0 = \{\alpha \in \omega_1 + 1 \mid \exists \nu [(\alpha, \nu) \in \mathscr{S}]\};$$

S^1 $$S^1 = \{\nu \in \omega_2 \mid \exists \alpha [(\alpha, \nu) \in \mathscr{S}]\};$$

S $$S = S^0 \cup S^1;$$

S_α $$S_\alpha = \{\nu \in S^1 \mid (\alpha, \nu) \in \mathscr{S}\}, \quad \text{for } \alpha \in S^0;$$

α_ν $$\alpha_\nu = \text{the unique ordial } \alpha \in S^0 \text{ such that } (\alpha, \nu) \in \mathscr{S}, \text{ for } \nu \in S^1.$$

Intuitively, S_{ω_1} is the ω_2-chain we are trying to determine, whilst each S_α, $\alpha < \omega_1$, is a countable approximation to S_{ω_1}. Fig. 3 illustrates the notation.

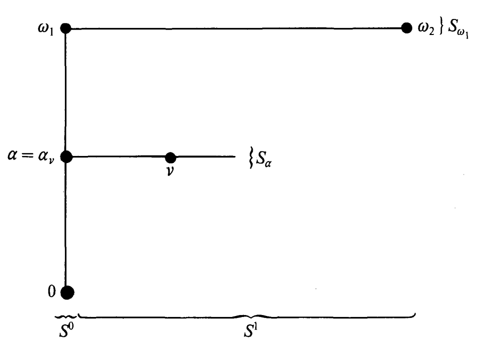


Fig. 3

Let $\dashv$ be a tree ordering on S^1 such that

$$\nu \dashv \tau \rightarrow \alpha_\nu < \alpha_\tau .$$

Let $(\pi_{\nu\tau} \mid \nu \dashv \tau)$ be a commutative system of maps

$$\pi_{\nu\tau}\colon (\nu + 1) \rightarrow (\tau + 1) .$$

Let

$$\mathcal{M} = \langle S, \mathcal{S}, \dashv, (\pi_{\nu\tau})_{\nu \dashv \tau} \rangle .$$

We say that $\mathcal{M}$ is an $(\omega_1, 1)$-*morass* (*morass*, from now on) iff the following axioms (M 0) through (M 7) are satisfied.

(M 0) (a) S_α is closed in $\sup(S_\alpha)$ for all $\alpha \in S^0$, and if $\alpha < \omega_1$, then $\sup(S_\alpha) \in S_\alpha$;

(b) $\omega_1 = \max(S^0) = \sup(S^0 \cap \omega_1)$ and $\omega_2 = \sup(S_{\omega_1})$.

(M 1) If $\nu \dashv \tau$, then

$$\pi_{\nu\tau} \upharpoonright \alpha_\nu = \mathrm{id} \upharpoonright \alpha_\nu, \qquad \pi_{\nu\tau}(\alpha_\nu) = \alpha_\tau, \qquad \pi_{\nu\tau}(\nu) = \tau$$

and $\pi_{\nu\tau}$ maps $S_{\alpha_\nu} \cap (\nu + 1)$ into $S_{\alpha_\tau} \cap (\tau + 1)$ in an order-preserving fashion so that:

(i) if γ is the first member of S_{α_ν}, then $\pi_{\nu\tau}(\gamma)$ is the first member of $S_{\alpha\tau}$;

(ii) if γ immediately succeeds β in $S_{\alpha_\nu} \cap (\nu + 1)$, then $\pi_{\nu\tau}(\gamma)$ immediately succeeds $\pi_{\nu\tau}(\beta)$ in S_{α_τ};

(iii) if γ is a limit point in $S_{\alpha_\nu} \cap (\nu + 1)$, then $\pi_{\nu\tau}(\gamma)$ is a limit point in S_{α_τ}.

Thus what (M 1) says is that the maps $\pi_{\nu\tau}$ embed each morass "interval" $S_{\alpha_\nu} \cap (\nu + 1)$ into the morass "interval" $S_{\alpha_\tau} \cap (\tau + 1)$ in a structure preserving fashion.

(M 2) If $\bar{\tau} \dashv \tau$ and $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$, and we set $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu})$, then

$$\bar{\nu} \dashv \nu \quad \text{and} \quad \pi_{\bar{\nu}\nu} \upharpoonright \bar{\nu} = \pi_{\bar{\tau}\tau} \upharpoonright \bar{\nu} .$$

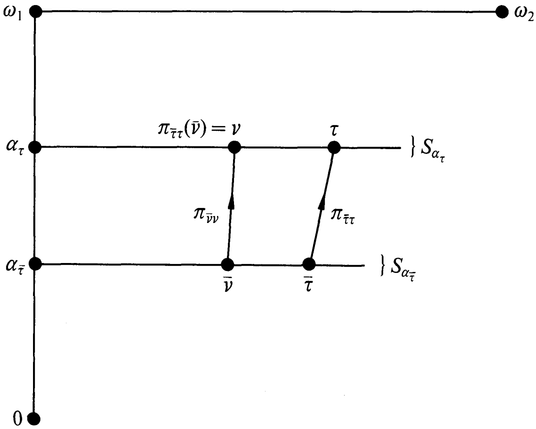
What (M 2) says is that the morass embeddings $\pi_{\bar{\nu}\nu}$ fit together nicely as we move right along each row S_α. Figure 4 provides the picture.

(M 3) $\{\alpha_\nu \mid \nu \dashv \tau\}$ is closed in α_τ for every $\tau \in S^1$.

(M 3) tells us that as we move up along a branch $\{\nu \mid \nu \dashv \tau\}$ of the morass tree, all limit points exist on this branch, or more precisely, the limit points are on the morass "levels" they "ought" to be.

(M 4) If τ is not maximal in S_{α_τ}, then the set $\{\alpha_\nu \mid \nu \dashv \tau\}$ is unbounded in α_τ.

(M 4) tells us that any point which is not at the extreme right hand end of its morass level is a limit point in the morass tree $\dashv$. This has the rather surprising



(M 2) asserts that $\bar\nu \dashv \nu$ and $\pi_{\bar\nu\nu} \upharpoonright \bar\nu = \pi_{\bar\tau\tau} \upharpoonright \bar\nu$.

Fig. 4

consequence that if α is a successor point in S^0, then S_α has only one member. Thus the approximations S_α, $\alpha \in S^0$, to S_{ω_1} do not "get better" monotonically as α increases: only at limit stages is there any chance of some progress in this sense.

(M 5) If $\{\alpha_\nu \mid \nu \dashv \tau\}$ is unbounded in α_τ, then

$$\tau = \bigcup_{\nu \dashv \tau} \pi_{\nu\tau}{}''\nu .$$

Used in conjunction, (M 3), (M 4), and (M 5) tell us that if τ is not the maximal point in its level, then the entire structure up to τ, in particular the morass interval $S_{\alpha_\tau} \cap \tau$, is the limit of the lower structure. For by these three axioms, together with (M 0), we see that if τ is not the maximal point of S_{α_τ}, then

$$S_{\alpha_\tau} \cap \tau = \bigcup_{\bar\tau \dashv \tau} \pi_{\bar\tau\tau}{}''(S_{\alpha_{\bar\tau}} \cap \bar\tau).$$

By (M 0), this applies in particular to any point $\tau \in S_{\omega_1}$. Thus the entire top level of the morass is determined by the structure below.

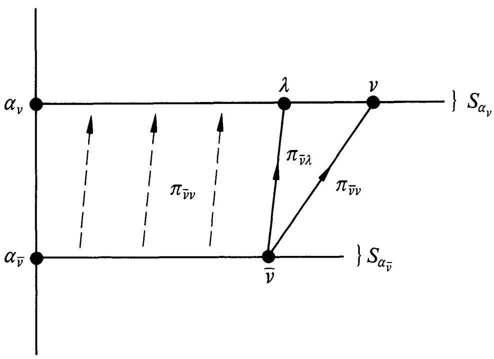


Fig. 5

(M 6) If $\bar{\nu}$ is a limit point of $S_{\alpha_{\bar{\nu}}}$ and $\bar{\nu} \dashv \nu$, and if we set $\lambda = \sup(\pi_{\bar{\nu}\nu}{}''\bar{\nu})$, then $\bar{\nu} \dashv \lambda$ and $\pi_{\bar{\nu}\lambda} \upharpoonright \bar{\nu} = \pi_{\bar{\nu}\nu} \upharpoonright \bar{\nu}$.

Loosely speaking, (M 6) says that although $\pi_{\bar{\nu}\nu}$ need not map $\bar{\nu}$ cofinally into ν, all of the morass maps $\pi_{\bar{\nu}\nu}$ are nevertheless cofinal maps in some sense. Figure 5 illustrates the situation.

(M 7) If $\bar{\nu}$ is a limit point of $S_{\alpha_{\bar{\nu}}}$, $\bar{\nu} \dashv \nu$, $\nu = \sup(\pi_{\bar{\nu}\nu}{}''\bar{\nu})$, and if

$$\alpha \in \bigcap_{\bar{\tau} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}} \{\alpha_\eta \mid \bar{\tau} \sqsupseteq \eta \sqsupseteq \pi_{\bar{\nu}\nu}(\bar{\tau})\},$$

then

$$(\exists \tau \in S_\alpha)(\bar{\nu} \sqsupseteq \tau \sqsupseteq \nu).$$

(Notice that by (M 2), if $\bar{\tau} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$, then $\bar{\tau} \dashv \pi_{\bar{\nu}\nu}(\bar{\tau})$.)

(M 7) says, more or less, that a level "intermediate" between two levels cannot peter out at some limit point. See Fig. 6.

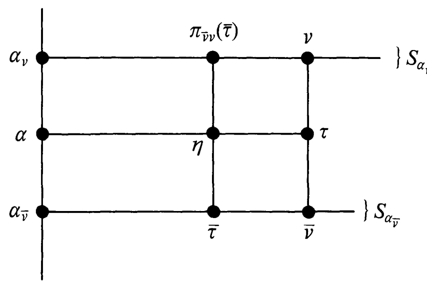


Fig. 6

That defines the notion of a morass. We should point out that it is known that such a structure cannot be constructed in ZFC. But if we assume $V = L$, as we do from now on, then we can construct a morass, though it will take some time to do so. We start with a simple model-theoretic notion.

If ν is a limit ordinal and $X \subseteq J_\nu$, we write

$$X \prec_Q J_\nu,$$

and say that X is a *Q-submodel* of J_ν, iff, for all Σ_0-formulas $\varphi(v_0, v_1)$ of $\mathscr{L}_X$,

$$\models_X (\forall \alpha)(\exists \beta > \alpha)\, \varphi(\beta, J_\beta) \quad \text{iff } \models_{J_\nu} (\forall \alpha)(\exists \beta > \alpha)\, \varphi(\beta, J_\beta).$$

Clearly, if $X \prec_Q J_\nu$ then $X \prec_1 J_\nu$, since by $\lim(\nu)$ we can bind any existential quantifier by some J_β for $\beta < \nu$. Conversely, if $X \prec_1 J_\nu$ is such that $X \cap \nu$ is cofinal in ν, then $X \prec_Q J_\nu$. Hence the notion of a Q-submodel lies between the notions of a Σ_1-submodel and a cofinal Σ_1-submodel. We shall use the notion of a Q-submodel, or rather the associated notion of a Q-embedding, when we define the morass tree relation $\dashv$.

As a first step to the construction of a morass, we define

$\mathscr{S}$ $$\mathscr{S} = \{(\alpha, \nu) \mid \alpha < \nu < \omega_2 \wedge \omega < \alpha \leqslant \omega_1 \wedge \nu \text{ is adequate} \wedge \models_{J_\nu} \text{“}\alpha \text{ is regular and is the largest cardinal”}\}.$$

Notice that if $(\alpha, \nu) \in \mathscr{S}$, then α is admissible, hence adequate.

Define $S^0, S^1, S, S_\alpha, \alpha_\nu$ now as before. Notice that $S_{\alpha_\nu} \cap \nu$ is uniformly $\Sigma_1^{J_\nu}(\{\alpha_\nu\})$ for $\nu \in S^1$. Notice also that S_α is always closed in $\sup(S_\alpha)$ and that $\sup(S_\alpha) \in S_\alpha$ for $\alpha < \omega_1$.

If $\nu \in S^1$ now, then clearly $\nu \neq \omega_1$, so we may define

$\beta(\nu)$ $$\beta(\nu) = \text{the least } \beta \geqslant \nu \text{ such that } \nu \text{ is singular over } J_\beta;$$
$n(\nu)$ $$n(\nu) = \text{the least } n \geqslant 1 \text{ such that } \nu \text{ is } \Sigma_n\text{-singular over } J_{\beta(\nu)};$$
$\varrho(\nu)$ $$\varrho(\nu) = \varrho_{\beta(\nu)}^{n(\nu)-1};$$
$A(\nu)$ $$A(\nu) = A_{\beta(\nu)}^{n(\nu)-1}.$$

Notice that as ν is Σ_{n-1}-regular over $J_{\beta(\nu)}$, we have $\varrho(\nu) \geqslant \nu$, and that if $\varrho(\nu) > \nu$, then, since $\varrho(\nu) \leqslant \beta(\nu)$, $\models_{J_{\rho(\nu)}}$ “ν is regular”. Notice also that if $\nu \in S_\alpha$, then $\models_{J_{\beta(\nu)}}$ “α is regular”. For if $\beta(\nu) = \nu$, this is true because $\nu \in S_\alpha$, whilst if $\beta(\nu) > \nu$ it follows from the two facts $\models_{J_{\beta(\nu)}}$ “ν is regular” and $\models_{J_\nu}$ “α is regular”. Also, if $\tau \in S_\alpha \cap \nu$, where $\nu \in S_\alpha$, then as α is the largest cardinal in J_ν, τ cannot be a cardinal in J_ν, so $\varrho(\tau) \leqslant \beta(\tau) < \nu \leqslant \varrho(\nu) \leqslant \beta(\nu)$.

We turn now to the definition of the tree relation $\dashv$ and the maps $\pi_{\nu\tau}$, $\nu \dashv \tau$. The idea is as follows. To each $\nu \in S^1$ we shall associate a certain parameter $p(\nu) \in J_{\varrho(\nu)}$, so that, in particular, every element of $J_{\varrho(\nu)}$ is Σ_1-definable from parameters in $\alpha_\nu \cup \{p(\nu)\}$ in the structure $\langle J_{\varrho(\nu)}, A(\nu)\rangle$. We shall then set $\nu \dashv \tau$ iff $\alpha_\nu < \alpha_\tau$ and there is an embedding

$$\sigma\colon \langle J_{\varrho(\nu)}, A(\nu)\rangle \prec_1 \langle J_{\varrho(\tau)}, A(\tau)\rangle$$

such that

$$(\sigma \upharpoonright J_\nu)\colon J_\nu \prec_Q J_\tau, \qquad \sigma \upharpoonright \alpha_\nu = \mathrm{id} \upharpoonright \alpha_\nu, \qquad \sigma(p(\nu)) = p(\tau).$$

The definability property of $p(\nu)$ just mentioned will ensure that σ is unique here, and we shall set

$$\pi_{\nu\tau} = (\sigma \upharpoonright \nu) \cup \{(\tau, \nu)\}.$$

Let us remark right away that the requirement that $(\sigma \upharpoonright J_\nu)\colon J_\nu \prec_Q J_\tau$ in the above definition is a minor technical matter connected with morass axiom (M 1), and otherwise plays no role in our development. So the reader can for the most part ignore this point.

The definition of the parameter $p(\nu)$ depends upon the nature of ν. There are two cases to consider. We partition S^1 into two sets thus:

P $$P = \{\nu \in S^1 \mid n(\nu) = 1 \wedge \operatorname{succ}(\beta(\nu))\};$$
R $$R = S^1 - P.$$

$\gamma(\nu)$ In case $\nu \in P$, let $\gamma(\nu)$ denote that ordinal γ such that $\beta(\nu) = \gamma + 1$.

Notice that if $\nu \in P$, then $\varrho(\nu) = \beta(\nu) = \gamma(\nu) + 1$, $A(\nu) = \emptyset$, and (since $\nu \leqslant \beta(\nu)$ and ν is adequate) $\nu \leqslant \gamma(\nu)$. Whilst if $\nu \in R$, then $\lim(\varrho(\nu))$.

Let $\nu \in S^1$ now, and set $\alpha = \alpha_\nu$, $\beta = \beta(\nu)$, $n = n(\nu)$, $\varrho = \varrho(\nu)$, $A = A(\nu)$, and if $\nu \in P$, $\gamma = \gamma(\nu)$. $\alpha, \beta, n, \varrho, A$ γ

2.1 Lemma. $\varrho_\beta^n \leqslant \alpha$.

Proof. Since α is the largest cardinal in J_ν, the definition of β, n ensures that there is a $\Sigma_n(J_\beta)$ map $f \subseteq \nu \times \alpha$ such that $f''\alpha$ is cofinal in ν. Since either $\beta = \nu$ or else $\models_{J_\beta}$ "ν is regular", $f \notin J_\beta$. But if $\varrho_\beta^n > \nu$, then by amenability, $f = f \cap (\nu \times \nu) \in J_{\varrho_\beta^n} \subseteq J_\beta$. Hence $\varrho_\beta^n \leqslant \nu$. It follows that there is a $\Sigma_n(J_\beta)$ map g such that $g''\nu = J_\beta$. Let $(A_\xi \mid \xi < \alpha)$ be a partition of α into α sets of cardinality α in J_ν, and let f_ξ be the $<_J$-least map from A_ξ onto $f(\xi)$ for each $\xi \in \mathrm{dom}(f)$. Since α is the largest cardinal in J_ν, we have $f_\xi \in J_\nu$ for all ξ. Let

$$k = \bigcup \{f_\xi \mid \xi \in \mathrm{dom}(f)\}.$$

Then k is a $\Sigma_n(J_\beta)$ function such that $k''\alpha = \nu$. Hence $g \circ k$ is a $\Sigma_n(J_\beta)$ function such that $g \circ k''\alpha = J_\beta$. Thus $\varrho_\beta^n \leqslant \alpha$. $\square$

Case 1. $\nu \in P$.

2.2 Lemma. *There is a $q \in J_\gamma$ such that every $x \in J_\gamma$ is J_γ-definable from parameters in $J_\alpha \cup \{q\}$.*

Proof. By 2.1 there is a $p \in J_\beta$ such that every $x \in J_\beta$ is Σ_1-definable in J_β from parameters in $J_\alpha \cup \{p\}$. Since $J_\beta = \mathrm{rud}(J_\gamma)$, there is a rudimentary function f and an element $q \in J_\gamma$ such that $p = f(J_\gamma, q)$. We show that q is as in the lemma.

Let $x \in J_\gamma$. For some Σ_0 formula φ of $\mathscr{L}$ and some $\vec{z} \in J_\alpha$, x is the unique x in J_β such that

$$(\exists y \in J_\beta)[\models_{J_\beta} \varphi(\mathring{y}, \vec{\mathring{z}}, \mathring{p}, \mathring{x})].$$

Pick $y \in J_\beta$ so that

$$\models_{J_\beta} \varphi(\mathring{y}, \vec{\mathring{z}}, \mathring{p}, \mathring{x}).$$

For some rudimentary function g and some $u \in J_\gamma$, we have $y = g(J_\gamma, u)$. So

$$(*) \qquad \models_{J_\beta} \varphi(g(J_\gamma, u)^\circ, \vec{\mathring{z}}, f(J_\gamma, q)^\circ, \mathring{x}).$$

Since g, f are rudimentary, hence simple, the formula $\varphi(g(y, u), \vec{z}, f(y, q), x)$ is Σ_0 in variables $y, u, \vec{z}, q, x$. So by VI.1.18 there is an $\mathscr{L}$-formula ψ such that $(*)$ is equivalent to

$$(**) \qquad \models_{J_\gamma} \psi(\mathring{x}, \mathring{u}, \vec{\mathring{z}}, \mathring{q}).$$

It follows at once that x is the unique element of J_γ such that $\models_{J_\gamma} \exists u\, \psi(\mathring{x}, \mathring{u}, \vec{\mathring{z}}, \mathring{q})$, and the lemma is proved. $\square$

$q(\nu)$ Let $q(\nu)$ be the $<_J$-least $q \in J_\gamma$ such that every $x \in J_\gamma$ is J_γ-definable from parameters in $\alpha \cup \{q\}$. (Since α is adequate, α is closed under Gödel's Pairing Function, so by VI.3.17, there is a $\Sigma_1^{J_\alpha}$ map from α onto J_α. But $\gamma \geqslant \nu > \alpha$, so this map is an element of J_γ. Hence $q(\nu)$ exists by virtue of 2.2.)

Set

$p(\nu)$ $$p(\nu) = (q(\nu), \gamma(\nu), \nu, \alpha_\nu).$$

That defines $p(\nu)$ in Case 1. We check that $p(\nu)$ has the property we mentioned earlier, that every element of $J_{\varrho(\nu)}$ is Σ_1 definable from parameters in $\alpha_\nu \cup \{p(\nu)\}$ in $\langle J_{\varrho(\nu)}, A(\nu)\rangle$. In this case, what this says is that every element of J_β is Σ_1 definable from parameters in $\alpha \cup \{p(\nu)\}$ in J_β.

Let $x \in J_\beta$. Then for some rudimentary function f and some $u \in J_\gamma$, $x = f(J_\gamma, u)$. Since J_β is rud closed, $f \upharpoonright J_\beta$ is $\Sigma_1^{J_\beta}$. So x is Σ_1 definable from γ and u in J_β. Since $\gamma = (p(\nu))_1$, we are done if we can show that u is Σ_1 definable from parameters in $\alpha \cup \{p(\nu)\}$ in J_β. Well, by choice of $q(\nu)$ there is an $\mathscr{L}$-formula φ and elements $\vec{z} \in \alpha$ such that

$$u = \text{the unique } u \in J_\gamma \text{ such that } \models_{J_\gamma} \varphi(\mathring{u}, \mathring{\vec{z}}, \mathring{q(\nu)}).$$

But this defines u in a $\Sigma_1^{J_\beta}$ fashion from $\gamma, \vec{z}, q(\nu)$. Hence as $\gamma = (p(\nu))_1$ and $q(\nu) = (p(\nu))_0$, we see that u is Σ_1 definable from $\vec{z}, p(\nu)$ in J_β, and we are done.

Case 2. $\nu \in R$.

$q(\nu)$ Let $q(\nu)$ be the $<_J$-least $q \in J_\varrho$ such that every $x \in J_\varrho$ is Σ_1-definable from parameters in $\alpha \cup \{q\}$ in $\langle J_\varrho, A\rangle$. Since $\varrho^1_{\varrho, A} = \varrho^n_\beta$, this is possible by virtue of 2.1. (An argument as in Case 1 allows us to write α in place of J_α in this definition.) Set

$p(\nu)$ $$p(\nu) = \begin{cases} (q(\nu), \nu, \alpha_\nu), & \text{if } \nu < \varrho, \\ (q(\nu), \alpha_\nu), & \text{if } \nu = \varrho. \end{cases}$$

That defines $p(\nu)$ in Case 2.

Note that as $q(\nu) = (p(\nu))_0$, it follows from the definition of $q(\nu)$ that every element of $J_{\varrho(\nu)}$ is Σ_1 definable from parameters in $\alpha_\nu \cup \{p(\nu)\}$ in $\langle J_{\varrho(\nu)}, A(\nu)\rangle$ in this case also.

Having now defined the parameters $p(\nu)$, $\nu \in S^1$, we establish a series of lemmas which will enable us to construct a morass in the manner outlined earlier.

2.3 Lemma. *The sequences* $\langle (J_{\varrho(\eta)}, A(\eta), J_\eta, p(\eta)) \mid \eta \in S_{\alpha_\nu} \cap \nu\rangle$ *is uniformly* $\Sigma_1^{J_\nu}(\{\alpha_\nu\})$ *for all* $\nu \in S^1$.

Proof. Since ν is adequate, this follows easily from the fact that if $\eta \in S_{\alpha_\nu} \cap \nu$, then $\beta(\eta) < \nu$, mentioned earlier. □

2.4 Lemma. *Let* $\nu, \tau \in S^1$. *Suppose that*

$$\sigma\colon \langle J_{\varrho(\nu)}, A(\nu)\rangle \prec_1 \langle J_{\varrho(\tau)}, A(\tau)\rangle$$

is such that $\sigma(p(\nu)) = p(\tau)$. *Then* σ is uniquely determined by $\sigma \upharpoonright \alpha_\nu$. *Moreover*:

(i) $\nu \in P \leftrightarrow \tau \in P$;
(ii) $\sigma(\alpha_\nu) = \alpha_\tau$;
(iii) $\nu < \varrho(\nu) \leftrightarrow \tau < \varrho(\tau)$;
(iv) $\nu < \varrho(\nu) \to \sigma(\nu) = \tau$;
(v) *if* $\nu \in P$, *then* $\sigma(\gamma(\nu)) = \gamma(\tau)$;
(vi) $\sigma(q(\nu)) = q(\tau)$.

Proof. The uniqueness of σ follows from the definability property of $p(\nu)$. The remaining assertions of the lemma all follow from the definitions of $p(\nu)$ and $p(\tau)$. □

2.5 Lemma. *Let* $\nu \in S^1$, $\bar{\varrho} \leqslant \varrho(\nu)$, $\bar{A} \subseteq J_{\bar{\varrho}}$. *Let*

$$\sigma\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 \langle J_{\varrho(\nu)}, A(\nu)\rangle$$ σ

be such that $p(\nu) \in \operatorname{ran}(\sigma)$. *Then there is a (necessarily unique)* $\bar{\nu} \in S^1$ *such that* $\bar{\varrho} = \varrho(\bar{\nu})$, $\bar{A} = A(\bar{\nu})$. *Moreover*, $\sigma(p(\bar{\nu})) = p(\nu)$.

Proof. To commence, notice that $\langle J_{\bar{\varrho}}, \bar{A}\rangle$ is amenable. For if $\nu \in P$, then $A(\nu) = \emptyset$, so $\bar{A} = \emptyset$ and amenability is trivial, and if $\nu \in R$, then $\lim(\varrho(\nu))$, so $\lim(\bar{\varrho})$, and for each $\eta < \bar{\varrho}$, we have

$$\models_{\langle J_{\rho(\nu)}, A(\nu)\rangle} \exists x\,[x = A(\nu) \cap J_{\sigma(\eta)}],$$

so

$$\models_{\langle J_{\bar{\rho}}, \bar{A}\rangle} \exists x\,[x = \bar{A} \cap J_\eta].$$

Set $\alpha = \alpha_\nu$, $\beta = \beta(\nu)$, $n = n(\nu)$, $\varrho = \varrho(\nu)$, $A = A(\nu)$, $p = p(\nu)$, $q = q(\nu)$, and, if $\nu \in P$, $\gamma = \gamma(\nu)$. $\alpha, \beta, n, \varrho, A, p, q, \gamma$

Case 1. $\nu \in P$.

Thus $\beta = \varrho = \gamma + 1$, $\bar{A} = A = \emptyset$, and ν is regular over J_γ. Since $p \in \operatorname{ran}(\sigma)$, we have $q, \gamma, \nu, \alpha \in \operatorname{ran}(\sigma)$. Let $\bar{q} = \sigma^{-1}(q)$, $\bar{\gamma} = \sigma^{-1}(\gamma)$, $\bar{\alpha} = \sigma^{-1}(\alpha)$, $\bar{\nu} = \sigma^{-1}(\nu)$. Let $\bar{\sigma} = \sigma \upharpoonright J_{\bar{\gamma}}$. Clearly, $\bar{\sigma}\colon J_{\bar{\gamma}} \prec J_\gamma$ and $q \in \operatorname{ran}(\bar{\sigma})$. $\bar{q}, \bar{\gamma}, \bar{\alpha}, \bar{\nu}$ $\bar{\sigma}$

Claim A. $\bar{\nu} \in S^1$ and $\alpha_{\bar{\nu}} = \bar{\alpha}$.

Since $\sigma\colon J_{\bar{\varrho}} \prec_1 J_\varrho$ and $\sigma(\bar{\nu}) = \nu$, we have $(\sigma \upharpoonright J_{\bar{\nu}})\colon J_{\bar{\nu}} \prec J_\nu$. Hence $\bar{\nu}$ is adequate. Moreover, since $\sigma(\bar{\alpha}) = \alpha$,

$$\models_{J_{\bar{\nu}}} \text{“}\bar{\alpha} \text{ is regular and is the largest cardinal”}.$$

Thus $\bar{\nu} \in S_{\bar{\alpha}}$. Claim A is proved.

Claim B. $\bar{\sigma}''\,\bar{\nu}$ is cofinal in ν.

For each $m < \omega$, set

$$X_m = \{x \in J_\gamma \mid x \text{ is } \Sigma_{m+1}\text{-definable from parameters in } \alpha \cup \{q\} \text{ in } J_\gamma\}.$$

Then $X_m \prec_m J_\gamma$, and there is a J_γ-definable map from α onto X_m. Since α is the largest cardinal in J_ν and $\alpha \subseteq X_m \prec_1 J_\gamma$, $X_m \cap \nu$ is transitive, so set $\nu_m = X_m \cap \nu$. Since ν is regular over J_γ and there is a J_γ-definable map from α onto ν_m, we must have $\nu_m < \nu$. But by choice of q,

$$\bigcup_{m<\omega} X_m = J_\gamma .$$

Thus $\sup_{m<\omega} \nu_m = \nu$. But for each m, ν_m is J_γ-definable from q, so $\{\nu_m \mid m < \omega\} \subseteq \operatorname{ran}(\bar\sigma)$. This proves Claim B.

For later use, we point out that the sequence $(\nu_m \mid m < \omega)$ is clearly $\Sigma_1^{J_{\gamma+1}}(\{p\})$.

Claim C. $\bar\nu$ is regular over $J_{\bar\gamma}$.

We know that ν is regular over J_γ. But $\bar\sigma$: $J_{\bar\gamma} \prec J_\gamma$ and

$$\bar\nu = \bar\gamma \to \nu = \gamma ,$$
$$\bar\nu < \bar\gamma \to \bar\sigma(\bar\nu) = \nu ,$$

so Claim C is immediate.

Claim D. $\bar q$ is the $<_J$-least element of $J_{\bar\gamma}$ such that every $x \in J_{\bar\gamma}$ is $J_{\bar\gamma}$-definable from parameters in $\bar\alpha \cup \{\bar q\}$.

Let $x \in J_{\bar\gamma}$. Then $\sigma(x) \in J_\gamma$, so for some $\vec\delta \in \alpha$, $\bar\sigma(x)$ is J_γ-definable from q, $\vec\delta$. Set $s = (\vec\delta)$, and let φ be a formula of $\mathscr{L}$ such that:

(i) $\models_{J_\gamma} \forall z \, \exists y \, \forall y' [y' = y \leftrightarrow \varphi(y', z, \mathring{q})]$;

(ii) $\models_{J_\gamma} \forall z \, \forall y [\varphi(y, z, \mathring{q}) \to (\exists \vec\xi)(z = (\vec\xi))]$;

(iii) $(\forall y \in J_\gamma)[y = \bar\sigma(x) \leftrightarrow \models_{J_\gamma} \varphi(\mathring{y}, \mathring{s}, \mathring{q})]$.

Let t be the $<_J$-least element of J_γ such that $\models_{J_\gamma} \varphi(\mathring{\bar\sigma}(x), \mathring{t}, \mathring{q})$. Then t is J_γ-definable from $\bar\sigma(x)$, q. But $\bar\sigma(x), q \in \operatorname{ran}(\bar\sigma) \prec J_\gamma$. Hence $t \in \operatorname{ran}(\bar\sigma)$. By choice of t, $t \leqslant_J s$, so $t \in J_\alpha$. Thus $t = (\vec\xi)$ for some $\vec\xi \in \alpha$. By (i) above,

$$(\forall y \in J_\gamma)[y = \bar\sigma(x) \leftrightarrow \models_{J_\gamma} \varphi(\mathring{y}, \mathring{t}, \mathring{q})].$$

Applying $\bar\sigma^{-1}$ and setting $\bar t = \bar\sigma^{-1}(t) = (\vec\zeta)$, we get,

$$(\forall y \in J_{\bar\gamma})[y = x \leftrightarrow \models_{J_{\bar\gamma}} \varphi(\mathring{y}, \mathring{\bar t}, \mathring{\bar q})] .$$

But $\vec\zeta \in \bar\alpha$. Thus x is $J_{\bar\gamma}$-definable from parameters in $\alpha \cup \{\bar q\}$.

Now suppose that $\bar q' <_J \bar q$ also has the property that every element of $J_{\bar\gamma}$ is $J_{\bar\gamma}$-definable from parameters in $\bar\alpha \cup \{\bar q'\}$. Then in particular there are $\vec\xi \in \bar\alpha$ such that $\bar q$ is $J_{\bar\gamma}$-definable from $\vec\xi$, $\bar q'$. Applying $\bar\sigma$ and setting $q' = \bar\sigma(\bar q')$, $\vec\xi = \bar\sigma(\vec\zeta)$, we see that $q' <_J q$ and that q is J_γ-definable from $\vec\xi, q'$. Hence every element of J_γ is J_γ-definable from parameters in $\bar\alpha \cup \{q'\}$, contrary to the choice of q. That completes the proof of Claim D.

Claim E. $\bar\nu$ is Σ_1-singular over $J_{\bar\gamma+1}$.

Using Claims A, C, D we may define $(\bar{v}_m | m < \omega)$ from $J_{\bar{\gamma}}, \bar{\alpha}, \bar{q}, \bar{v}$ exactly as we defined $(v_m | m < \omega)$ from J_γ, α, q, v in the proof of Claim B. Then $(\bar{v}_m | m < \omega)$ is a $\Sigma_1(J_{\bar{\gamma}+1})$ sequence which is cofinal in $\bar{v}$, proving Claim E.

Claim F. $\beta(\bar{v}) = \bar{\gamma} + 1, n(\bar{v}) = 1, \bar{v} \in P, \varrho(\bar{v}) = \bar{\gamma} + 1, \gamma(\bar{v}) = \bar{\gamma}$.

By Claims C and E.

Claim G. $q(\bar{v}) = \bar{q}, \sigma(p(\bar{v})) = p(v)$.

By Claims F and D, $q(\bar{v}) = \bar{q}$. Thus by Claims A and F, $p(\bar{v}) = (\bar{q}, \bar{\gamma}, \bar{v}, \bar{\alpha})$. Hence $\sigma(p(\bar{v})) = p(v)$. Claim G is proved.

That completes the proof of the lemma in Case 1.

Case 2. $v \in R$.

Set $\bar{q} = \sigma^{-1}(q), \bar{\alpha} = \sigma^{-1}(\alpha)$. Set $\bar{v} = \sigma^{-1}(v)$ if $v < \varrho$ and set $\bar{v} = \bar{\varrho}$ if $v = \varrho$. $\bar{q}, \bar{\alpha}, \bar{v}$
By VI.5.6 there is a unique $\bar{\beta} \geqslant \bar{\varrho}$ such that $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, and an embedding $\tilde{\sigma}: J_{\bar{\beta}} \prec_n J_\beta, \sigma \subseteq \tilde{\sigma}$. $\bar{\beta}$ $\tilde{\sigma}$

Claim H. $\bar{v} \in S^1$ and $\alpha_{\bar{v}} = \bar{\alpha}$.

If $v = \varrho$, then $\bar{v} = \bar{\varrho}$ and $\sigma: J_{\bar{v}} \prec_1 J_v$. And if $v < \varrho$, then $\sigma(\bar{v}) = v$, so $(\sigma \restriction J_{\bar{v}}): J_{\bar{v}} \prec J_v$. In either case, $\bar{v}$ is adequate. Since we always have $(\sigma \restriction J_{\bar{v}}): J_{\bar{v}} \prec_1 J_v$ and $\sigma(\bar{\alpha}) = \alpha$, $\bar{\alpha}$ is regular inside $J_{\bar{v}}$ and is the largest cardinal inside $J_{\bar{v}}$. Hence $\bar{v} \in S_{\bar{\alpha}}$. Claim H is proved.

Claim I. $\bar{v}$ is Σ_{n-1}-regular over $J_{\bar{\beta}}$.

Suppose not. Then, since $\bar{\alpha}$ is the largest cardinal in $J_{\bar{v}}$, we can find a $\Sigma_{n-1}(J_{\bar{\beta}})$ map f such that $f''\bar{\alpha}$ is cofinal in $\bar{v}$. There are now two cases to consider.

Suppose first that $v < \varrho$. Thus $\bar{v} < \bar{\varrho}$ and $\sigma(\bar{v}) = v$. If $f \in J_{\bar{\beta}}$, then by applying $\tilde{\sigma}: J_{\bar{\beta}} \prec_n J_\beta$, we see that $\tilde{\sigma}(f)$ maps a subset of α cofinally into v, contrary to v being regular inside J_β. Hence $f \notin J_{\bar{\beta}}$. But by using Gödel's pairing function we can code f as a $\Sigma_{n-1}(J_{\bar{\beta}})$ subset of $\bar{v}$. Thus $\mathscr{P}(\bar{v}) \cap \Sigma_{n-1}(J_{\bar{\beta}}) \nsubseteq J_{\bar{\beta}}$. Thus $\varrho_{\bar{\beta}}^{n-1} \leqslant \bar{v}$. But $\varrho_{\bar{\beta}}^{n-1} = \bar{\varrho} > \bar{v}$, so we have a contradiction. That proves the claim in the first case.

Now suppose that $v = \varrho$. Thus $\bar{v} = \bar{\varrho}$. In $J_{\bar{v}}$, let $(A_\xi | \xi < \bar{\alpha})$ be a partition of $\bar{\alpha}$ into $\bar{\alpha}$ many sets of cardinality $\bar{\alpha}$. For each $\xi \in \mathrm{dom}(f)$, let $k_\xi \in J_{\bar{v}}$ be the $<_J$-least map from A_ξ onto $f(\xi)$. (Since $\bar{\alpha}$ is the largest cardinal in $J_{\bar{v}}$, k_ξ is well-defined here.) Set

$$k = \bigcup \{k_\xi | \xi \in \mathrm{dom}(f)\}.$$

Clearly, k is a $\Sigma_{n-1}(J_{\bar{\beta}})$ function such that $k''\bar{\alpha} = \bar{v}$. But $\bar{v} = \bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$ and $\bar{\alpha} < \bar{v}$, so this contradicts the definition of the Σ_{n-1}-projectum. Claim I is proved.

Claim J. $\bar{q}$ is the $<_J$-least element of $J_{\bar{\varrho}}$ such that every element of $J_{\bar{\varrho}}$ is Σ_1-definable from parameters in $\bar{\alpha} \cup \{\bar{q}\}$ in $\langle J_{\bar{\varrho}}, \bar{A} \rangle$.

Let $x \in J_{\bar{\varrho}}$. Then $\sigma(x) \in J_\varrho$ so for some $\vec{\delta} \in \alpha$, $\sigma(x)$ is Σ_1-definable from $q, \vec{\delta}$ in $\langle J_\varrho, A \rangle$. Set $s = (\vec{\delta})$. Let φ be a Σ_0-formula of $\mathscr{L}$ such that:

(i) $\models_{\langle J_\rho, A \rangle} \forall z \, \exists y \, \forall y' [y' = y \leftrightarrow \exists u \varphi(u, y', z, \mathring{q})]$;

(ii) $\models_{\langle J_\rho, A \rangle} \forall z \, \forall y [\exists u \varphi(u, y, z, \mathring{q}) \rightarrow (\exists \vec{\xi})(z = (\vec{\xi}))]$;

(iii) $(\forall y \in J_\varrho)[y = \sigma(x) \leftrightarrow \models_{\langle J_\rho, A \rangle} \exists u \, \varphi(u, \mathring{y}, \mathring{s}, \mathring{q})]$.

Let $<^*$ be the lexicographic ordering on $L \times L$ induced by $<_J$. Clearly, $<^*$ is Σ_1 definable, and $<^* \cap (J_\varrho \times J_\varrho)$ is uniformly $\Sigma_1^{J_\rho}$ for all limit $\varrho > 0$. Let (t, u_0) be the $<^*$-least pair such that

$$\models_{\langle J_\rho, A\rangle} \varphi(\mathring{u}_0, \mathring{\sigma}(x)\, \mathring{t}, \mathring{q}).$$

Then (t, u_0) is Σ_1-definable from $\sigma(x), q$ in $\langle J_\varrho, A\rangle$. Hence $t, u_0 \in \operatorname{ran}(\sigma)$. Since $t \leqslant_J s$, $t \in J_\alpha$. Thus $t = (\vec{\xi})$ for some $\vec{\xi} \in \alpha$. By (i) above,

$$(\forall y \in J_\varrho)[y = \sigma(x) \leftrightarrow \models_{\langle J_\rho, A\rangle} \exists u \varphi(u, \mathring{y}, \mathring{t}, \mathring{q})].$$

Applying σ^{-1}, we obtain

$$(\forall y \in J_{\bar{\varrho}})[y = x \leftrightarrow \models_{\langle J_{\bar{\rho}}, A\rangle} \exists u \varphi(u, \mathring{y}, \mathring{\bar{t}}, \mathring{\bar{q}})],$$

where $\bar{t} = \sigma^{-1}(t) = (\vec{\zeta})$, $\vec{\zeta} \in \bar{\alpha}$. Hence x is Σ_1-definable from parameters in $\alpha \cup \{\bar{q}\}$ in $\langle J_{\bar{\varrho}}, \bar{A}\rangle$. The rest of the proof of Claim J is entirely similar to the argument used in proving the minimality of $\bar{q}$ in Claim D (for Case 1). So Claim J is established.

Claim K. $\bar{\nu}$ is Σ_n-singular over $J_{\bar{\beta}}$.

By Claim J,

$$J_{\bar{\varrho}} = h^*_{\bar{\varrho}, \bar{A}}(J_{\bar{\alpha}} \times \{q\}).$$

So there is a $\Sigma_1(\langle J_{\bar{\varrho}}, \bar{A}\rangle)$ map from a subset of $\bar{\alpha}$ onto (in particular) $\bar{\nu}$. Since $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, this map is $\Sigma_n(J_{\bar{\beta}})$. Claim K is proved.

Claim L. $\bar{\beta} = \beta(\bar{\nu})$, $n = n(\bar{\nu})$, $\bar{\nu} \in R$, $\bar{\varrho} = \varrho(\bar{\nu})$, $\bar{A} = A(\bar{\nu})$.

Directly from Claims I and K. (For $\bar{\nu} \in R$, notice that as $\tilde{\sigma}\colon J_{\bar{\beta}} \prec_1 J_\beta$, $\lim(\bar{\beta})$ follows from $\lim(\beta)$.)

Claim M. $\bar{q} = q(\bar{\nu})$ and $\sigma(p(\bar{\nu})) = p(\nu)$.

By Claims L and J, $\bar{q} = q(\bar{\nu})$. If $\nu < \varrho$ now, then $\bar{\nu} < \bar{\varrho}$ and we have $p(\bar{\nu}) = (q(\bar{\nu}), \bar{\nu}, \alpha_{\bar{\nu}})$, so $\sigma(p(\bar{\nu})) = p(\nu)$ by Claim H and the equality $\sigma(\bar{\nu}) = \nu$. If $\nu = \varrho$, then $\bar{\nu} = \bar{\varrho}$ and $p(\bar{\nu}) = (q(\bar{\nu}), \alpha_{\bar{\nu}})$, so again $\sigma(p(\bar{\nu})) = p(\nu)$.

That completes the proof of the lemma. □

$\bar{\nu}, \nu, \bar{\alpha}$ **2.6 Lemma.** *Let $\bar{\nu} \in S_{\bar{\alpha}}$, $\nu \in S_\alpha$, $\bar{\alpha} < \alpha$, and suppose that $\bar{\nu}$ is a limit point of $S_{\bar{\alpha}}$. Let*

σ

$$\sigma\colon \langle J_{\varrho(\bar{\nu})}, A(\bar{\nu})\rangle \prec_1 \langle J_{\varrho(\nu)}, A(\nu)\rangle$$

be such that $\sigma(p(\bar{\nu})) = p(\nu)$.

ν' *Let $\nu' = \sup(\sigma''\bar{\nu})$. Then $\nu' \in S_\alpha$ and there is an embedding*

$$\sigma'\colon \langle J_{\varrho(\bar{\nu})}, A(\bar{\nu})\rangle \prec_1 \langle J_{\varrho(\nu')}, A(\nu')\rangle$$

such that $\sigma \restriction J_{\bar{\nu}} \subseteq \sigma'$, $(\sigma' \restriction J_{\bar{\nu}})\colon J_{\bar{\nu}} \prec_Q J_{\nu'}$, and $\sigma'(p(\bar{\nu})) = p(\nu')$.

Proof. Case 1. $\nu \in P$.

Let $(\nu_m \mid m < \omega)$ be the sequence defined in Claim B of 2.5. This sequence is $\Sigma_1^{J_{\rho(\nu)}}(\{p(\nu)\})$ and is cofinal in ν. Hence $\{\nu_m \mid m < \omega\} \subseteq \operatorname{ran}(\sigma)$, and so $\nu' = \nu$. There is nothing to prove in this case.

Case 2. $\nu \in R$.

Set $\beta = \beta(\nu)$, $n = (\nu)$, $\varrho = \varrho(\nu)$, $A = A(\nu)$, $q = q(\nu)$, $p = p(\nu)$, $\bar{\varrho} = \varrho(\bar{\nu})$, $\bar{A} =$ $A(\bar{\nu})$, $\bar{q} = q(\bar{\nu})$, $\bar{p} = p(\bar{\nu})$. $\beta, n, \varrho, A, q,$ $p, \bar{\varrho}, \bar{A}, \bar{q}, \bar{p}$

Now, $S_{\bar{\alpha}} \cap \bar{\nu}$ is $\Sigma_1^{J_{\bar{\nu}}}(\{\bar{\alpha}\})$ and $S_\alpha \cap \nu$ is $\Sigma_1^{J_\nu}(\{\alpha\})$ by the same definition. And by 2.4, $\sigma(\bar{\alpha}) = \alpha$. Hence as $S_{\bar{\alpha}} \cap \bar{\nu}$ is cofinal in $\bar{\nu}$ and $\nu' = \sup(\sigma''\bar{\nu})$, applying $(\sigma \restriction J_{\bar{\nu}})$: $J_{\bar{\nu}} \prec_1 J_\nu$ gives $\nu' = \sup(S_\alpha \cap \nu')$. But S_α is closed in $\sup(S_\alpha)$ and $\nu' \leqslant \nu \in S_\alpha$. Hence $\nu' \in S_\alpha$.

Set $\eta = \sup(\sigma''\bar{\varrho})$, $\tilde{A} = A \cap J_\eta$. Since $\operatorname{ran}(\sigma) \subseteq J_\eta$, $p, \alpha \in J_\eta$. By Σ_0-absoluteness, $\eta, \tilde{A}$

$$\sigma\colon \langle J_{\bar{\varrho}} \bar{A} \rangle \prec_0 \langle J_\eta, \tilde{A} \rangle .$$

But σ is cofinal in η. Hence

$$\sigma\colon \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_\eta, \tilde{A} \rangle .$$

Set

$$X = h^*_{\eta, \tilde{A}}(J_\alpha \times \{q\}) .$$

Let

$$\pi\colon \langle J_\gamma, B \rangle \cong \langle X, \tilde{A} \cap x \rangle .$$ π, γ, B

Thus

$$\pi\colon \langle J_\gamma, B \rangle \prec_1 \langle J_\eta, \tilde{A} \rangle .$$

Claim A. $\operatorname{ran}(\sigma) \subseteq X$.

Let $x \in \operatorname{ran}(\sigma)$. Then $x \in J_\varrho$, so x is Σ_1-definable from parameters in $\alpha \cup \{q\}$ in $\langle J_\varrho, A \rangle$. Let $\bar{x} = \sigma^{-1}(x)$. An argument as used in the proof of Claim J in 2.5 shows that $\bar{x}$ is Σ_1-definable from parameters in $\bar{\alpha} \cup \{\bar{q}\}$ in $\langle J_{\bar{\varrho}}, \bar{A} \rangle$. Hence for some $i \in \omega$, $\bar{z} \in J_{\bar{\alpha}}$,

$$\bar{x} = h_{\bar{\varrho}, \bar{A}}(i, (\bar{z}, \bar{q})) .$$

Applying $\sigma\colon \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_\eta, \tilde{A} \rangle$, and setting $z = \sigma(\bar{z})$,

$$x = h_{\eta, \tilde{A}}(i, (z, q)) .$$

Hence $x \in X$, which proves Claim A.

Claim B. $X \cap \nu = \nu'$.

Let $\xi \in X \cap \nu$. Then for some $z \in J_\alpha$ and some $i \in \omega$,

$$\xi = h_{\eta, \tilde{A}}(i, (z, q)) .$$

Since $\lim(\eta)$, there is a $\tau < \eta$ such that

$$\xi = h_{\tau, \tilde{A} \cap J_\tau}(i, (z, q)).$$

Since $\eta = \sup(\sigma'' \bar{\varrho})$, we can pick τ here so that $\tau = \sigma(\bar{\tau})$ for some $\bar{\tau} < \bar{\varrho}$. Set

$$\theta = \sup[\nu \cap h^*_{\tau, \tilde{A} \cap J_\tau}(J_\alpha \times \{q\})],$$
$$\bar{\theta} = \sup[\bar{\nu} \cap h^*_{\bar{\tau}, \bar{A} \cap J_{\bar{\tau}}}(J_{\bar{\alpha}} \times \{\bar{q}\})].$$

Now, $\tilde{A} \cap J_\tau = A \cap J_\eta \cap J_\tau = A \cap J_\tau$, so $h_{\tau, \tilde{A} \cap J_\tau} \in J_\varrho$ by amenability. Since $\alpha < \nu$ and ν is regular inside J_ϱ, it follows that $\theta < \nu$. Similarly, $\bar{\theta} < \bar{\nu}$. But clearly, $\sigma(\bar{\theta}) = \theta$. Hence

$$\xi < \theta = \sigma(\bar{\theta}) < \sup(\sigma'' \bar{\nu}) = \nu'.$$

Thus $X \cap \nu \subseteq \nu'$.

Now let $\xi \in \nu'$. For some $\bar{\delta} < \bar{\nu}$, $\xi \in \delta = \sigma(\bar{\delta})$. Since $\bar{\delta} < \bar{\nu}$, there is an $\bar{f} \in J_{\bar{\nu}}$ $\bar{f}: \bar{\alpha} \xrightarrow{\text{onto}} \bar{\delta}$. Since $(\sigma \upharpoonright J_{\bar{\nu}}): J_{\bar{\nu}} \prec_0 J_{\nu'}, f = \sigma(\bar{f}) \in J_{\nu'}$ and $f: \alpha \xrightarrow{\text{onto}} \delta$. But by Claim A, $f \in X$. So as $\alpha \subseteq X$, $\delta = f'' \alpha \subseteq X$. Hence $\xi \in X$. This shows that $\nu' \subseteq X \cap \nu$. Claim B is proved.

We have

$$\operatorname{ran}(\sigma) \prec_1 \langle J_\eta, \tilde{A} \rangle,$$
$$X \prec_1 \langle J_\eta, \tilde{A} \rangle,$$
$$\operatorname{ran}(\sigma) \subseteq X.$$

Thus

$$\operatorname{ran}(\sigma) \prec_1 \langle X, \tilde{A} \cap X \rangle.$$

So, if we set $\sigma' = \pi^{-1} \circ \sigma$, we have

$$\sigma': \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_\gamma, B \rangle.$$

By Claim B, $\pi^{-1} \upharpoonright \nu' = \mathrm{id} \upharpoonright \nu'$, so $\sigma' \upharpoonright \bar{\nu} = \sigma \upharpoonright \bar{\nu}$. Hence $\sigma \upharpoonright J_{\bar{\nu}} \subseteq \sigma'$. Moreover, $(\sigma \upharpoonright J_{\bar{\nu}}): J_{\bar{\nu}} \prec_0 J_{\nu'}$ cofinally, so $(\sigma' \upharpoonright J_{\bar{\nu}}): J_{\bar{\nu}} \prec_Q J_{\nu'}$. So in order to complete the proof of the lemma it remains to show that $\gamma = \varrho(\nu')$, $B = A(\nu')$, and $\pi^{-1}(p) = p(\nu')$.

By Σ_0-absoluteness,

$$\pi: \langle J_\gamma, B \rangle \prec_0 \langle J_\varrho, A \rangle.$$

β' So by VI.5.6 there is a unique β' such that $\gamma = \varrho^{n-1}_{\beta'}$, $B = A^{n-1}_{\beta'}$, and a mapping
$\tilde{\pi}, q', p'$ $\tilde{\pi} \supseteq \pi$ such that $\tilde{\pi}: J_{\beta'} \prec_{n-1} J_\beta$. Set $q' = \pi^{-1}(q)$, $p' = \pi^{-1}(p)$. Notice that if $\nu < \varrho$, then $\bar{\nu} < \bar{\varrho}$ and $p = (q, \nu, \alpha)$, $\bar{p} = (\bar{q}, \bar{\nu}, \bar{\alpha})$, so $p' = (q', \pi^{-1}(\nu), \pi^{-1}(\alpha))$. But by Claim B, $\pi^{-1}(\nu) = \nu'$. And since $\pi^{-1} \upharpoonright \nu' = \mathrm{id} \upharpoonright \nu'$, $\pi^{-1}(\alpha) = \alpha$. Thus $p' = (q', \nu', \alpha)$, and $\nu' < \gamma$. Again, if $\nu = \varrho$, then $\bar{\nu} = \bar{\varrho}$ and $p = (q, \alpha)$, $\bar{p} = (\bar{q}, \bar{\alpha})$, so $p' = (q', \alpha)$, and (using Claim B) $\nu' = \gamma$. Hence in order to show that $\pi^{-1}(p) = p(\nu')$, it suffices to show that $\pi^{-1}(q) = q(\nu')$, i.e. that $q' = q(\nu')$.

Claim C. v' is Σ_{n-1} regular over $J_{\beta'}$.

This is proved exactly as in Claim I in the proof of 2.5, so we do not give any details here.

Claim D. v' is Σ_n singular over $J_{\beta'}$.

By Claim B,

$$v' = v' \cap h^*_{\eta, \tilde{A}}(J_\alpha \times \{q\}).$$

So as $\pi: \langle J_\gamma, B\rangle \prec_1 \langle J_\eta, \tilde{A}\rangle$ and $v' \subseteq X = \operatorname{ran}(\pi)$, we have

$$v' = v' \cap h^*_{\gamma, B}(J_\alpha \times \{q'\}).$$

So there is a $\Sigma_1(\langle J_\gamma, B\rangle)$ map from α onto v'. But $\gamma = \varrho^{n-1}_{\beta'}$, $B = A^{n-1}_{\beta'}$, so this map is $\Sigma_n(J_{\beta'})$. Claim D is proved.

Claim E. $\beta' = \beta(v')$, $n = n(v')$, $v' \in R$, $\gamma = \varrho(v')$, $B = A(v')$.

By Claims C and D. (For $v' \in R$, notice that as $\tilde{\pi}: J_{\beta'} \prec_{n-1} J_\beta$, $\lim(\beta')$ follows from $\lim(\beta)$.)

Claim F. $q' = q(v')$.

By definition,

$$X = h^*_{\eta, \tilde{A}}(J_\alpha \times \{q\}).$$

So, applying π^{-1},

$$J_\gamma = h^*_{\gamma, B}(J_\alpha \times \{q'\}).$$

Hence every member of J_γ is Σ_1-definable from parametes in $\alpha \cup \{q'\}$ in $\langle J_\gamma, B\rangle$. An argument as in the proof of Claim D od 2.5 now completes the proof of Claim F.

The lemma is proved. □

We are now in a position to commence the construction of our morass.

For $v, \tau \in S^1$, set $v \dashv \tau$ iff $\alpha_v < \alpha_\tau$ and there is an embedding

$$\sigma: \langle J_{\varrho(v)}, A(v)\rangle \prec_1 \langle J_{\varrho(\tau)}, A(\tau)\rangle$$

such that

(i) $\sigma \restriction \alpha_v = \mathrm{id} \restriction \alpha_v$;
(ii) $\sigma(p(v)) = p(\tau)$;
(iii) $(\sigma \restriction J_v): J_v \prec_Q J_\tau$.

Clearly, $\dashv$ is a partial ordering on S^1. And since $v \dashv \tau$ implies $\alpha_v < \alpha_\tau$, $\dashv$ is well-founded. We show that $\dashv$ is a tree. It suffices to show that if $\tau \in S_\alpha$ and $\bar{\alpha} < \alpha$, there is at most one $v \dashv \tau$ with $\alpha_v = \bar{\alpha}$. Let $v \dashv \tau$, $\alpha_v = \bar{\alpha}$. Then there is an embedding

$$\sigma: \langle J_{\varrho(v)}, A(v)\rangle \prec_1 \langle J_{\varrho(\tau)}, A(\tau)\rangle$$

as above. Now, $J_{\varrho(\nu)} = h^*_{\varrho(\nu), A(\nu)}(J_{\bar{\alpha}} \times \{p(\nu)\})$, so by applying σ and using properties (i) and (ii) above,

$$\operatorname{ran}(\sigma) = h^*_{\varrho(\tau), A(\tau)}(J_{\bar{\alpha}} \times \{p(\tau)\}).$$

Thus ran (ϱ) is entirely determined by τ and $\bar{\alpha}$. Since σ^{-1} is a collapsing map, it follows that $\varrho(\nu)$ is completely determined by τ and $\bar{\alpha}$. But if $\nu_1, \nu_2 \in S_{\bar{\alpha}}$ and $\nu_1 < \nu_2$, then $\varrho(\nu_1) < \nu_2 \leqslant \varrho(\nu_2)$. Hence $\nu \in S_{\bar{\alpha}}$ is unique here. Thus $\dashv$ is a tree on S^1. (It should be noted that the morass levels S_α are *not* the levels of this tree.)

By 2.4, if $\nu \dashv \tau$, the map σ testifying this fact is unique, so it may be denoted by
$\sigma_{\nu\tau}$ $\sigma_{\nu\tau}$. Clearly, the system of embeddings $(\sigma_{\nu\tau} | \nu \dashv \tau)$ is commutative. Set

$\pi_{\nu\tau}$ $$\pi_{\nu\tau} = (\sigma_{\nu\tau} \restriction \nu) \cup \{(\tau, \nu)\}.$$

Then $(\pi_{\nu\tau} | \nu \dashv \tau)$ is a commutative system of maps

$$\pi_{\nu\tau}\colon (\nu + 1) \to (\tau + 1).$$

We show that the structure

$$\mathscr{M} = \langle S, \mathscr{S}, \dashv, (\pi_{\nu\tau})_{\nu \dashv \tau} \rangle$$

so defined is a morass.

(M 0) This is immediate. (To show that $\sup(S^0 \cap \omega_1) = \omega_1$ only requires a simple application of the Condensation Lemma. All other parts of (M 0) really *are* immediate.)

(M 1) If $\nu \dashv \tau$, then $S_{\alpha_\nu} \cap \nu$ is $\Sigma_1^{J_\nu}(\{\alpha_\nu\})$ and $S_{\alpha_\tau} \cap \tau$ is $\Sigma_1^{J_\tau}(\{\alpha_\tau\})$ by the same definition. But $(\sigma_{\nu\tau} \restriction J_\nu)\colon J_\nu \prec_Q J_\tau$ and $\sigma_{\nu\tau}(\alpha_\nu) = \alpha_\tau$, so the assertions of (M 1) are immediate. (It is precisely in order to obtain (M 1) that we introduced the notion of a Q-embedding. And we only need this notion in order to prove (ii) and (iii) of (M 1) in the particular case $\gamma = \nu$.)

(M 2) Let $\bar{\tau} \in S_{\bar{\alpha}}$, $\tau \in S_\alpha$, $\bar{\tau} \dashv \tau$, $\bar{\nu} \in S_{\bar{\alpha}} \cap \bar{\tau}$, $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu})$. We must show that $\bar{\nu} \dashv \nu$ and $\pi_{\bar{\nu}\nu} \restriction \bar{\nu} = \pi_{\bar{\tau}\tau} \restriction \bar{\nu}$.

Let $\sigma = \sigma_{\bar{\tau}\tau} \restriction J_{\bar{\tau}}$. Thus $\sigma\colon J_{\bar{\tau}} \prec_1 J_\tau$, $\sigma(\bar{\alpha}) = \alpha$, $\sigma(\bar{\nu}) = \nu$. By 2.3,

$$(\sigma \restriction J_{\varrho(\bar{\nu})})\colon \langle J_{\varrho(\bar{\nu})}, A(\bar{\nu}), J_{\bar{\nu}}, \{p(\bar{\nu})\}\rangle \prec \langle J_{\varrho(\nu)}, A(\nu), J_\nu, (p(\nu))\rangle.$$

Hence $\bar{\nu} \dashv \nu$ and $\pi_{\bar{\nu}\nu} \restriction \bar{\nu} = (\sigma \restriction J_{\bar{\nu}}) \restriction \bar{\nu} = \sigma \restriction \bar{\nu} = \sigma_{\bar{\tau}\tau} \restriction \bar{\nu} = \pi_{\bar{\tau}\tau} \restriction \bar{\nu}$.

(M 3) Let $\tau \in S_\alpha$. Let $\bar{\alpha} < \alpha$ be a limit ordinal such that the set $\{\alpha_\nu | \nu \dashv \tau \wedge \alpha_\nu < \bar{\alpha}\}$ is unbounded in $\bar{\alpha}$. We must show that there is a $\nu \dashv \tau$ such that $\bar{\alpha} = \alpha_\nu$.

For each $\eta \dashv \tau$, let $X_\eta = \operatorname{ran}(\sigma_{\eta\tau})$. Thus $(X_\eta | \eta \dashv \tau)$ is an increasing sequence of Σ_1 submodels of $\langle J_{\varrho(\tau)}, A(\tau)\rangle$. Set

$$X = \bigcup \{X_\eta | \eta \dashv \tau \wedge \alpha_\eta < \bar{\alpha}\}.$$

Then $X \prec_1 \langle J_{\varrho(t)}, A(\tau)\rangle$. Let

$$\sigma\colon \langle J_\varrho, A\rangle \cong \langle X, A(\tau)\cap X\rangle.$$

By 2.5 there is a unique $v \in S^1$ such that $\varrho = \varrho(v)$, $A = A(v)$, $\sigma(p(v)) = p(\tau)$. Since $\bar\alpha = \sup\{\alpha_\eta \mid \eta \dashv \tau \wedge \alpha_\eta < \bar\alpha\}$, it is easily seen that we must have $v \in S_{\bar\alpha}$. And since $X_\eta \cap J_\tau \prec_Q J_\tau$ for all $\eta \dashv \tau$, we have $X \cap J_\tau \prec_Q J_\tau$. It follows that $v \dashv \tau$ and $\sigma_{v\tau} = \sigma$.

(M 4) Let $\tau \in S_\alpha$, and suppose that τ is not maximal in S_α. We must show that the set $\{\alpha_v \mid v \dashv \tau\}$ is unbounded in α (i.e. that τ is a limit point in $\dashv$).

Pick $\lambda \in S_\alpha$, $\lambda > \tau$, λ admissible. Let $\theta < \alpha$ be given. Let X be the smallest Σ_1 elementary submodel of $\langle J_{\varrho(\tau)}, A(\tau)\rangle$ which contains $p(\tau)$ and θ and is such that $X \cap \alpha$ is transitive. Now, $\langle J_{\varrho(\tau)}, A(\tau)\rangle$ is an element of J_λ, and λ is admissible, so $X \in J_\lambda$. But α is regular inside J_λ. Hence $\bar\alpha = X \cap \alpha \in \alpha$. Let

$$\sigma\colon \langle J_\varrho, A\rangle \cong \langle X, A(\tau)\cap X\rangle.$$

Using 2.5 we see that there is a unique $v \in S_{\bar\alpha}$ such that $\varrho = \varrho(v)$, $A = A(v)$, $v \dashv \tau$, and $\sigma = \sigma_{v\tau}$. Since $\theta < \bar\alpha = \alpha_v$, we are done.

(M 5) Let $\tau \in S_\alpha$, and suppose that $\{\alpha_v \mid v \dashv \tau\}$ is unbounded in α. We must show that $\tau = \bigcup_{v \dashv \tau} [\pi_{v\tau}{}''v]$.

In fact we show that $J_\tau = \bigcup_{v\dashv\tau} [\sigma_{v\tau}{}''J_v]$. Since $\supseteq$ is trivial, we only have to worry about $\subseteq$. Let $x \in J_\tau$. Then for some $\vec\delta \in \alpha$, x is Σ_1-definable from $p(\tau)$, $\vec\delta$ in $\langle J_{\varrho(\tau)}, A(\tau)\rangle$. Pick $v \dashv \tau$ such that $\vec\delta \in \alpha_v$. Then, since $\vec\delta, p(\tau) \in \operatorname{ran}(\sigma_{v\tau}) \prec_1 \langle J_{\varrho(\tau)}, A(\tau)\rangle$, we have

$$x \in \operatorname{ran}(\sigma_{v\tau}) \cap J_\tau = \sigma_{v\tau}{}''J_v,$$

as required.

(M 6) Let $\bar v$ be a limit point of $S_{\alpha_{\bar v}}$, $\bar v \dashv v$, $v' = \sup(\pi_{\bar v v}{}''\bar v)$. We must show that $\bar v \dashv v'$ and $\pi_{\bar v v'} \restriction \bar v = \pi_{\bar v v} \restriction \bar v$.

But this is immediate by 2.6.

(M 7) Let $\bar v$ be a limit point of $S_{\bar\alpha}$, $v \in S_\alpha$, $\bar v \dashv v$, $v = \sup[\pi_{\bar v v}{}''\bar v]$. Let $\bar\alpha < \theta < \alpha$ be such that for each $\bar\tau \in S_{\bar\alpha} \cap \bar v$, S_θ contains an $\eta \in S^1$ such that $\bar\tau \dashv \eta \dashv \pi_{\bar v v}(\bar\tau)$. We must show that S_θ contains an η such that $\bar v \dashv \eta \dashv v$.

For each $\bar\tau \in S_{\bar\alpha} \cap \bar v$, set $\tau = \pi_{\bar v v}(\bar\tau)$ and let $\eta(\bar\tau)$ denote the unique $\eta \in S_\theta$ such $\eta(\bar\tau)$
that $\bar\tau \dashv \eta \dashv \tau$. Note that the function η is monotone increasing. Set $\eta =$ η
$\sup\{\eta(\bar\tau) \mid \bar\tau \in S_{\bar\alpha} \cap \bar v\}$. Since S_θ is closed, $\eta \in S_\theta$. We show that $\bar v \dashv \eta \dashv v$. Since $\dashv$ is a tree and $\bar v \dashv v$, it suffices to show that $\eta \dashv v$. This will take some time.

By the verification of (M 2), if $\bar\tau, \bar\tau' \in S_{\bar\alpha} \cap \bar v$, $\bar\tau < \bar\tau'$, we have:

$$\sigma_{\bar\tau,\eta(\bar\tau)} \restriction J_{\bar\tau} = \sigma_{\bar\tau',\eta(\bar\tau')} \restriction J_{\bar\tau} \quad\text{and}\quad \sigma_{\eta(\bar\tau),\tau} \restriction J_{\eta(\bar\tau)} = \sigma_{\eta(\bar\tau'),\tau'} \restriction J_{\eta(\bar\tau)}.$$

Hence we can define functions σ_0, σ_1 by

$$\sigma_0 = \bigcup\{\sigma_{\bar\tau,\eta(\bar\tau)} \restriction J_{\bar\tau} \mid \bar\tau \in S_{\bar\alpha} \cap \bar v\}, \qquad \sigma_1 = \bigcup\{\sigma_{\eta(\bar\tau),\tau} \restriction J_{\eta(\bar\tau)} \mid \bar\tau \in S_{\bar\alpha} \cap \bar v\}.$$ σ_0, σ_1

Clearly,

$$\sigma_0\colon J_{\bar\nu} \prec_0 J_\eta, \qquad \sigma_1\colon J_\eta \prec_0 J_\nu .$$

But σ_0, σ_1 are cofinal. Hence

$$\sigma_0\colon J_{\bar\nu} \prec_1 J_\eta, \qquad \sigma_1\colon J_\eta \prec_1 J_\nu .$$

2.7 Lemma. $\operatorname{ran}(\sigma_{\bar\nu\nu}) \cap J_\nu \subseteq \operatorname{ran}(\sigma_1)$.

Proof. If $\bar\tau \in S_{\bar\alpha} \cap \bar\nu$ and $x \in J_{\bar\tau}$, then $\sigma_1 \circ \sigma_0(x) = \sigma_{\eta(\bar\tau),\tau} \circ \sigma_{\bar\tau,\eta(\bar\tau)}(x) = \sigma_{\bar\tau\tau}(x)$, so $\sigma_1 \circ \sigma_0 \upharpoonright J_{\bar\tau} = \sigma_{\bar\tau\tau} \upharpoonright J_{\bar\tau} = \sigma_{\bar\nu\nu} \upharpoonright J_{\bar\tau}$ (using the verification of (M 2)). But $S_{\bar\alpha} \cap \bar\nu$ is cofinal in $\bar\nu$. Hence $\sigma_1 \circ \sigma_0 = \sigma_{\bar\nu\nu} \upharpoonright J_{\bar\nu}$. Thus we have

$$\operatorname{ran}(\sigma_{\bar\nu\nu}) \cap J_\nu = \sigma_{\bar\nu\nu}{}''J_{\bar\nu} = \sigma_1 \circ \sigma_0{}''J_{\bar\nu} \subseteq \operatorname{ran}(\sigma_1).$$

Since $\sigma_1 | \theta = \mathrm{id} \upharpoonright \theta$ and (by confinality) $\sigma_1\colon J_\eta \prec_Q J_\nu$, we obtain $\eta \dashv \nu$ (and hence the verification of (M 7)) as an immediate consequence of the next lemma.

2.8 Lemma. *There is a* $\tilde\sigma \supseteq \sigma_1$ *such that*

$$\tilde\sigma\colon \langle J_{\varrho(\eta)}, A(\eta)\rangle \prec_1 \langle J_{\varrho(\nu)}, A(\nu)\rangle$$

and $\tilde\sigma(p(\eta)) = p(\nu)$.

$\sigma, \beta, n, \varrho,$ A, q, p, γ *Proof.* Set $\sigma = \sigma_1$, $\beta = \beta(\nu)$, $n = n(\nu)$, $\varrho = \varrho(\nu)$, $A = A(\nu)$, $q = q(\nu)$, $p = p(\nu)$, and if $\nu \in P$, $\gamma = \gamma(\nu)$.

Case 1. $\nu \in P$.

Thus $\varrho = \beta = \gamma + 1$ and $A = \emptyset$. Set

M $$M = \{x \in J_\gamma \mid x \text{ is } J_\gamma\text{-definable from parameters in } \operatorname{ran}(\sigma) \cup \{q\}\}.$$

Thus $M \prec J_\gamma$.

Claim A. $M \cap J_\nu = \operatorname{ran}(\sigma)$.

Let $x \in M \cap J_\nu$. Thus for some r and some Σ_r formula φ, and for some $y \in \operatorname{ran}(\sigma)$, x is the unique $x \in J_\gamma$ such that $\models_{J_\gamma} \varphi(\mathring{x}, \mathring{y}, \mathring{q})$. Define $X_m \prec_m J_\gamma$ just as in the proof of Claim B of 2.5, and, as there, set $\nu_m = X_m \cap \nu$. Let $\pi_m\colon X_m \cong J_{\gamma_m}$, and set $\pi_m(q) = q_m$. Since there is a J_γ-definable map from α onto γ_m, $\gamma_m < \nu$. Hence $(\gamma_m, q_m) \in J_\nu$. Now, (γ_m, q_m) is clearly Σ_1-definable from p in J_ϱ. So as $p \in \operatorname{ran}(\sigma_{\bar\nu\nu}) \prec_1 J_\varrho$, $(\gamma_m, q_m) \in \operatorname{ran}(\sigma_{\bar\nu\nu})$. So by 2.7, $(\gamma_m, q_m) \in \operatorname{ran}(\sigma)$. Pick $m \geqslant r$ so that $x, y \in J_{\nu_m}$. Since $\pi_m^{-1}\colon J_{\gamma_m} \prec_m J_\gamma$ and $\pi_m^{-1} \upharpoonright J_{\nu_m} = \mathrm{id} \upharpoonright J_{\nu_m}$, x is the unique $x \in J_{\gamma_m}$ such that $\models_{J_{\gamma_m}} \varphi(\mathring{x}, \mathring{y}, \mathring{q}_m)$. This provides us with a Σ_1 definition of x from γ_m, y, q_m in J_ν. But $\gamma_m, y, q_m \in \operatorname{ran}(\sigma) \prec_1 J_\nu$. Hence $x \in \operatorname{ran}(\sigma)$, which proves Claim A.

Let

$\tilde\sigma, \bar\gamma$ $$\tilde\sigma\colon J_{\bar\gamma} \cong M.$$

Thus

$$\tilde\sigma\colon J_{\bar\gamma} \prec J_\gamma .$$

$\bar q$ By Claim A, $\sigma \subseteq \tilde\sigma$. In particular, $\tilde\sigma(\theta) = \alpha$. Set $\bar q = \tilde\sigma^{-1}(q)$.

Claim B. (i) $\nu < \gamma \to \eta < \bar{\gamma} \wedge \tilde{\sigma}(\eta) = \nu$;
(ii) $\nu = \gamma \to \eta = \bar{\gamma}$.

Suppose $\nu < \gamma$. Then, since $\nu \in S_\alpha$ and $\gamma = \gamma(\nu)$, we have $\nu = [\alpha^+]^{J_\nu}$. But $\alpha = \sigma(\theta) \in \operatorname{ran}(\sigma) \subseteq M \prec J_\gamma$. Hence $\nu \in M$. But $\sigma''\eta$ is cofinal in ν. Hence $\eta = \tilde{\sigma}^{-1}(\nu)$. Thus $\eta < \bar{\gamma}$ and $\tilde{\sigma}(\eta) = \nu$.

Now suppose $\eta \neq \bar{\gamma}$. Thus $\eta < \bar{\gamma}$, and so $\tilde{\sigma}(\eta)$ is defined. Since $\sigma''\eta$ is cofinal in ν, $\tilde{\sigma}(\eta) \geqslant \nu$. Thus $\gamma > \nu$. Hence $\nu = \gamma \to \eta = \bar{\gamma}$. Claim B is proved.

Claim C. η is regular over $J_{\bar{\gamma}}$.

We know that ν is regular over J_γ. But $\tilde{\sigma}\colon J_{\bar{\gamma}} \prec J_\gamma$ and $\tilde{\sigma}''\eta$ is cofinal in ν, so this claim follows from Claim B.

Claim D. $\bar{q}$ is the $<_J$-least element of $J_{\bar{\gamma}}$ such that every element of $J_{\bar{\gamma}}$ is $J_{\bar{\gamma}}$-definable from parameters in $\theta \cup \{\bar{q}\}$.

Argue just as in Claim D of 2.5.

Claim E. η is Σ_1-singular over $J_{\bar{\gamma}+1}$.

By Claims C and D we may define $(\eta_m \mid m < \omega)$ from $J_{\bar{\gamma}}, \theta, \bar{q}, \eta$ exactly as $(\nu_m \mid m < \omega)$ was defined from J_γ, α, q, ν in Claim B of 2.5, thereby obtaining a $\Sigma_1(J_{\bar{\gamma}+1})$ ω-sequence cofinal in η, which proves Claim E.

Claim F. $\eta \in P$, $\gamma(\eta) = \bar{\gamma}$, $\varrho(\eta) = \beta(\eta) = \bar{\gamma} + 1$, $q(\eta) = \bar{q}$, $p(\eta) = (\bar{q}, \bar{\gamma}, \eta, \theta)$.

By Claims C, D, E.

Since $\tilde{\sigma}\colon J_{\bar{\gamma}} \prec J_\gamma$, by VI.1.19 there is a unique extension of $\tilde{\sigma}$ to an embedding $\tilde{\sigma}\colon J_{\varrho(\eta)} \prec_1 J_\varrho$. Using Claims B and F, $\tilde{\sigma}(p(\eta)) = p$. That completes the proof in this case.

Case 2. $\nu \in R$.

Let $h = h_{\varrho, A}$, and for $\tau < \varrho$, set $h_\tau = h_{\tau, A \cap J_\tau}$. Let $\delta = \delta(\nu) =$ the least $\delta < \varrho$ such that $q \in J_\delta$ and $\alpha \in h_\delta^*(J_\alpha \times \{p\})$, and such that $\nu \in h_\delta^*(J_\alpha \times \{p\})$ in case $\nu < \varrho$. Since $\lim(\varrho)$ and $J_\varrho = h^*(J_\alpha \times \{p\})$, such a δ can always be defined. $h, h_\tau, \delta, \delta(\nu)$

For $\delta \leqslant \tau < \varrho$, let

$$X_\tau = h_\tau^*(J_\alpha \times \{p\}). \qquad X_\tau$$

Then $X_\tau \prec_1 \langle J_\tau, A \cap J_\tau \rangle$. Moreover, by choice of p, $\bigcup_{\delta \leqslant \tau < \varrho} X_\tau = J_\varrho$. Since α is the largest cardinal inside J_ν, $X_\tau \cap \nu$ is transitive, so set $\nu_\tau = X_\tau \cap \nu$. Let ν_τ

$$\pi_\tau\colon \langle X_\tau, A \cap X_\tau \rangle \cong \langle J_{\gamma_\tau}, A_\tau \rangle. \qquad \pi_\tau, \gamma_\tau, A_\tau$$

Set $\pi_\tau(p) = p_\tau$. Notice that $\pi_\tau \restriction \nu_\tau = \mathrm{id} \restriction \nu_\tau$, and that if $\nu < \varrho$, then $\pi_\tau(\nu) = \nu_\tau$. p_τ

Claim G. Let $\delta \leqslant \tau < \varrho$. Then $\langle J_{\gamma_\tau}, A_\tau \rangle \in J_\nu$.

Since $\langle J_\varrho, A \rangle$ is amenable, $h_\tau \in J_\varrho$. Hence $X_\tau \in J_\varrho$, and there is an $f \in J_\varrho$ such that $f\colon \alpha \leftrightarrow X_\tau$. Set

$$E = \{(\xi, \zeta) \in \alpha^2 \mid f(\xi) \in f(\zeta)\},$$
$$B = \{\xi \in \alpha \mid f(\xi) \in A\}.$$

Thus $(\alpha, E, B) \in J_\varrho$. So if $\nu = \varrho$, then $(\alpha, E, B) \in J_\gamma$. If $\nu < \varrho$, then ν is a cardinal inside J_ϱ, so $\mathscr{P}^{J_\rho}(\alpha) \subseteq J_\nu$, by applying II.5.5 within J_ν, so again $(\alpha, E, B) \in J_\nu$. But ν is adequate, so the transitive realisation of the well-founded, extensional structure $\langle \alpha, E, B \rangle$ is also in J_ν. In other words, $\langle J_{\gamma_\tau}, \in, A_\tau \rangle \in J_\nu$, which proves Claim G.

By Claim G, $\nu_\tau < \nu$ for all τ. Hence $(\nu_t \mid \delta \leqslant \tau < \varrho)$ is a cofinal sequence in ν.

Claim H. $((\gamma_\tau, A_\tau, \nu_\tau, p_\tau) \mid \delta \leqslant \tau < \varrho)$ is $\Sigma_1^{\langle J_\rho, A \rangle}(\{p\})$.

This is immediate from the definition.

Define

M $$M = \{x \in J_\varrho \mid x \text{ is } \Sigma_1\text{-definable from parameters in } (\operatorname{ran}(\sigma) \cap J_\alpha) \cup \{p\} \text{ in } \langle J_\varrho, A \rangle\}.$$

Of course, $\operatorname{ran}(\sigma) \cap J_\alpha = \sigma'' J_\theta = J_\theta$ here, but we have given the definition of M in the form required for the proof. We have

$$\langle M, A \cap M \rangle \prec_1 \langle J_\varrho, A \rangle.$$

Claim I. $M \cap J_\nu = \operatorname{ran}(\sigma)$.

Let $x \in M \cap J_\nu$. Then for some Σ_0-formula φ of $\mathscr{L}$ and some $y \in \operatorname{ran}(\sigma) \cap J_\alpha$, x is the unique $x \in J_\varrho$ such that $\models_{\langle J_\rho, A \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p})$. Pick τ so that $x, y \in X_\tau$, $x \in J_{\nu_\tau}$, and for some $u \in J_\tau$, $\models_{\langle J_\tau, A \cap J_\tau \rangle} \varphi(\mathring{u}, \mathring{x}, \mathring{y}, \mathring{p})$. Then x is the unique $x \in J_\tau$ such that $\models_{\langle J_\tau, A \cap J_\tau \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p})$. But $x, y, p \in X_\tau \prec_1 \langle J_\tau, A \cap J_\tau \rangle$, so applying π_τ and noting that $\pi_\tau \upharpoonright J_{\nu_\tau} = \operatorname{id} \upharpoonright J_{\nu_\tau}$, we see that x is the unique $x \in J_{\gamma_\tau}$ such that $\models_{\langle J_{\gamma_\tau}, A_\tau \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p})$. This gives us a Σ_1 definition of x from $\gamma_\tau, A_\tau, y, p_\tau$ in J_ν.

We may assume that τ was chosen above so that $\tau \in \operatorname{ran}(\sigma_{\bar{\nu}\nu})$. To see this, pick $\xi \in \sigma_{\bar{\nu}\nu}'' \bar{\nu}$ large enough so that whenever τ is such that $\nu_\tau \geqslant \xi$, then τ has the properties used above. Since $\sigma_{\bar{\nu}\nu}'' \bar{\nu}$ is cofinal in ν, such a ξ can be found. The smallest τ with $\nu_\tau \geqslant \xi$ is now Σ_1-definable from ξ, p in $\langle J_\varrho, A \rangle$, by virtue of Claim H. So as $\xi, p \in \operatorname{ran}(\sigma_{\bar{\nu}\nu}) \prec_1 \langle J_\varrho, A \rangle$, we have $t \in \operatorname{ran}(\sigma_{\bar{\nu}\nu})$.

By Claim H, it now follows that $\gamma_\tau, A_\tau, p_\tau \in \operatorname{ran}(\sigma_{\bar{\nu}\nu})$. So by 2.7, $\gamma_\tau, A_\tau, p_\tau \in \operatorname{ran}(\sigma)$. Hence $\gamma_\tau, A_\tau, y, p_\tau \in \operatorname{ran}(\sigma)$. But $\operatorname{ran}(\sigma) \prec_1 J_\nu$. Thus $x \in \operatorname{ran}(\sigma)$.

Now let $x \in \operatorname{ran}(\sigma)$. Then $x \in J_\nu$. We show that $x \in M$. Since $x \in J_\varrho$, x is Σ_1-definable from parameters in $\alpha \cup \{p\}$ in $\langle J_\varrho, A \rangle$. So for some Σ_0 formula φ of $\mathscr{L}$ and some $y \in J_\alpha$, x is the unique $x \in J_\varrho$ such that

$$\models_{\langle J_\rho, A \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p}).$$

Let y be the $\prec_J$-least such parameter. We show that $x \in M$ by proving that $y \in \operatorname{ran}(\sigma)$. Pick $\tau < \varrho$ such that $x, y \in X_\tau$, $x \in J_{\nu_\tau}$, so that for some $u \in J_\tau$,

$$\models_{\langle J_\tau, A \cap J_\tau \rangle} \varphi(\mathring{u}, \mathring{x}, \mathring{y}, \mathring{p}).$$

Then

$$\models_{\langle J_\tau, A \cap J_\tau \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p}),$$

so applying π_τ, much as before, we see that x is the unique $x \in J_{\gamma_\tau}$ such that

$$\models_{\langle J_{\gamma_\tau}, A_\tau \rangle} \exists u \varphi(u, \mathring{x}, \mathring{y}, \mathring{p}_\tau).$$

Moreover, y is the $<_J$-least such parameter. This provides us with a Σ_1 definition of x from $\gamma_\tau, A_\tau, y, p_\tau$ in J_ν. As above, we can assume that τ has been chosen so that $\gamma_\tau, A_\tau, p_\tau \in \operatorname{ran}(\sigma)$. Hence as $x \in \operatorname{ran}(\sigma)$ and $\operatorname{ran}(\sigma) \prec_1 J_\nu$, the minimality of y gives $y \in \operatorname{ran}(\sigma)$, as required. Claim I is proved.

Let

$$\tilde{\sigma}: \langle J_{\bar{\varrho}}, \bar{A} \rangle \cong \langle M, A \cap M \rangle.$$ $\tilde{\sigma}, \bar{\varrho}, \bar{A}$

Thus

$$\tilde{\sigma}: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_\varrho, A \rangle.$$

By Claim I, $\sigma \subseteq \tilde{\sigma}$. In particular, $\tilde{\sigma}(\theta) = \alpha$. Set $\bar{p} = \tilde{\sigma}^{-1}(p)$. $\bar{p}$

By VI.5.6 there is a $\bar{\beta} \geqslant \bar{\varrho}$ such that $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, and an extension $\tilde{\sigma}'$ of $\tilde{\sigma}$ such that $\tilde{\sigma}': J_{\bar{\beta}} \prec_n J_\beta$. $\bar{\beta}$ $\tilde{\sigma}'$

Claim J. η is Σ_{n-1} regular over $J_{\bar{\beta}}$.

This follows immediately from the fact that $\tilde{\sigma}': J_{\bar{\beta}} \prec_n J_\beta$ and $\sup[\tilde{\sigma}'{}''\eta] = \nu$.

Claim K. η is Σ_n singular over $J_{\bar{\beta}}$.

By definition of M, we have

$$J_{\bar{\varrho}} = h^*_{\bar{\varrho}, \bar{A}}(J_\theta \times \{\bar{p}\}).$$

Hence there is a $\Sigma_1(\langle J_{\bar{\varrho}}, \bar{A} \rangle)$ map from θ onto η. Since $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, this map is $\Sigma_n(J_{\bar{\beta}})$.

Claim L. $\bar{\beta} = \beta(\eta)$, $n = n(\eta)$, $\eta \in R$, $\bar{\varrho} = \varrho(\eta)$, $\bar{A} = A(\eta)$.

By Claims J and K.

Hence

$$\tilde{\sigma}: \langle J_{\varrho(\eta)}, A(\eta) \rangle \prec \langle J_\varrho, A \rangle.$$

But $p \in \operatorname{ran}(\tilde{\sigma})$, so by 2.5, $\tilde{\sigma}(p(\eta)) = p$. The lemma is proved now. □

3. The Gap-2 Cardinal Transfer Theorem

In this section we prove the following theorem.

3.1 Theorem. *Assume there is a morass (i.e. an $(\omega_1, 1)$-morass). Let $\mathscr{A}$ be a K-structure of type (κ^{++}, κ) for some uncountable cardinal κ. Assume that $2^\kappa = \kappa^+$. Then there is a K-structure $\mathscr{B}$ of type (ω_2, ω) such that $\mathscr{B} \equiv \mathscr{A}$.* □

By virtue of the results of the last section, this implies that the Gap-2 Property is valid in L. In the exercises we indicate how the result may be extended to cover any type (λ^{++}, λ) in place of (ω_2, ω).

We fix $\mathscr{M} = \langle S, \mathscr{S}, \dashv, (\pi_{\nu\tau})_{\nu \dashv \tau} \rangle$ a morass from now on. We are given a K-structure $\mathscr{A}$ of type (κ^{++}, κ). We may assume that $\mathscr{A}$ has the form

$$\mathscr{A} = \langle \kappa^{++}, \kappa, <, \dots \rangle,$$

where $\kappa = U^{\mathscr{A}}$ and $<$ is the usual ordering of κ^{++}. If $\mathscr{B} \equiv \mathscr{A}$ and $e \in B$, we shall denote by $\mathrm{Pr}^{\mathscr{B}}(e)$ the set of all $<$-predecessors of e in the sense of $\mathscr{B}$, i.e.

$$\mathrm{Pr}^{\mathscr{B}}(e) = \{b \in B \mid \mathscr{B} \models b < e\}.$$

The key model-theoretic fact required for our proof is supplied by the following lemma.

3.2 Lemma. *Assume $2^\kappa = \kappa^+$. Then there are K-structures $\mathscr{B}, \mathscr{C}$ such that:*

(i) $\mathscr{B} \equiv \mathscr{C} \equiv \mathscr{A}$;

(ii) $\mathscr{B} \prec \mathscr{C}$ *and* $U^{\mathscr{B}} = U^{\mathscr{C}}$;

(iii) *there is an embedding* $\sigma\colon \mathscr{B} \prec \mathscr{C}$ *and an element* $e \in B$ *such that*:

 (a) $U^{\mathscr{B}} \subseteq \mathrm{Pr}^{\mathscr{B}}(e)$;

 (b) $\sigma \restriction \mathrm{Pr}^{\mathscr{B}}(e) = \mathrm{id} \restriction \mathrm{Pr}^{\mathscr{B}}(e)$;

 (c) $B \subseteq \mathrm{Pr}^{\mathscr{C}}(\sigma(e))$.

Proof. For those familiar with the term, we remark that the proof is by means of a "Δ-system" argument.

For each $\alpha < \kappa^{++}$, let $\mathscr{A}_\alpha = \langle A_\alpha, \kappa, <, \dots \rangle$ be the smallest $\mathscr{A}_\alpha \prec \mathscr{A}$ such that $\kappa \cup \{\alpha\} \subseteq A_\alpha$. (Since $<$ well-orders $\mathscr{A}$, this definition makes sense.) We can clearly find a cofinal set $X \subseteq \kappa^{++}$ such that $A_\alpha \neq A_\beta$ whenever $\alpha, \beta \in X$, $\alpha \neq \beta$. Since $|A_\alpha| = \kappa$ for all $\alpha \in X$, we may assume that $\mathrm{otp}(A_\alpha) = \theta$ for all $\alpha \in X$, where θ is a fixed ordinal, $\kappa < \theta < \kappa^+$. Let $(a_\nu^\alpha \mid \nu < \theta)$ be the monotone enumeration of A_α for each $\alpha \in X$. Since $\alpha \in A_\alpha$, there is a least ordinal $\varrho < \theta$ such that $(a_\varrho^a \mid a \in X)$ is cofinal in κ^{++}. Since $(\kappa^+)^\kappa = \kappa^+$, we may assume that $(a_\nu^\alpha \mid \nu < \varrho) = (a_\nu^\beta \mid \nu < \varrho)$ for all $\alpha, \beta \in X$. We may further assume that for all $\alpha, \beta \in X$, if $\alpha < \beta$ then $a_\varrho^\beta > a_\nu^\alpha$ for all $\nu < \theta$. Thus if we set

$$Y = \{a_\nu^\alpha \mid \nu < \varrho\} \qquad \text{(for any } \alpha \in X),$$
$$Z_\alpha = \{a_\nu^\alpha \mid \varrho \leqslant \nu < \theta\} \qquad \text{(each } \alpha \in X),$$

we have:

$$A_\alpha = Y \cup Z_\alpha \qquad \text{(all } \alpha \in X),$$
$$Y \cap Z_\alpha = \emptyset \qquad \text{(all } \alpha \in X),$$
$$Y < Z_\alpha < Z_\beta \qquad \text{(all } \alpha, \beta \in X,\ a < \beta).$$

(i.e. $\{A_\alpha \mid \alpha \in X\}$ forms a Δ-*system*.)

Now there are at most $\kappa^\kappa = \kappa^+$ non-isomorphic K-structures of cardinality κ. So we can find $\alpha, \beta \in X, \alpha < \beta$, such that $\mathscr{A}_\alpha \cong \mathscr{A}_\beta$. It is clear that the only possible isomorphism $\sigma\colon \mathscr{A}_\alpha \cong \mathscr{A}_\beta$ is the unique order-isomorphism of A_α onto A_β (as sets of ordinals). Thus if we take

$$\mathscr{B} = \mathscr{A}_\alpha, \qquad \mathscr{C} = \mathscr{A}, \qquad e = a_\varrho^a,$$

then $\mathscr{B}, \mathscr{C}, \sigma, e$ are clearly as required for the lemma. □

By means of an argument almost identical to that used in 1.5, we can use 3.2 in order to prove the following sharper result.

3.3 Lemma. *Assume $2^\kappa = \kappa^+$. Then there are countable homogeneous K-structures $\mathscr{B}_0, \mathscr{C}_0$ such that:*

(i) $\mathscr{B}_0 \equiv \mathscr{C}_0 \equiv \mathscr{A}$;

(ii) $\mathscr{B}_0 \prec \mathscr{C}_0$ *and* $U^{\mathscr{B}_0} = U^{\mathscr{C}_0}$;

(iii) *there is an embedding* $\sigma_0\colon \mathscr{B}_0 \prec \mathscr{C}_0$ *and an element* $e_0 \in B_0$ *such that:*

  (a) $U^{\mathscr{B}_0} \subseteq \mathrm{Pr}^{\mathscr{B}_0}(e_0)$;

  (b) $\sigma_0 \restriction \mathrm{Pr}^{\mathscr{B}_0}(e_0) = \mathrm{id} \restriction \mathrm{Pr}^{\mathscr{B}_0}(e_0)$;

  (c) $B_0 \subseteq \mathrm{Pr}^{\mathscr{C}_0}(\sigma_0(e_0))$;

(iv) $\langle \mathscr{B}_0, e_0 \rangle \cong \langle \mathscr{C}_0, \sigma_0(e_0) \rangle$.

Proof. Since the proof is virtually the same as in 1.5 we give only a brief sketch. Commence with $\mathscr{B}, \mathscr{C}, \sigma, e$ as in 3.2. By replacing $\mathscr{C}$ by its skolem hull around $B \cup \sigma''B$ if necessary, we may assume that $|C| = |B|$. Let $h\colon C \leftrightarrow B$. Let

$$\langle \mathscr{C}', \mathscr{B}', \sigma', e', \sigma'(e'), h' \rangle \equiv \langle \mathscr{C}, \mathscr{B}, \sigma, e, \sigma(e), h \rangle$$

be special. Then, in particular, $\langle \mathscr{C}', \sigma'(e') \rangle$ and $\langle \mathscr{B}', e' \rangle$ are special structures of the same cardinality, so let

$$k'\colon \langle \mathscr{C}', \sigma'(e') \rangle \cong \langle \mathscr{B}', e' \rangle.$$

Let

$$\langle \mathscr{C}_0, \mathscr{B}_0, \sigma_0, e_0, \sigma_0(e_0), k_0 \rangle \equiv \langle \mathscr{C}', \mathscr{B}', \sigma', e', \sigma'(e'), k' \rangle$$

be countable and homogeneous. Then $\mathscr{B}_0, \mathscr{C}_0, \sigma_0, e_0$ are as required by the lemma. □

We are now ready to commence our construction of an (ω_2, ω)-model $\mathscr{B} \equiv \mathscr{A}$. We shall obtain $\mathscr{B}$ as a limit of a certain directed, elementary system.

To each $\tau \in S^1$ we shall attach a K-structure $\mathscr{B}_\tau \equiv \mathscr{A}$ and an element $e_\tau \in B_\tau$. If $\nu, \tau \in S^1$ and $\nu < \tau$ (as ordinals) we shall have $\mathscr{B}_\nu \prec \mathscr{B}_\tau$. We shall set, for each $\tau \in S^1$,

$$\mathscr{B}_\tau^0 = \bigcup_{\nu < \tau} \mathscr{B}_\nu; \qquad \mathscr{B}_\tau^+ = \bigcup_{\nu \in S_{\alpha_\tau}} \mathscr{B}_\nu,$$

it being understood that ν, τ, etc. vary over S^1 in such situations. The directed system we construct will be called an *$\mathscr{M}$-complex*. We begin with an axiomatic description of the system. We fix $\mathscr{B}_0, \mathscr{C}_0, \sigma_0, e_0$ as in 3.3.

An $\mathscr{M}$-*complex* (for $\mathscr{A}$) is a structure

$$\mathscr{D} = \langle (\mathscr{B}_\tau)_{\tau \in S^1}, (e_\tau)_{\tau \in S^1}, (\sigma_{\bar{\tau}\tau})_{\bar{\tau} \dashv \tau} \rangle$$

such that:

(C 1) $\tau \in S^1 \to \mathscr{B}_\tau \equiv \mathscr{A}$ & $e_\tau \in B_\tau$;

(C 2) $\tau \in S^1 \cap \omega_1 \to \mathscr{B}_\tau$ is countable homogeneous and $\langle \mathscr{B}_\tau, e_\tau \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$;

(C 3) $\nu, \tau \in S^1$ & $\nu < \tau \to \mathscr{B}_\nu \prec \mathscr{B}_\tau$ & $U^{\mathscr{B}_\nu} = U^{\mathscr{B}_\tau}$;

(C 4) $\tau \in S^1 \to B^0_\tau \subseteq \mathrm{Pr}^{\mathscr{B}_\tau}(e_\tau)$;

(C 5) the embeddings $\sigma_{\bar{\tau}\tau}\colon \mathscr{B}^+_{\bar{\tau}} \prec \mathscr{B}_\tau$ for $\bar{\tau} \dashv \tau$, form a commutative system;

(C 6) $\bar{\tau} \dashv \tau \to \sigma_{\bar{\tau}\tau}(e_{\bar{\tau}}) = e_\tau$;

(C 7) $\bar{\tau} \dashv \tau$ & $\bar{\nu} \in S_{\alpha_\tau} \cap \bar{\tau}$ & $\nu = \pi_{\bar{\tau}\tau}(\bar{\nu}) \to \sigma_{\bar{\tau}\tau} \upharpoonright B_{\bar{\nu}} = \sigma_{\bar{\nu}\nu} \upharpoonright B_{\bar{\nu}}$;

(C 8) $\bar{\tau} \dashv \tau$ & $\nu \in S^1$ & $\alpha_\nu < \alpha_{\bar{\tau}} \to \sigma_{\bar{\tau}\tau} \upharpoonright B^+_\nu = \mathrm{id} \upharpoonright B^+_\nu$;

(C 9) if τ is a limit point of $\dashv$, then $\mathscr{B}_\tau = \bigcup_{\bar{\tau} \dashv \tau} \sigma_{\bar{\tau}\tau}{}''\mathscr{B}_{\bar{\tau}}$.

Given an $\mathscr{M}$-complex as above, the Gap-2 Theorem follows at once. For if we set

$$\mathscr{B} = \bigcup_{\nu \in S_{\omega_1}} \mathscr{B}_\beta,$$

then $\mathscr{B} \equiv \mathscr{A}$ and by (C 3) and (C 4), $\mathscr{B}$ has type (ω_2, ω).

The construction of an $\mathscr{M}$-complex proceeds by recursion on $\tau \in S^1$. To commence, if τ_0 is the least ordinal in S^1 we take $\mathscr{B}_{\tau_0} = \mathscr{B}_0, e_{\tau_0} = e_0$. The induction step in the construction splits into three cases.

Case 1. τ is minimal in $\dashv$.

By morass axiom (M 4), since τ is not a limit point in $\dashv$, we must have $\alpha_\tau \neq \omega_1$. So $\mathscr{B}^0_\tau$ is a union of a countable elementary chain of countable homogeneous structures. Thus, using 1.6 in case this chain is of limit length, $\mathscr{B}^0_\tau$ is countable homogeneous and $\mathscr{B}^0_\tau \cong \mathscr{B}_0$. Let $e^0_\tau \in B^0_\tau$ correspond to $e_0 \in B_0$ under such an isomorphism. Then, by the properties of $\mathscr{B}_0, \mathscr{C}_0, \sigma_0, e_0$ we can find a structure $\langle \mathscr{B}_\tau, e_\tau \rangle$ such that the relationship between $\langle \mathscr{B}_\tau, e_\tau \rangle$ and $\langle \mathscr{B}^0_\tau, e^0_\tau \rangle$ is the same as that between $\langle \mathscr{C}_0, \sigma_0(e_0) \rangle$ and $\langle \mathscr{L}_0, e_0 \rangle$. In particular, we have:

$$\mathscr{B}^0_\tau \prec \mathscr{B}_\tau; \quad U^{\mathscr{B}^0_\tau} = U^{\mathscr{B}_\tau}; \quad \langle \mathscr{B}_\tau, e_\tau \rangle \cong \langle \mathscr{B}^0_\tau, e^0_\tau \rangle; \quad B^0_\tau \subseteq \mathrm{Pr}^{\mathscr{B}_\tau}(e_\tau).$$

Thus $\mathscr{B}_\tau, e_\tau$ satisfy (C 1)–(C 4), whilst no new cases of (C 5)–(C 9) arise.

Case 2. τ is a limit point of $\dashv$.

Consider the directed elementary system

$$\langle \mathscr{B}^+_\nu)_{\nu \dashv \tau}, (\sigma_{\bar{\nu}\nu})_{\bar{\nu} \dashv \nu \dashv \tau} \rangle.$$

Let

$$\langle \mathscr{C}, (\sigma_\nu)_{\nu \dashv \tau} \rangle$$

be its direct limit. We may define an embedding

$$j \colon \mathscr{B}_\tau^0 \prec \mathscr{C}$$

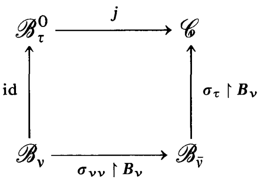
as follows. Let $x \in B_\tau^0$. For some $\nu < \tau$, $x \in B_\nu$. Suppose first that $\nu \in S_{\alpha_\tau}$. By (M 5) we have

$$S_{\alpha_\tau} \cap \tau = \bigcup_{\bar\tau \dashv \tau} \pi_{\bar\tau\tau}{}''(S_{\alpha_\tau} \cap \bar\tau).$$

So we can find a $\bar\tau \dashv \tau$ and a $\bar\nu \in S_{\alpha_\tau} \cap \bar\tau$ such that $\nu = \pi_{\bar\tau\tau}(\bar\nu)$. By (M 4), ν is a limit point of $\dashv$, so (C 9) tells us that

$$B_\nu = \bigcup_{\bar\nu \dashv \nu} \sigma_{\bar\nu\nu}{}''\mathrm{B}_{\bar\nu}.$$

So we can find a $\bar\tau \dashv \tau$ sufficiently high in $\dashv$ so that $x = \sigma_{\bar\nu\nu}(\bar x)$ for some $\bar x \in B_{\bar\nu} \subseteq B_{\bar\tau}$. Set $j(x) = \sigma_{\bar\tau}(\bar x)$ in this case. On the other hand, if $\alpha_\nu < \alpha_\tau$, then if we pick $\bar\tau \dashv \tau$ so that $\alpha_{\bar\tau} > \alpha_\nu$, we have $x \in B_{\bar\tau}$, and we can set $j(x) = \sigma_{\bar\tau}(x)$. Using (C 5), (C 7), and (C 8) it is routine to verify that j is well-defined and elementary from $\mathscr{B}_\tau^0$ into $\mathscr{C}$, and that for any $\bar\tau \dashv \tau$, $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, $\nu = \pi_{\bar\tau\tau}(\bar\nu)$, the following diagram commutes:



We may thus choose $\mathscr{C}$ specifically so that $j = \mathrm{id} \restriction B_\tau^0$. Let $\mathscr{B}_\tau = \mathscr{C}$ and set $\sigma_{\bar\tau\tau} = \sigma_{\bar\tau}$ for all $\bar\tau \dashv \tau$. By (C 6) below τ, there is a unique element $e_\tau \in B_\tau$ so that $\sigma_{\bar\tau\tau}(e_{\bar\tau}) = e_\tau$ for all $\bar\tau \dashv \tau$. We check that (C 1)–(C 9) hold for $\mathscr{B}_\tau$, e_τ under these definitions. The only one that is not immediate is (C 2).

Using an obvious notation we have:

$$\langle \mathscr{B}_\tau, e_\tau \rangle = \bigcup_{\bar\tau \dashv \tau} \langle \sigma_{\bar\tau\tau}{}''\mathscr{B}_{\bar\tau}, e_\tau \rangle.$$

Since $(\sigma_{\bar\tau\tau} \restriction B_{\bar\tau}) \colon \langle \mathscr{B}_{\bar\tau}, e_{\bar\tau} \rangle \cong \langle \sigma_{\bar\tau\tau}{}''\mathscr{B}_{\bar\tau}, e_\tau \rangle$, the structures $\langle \sigma_{\bar\tau\tau}{}''\mathscr{B}_{\bar\tau}, e_\tau \rangle$, for $\bar\tau \dashv \tau$, form an elementary chain of isomorphic, countable homogeneous structures. Thus by 1.6, $\langle \mathscr{B}_\tau, e_\tau \rangle$ is countable homogeneous and for any $\bar\tau \dashv \tau$ we have $\langle \mathscr{B}_\tau, e_\tau \rangle \cong \langle \sigma_{\bar\tau\tau}{}''\mathscr{B}_{\bar\tau}, e_\tau \rangle \cong \langle \mathscr{B}_{\bar\tau}\, e_{\bar\tau} \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$. This proves (C 2).

Before we commence Case 3, we observe the following consequence of the axioms for an $\mathscr{M}$-complex:

(C 10) $\nu, \tau \in S^1 \cap \omega_1 \;\&\; \nu < \tau \to \langle \mathscr{B}_\tau, e_\nu \rangle \cong \langle \mathscr{B}_\tau, e_\tau \rangle$.

To see this, we first note that by (C 3),

$$\langle \mathscr{B}_\nu, e_\nu \rangle \prec \langle \mathscr{B}_\tau, e_\nu \rangle .$$

Also by (C 2),

$$\langle \mathscr{B}_\nu, e_\nu \rangle \cong \langle \mathscr{B}_\tau, e_\tau \rangle .$$

Thus

$$\langle \mathscr{B}_\tau, e_\nu \rangle \equiv \langle \mathscr{B}_\tau, e_\tau \rangle .$$

So as $\mathscr{B}_\tau$ is homogeneous,

$$\langle \mathscr{B}_\tau, e_\nu \rangle \cong \langle \mathscr{B}_\tau, e_\tau \rangle ,$$

which is (C 10).

We shall make use of (C 10) in dealing with Case 3.

Case 3. τ immediately succeeds $\bar{\tau}$ in $\dashv$.

Note that by (M 4), $\alpha_\tau \neq \omega_1$, so $\tau < \omega_1$. There are three subcases to consider.

Case 3.1. τ is minimal in S_{α_τ}.

Thus $\bar{\tau}$ is minimal in $S_{\alpha_{\bar{\tau}}}$ (by (M 1)). Using (C 10) and (possibly) Lemma 1.6, $\langle \mathscr{B}^0_\tau, e_{\bar{\tau}} \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$. Thus there is a countable homogeneous structure $\mathscr{B}_\tau$ and an embedding $\sigma\colon \mathscr{B}^0_\tau \prec \mathscr{B}_\tau$ such that:

$$\mathscr{B}^0_\tau \prec \mathscr{B}_\tau, \quad U^{\mathscr{B}^0_\tau} = U^{\mathscr{B}_\tau}, \quad \sigma \restriction \mathrm{Pr}^{\mathscr{B}^0_\tau}(e_{\bar{\tau}}) = \mathrm{id} \restriction \mathrm{Pr}^{\mathscr{B}^0_\tau}(e_{\bar{\tau}}), \quad B^0_\tau \subseteq \mathrm{Pr}^{\mathscr{B}^0_\tau}(\sigma(e_{\bar{\tau}})) .$$

Let $\sigma_{\bar{\tau}\tau} = \sigma$, $e_\tau = \sigma(e_{\bar{\tau}})$. It is routine to check (C 1)–(C 9) for τ.

Case 3.2. τ immediately succeeds η in S_{α_τ}.

Thus by (M 1), $\bar{\tau}$ immediately succeeds $\bar{\eta}$ in $S_{\alpha_{\bar{\tau}}}$, where $\pi_{\bar{\tau}\tau}(\bar{\eta}) = \eta$. Moreover, we have $\mathscr{B}^0_\tau = \mathscr{B}_\eta$. Let $b = \sigma_{\bar{\eta}\eta}(e_{\bar{\tau}})$.

By (C 10) and (C 2) we have

$$\langle \mathscr{B}^+_{\bar{\eta}}, e_{\bar{\tau}} \rangle \cong \langle \mathscr{B}_{\bar{\tau}}, e_{\bar{\tau}} \rangle \cong \langle \mathscr{B}_0, e_0 \rangle .$$

Thus, applying $\sigma_{\bar{\eta}\eta}\colon \mathscr{B}^+_{\bar{\eta}} \prec \mathscr{B}_\eta$,

$$\langle \sigma_{\bar{\eta}\eta}{}'' \mathscr{B}^+_{\bar{\eta}}, b \rangle \cong \langle \mathscr{B}_0, e_0 \rangle .$$

So as $\sigma_{\bar{\eta}\eta}{}'' \mathscr{B}^+_{\bar{\eta}} \prec \mathscr{B}_\eta$,

$$\langle \mathscr{B}_\eta, b \rangle \equiv \langle \mathscr{B}_0, e_0 \rangle .$$

Thus by (C 2),

$$\langle \mathscr{B}_\eta, b \rangle \equiv \langle \mathscr{B}_\eta, e_\eta \rangle .$$

So as $\mathscr{B}_\eta$ is homogeneous,

$$\langle \mathscr{B}_\eta, b \rangle \cong \langle \mathscr{B}_\eta, e_\eta \rangle .$$

Thus by (C 2),

$$\langle \mathscr{B}_\eta, b\rangle \cong \langle \mathscr{B}_0, e_0\rangle .$$

It follows that we can find a countable homogeneous structure $\mathscr{B}_\tau$ and an embedding $\sigma\colon \mathscr{B}_\tau^0 \prec \mathscr{B}_\tau$ such that

$$\mathscr{B}_\tau^0 \prec \mathscr{B}_\tau,\quad U^{\mathscr{B}_\tau^0} = U^{\mathscr{B}_\tau},\quad \sigma \upharpoonright \mathrm{Pr}^{\mathscr{B}_\tau^0}(b) = \mathrm{id} \upharpoonright \mathrm{Pr}^{\mathscr{B}_\tau^0}(b),\quad B_\tau^0 \subseteq \mathrm{Pr}^{\mathscr{B}_\tau}(\sigma(b)),$$
$$\langle \mathscr{B}_\tau^0, b\rangle \cong \langle \mathscr{B}_\tau, \sigma(b)\rangle .$$

Let $e_\tau = \sigma(b)$, $\sigma_{\bar\tau\tau} = \sigma \circ \sigma_{\bar\eta\eta}$.

It is immediate that (C 1)–(C 6) are preserved by this definition. Also, (C 9) does not apply in this case, and (C 8) follows easily from (C 7) (and the induction hypothesis). So we need to check (C 7) for τ. It clearly suffices to prove this for the case $v = \eta$ only, i.e. we show that

$$\sigma_{\bar\tau\tau} \upharpoonright B_{\bar\eta} = \sigma_{\bar\eta\eta} \upharpoonright B_{\bar\eta} .$$

Well, we have

$$B_{\bar\eta} \subseteq \mathrm{Pr}^{\mathscr{B}_{\bar\tau}}(e_{\bar\tau}) \subseteq \mathrm{Pr}^{\mathscr{B}_{\bar\eta}^+}(e_{\bar\tau}) .$$

So, applying $\sigma_{\bar\eta\eta}$,

$$\sigma_{\bar\eta\eta}{}'' B_{\bar\eta} \subseteq \mathrm{Pr}^{\mathscr{B}_\eta}(b) .$$

But

$$\sigma \upharpoonright \mathrm{Pr}^{\mathscr{B}_\eta}(b) = \mathrm{id} \upharpoonright \mathrm{Pr}^{\mathscr{B}_\eta}(b) .$$

Thus

$$\sigma \circ \sigma_{\bar\eta\eta}{}'' B_{\bar\eta} = \sigma_{\bar\eta\eta}{}'' B_{\bar\eta} ,$$

i.e.

$$\sigma_{\bar\tau\tau} \upharpoonright B_{\bar\eta} = \sigma_{\bar\eta\eta} \upharpoonright B_{\bar\eta} .$$

For future use, we note that for any K-formula $\varphi(\vec{y}, \vec{x})$:

(**) if $\vec{y} \in B_\eta^0$, $\vec{x} \in B_{\bar\tau}^+$, then $\mathscr{B}_\eta \models \varphi(\vec{y}, \sigma_{\bar\eta\eta}(\vec{x}))$ iff $\mathscr{B}_\tau \models \varphi(\vec{y}, \sigma_{\bar\tau\tau}(\vec{x}))$.

To see this, apply σ to the left-hand side and note that as $e_{\bar\eta} < e_{\bar\tau}$ in $\mathscr{B}_{\bar\eta}^+$, an application of $\sigma_{\bar\eta\eta}$ yields $e_\eta < b$ in $\mathscr{B}_\eta$, so $B_\eta^0 \subseteq \mathrm{Pr}^{\mathscr{B}_\tau^0}(b)$.

Case 3.3. τ is a limit point in S_{α_τ}.

Thus $\bar\tau$ is a limit point in $S_{\alpha_{\bar\tau}}$. There are two subcases to consider.

Case 3.3.1. $\lambda = \sup_{\bar v < \bar\lambda} \pi_{\bar\tau\tau}(\bar v) < \tau$.

In this case, $\langle \mathscr{B}_\tau^0, e_\lambda\rangle$ is the union of the elementary chain

$$(\langle \mathscr{B}_v, e_\lambda\rangle \mid v \in S_{\alpha_\tau} \;\&\; \lambda \leqslant v < \tau) .$$

By (C 10), this is a chain of isomorphic, countable homogeneous structures. So by 1.6,

$$\langle \mathscr{B}_\tau^0, e_\lambda \rangle \cong \langle \mathscr{B}_\lambda, e_\lambda \rangle \cong \langle \mathscr{B}_0, e_0 \rangle .$$

Thus we can find a countable homogeneous structure $\mathscr{B}_\tau$ and an embedding $\sigma\colon \mathscr{B}_\tau^0 \prec \mathscr{B}_\tau$ such that:

$$\mathscr{B}_\tau^0 \prec \mathscr{B}_\tau,\ U^{\mathscr{B}_\tau^0} = U^{\mathscr{B}_\tau},\ \sigma \restriction \mathrm{Pr}^{\mathscr{B}_\tau^0}(e_\lambda) = \mathrm{id} \restriction \mathrm{Pr}^{\mathscr{B}_\tau^0}(e_\lambda),\quad B_\tau^0 \subseteq \mathrm{Pr}^{\mathscr{B}_\tau}(\sigma(e_\lambda)),$$
$$\langle \mathscr{B}_\tau^0, e_\lambda \rangle \cong \langle B_\tau, \sigma(e_\lambda) \rangle .$$

Let $e_\tau = \sigma(e_\lambda)$, $\sigma_{\bar\tau\tau} = \sigma \circ \sigma_{\bar\tau\lambda}$. Much as in Case 3.2, we see that (C 1)–(C 9) continue to hold, and that (for later use):

if $\varphi(\vec{y}, \vec{x})$ is any K-formula, then

(**) if $\vec{y} \in B_\lambda^0$, $\vec{x} \in B_{\bar\tau}^+$, then $\mathscr{B}_\lambda \models \varphi(\vec{y}, \sigma_{\bar\tau\lambda}(\vec{x}))$ iff $\mathscr{B}_\tau \models \varphi(\vec{y}, \sigma_{\bar\tau\tau}(\vec{x}))$.

The final case is by far the most complicated one, though as will be seen, we have already "done all of the work" for this case, in the sense that our construction is a "limit" construction.

Case 3.3.2. $\sup_{\bar\nu < \bar\tau} \pi_{\bar\tau\tau}(\bar\nu) = \tau$.

For each $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, let $\eta(\bar\nu)$ be the $\dashv$-least η such that $\bar\nu \dashv \eta \dashv \pi_{\bar\tau\tau}(\bar\nu)$. [Notice that as $\pi_{\bar\tau\tau}(\bar\nu)$ is not maximal in S_{α_τ}, there is no possibility that $\eta = \pi_{\bar\tau\tau}(\bar\nu)$ here.] Clearly, $(\alpha_{\eta(\bar\nu)} \mid \bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau)$ is non-decreasing. Also, by morass axiom (M 4), it is in fact strictly increasing. Set

$$\alpha = \sup \{\alpha_{\eta(\bar\nu)} \mid \bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau\} .$$

By (M 3), $\alpha \in S^1$, and in fact whenever $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, there is a $\nu' \in S_\alpha$ such that $\bar\nu \dashv \nu' \,\underline{\dashv}\, \pi_{\bar\tau\tau}(\bar\nu)$. So by (M 7) and the fact that τ immediately succeeds $\bar\tau$ in $\dashv$, we see that $\alpha = \alpha_\tau$. We shall define $\mathscr{B}_\tau$ as a "diagonal limit" of the structures $\mathscr{B}_{\eta(\bar\nu)}$, for $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$.

For $\bar\nu, \bar\gamma \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, $\bar\nu \leqslant \bar\gamma$, let $\eta(\bar\nu, \bar\gamma) = \pi_{\bar\gamma, \eta(\bar\gamma)}(\bar\nu)$. Thus $\eta(\bar\nu, \bar\gamma)$ is the unique $\eta \in S_{\alpha_{\eta(\bar\gamma)}}$ such that $\bar\nu \dashv \eta \dashv \pi_{\bar\tau\tau}(\bar\nu)$. (See Fig. 7.)

Notice that:

(i) $\bar\nu \dashv \eta(\bar\nu) \dashv \eta(\bar\nu, \bar\gamma) \dashv \pi_{\bar\tau\tau}(\bar\nu)$;

(ii) $\sigma_{\eta(\bar\nu), \eta(\bar\nu,\bar\gamma)}{}''B_{\eta(\bar\nu)}^0 \subseteq B_{\eta(\bar\nu,\bar\gamma)}^0$.

Also, since $\pi_{\bar\tau\tau}$ is cofinal in τ on $\bar\tau$ and the sequence $(\alpha_{\eta(\bar\nu)} \mid \bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau)$ is cofinal in α_τ, we have, setting $\nu = \pi_{\bar\tau\tau}(\bar\nu)$:

(iii) $\mathscr{B}_\tau^0 = \bigcup \{\sigma_{\eta(\bar\nu), \nu}{}''\mathscr{B}_{\eta(\nu)}^0 \mid \bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau\}$.

Claim. If $\bar\nu, \bar\gamma \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, $\bar\nu \leqslant \bar\gamma$, $\vec{y} \in B_{\eta(\bar\nu)}^0$, $\vec{x} \in B_{\bar\tau}^+$, then for any K-formula φ:

$$\mathscr{B}_{\eta(\bar\nu)} \models \varphi(\vec{y}, \sigma_{\bar\nu, \eta(\bar\nu)}(\vec{x})) \quad \text{iff } \mathscr{B}_{\eta(\bar\gamma)} \models \varphi(\sigma_{\eta(\bar\nu), \eta(\bar\nu,\bar\gamma)}(\vec{y}), \sigma_{\bar\gamma, \eta(\bar\gamma)}(\vec{x})) .$$

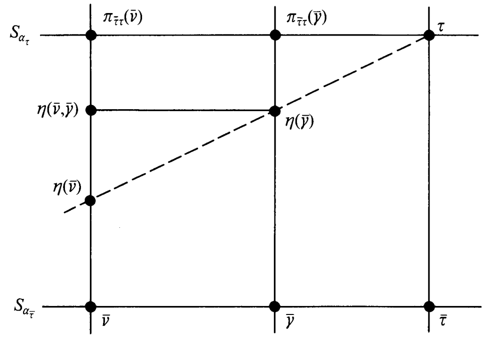


Fig. 7

We postpone the proof of this claim for a moment and complete the definition of $\mathscr{B}_\tau$.

Let X be an arbitrary, countable, infinite set disjoint from B_τ^0, and let

$$h\colon (B_{\bar\tau}^{+} - B_{\bar\tau}^0) \leftrightarrow X\,.$$

Let $B_\tau = B_\tau^0 \cup X$, and define a function $\sigma\colon B_{\bar\tau}^{+} \to B_\tau$ by

$$\sigma(x) = \begin{cases} h(x), & \text{if } x \notin B_{\bar\tau}^0 \\ \sigma_{\bar\nu\nu}(x), & \text{if } x \in B_{\bar\nu}, \text{ where } \bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau \text{ and } \nu = \pi_{\bar\tau\tau}(\bar\nu). \end{cases}$$

By (C 7) and the induction hypothesis, this clearly defines a function on $B_{\bar\tau}^{+}$.

We define a K-structure $\mathscr{B}_\tau$ on B_τ so that $\sigma\colon \mathscr{B}_{\bar\tau}^{+} \prec \mathscr{B}_\tau$. We can do this in a unique way so that for all K-formulas φ and all $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$, $\vec y \in B_{\eta(\bar\nu)}$, $\vec x \in B_{\bar\tau}^{+}$, we have

$$\text{(iv)}\quad \mathscr{B}_\tau \models \varphi(\sigma_{\eta(\bar\nu),\nu}(\vec y), \sigma(\vec x)) \quad \text{iff } \mathscr{B}_{\eta(\bar\nu)} \models \varphi(\vec y, \sigma_{\bar\nu,\eta(\bar\nu)}(\vec x)),$$

where $\nu = \pi_{\bar\tau\tau}(\bar\nu)$. By (iii) above, the $\vec y$'s take care of the whole of B_τ^0, and the $\vec x$'s take care of X, so all of B_τ is covered. By the claim there is no conflict between different choices of $\bar\nu$. Hence $\mathscr{B}_\tau$ is uniquely determined. And by means of an argument as in Case 2 there is an embedding $j\colon \mathscr{B}^0 \prec \mathscr{B}_\tau$. By choosing X suitably we can clearly ensure that $j = \mathrm{id} \restriction B_\tau^0$. Thus $\mathscr{B}_\tau^0 \prec \mathscr{B}_\tau$. We set $\sigma_{\bar\tau\tau} = \sigma$, $e_\tau = \sigma_{\bar\tau\tau}(e_{\bar\tau})$.

Now, by equivalence (iv) above, $\langle \mathscr{B}_\tau, e_\tau \rangle$ is a sort of limit of the isomorphic (to $\langle \mathscr{B}_0, e_0 \rangle$) countable homogeneous structures $\langle \mathscr{B}_{\eta(\bar\nu)}, \sigma_{\bar\nu,\eta(\bar\nu)}(e_{\bar\tau}) \rangle$, for $\bar\nu \in S_{\alpha_\tau} \cap \bar\tau$. By means of arguments which are in essence the same as those used to prove 1.3 and 1.6, it is easily seen that $\mathscr{B}_\tau$ is countable homogeneous and that $\langle \mathscr{B}_\tau, e_\tau \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$. (In fact we did not give the proof of 1.3, since this is a "standard" result of model theory, but the details are easily worked out. The idea is to construct the desired homogeneity automorphism by means of a "back and forth" procedure as used in the proof of 1.6. This is similar to, but a little easier than the argument in 1.6 itself.) Thus (C 2) is preserved. The verification of (C 1) and (C 3)–(C 9) is routine.

There remains the verification of the claim. This is done by induction on $\bar{\gamma} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$. For $\bar{\gamma}$ the minimal member of $S_{\alpha_{\bar{\tau}}}$, the claim is trivially valid. Suppose next that $\bar{\gamma}$ immediately succeeds $\bar{\nu}$ in $S_{\alpha_{\bar{\tau}}}$. Then by induction it suffices to prove the claim for this one pair $\bar{\nu}, \bar{\gamma}$. We have:

$$\sigma_{\bar{\nu}, \eta(\bar{\nu})}(\vec{x}),\ \vec{y} \in B_{\eta(\bar{\nu})},$$
$$\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}\colon \mathscr{B}^+_{\eta(\bar{\nu})} \prec \mathscr{B}_{\eta(\bar{\nu}, \bar{\gamma})},$$
$$\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})} \circ \sigma_{\bar{\nu}, \eta(\bar{\nu})} = \sigma_{\bar{\nu}, \eta(\bar{\nu}, \bar{\gamma})}.$$

So for any φ, we clearly have

$$\mathscr{B}_{\eta(\bar{\nu})} \models \varphi(\vec{y}, \sigma_{\bar{\nu}, \eta(\bar{\nu})}(\vec{x})) \quad \text{iff } \mathscr{B}_{\eta(\bar{\nu}, \bar{\gamma})} \models \varphi(\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}(\vec{y}), \sigma_{\bar{\nu}, \eta(\bar{\nu}, \bar{\gamma})}(\vec{x})).$$

Now, $\eta(\bar{\gamma})$ immediately succeeds γ in $\multimap$ and $\eta(\bar{\gamma})$ immediately succeeds $\eta(\bar{\nu}, \bar{\eta})$ in $S_{\alpha_{\eta(\gamma)}}$. Hence Case 3.2 applies to $\eta(\bar{\gamma})$. Note that as $\vec{y} \in B^0_{\eta(\bar{\nu})}$, (ii) above gives

$$\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}(\vec{y}) \in B^0_{\eta(\bar{\nu}, \bar{\gamma})}.$$

So by (*) above,

$$\mathscr{B}_{\eta(\bar{\nu}, \bar{\gamma})} \models \varphi(\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}(\vec{y}), \sigma_{\bar{\nu}, \eta(\bar{\nu}, \bar{\gamma})}(\vec{x})) \quad \text{iff } \mathscr{B}_{\eta(\bar{\gamma})} \models \varphi(\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}(\vec{y}), \sigma_{\bar{\gamma}, \eta(\bar{\gamma})}(\vec{x})).$$

The claim follows from the above two equivalences.

Finally, suppose that $\bar{\gamma}$ is a limit point in $S_{\alpha_{\bar{\tau}}}$. Let

$$\lambda = \sup_{\bar{\nu} < \bar{\gamma}} \pi_{\bar{\gamma}, \eta(\bar{\gamma})}(\bar{\nu}).$$

Either $\lambda = \eta(\bar{\gamma})$ or else $\lambda < \eta(\bar{\gamma})$. Thus, either by identity or else by (**), respectively, we have, for all $\vec{y} \in B^0_\lambda$, $\vec{x} \in B^+_{\bar{\gamma}}$,

(v) $\quad \mathscr{B}_{\eta(\bar{\gamma})} \models \varphi(\vec{y}, \sigma_{\bar{\gamma}, \eta(\bar{\gamma})}(\vec{x})) \quad \text{iff } \mathscr{B}_\lambda \models \varphi(\vec{y}, \sigma_{\bar{\gamma}, \lambda}(\vec{x})).$

Let $\alpha = \sup_{\bar{\nu} < \bar{\gamma}} \alpha_{\eta(\bar{\nu})}$. Let $\bar{\eta}$ be the unique $\bar{\eta} \in S_\alpha$ such that $\bar{\gamma} \multimap \bar{\eta} \sqsupseteq \lambda$. Then $\bar{\eta}$ immediately succeeds $\bar{\gamma}$ in $\multimap$. For if $\bar{\gamma} \multimap \bar{\bar{\eta}} \multimap \bar{\eta}$, then for $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\tau}$, we would have $\alpha_{\eta(\bar{\nu})} \leqslant \alpha_{\bar{\bar{\eta}}}$, contrary to $\bar{\eta} \in S_\alpha$ and the choice of α. So, recalling the definition of λ we see that Case 3.3.2 applies to $\bar{\eta}$. So by induction, for $\vec{y} \in B^0_{\eta(\bar{\nu})}$, $\bar{\nu} \in S_{\alpha_{\bar{\tau}}} \cap \bar{\gamma}$, $\vec{x} \in B^+_{\bar{\gamma}}$, we have, by (iv),

$$\mathscr{B}_{\eta(\bar{\nu})} \models \varphi(\vec{y}, \sigma_{\bar{\nu}, \eta(\bar{\nu})}(\vec{x})) \quad \text{iff } \mathscr{B}_{\bar{\eta}} \models \varphi(\sigma_{\eta(\bar{\nu}), \nu}(\vec{y}), \sigma_{\bar{\gamma}\bar{\eta}}(\vec{x})),$$

where $\nu = \pi_{\bar{\gamma}, \bar{\eta}}(\bar{\nu})$. Applying $\sigma_{\bar{\eta}\lambda}\colon \mathscr{B}_{\bar{\eta}} \prec \mathscr{B}_\lambda$ to the right hand side we obtain:

$$\mathscr{B}_{\eta(\bar{\nu})} \models \varphi(\vec{y}, \sigma_{\bar{\nu}, \eta(\bar{\nu})}(\vec{x})) \quad \text{iff } \mathscr{B}_\lambda \models \varphi(\sigma_{\eta(\bar{\nu}), \nu}(\vec{y}), \sigma_{\bar{\gamma}\lambda}(\vec{x})),$$

where this time $\nu = \pi_{\bar{\gamma}, \lambda}(\bar{\nu})$. But $\vec{y} \in B^0_{\eta(\bar{\nu})}$, so $\sigma_{\eta(\bar{\nu}), \nu}(\vec{y}) \in B^0_\lambda$ here. Thus by (v) we obtain

$$\mathscr{B}_{\eta(\bar{\nu})} \models \varphi(\vec{y}, \sigma_{\bar{\nu}, \eta(\bar{\nu})}(\vec{x})) \quad \text{iff } \mathscr{B}_{\eta(\bar{\gamma})} \models \varphi(\sigma_{\eta(\bar{\nu}), \eta(\bar{\nu}, \bar{\gamma})}(\vec{y}), \sigma_{\bar{\gamma}, \eta(\bar{\gamma})}(\vec{x})),$$

as required.

The claim is proved, and with it the Gap-2 Theorem.

4. Simplified Morasses

The morass defined in section 2 and used in section 3 provides us with some insight into the structure of the constructible hierarchy. But as we have seen, it is not particularly easy to use, requiring the consideration of many separate cases (five in the proof of the Gap-2 Theorem), one of them (Case 3.3.2) quite complicated. If one's main interest is simply to use morasses to prove theorems like the Gap-2 Theorem, this complexity is a nuisance. In this section we show that it is an avoidable nuisance. We shall describe a "simplified morass" structure. The existence of a simplified morass is provably (in ZFC) equivalent to the existence of a full morass in the sense of section 2. (We shall give one half of the proof, the half of relevance to us here.) And as we shall see, it is considerably easier to prove the Gap-2 Theorem using a simplified morass.

As with the morass of section 2, the motivation is the approximation of a structure of cardinality ω_2 by means of a system of ω_1 many countable structures. (As before, we consider the case of an ω_1 morass for definiteness, but everything generalises quite easily to and arbitrary uncountable regular cardinal κ.)

A simplified morass (morass precisely, a simplified $(\omega_1, 1)$-morass) consists of a structure

$$\mathcal{M} = \langle(\theta_\alpha \mid \alpha \leqslant \omega_1), (\mathcal{F}_{\alpha\beta} \mid \alpha < \beta \leqslant \omega_1)\rangle$$

satisfying the following six conditions (which we examine below):

(P 0) (a) $\theta_0 = 1$, $\theta_{\omega_1} = \omega_2$, $(\forall \alpha < \omega_1)(0 < \theta_\alpha < \omega_1)$;

(b) $\mathcal{F}_{\alpha\beta}$ is a set of order-preserving functions $f\colon \theta_\alpha \to \theta_\beta$;

(P 1) $|\mathcal{F}_{\alpha\beta}| \leqslant \omega$ for all $\alpha < \beta < \omega_1$;

(P 2) if $\alpha < \beta < \gamma$, then $\mathcal{F}_{\alpha\gamma} = \{f \circ g \mid f \in \mathcal{F}_{\beta\gamma} \;\&\; g \in \mathcal{F}_{\alpha\beta}\}$;

(P 3) if $\alpha < \omega_1$, then $\mathcal{F}_{\alpha,\alpha+1} = \{\mathrm{id} \restriction \theta_\alpha, f_\alpha\}$, where f_α is such that for some $\delta < \theta_\alpha$, $f_\alpha \restriction \delta = \mathrm{id} \restriction \delta$ and $f_\alpha(\delta) \geqslant \theta_\alpha$;

(P 4) if $\alpha \leqslant \omega_1$ is a limit ordinal, if $\beta_1, \beta_2 < \alpha$, and if $f_1 \in \mathcal{F}_{\beta_1\alpha}$, $f_2 \in \mathcal{F}_{\beta_2\alpha}$, then there is a $\gamma < \alpha$, $\gamma > \beta_1, \beta_2$, and there are $f_1' \in \mathcal{F}_{\beta_1\gamma}$, $f_2' \in \mathcal{F}_{\beta_2\gamma}$, $g \in \mathcal{F}_{\gamma\alpha}$, such that $f_1 = g \circ f_1'$, $f_2 = g \circ f_2'$.

(P 5) for all $\alpha > 0$, $\theta_\alpha = \bigcup \{f'' \theta_\beta \mid \beta < \alpha \;\&\; f \in \mathcal{F}_{\beta\alpha}\}$.

The idea of the above definition is this. We approximate $\theta_{\omega_1} = \omega_2$ by means of the countable ordinals θ_α, $\alpha < \omega_1$. To do this we need to know how the intervals θ_α "fit inside" θ_{ω_1}. $\mathcal{F}_{\alpha\beta}$ consists of a set of order-preserving maps from θ_α into θ_β. Each map f in $\mathcal{F}_{\alpha\beta}$ gives one way in which θ_α "fits inside" θ_β as an approximation to it. (P 1) tells us that there are not too many ways in which this can happen for any given pair $\alpha, \beta < \omega_1$. (P 2) is self-explanatory. (P 3) (together with (P 0)(b)) says that at successor steps in the approximation procedure there are just two ways in which θ_α fits inside $\theta_{\alpha+1}$, both very simple (P 4) tells us that the "approximation tree" going up to θ_{ω_1} does not have branches which "split" at limit levels.

A particular consequence of (P 5) is that θ_{ω_1} is entirely determined by the countable parts of the simplified morass.

We shall use the simplified morass to prove the Gap-2 Theorem. We are given a K-structure $\mathscr{A} = \langle A, U, \dots \rangle$ of type (κ^{++}, κ) and wish to construct a K-structure $\mathscr{B}$ of type (ω_2, ω) such that $\mathscr{B} \equiv \mathscr{A}$. We commence as in section 3. In particular, let $\mathscr{B}_0, \mathscr{C}_0, e_0, \sigma_0$ be as in lemma 3.3. We construct (instead of an $\mathscr{M}$-complex) sequences

$$(\mathscr{B}_\alpha | \alpha \leqslant \omega_1), \quad (h_\alpha | \alpha \leqslant \omega_1), \quad (f^* | f \in \mathscr{F}_{\beta\alpha}, \beta < \alpha \leqslant \omega_1)$$

so that:

(C 1) $\mathscr{B}_\alpha \equiv \mathscr{A}$;

(C 2) $h_\alpha \colon \theta_\alpha \to B_\alpha$ is order-preserving (where B_α is ordered by the linear ordering which is part of $\mathscr{B}_\alpha$);

(C 3) $f^* \colon \mathscr{B}_\beta \prec \mathscr{B}_\alpha$ and $U^{\mathscr{B}_\alpha} \subseteq \operatorname{ran}(f^*)$ for $\beta < \alpha \leqslant \omega_1$, $f \in \mathscr{F}_{\beta\alpha}$;

(C 4) if $\alpha < \omega_1$, then $\langle \mathscr{B}_\alpha, h_\alpha(\delta) \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$ for all $\delta < \theta_\alpha$;

(C 5) $(f \circ g)^* = f^* \circ g^*$, whenever $f \in \mathscr{F}_{\gamma\alpha}$, $g \in \mathscr{F}_{\beta\gamma}$, $\beta < \gamma < \alpha$;

(C 6) $h_\alpha \circ f = f^* \circ h_\beta$ for each $f \in \mathscr{F}_{\beta\alpha}$;

(C 7) if $f \in \mathscr{F}_{\beta\alpha}$ and $\operatorname{ran}(f) \subseteq \delta < \theta_\alpha$, then $\operatorname{ran}(f^*) \subseteq \operatorname{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta))$.

Provided we can carry out this construction we shall be done, since then $\mathscr{B}_{\omega_1}$ is of type (ω_2, ω) as required. ($|B_{\omega_1}| = \omega_2$ by (C 2) and $|U^{\mathscr{B}_{\omega_1}}| = \omega$ by (C 3).) We construct the above sequences by recursion on α.

$\mathscr{B}_0$ has been defined already. We set $h_0(0) = e_0$. Now suppose that we are at a successor step, $\alpha + 1$. By (P 3), $\mathscr{F}_{\alpha, \alpha+1} = \{\operatorname{id} \restriction \theta_\alpha, f\}$, where for some $\delta < \theta_\alpha$, $f_\alpha \restriction \delta = \operatorname{id} \restriction \delta$ and $f_\alpha(\delta) \geqslant \theta_\alpha$. By (C 4), $\langle \mathscr{B}_\alpha, h_\alpha(\delta) \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$, so we can find $\mathscr{B}_{\alpha+1}, \sigma$ such that:

$$\begin{gathered}\mathscr{B}_\alpha \prec \mathscr{B}_{\alpha+1}, \quad U^{\mathscr{B}_\alpha} = U^{\mathscr{B}_{\alpha+1}}, \quad \sigma \colon \mathscr{B}_\alpha \prec \mathscr{B}_{\alpha+1}, \quad B_\alpha \subseteq \operatorname{Pr}^{\mathscr{B}_{\alpha+1}}(\sigma(h_\alpha(\delta))), \\ \sigma \restriction \operatorname{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta)) = \operatorname{id} \restriction \operatorname{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta)), \quad \langle \mathscr{B}_{\alpha+1}, \sigma(h_\alpha(\delta)) \rangle \cong \langle \mathscr{B}_\alpha, h_\alpha(\delta) \rangle.\end{gathered}$$

Set $(\operatorname{id} \restriction \theta_\alpha)^* = \operatorname{id} \restriction B_\alpha$, $f_\alpha^* = \sigma$.

Suppose now that $h \in \mathscr{F}_{\beta, \alpha+1}$, $\beta < \alpha$. To define h^*, choose $f \in \mathscr{F}_{\alpha, \alpha+1}$, $g \in \mathscr{F}_{\beta\alpha}$, so that $h = f \circ g$ (by (P 2)) and let $h^* = f^* \circ g^*$. Now, g is clearly uniquely determined by h here, but if $\operatorname{ran}(h) \subseteq \delta$, then f is not. However, by (C 7) we have $\operatorname{ran}(g^*) \subseteq \operatorname{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta))$, so by choice of σ, h^* does not depend upon the choice of f. Hence h^* is well-defined in all cases.

Define $h_{\alpha+1} \colon \theta_{\alpha+1} \to B_{\alpha+1}$ by

$$h_{\alpha+1}(v) = \begin{cases} h_\alpha(v), & \text{if } v < \theta_\alpha, \\ \sigma(h_\alpha(\bar{v})), & \text{if } v = f_\alpha(\bar{v}) \geqslant \theta_\alpha. \end{cases}$$

(Using (P 5), it is easy to see that $h_{\alpha+1}$ is well-defined on $\theta_{\alpha+1}$.)

We must check that (C 1)–(C 7) are preserved. (C 1) is clear. (C 2) holds because $B_\alpha \subseteq \operatorname{Pr}^{\mathscr{B}_{\alpha+1}}(\sigma(h_\alpha(\delta)))$. For (C 3), note that since $\langle \mathscr{B}_\alpha, h_\alpha(\delta) \rangle \cong \langle \mathscr{B}_0, e_0 \rangle$, we have

$U^{\mathscr{B}_\alpha} \subseteq \mathrm{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta))$, and that by choice of σ, $\sigma \restriction \mathrm{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta)) = \mathrm{id} \restriction \mathrm{Pr}^{\mathscr{B}_\alpha}(h_\alpha(\delta))$, so $U^{\mathscr{B}_{\alpha+1}} = U^{\mathscr{B}_\alpha} \subseteq \mathrm{ran}((\mathrm{id} \restriction \theta_\alpha)^*)$ and $U^{\mathscr{B}_{\alpha+1}} = U^{\mathscr{B}_\alpha} \subseteq \mathrm{ran}(f_\alpha^*)$. (C 4) is a simple consequence of the fact that $\mathscr{B}_{\alpha+1}$ is countable homogeneous (cf. the corresponding arguments in section 3, in particular the proof of (C 10) there). (C 5) holds by definition. (C 6) need only be verified for $\mathscr{F}_{\alpha,\alpha+1}$, i.e. it must be shown that if $f \in \mathscr{F}_{\alpha,\alpha+1}$, then $h_{\alpha+1} \circ f = f^* \circ h_\alpha$. But this is immediate. Finally, (C 7) also only requires verification for $\mathscr{F}_{\alpha,\alpha+1}$, which is a triviality.

There remains the limit case (i.e. $\lim(\alpha)$). Let

$$\mathscr{F} = \bigcup_{\beta<\alpha} \mathscr{F}_{\beta\alpha}$$

For each $f \in \mathscr{F}$, let $d(f)$ be that β such that $f \in \mathscr{F}_{\beta\alpha}$. For $f, f' \in \mathscr{F}$, set $f <^* f'$ iff $d(f) < d(f')$ and there is a $g \in \mathscr{F}_{d(f),d(f')}$ such that $f = f' \circ g$. Note that, if it exists, the g here is uniquely determined by f, f'. Hence for $f <^* f'$ we may define an embedding

$$\pi_{ff'}\colon \mathscr{B}_{d(f)} \prec \mathscr{B}_{d(f')}$$

by

$$\pi_{ff'} = g^*,$$

where $f = f' \circ g$. By (P 2) and (P 4), $<^*$ is a transitive, directed relation on $\mathscr{F}$. Clearly,

$$\langle (\mathscr{B}_{d(f)})_{f\in\mathscr{F}}, (\pi_{ff'})_{f<^*f'} \rangle$$

is a commutative, directed elementary system. Let

$$\langle \mathscr{B}_\alpha, (f^*)_{f\in\mathscr{F}} \rangle$$

be a direct limit. Using (C 6) we may define $h_\alpha\colon \theta_\alpha \to B_\alpha$ by requiring commutativity of the following diagram for all $\beta < \alpha$:

$$\begin{array}{ccc} \theta_\alpha & \xrightarrow{h_\alpha} & B_\alpha \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f^*} \\ \theta_\beta & \xrightarrow[h_\beta]{} & B_\beta \end{array}$$

(By (P 5), this does define h_α on all of θ_α.)

We must verify (C 1)–(C 7). The only one that is not entirely trivial is (C 4). But if α is a countable limit ordinal, then $\langle \mathscr{F}, <^* \rangle$ has a cofinal subset of order-type ω, so (C 4) follows from lemma 1.6. That completes the proof of the Gap-2 Theorem using a simplified morass.

We turn now to the question of the existence of a simplified morass. It should be stressed that the definition of the simplified morass is designed to make applications easy. The simplified morass structure is not particularly closely related to the constructible hierarchy in the way that the "standard" morass is. In fact, in order to construct a simplified morass, what we shall in fact do is start with a standard morass and use it to construct the new morass, rather than the fine structure theory. This construction is not at all intuitive, and is motivated solely by the aim of obtaining the various properties of a simplified morass.

The general idea is to define the ordinals θ_α of the simplified morass as the order-types of certain well-ordered sets $(W_\alpha, <_\alpha)$ of finite tuples of elements of the standard morass, and to obtain the embeddings in $\mathscr{F}_{\alpha\beta}$ as compositions of some specific maps from W_α into W_β. In order to make this work we first of all have to add some extra points to the morass to enable us to "smooth out" the irregularities in the morass structure which manifested themselves in the large number of cases required to prove the Gap-2 Theorem using the standard morass.

Beyond this very rough outline, the rest is, unfortunately, highly technical, so you may expect a somewhat rough ride. Best of luck!

We fix a standard morass

$$\mathscr{M} = \langle S, \mathscr{S}, \dashv, (\pi_{\nu\tau})_{\nu \dashv \tau} \rangle$$

as in section 2. We shall write

$$\bar{\nu} \dashv_* \nu \quad \text{iff } \nu \text{ immediately succeeds } \bar{\nu} \text{ in } \dashv;$$

and (see Fig. 8)

$$\mu \dashv \nu \quad \text{iff there are } \bar{\nu}, \bar{\mu} \text{ such that } \bar{\nu} \dashv_* \nu,\ \bar{\mu} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}, \text{ and } \bar{\mu} \dashv_* \mu \dashv \pi_{\bar{\nu}\nu}(\bar{\mu}).$$

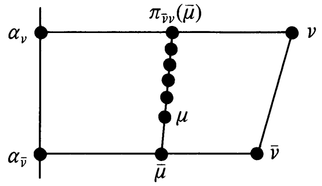


Fig. 8

We may (and shall) assume that if $\bar{\alpha} < \alpha$ and $\bar{\nu}$ is minimal in $S_{\bar{\alpha}}$ and ν is minimal in S_α, then $\bar{\nu} \dashv \nu$. (Simply extend $\dashv$ to achieve this. None of the morass axioms are effected by this.)

For each $\alpha \in S^0 \cap \omega_1$, we set $\nu_\alpha = \max(S_\alpha)$. Let

$$A = \{\alpha \in S^0 \cap \omega_1 \mid \nu_\alpha \text{ is a successor in } \dashv\};$$
$$A_0 = \{\alpha \in A \mid \nu_\alpha \text{ is a successor in } S_\alpha\};$$
$$A_1 = \{\alpha \in A \mid \nu_\alpha \text{ is a limit in } S_\alpha \text{ but } \pi_{\bar{\nu}\nu_\alpha} \restriction \bar{\nu} \text{ is not cofinal in } \nu_\alpha, \text{ where } \bar{\nu} \dashv_* \nu_\alpha\}.$$

We now add some more points to the morass. For $\alpha \in A$, set

$$S_\alpha^+ = \{\nu_a + \tau \mid \tau \in S_\alpha \cap \nu_\alpha\}.$$

Extend the relation $\multimap$ to $'$ by setting, for $\nu_\alpha + \tau \in S_\alpha^+$,

$$\bar{\tau} \multimap' \nu_\alpha + \tau \quad \text{iff } \bar{\tau} \multimap \tau,$$

and then extending to obtain transitivity.

Let

$$\bar{S} = S^0 \cup \{\alpha + 1 \mid \alpha \in A_0 \cup A_1\}.$$

For $\alpha \in \bar{S}$, let

$$\bar{S}_\alpha = \begin{cases} S_\alpha, & \text{if } \alpha \in S^0 - A, \\ S_\alpha \cup S_\alpha^+, & \text{if } \alpha \in A - (A_0 \cup A_1), \\ S_\alpha - \{\nu_\alpha\}, & \text{if } \alpha \in A_0 \cup A_1, \\ S_{\bar{\alpha}} \cup S_{\bar{\alpha}}^+, & \text{if } \alpha = \bar{\alpha} + 1,\ \bar{\alpha} \in A_0 \cup A_1. \end{cases}$$

For $\nu \in \bigcup_{\alpha \in \bar{S}} \bar{S}_\alpha$, let

$$\alpha'_\nu = \text{the largest } \alpha \text{ such that } \nu \in \bar{S}_\alpha,$$
$$\alpha''_\nu = \text{the smallest } \alpha \text{ such that } \nu \in \bar{S}_\alpha.$$

(Notice that there are at most two α such that $\nu \in \bar{S}_\alpha$.)

Let $(\gamma_\nu \mid \nu \leqslant \omega_1)$ be the monotone enumeration of $\bar{S}$. For $\nu \leqslant \omega_1$, let W_ν be the set of all finite tuples $(\eta_0, \ldots, \eta_n)$ such that n is odd and:

(i) $\eta_0 \in \bar{S}_{\gamma_\nu}$;

(ii) $\eta_{2k+1} \multimap' \eta_{2k}$;

(iii) $\eta_{2k+2} \in S_{\alpha'_{\eta_{2k+1}}}$, $\eta_{2k+2} > \eta_{2k+1}$.

Let $<_\nu$ be the Kleene-Brouwer ordering on W_ν; that is, $(\vec{\eta}) <_\nu (\vec{\mu})$ iff $\vec{\mu}$ is an initial segment of $\vec{\eta}$ or else $\vec{\eta}$ precedes $\vec{\mu}$ lexicographically. It is easily seen that W_ν is well-ordered by $<_\nu$, so let $\theta_\nu = \operatorname{otp}(W_\nu, <_\nu)$. In the following we shall identify θ_ν with $(W_\nu, <_\nu)$.

Clearly, $0 < \theta_\nu < \omega_1$ for $\nu < \omega_1$, and $\theta_{\omega_1} = \omega_2$. As a prelude to defining the sets $\mathscr{F}_{\alpha\beta}$ of embeddings we define some special maps.

First of all, let $\bar{\nu} \multimap \nu$, $\alpha'_{\bar{\nu}} = \gamma_\alpha$, $\alpha''_\nu = \gamma_\beta$. We define $\tilde{\pi}_{\bar{\nu}\nu}\colon W_\alpha \to W_\beta$ by

$$\tilde{\pi}_{\bar{\nu}\nu}((\eta_0, \ldots, \eta_n)) = \begin{cases} (\pi_{\bar{\nu}\nu}(\eta_0), \eta_1, \ldots, \eta_n), & \text{if } \eta_0 \leqslant \bar{\nu}, \\ (\nu, \bar{\nu}, \eta_0, \ldots, \eta_n), & \text{if } \eta_0 > \nu. \end{cases}$$

Now suppose $\eta \dashv \nu$. Then there are $\bar{\nu}, \bar{\eta}$ such that $\bar{\nu} \multimap_* \nu$, $\bar{\eta} \in S_{\alpha_\nu} \cap \bar{\nu}$ and $\bar{\eta} \multimap_* \eta \multimap \pi_{\bar{\nu}\nu}(\bar{\eta})$. Let $\eta' = \pi_{\bar{\nu}\nu}(\bar{\eta})$. For some α, β, we have $\alpha'_\eta = \gamma_\alpha$, $\alpha''_\nu = \gamma_\beta$. Let $\varrho = \alpha'_{\bar{\nu}}$. Notice that by (M 4), $\eta = \max(S_{\gamma_\alpha})$, $\nu = \max(S_{\gamma_\beta})$, and (hence) $\gamma_\alpha, \gamma_\beta \in A$. We define $\sigma_{\eta\nu}\colon W_\alpha \to W_\beta$ by:

$$\sigma_{\eta\nu}((\eta_0, \ldots, \eta_n)) = \begin{cases} (\pi_{\eta\eta'}(\eta_0), \eta_1, \ldots, \eta_n), & \text{if } \eta_0 < \eta \text{ or } (\eta_0 = \eta \text{ and } \eta_1 \neq \bar{\eta}), \\ (\nu + \pi_{\eta\eta'}(\tau), \eta_1, \ldots, \eta_n), & \text{if } \eta_0 = \eta + \tau, \tau > 0, \\ \pi_{\bar{\nu}\nu}((\eta_2, \ldots, \eta_n)), & \text{if } (\eta_0, \eta_1) = (\eta, \bar{\eta}) \text{ and } \eta_2 \in \bar{S}_\varrho, \\ (\nu, \bar{\nu}), & \text{if } (\eta_0, \ldots, \eta_n) = (\eta, \bar{\eta}). \end{cases}$$

Again, let $\alpha \in A_0$. Then we can find $\bar{\nu}, \varrho$ so that $\bar{\nu} \dashv\!\!\dashv_* \nu_\alpha$ and ν_α immediately succeeds ϱ in S_α. Let $\nu = \nu_\alpha$, $\pi_{\bar{\nu}\nu}(\bar{\varrho}) = \varrho$, $\alpha = \gamma_\delta$. Define $g_0^\delta \colon W_\delta \to W_{\delta+1}$ and $g_1^\delta \colon W_\delta \to W_{\delta+1}$ by $g_0^\delta = \mathrm{id} \restriction W_\delta$ and

$$g_1^\delta((\eta_0, \ldots, \eta_n)) = \begin{cases} (\eta_0, \ldots, \eta_n), & \text{if } \eta_0 < \varrho \text{ or } (\eta_0 = \varrho \text{ and } \eta_1 < \bar{\varrho}), \\ (\nu, \bar{\nu}, \eta_2, \ldots, \eta_n), & \text{if } \eta_0 = \varrho, \eta_1 = \bar{\varrho}, \text{ and either } \eta_2 \text{ does not} \\ & \qquad \text{exist or else } \eta_2 \neq \bar{\nu}, \\ (\nu, \eta_3, \ldots, \eta_n), & \text{if } \eta_0 = \varrho, \eta_1 = \bar{\varrho}, \eta_2 = \bar{\nu}, \\ (\nu + \varrho, \eta_1, \ldots, \eta_n), & \text{if } \eta_0 = \varrho \text{ and } \eta_1 > \bar{\varrho}. \end{cases}$$

Finally, let $\alpha \in A_1$. Then there are $\bar{\nu}, \lambda$ such that $\bar{\nu} \dashv\!\!\dashv_* \nu_\alpha$ and $\lambda = \sup \pi_{\bar{\nu}\nu}''\bar{\nu} < \nu_\alpha$. By (M 6), $\bar{\nu} \dashv\!\!\dashv \lambda$. Let $\alpha = \gamma_\delta$. Define $g_0^\delta \colon W_\delta \to W_{\delta+1}$ and $g_1^\delta \colon W_\delta \to W_{\delta+1}$ by $g_0^\delta = \mathrm{id} \restriction W_\delta$ and

$$g_1^\delta((\eta_0, \ldots, \eta_n)) = \begin{cases} (\eta_0, \ldots, \eta_n), & \text{if } \eta_0 < \lambda \text{ or } (\eta_0 = \lambda \text{ and } \eta_1 < \bar{\nu}), \\ (\nu, \eta_1, \ldots, \eta_n), & \text{if } \eta_0 = \lambda \text{ and } \eta_1 = \bar{\nu}, \\ (\nu + \eta_0, \eta_1, \ldots, \eta_n), & \text{if } \eta_0 > \lambda \text{ or } (\eta_0 = \lambda \text{ and } \eta_1 > \bar{\nu}). \end{cases}$$

The proofs of all parts of the following lemma are routine (and hence omitted).

4.1 Lemma.

(i) *$\pi_{\bar{\nu}\nu}, \sigma_{\eta\nu}, g_i^\delta$ are all order-preserving.*

(ii) *For some $\alpha < \delta$, $g_1^\delta \restriction \alpha = \mathrm{id} \restriction \alpha$ and $g_1^\delta(\alpha) \geqslant \theta_\delta$. (Recall that we identify θ_δ with $(W_\delta, <_\delta)$.)*

(iii) $\bar{\nu} \dashv\!\!\dashv \nu' \dashv\!\!\dashv \nu \to \pi_{\bar{\nu}\nu} = \tilde{\pi}_{\nu'\nu} \circ \tilde{\pi}_{\bar{\nu}\nu'}$.

(iv) $\eta \dashv \varrho \dashv \nu \to \sigma_{\eta\nu} = \sigma_{\varrho\nu} \circ \sigma_{\eta\varrho}$

(v) *If $\eta \dashv \nu$, where $\bar{\nu} \dashv\!\!\dashv_* \nu$ and $\bar{\eta} \dashv\!\!\dashv_* \eta$, then $\tilde{\pi}_{\bar{\nu}\nu} = \sigma_{\eta\nu} \circ \tilde{\pi}_{\bar{\eta}\eta}$.*

(vi) *Let $\bar{\nu} \dashv\!\!\dashv_* \nu = \nu_\alpha$, $\alpha \in A_0 \cup A_1$, $\alpha = \gamma_\delta$, and let $\lambda = \sup \pi_{\bar{\nu}\nu}''(S_{\alpha_\nu} \cap \bar{\nu})$. Let $\bar{\lambda} \dashv\!\!\dashv \lambda$ be such that $\alpha_{\bar{\lambda}} = \alpha_{\bar{\nu}}$. (Thus either $\bar{\nu} = \bar{\lambda}$ or else $\bar{\nu}$ immediately succeeds $\bar{\lambda}$ in $S_{\alpha_{\bar{\nu}}}$.) Let $\bar{\lambda} \dashv\!\!\dashv_* \lambda' \dashv\!\!\dashv \lambda$. Then $\tilde{\pi}_{\bar{\nu}\nu} = g_1^\delta \circ \tilde{\pi}_{\bar{\lambda}\lambda}$ and for all $\eta \dashv \nu$ we have either $\sigma_{\eta\nu} = g_1^\delta \circ \tilde{\pi}_{\lambda\lambda'} \circ \sigma_{\eta\lambda'}$ or else $\eta = \lambda'$ and $\sigma_{\eta\nu} = g_1^\delta \circ \tilde{\pi}_{\eta\lambda}$.*

(vii) *Let $\tau \in S_\alpha \cap \nu_\alpha$, $\alpha \in A_0 \cup A_1$, $\alpha = \gamma_\delta$. Then:*

$$\bar{\tau} \dashv\!\!\dashv \tau \to \tilde{\pi}_{\bar{\tau}\tau} = g_0^\delta \circ \tilde{\pi}_{\bar{\tau}\tau},$$
$$\eta \dashv \nu \to \sigma_{\eta\nu} = g_0^\delta \circ \sigma_{\eta\nu}. \quad \square$$

Let $\bar{\mathscr{F}}$ be the set of all the maps $\pi_{\bar{\nu}\nu}, \sigma_{\eta\nu}, g_i^\delta$, and let $\mathscr{F}$ be the closure of $\bar{\mathscr{F}}$ under finite compositions. For $\alpha < \beta \leqslant \omega_1$, set

$$\mathscr{F}_{\alpha\beta} = \{f \in \mathscr{F} \mid \mathrm{dom}(f) = W_\alpha \text{ and } \mathrm{ran}(f) \subseteq W_\beta\}.$$

The structure

$$\mathscr{M}_0 = \langle (\theta_\alpha \mid \alpha \leqslant \omega_1), (\mathscr{F}_{\alpha\beta} \mid \alpha < \beta \leqslant \omega_1) \rangle$$

is not yet a simplified morass, but as we show below it already has most of the properties we require.

4.2 Lemma. *$\mathscr{M}_0$ has property* (P 1), *i.e.* $|\mathscr{F}_{\alpha\beta}| \leqslant \omega$ *for* $\alpha < \beta < \omega_1$.

Proof. Clear. □

4.3 Lemma. *$\mathscr{M}_0$ has property* (P 2), *i.e. if* $\alpha < \beta < \gamma \leqslant \omega_1$, *then*

$$\mathscr{F}_{\alpha\gamma} = \{f \circ g \mid f \in \mathscr{F}_{\beta\gamma} \,\&\, g \in \mathscr{F}_{\alpha\beta}\}.$$

Proof. ($\supseteq$) By definition.

($\subseteq$). By induction on γ. There are many different cases. As an example we deal with the case $\bar{\nu} \dashv_* \nu$, $\bar{\nu}$ a limit point in $S_{\alpha_{\bar{\gamma}}}$, $\pi_{\bar{\nu}\nu}$ is cofinal. Then γ is a limit ordinal. We must use the induction hypothesis to show that (where $\mathrm{dom}(\tilde{\pi}_{\bar{\nu}\nu}) = W_\alpha$):

(*) for cofinally many $\beta < \gamma$, there are $f \in \mathscr{F}_{\beta\gamma}$, $g \in \mathscr{F}_{\alpha\beta}$ such that $\tilde{\pi}_{\bar{\nu}\nu} = f \circ g$.

By 4.1 (v) we have

$$\tilde{\pi}_{\bar{\nu}\nu} = \sigma_{\eta\nu} \circ \tilde{\pi}_{\bar{\eta}\eta} \quad \text{for } \eta \dashv \nu.$$

By morass properties (see the argument used in handling Case 3.3.2 in the proof of the Gap-2 Theorem in section 3) we have:

$$\alpha_\nu = \sup\{\alpha_\eta \mid \eta \dashv \nu\}.$$

Thus (*) follows.

For the other cases, use 4.1 (iii), (iv), (vi), (vii). □

4.4 Lemma. *Let* $\alpha < \omega_1$. *Then* $\mathscr{F}_{\alpha,\alpha+1}$ *is either a singleton or else consists of a pair* $\{\mathrm{id} \restriction \theta_\alpha, f_\alpha\}$ *such that for some* $\delta < \theta_\alpha$, $f_\alpha \restriction \delta = \mathrm{id} \restriction \delta$ *and* $f_\alpha(\delta) \geqslant \theta_\alpha$.

Proof. If $\gamma_\alpha \in A_0 \cup A_1$, then we are done by 4.1 (ii). If $\gamma_\alpha \notin A_0 \cup A_1$, then $\bar{\gamma} = \gamma_{\alpha+1}$ is a successor in S^0, so by morass properties $S_{\bar{\gamma}} = \{\nu\}$ for some ν. By our initial special assumption on the morass tree, we have $\bar{\nu} \dashv_* \nu$ for some $\bar{\nu}$. Thus $\mathscr{F}_{\alpha,\alpha+1} = \{\tilde{\pi}_{\bar{\nu}\nu}\}$, and again we are done. □

4.5 Lemma. *$\mathscr{M}_0$ has property* (P 4).

Proof. Let $\alpha \leqslant \omega_1$ be a limit ordinal. We define a certain subset $\mathscr{G}_\alpha \subseteq \bigcup_{\gamma < \alpha} \mathscr{F}_{\gamma\alpha}$ and leave it to the reader to check that it is always possible to find a $g \in \mathscr{G}_\alpha$ which verifies (P 4). Let $\gamma = \gamma_\alpha$, and if $\alpha < \omega_1$, let $\nu = \nu_\alpha$.

Case 1. $\gamma \in A_1$ or $\alpha = \omega_1$.
Set $\mathscr{G}_\alpha = \{\tilde{\pi}_{\bar{\tau}\tau} \mid \bar{\tau} \dashv \tau, \tau \in S_\gamma, \tau < \sup(S_\gamma)\}$.

Case 2. $\gamma \in A_0$.
Let ν immediately succeed τ in S_{γ_α}. Set $\mathscr{G}_\alpha = \{\tilde{\pi}_{\bar{\tau}\tau} \mid \bar{\tau} \dashv \tau\}$.

Case 3. $\gamma \in (S^0 \cap \omega_1) - A$.
Set $\mathscr{G}_\alpha = \{\tilde{\pi}_{\bar{\nu}\nu} \mid \bar{\nu} \dashv \nu\}$.

Case 4. $\gamma \in A - (A_0 \cup A_1)$.
Set $\mathscr{G}_\alpha = \{\sigma_{\eta\nu} \mid \eta \dashv \nu\}$. □

4.6 Lemma. $\omega_2 = \bigcup \{f''\theta_\alpha \mid \alpha < \omega_1 \;\&\; f \in \mathscr{F}_{\alpha\omega_1}\}$.

Proof. Obvious. □

Our task now is to modify $\mathscr{M}_0$ so that (P 4), (P 5) and (P 0) are satisfied, as well as (P 1), (P 2) and (P 3). The only part of (P 0) that we do not have so far is $\theta_0 = 1$. Lemma 4.4 tells us that we are part way to having (P 3) already. And lemma 4.6 gives us (P 5) for the case $\alpha = \omega_1$.

By 4.6,

$$\omega_2 = \bigcup_{\gamma < \omega_1} \left(\bigcup \{f''\theta_\gamma \mid f \in \mathscr{F}_{\gamma\omega_1}\}\right).$$

So we can find a $\gamma < \omega_1$ such that

$$\left|\bigcup \{f''\theta_\gamma \mid f \in \mathscr{F}_{\gamma\omega_1}\}\right| = \omega_2 .$$

For $\gamma < \alpha \leqslant \omega_1$, let

$$S'_\alpha = \bigcup \{f''\theta_\gamma \mid f \in \mathscr{F}_{\gamma\alpha}\}.$$

Notice that by choice of γ,

$$|S'_{\omega_1}| = \omega_2 .$$

4.7 Lemma. *If $\gamma < \beta < \alpha \leqslant \omega_1$, then*

$$S'_\alpha = \bigcup \{f''S'_\beta \mid f \in \mathscr{F}_{\beta\alpha}\}.$$

Proof.
$$\begin{aligned} S'_\alpha &= \bigcup \{h''\theta_\gamma \mid h \in \mathscr{F}_{\gamma\alpha}\} = \bigcup \{(f \circ g)''\theta_\gamma \mid f \in \mathscr{F}_{\beta\alpha}, g \in \mathscr{F}_{\gamma\beta}\} \\ &= \bigcup \{f''(\bigcup \{g''\theta_\gamma \mid g \in \mathscr{F}_{\gamma\beta}\}) \mid f \in \mathscr{F}_{\beta\alpha}\} = \bigcup \{f''S'_\beta \mid f \in \mathscr{F}_{\beta\alpha}\}. \quad \square \end{aligned}$$

Before we state our next lemma, we note that if $\alpha < \beta \leqslant \omega_1$ and $\tau_1, \tau_2 < \theta_\alpha$, $f_1, f_2 \in \mathscr{F}_{\alpha\beta}$, and if $f_1(\tau_1) = f_2(\tau_2)$, then $\tau_1 = \tau_2$ and $f_1 \upharpoonright \tau_1 = f_2 \upharpoonright \tau_2$. (This is easily proved by induction on β, using 4.4 for the initial step $\beta = \alpha + 1$, 4.3 for the successor step, and 4.5 for the limit step.)

4.8 Lemma. *Let $\gamma < \beta < \alpha \leqslant \omega_1$. The following are equivalent*:

(i) $(\exists f \in \mathscr{F}_{\beta\alpha})(f''S'_\beta = S'_\alpha)$;
(ii) $(\forall f, g \in \mathscr{F}_{\beta\alpha})(f \upharpoonright S'_\beta = g \upharpoonright S'_\beta)$;
(iii) $(\forall f \in \mathscr{F}_{\beta\alpha})(f''S'_\beta = S'_\alpha)$.

Proof. (i) → (ii). Choose $f \in \mathscr{F}_{\beta\alpha}$ such that $f''S'_\beta = S'_\alpha$. Suppose there is a $g \in \mathscr{F}_{\beta\alpha}$ such that $g(\tau) \neq f(\tau)$ for some $\tau \in S_\beta$. By (i), $g''S'_\beta \subseteq S'_\alpha = f''S'_\beta$, so we can choose $\tau' \in S'_\beta$ such that $f(\tau') = g(\tau)$. But clearly, $\tau' \neq \tau$, so this contradicts the observation made above.

(ii) → (iii). By 4.7.

(iii) → (i). Trivial. □

We shall call an ordinal $\alpha \leqslant \omega_1$ *redundant* if there are $\beta < \alpha$, $f \in \mathscr{F}_{\beta\alpha}$ such that $\beta > \gamma$ and $f'' S'_\beta = S'_\alpha$. Let

$$N = \{\alpha \leqslant \omega_1 \mid \alpha > \gamma \ \& \ \alpha \text{ is not redundant}\}.$$

Clearly, $\omega_1 \in N$.

4.9 Lemma. *$N \cap \omega_1$ is a club subset of ω_1.*

Proof. To prove closure, suppose that α is a limit point of $N \cap \omega_1$ and that α is redundant. Choose $\beta < \alpha$ and $f \in \mathscr{F}_{\beta\alpha}$ such that $\beta > \gamma$ and $f'' S'_\beta = S'_\alpha$. Since α is a limit point of $N \cap \omega_1$ we can find a $\delta \in N \cap \omega_1$ such that $\beta < \delta < \alpha$. Choose $g \in \mathscr{F}_{\delta\alpha}$ and $h \in \mathscr{F}_{\beta\delta}$ such that $f = g \circ h$. Since δ is not redundant, $h'' S'_\beta \subset S'_\delta$. But then $f'' S'_\beta = (g \circ h)'' S'_\beta \subset g'' S_\delta \subseteq S_\alpha$, contrary to the choice of β and f.

To prove unboundedness, let $\gamma < \beta < \omega_1$. We find the least element of $N \cap \omega_1$ greater than β. Let α be the least ordinal such that $\beta < \alpha \leqslant \omega_1$ and $(\forall f \in \mathscr{F}_{\beta\alpha})(f'' S'_\beta \neq S'_\alpha)$. Such an ordinal exists, since we clearly have $(\forall f \in \mathscr{F}_{\beta\omega_1}) \cdot (f'' S'_\beta \neq S'_{\omega_1})$. By 4.8 we can choose $f_1, f_2 \in \mathscr{F}_{\beta\alpha}$ such that $f_1 \upharpoonright S'_\beta \neq f_2 \upharpoonright S'_\beta$. If α is a limit ordinal, then by applying 4.5 we can get a counterexample to the minimality of α. Thus $\alpha = \delta + 1 < \omega_1$ for some $\delta \geqslant \beta$. If $\delta > \beta$, then by the minimality of α there is an $f \in \mathscr{F}_{\beta\delta}$ such that $f'' S'_\beta = S'_\delta$. If $g \in \mathscr{F}_{\delta\alpha}$ and $g'' S'_\delta = S'_\alpha$, then $g \circ f \in \mathscr{F}_{\beta\alpha}$ and $(g \circ f)'' S'_\beta = S_\alpha$, contradicting the choice of α. Thus $(\forall g \in \mathscr{F}_{\delta\alpha}) \cdot (g'' S'_\delta \neq S'_\alpha)$. Clearly, the same conclusion holds if $\delta = \beta$. It follows easily that α is not redundant, so $\alpha \in N \cap \omega_1$. Note that we have shown that if $\gamma < \beta < \omega_1$, then the least element of $N - \beta$ is a successor ordinal $\delta + 1$ for some $\delta \geqslant \beta$, and if $\delta > \beta$ then $(\exists f \in \mathscr{F}_{\beta\delta})(f'' S'_\beta = S'_\delta)$. $\square$

Now let $(\eta_\nu \mid \nu \leqslant \omega_1)$ be the monotone enumeration of N, and for $\nu \leqslant \omega_1$, let $\theta'_\nu = \operatorname{otp}(S'_{\eta_\nu})$. Note that for $\nu < \omega_1$, $\eta_\nu < \omega_1$, so $0 < \theta'_\nu < \omega_1$. Also, $\eta_{\omega_1} = \omega_1$ and $|S'_{\omega_1}| = \omega_2$, so $\theta'_{\omega_1} = \omega_2$. We identify S'_{η_ν} with θ'_ν from now on. Subject to this identification, let $\mathscr{F}'_{\nu\tau}$ denote $\mathscr{F}_{\eta_\nu \eta_\tau}$.

4.10 Lemma. *Except for the fact that θ'_0 may not equal* 1, *the structure*

$$\mathscr{M}_1 = \langle (\theta'_\nu \mid \nu \leqslant \omega_1), (\mathscr{F}'_{\nu\tau} \mid \nu < \tau \leqslant \omega_1) \rangle$$

is a simplified morass.

Proof. Most of this is quite straightforward, and is left for the reader to check. To prove (P 4), use the fact that N is closed. For (P 3), note that for any $\nu < \omega_1$, $\eta_{\nu+1}$ is the least element of $N - \eta_\nu$, so by the proof of 4.9, $\eta_{\nu+1} = \delta + 1$ for some $\delta \geqslant \eta_\nu$, $(\forall f \in \mathscr{F}_{\delta\eta_{\nu+1}})(f'' S'_\delta \neq S'_{\eta_{\nu+1}})$, and if $\delta > \eta_\nu$ then $(\exists f \in \mathscr{F}_{\eta_\nu\delta})(f'' S'_{\eta_\nu} = S'_\delta)$, so by 4.8, $(\forall f, g \in \mathscr{F}_{\eta_\nu\delta})(f \upharpoonright S'_{\eta_\nu} = g \upharpoonright S'_{\eta_\nu})$. Clearly, $\mathscr{F}_{\delta\alpha_{\nu+1}} = \mathscr{F}_{\delta,\delta+1}$ is as in (P 3), and it is now not hard to show that $\mathscr{F}'_{\nu,\nu+1}$ is too. (P 5) follows easily from 4.7. $\square$

Finally, if $\theta'_0 > 1$, we simply add an initial segment to the structure obtained above and reindex. All that is required is to build up to the existing θ'_0 by a simplified morass-like structure consisting of θ'_0 levels, starting with $\{0\}$. This is easily achieved. We are done.

5. Gap-n Morasses

In order to prove the Gap-$(n+1)$ Cardinal Transfer Theorem, we need a morass structure which enables us to construct a model of cardinality ω_{n+1} using only countable structures. The morass required to do this is a gap-n morass, or more precisely a (ω_1, n)-morass. Assuming $V = L$, such morasses can be developed, and thus, assuming $V = L$ the Gap-n Theorem is valid for all n. Unfortunately, for $n > 1$, the definition and construction of a gap-n morass is little short of horrendous, and would require for a reasonable treatment a volume comparable to the present one. However, although it is not possible to even give the definition of a gap-n morass here (for $n > 1$), it is possible to indicate why one might expect that such a structure exists, and what it should look like.

The simplest type of system for building models is an elementary chain, which we may regard as a one-dimensional system. Then come gap-1 morasses (together with their associated model complexes), which we may think of as two-dimensional systems. A gap-2 morass would then be a three-dimensional system, and in general a gap-n morass would be an $n+1$-dimensional system. The formal definition of a gap-n morass would then proceed in the "obvious fashion". Just as a gap-1 morass was defined on a set, $\mathscr{S}$, of ordered pairs (α, ν) of ordinals, with $\alpha \leqslant \omega_1$, $\nu < \omega_2$, so a gap-2 morass is defined on a set, $\mathscr{S}$, of ordered triples (α, τ, ν) of ordinals such that $\alpha \leqslant \omega_1$, $\tau \leqslant \omega_2$, $\nu < \omega_3$, and so on. Indeed, the construction of such a structure is, in principal the same as in the gap-1 case, using the fine structure theory. Unfortunately, matters rapidly become very complicated, and so we must end our rather brief account at this point.

Exercises

1. Morasses and the Kurepa Hypothesis

Prove that the existence of a κ^+-morass implies $KH(\kappa^+)$. (Hint. For each $\nu \in S_{\kappa^+}$, let $X_\nu = \{\bar{\nu} \in S^1 \cap \kappa^+ \mid \bar{\nu} \dashv \nu\}$, and show that the family $\mathscr{F} = \{X_\nu \mid \nu \in S_{\kappa^+}\}$ is a κ^+-Kurepa family.)

2. Morasses and the Combinatorial Principle $\square$

Prove that if there is an ω_1-morass, then $\square_{\omega_1}$ is valid. (Hint. For each limit point ν of S_{ω_1}, let $C_\nu = \{\sup(\pi_{\bar{\nu}\nu}{}''\bar{\nu}) \mid \bar{\nu} \dashv \nu\} \cap \nu$, and investigate the properties of the sets C_ν.) Does this generalise to arbitrary successor cardinals κ^+ in place of ω_1?

3. Cardinal Transfer Theorems

The first result to be proved is that, assuming GCH, if $\mathscr{A}$ is a K-structure of type (ω_1, ω), then for any uncountable regular cardinal κ there is a K-structure $\mathscr{B}$ of type (κ^+, κ) such that $\mathscr{B} \equiv \mathscr{A}$. The general idea is to proceed much as in 1.7, using saturated structures of size κ instead of countable homogeneous structures. 1.1 (i) guarantees that all of the structures in the chain are isomorphic. The difficulty lies in ensuring that limit stages preserve saturation. This requires the use of a clever trick. Fix $\mathscr{A}$ as above now.

A K-structure $\mathscr{B}$ is said to be *U-saturated* if it satisfies the definition of saturatedness for all element types $\Sigma(x)$ which contain the formula $U(x)$.

3 A. Show that if $\mathscr{B}$ is a saturated K-structure of cardinality κ such that $\mathscr{B} \equiv \mathscr{A}$, then there is a saturated K-structure $\mathscr{B}'$ of cardinality κ such that $\mathscr{B} \prec \mathscr{B}'$, $\mathscr{B} \neq \mathscr{B}'$, $U^{\mathscr{B}} = U^{\mathscr{B}'}$, $\mathscr{B} \cong \mathscr{B}'$.

3 B. Show that if $\mathscr{B}$ is a U-saturated K-structure of cardinality κ, there is a saturated K-structure $\mathscr{B}'$ of cardinality κ such that $\mathscr{B} \prec \mathscr{B}'$ and $U^{\mathscr{B}} = U^{\mathscr{B}'}$.

There is no less of generality in assuming that the given model $\mathscr{A}$ has a binary predicate E with the property that for each finite set $H \subseteq U^{\mathscr{A}}$, there is an element $a \in U^{\mathscr{A}}$ such that $H = \{x \mid x E^{\mathscr{A}} a\}$. Using this assumption, the following key step of the proof can be established.

3 C. Show that if $\lambda < \kappa^+$ and $(\mathscr{B}_\nu \mid \nu < \lambda)$ is an elementary chain of U-saturated structures elementarily equivalent to $\mathscr{A}$, each of cardinality κ and all having the same distinguished subset U, then $\bigcup_{\nu < \lambda} \mathscr{B}_\nu$ is U-saturated.

3 D. Show that there is a K-structure $\mathscr{B}$ of type (κ^+, κ) such that $\mathscr{B} \equiv \mathscr{A}$.

The second result to be proved is that, assuming $V = L$, if $\mathscr{A}$ is a K-structure of type (ω_1, ω), then for any singular cardinal κ there is a K-structure $\mathscr{B}$ of type (κ^+, κ) such that $\mathscr{B} \equiv \mathscr{A}$. (What we actually require is $\square_\kappa$ together with GCH.)

Fix κ a singular cardinal from now on, and let $\mu = \mathrm{cf}(\kappa)$. Let $G\colon \mu \to \kappa$ be an increasing sequence of regular cardinals such that $G(0) = 0$, $G(1) > \omega$, and $\sup(G''\mu) = \kappa$. By $\square_\kappa$, let $(S_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ be such that:

(i) S_α is a closed subset of α;

(ii) if $\mathrm{cf}(\alpha) > \omega$, then S_α is unbounded in α;

(iii) $|S_\alpha| < \kappa$;

(iv) if $\gamma \in S_\alpha$, then $S_\gamma = \gamma \cap S_\alpha$.

Modifying the previously defined notion of "special" a little, let us now agree to call a K-structure $\mathscr{B}$ of cardinality κ *special* iff there is an elementary chain $(\mathscr{B}_\alpha \mid \alpha < \mu)$ of saturated structures such that $\mathscr{B} = \bigcup_{\alpha<\mu} \mathscr{B}_\alpha$ and $|B_\alpha| = G(\alpha)$ for each $\alpha < \mu$. A mapping $r\colon B \to \mu$ is called a *ranking* of $\mathscr{B}$ iff there is such a chain with

$$r(x) = \text{the least } \alpha \text{ such that } x \in B_{\alpha+1}$$

for all $x \in B$. Similarly we define the notions of *U-special* and *U-ranking* by replacing "saturated" by "U-saturated".

3 E. Show that if $\mathscr{A}$ is any K-structure, there is a special structure $\mathscr{B}$ such that $\mathscr{B} \equiv \mathscr{A}$. (We are assuming GCH throughout.)

3 F. Show that if $\mathscr{A}$, $\mathscr{B}$ are special structures with rankings r, s, respectively, then $\mathscr{A} \cong \mathscr{B}$ and there is an isomorphism $f\colon \mathscr{A} \cong \mathscr{B}$ such that $s(f(x)) = r(x)$ for all $x \in A$.

3 G. Let $\mathscr{A}$ be U-special with U-ranking r, and let $\mathscr{B}$ be special with ranking s, and suppose that $\mathscr{A} \equiv \mathscr{B}$. Show that there is an embedding $f\colon \mathscr{A} \prec \mathscr{B}$ such that $U^{\mathscr{B}} \subseteq \operatorname{ran}(f)$ and $s(f(x)) = r(x)$ for all $x \in A$.

3 H. Let $\mathscr{A}$ be a K-structure of type (ω_1, ω). Let $\mathscr{B}$ be a U-special structure with U-ranking r, $\mathscr{B} \equiv \mathscr{A}$. Show that there is a special structure $\mathscr{B}'$ with ranking r' such that $\mathscr{B} \prec \mathscr{B}'$, $\mathscr{B} \neq \mathscr{B}'$, $U^{\mathscr{B}} = U^{\mathscr{B}'}$, and $r \subseteq r'$.

Given U-special structures $\mathscr{B}, \mathscr{B}'$ with U-ranking r, r', respectively, we write:

$$(\mathscr{B}, r) \lhd (\mathscr{B}', r') \quad \text{iff } \mathscr{B} \prec \mathscr{B}' \ \&\ r \subseteq r' \ \&\ U^{\mathscr{B}} = U^{\mathscr{B}'};$$

$$(\mathscr{B}, r) \propto_\gamma (\mathscr{B}', r') \quad \text{iff } \mathscr{B} \prec \mathscr{B}' \ \&\ r \restriction U^{\mathscr{B}} = r' \restriction U^{\mathscr{B}} \ \&\ U^{\mathscr{B}} = U^{\mathscr{B}'}$$
$$\&\, (r(x) < \gamma \rightarrow r'(x) < \gamma) \ \&\ (r(x) \geqslant \gamma \rightarrow r(x) = r'(x));$$

$$(\mathscr{B}, r) \propto (\mathscr{B}', r') \quad \text{iff } (\exists \gamma)[(\mathscr{B}, r) \propto_\gamma (\mathscr{B}', r')].$$

Fix $\mathscr{A}$ a given K-structure of type (ω_1, ω) now. As before, assume that $\mathscr{A}$ has a binary predicate E which codes the finite subsets of $U^{\mathscr{A}}$ by elements of $U^{\mathscr{A}}$. To obtain a K-structure $\mathscr{B}$ of type (κ^+, κ) such that $\mathscr{B} \equiv \mathscr{A}$, the idea is to construct an elementary chain $(\mathscr{B}_\nu \mid \nu < \kappa^+)$ of U-special structures, all having the same U-set, and a sequence $(r_\nu \mid \nu < \kappa^+)$ such that r_ν is a U-ranking of $\mathscr{B}_\nu$, with $\mathscr{B}_0 \equiv \mathscr{A}$. The construction is carried out to preserve the following conditions:

(A) $\quad \alpha < \beta \rightarrow (\mathscr{B}_\alpha, r_\alpha) \propto (\mathscr{B}_\beta, r_\beta);$

(B) $\quad \alpha \in S_\beta \rightarrow (\mathscr{B}_\alpha, r_\alpha) \lhd (\mathscr{B}_\beta, r_\beta);$

(C) $\quad$ if $\alpha = S_\gamma(G(\beta))$, then $x \in A_\gamma - A_\alpha \rightarrow r_\gamma(x) \geqslant \beta$.

The only difficulty lies in the limit step in the definition. In case S_α is cofinal in α (for a limit stage α), set:

$$\mathscr{B}_\alpha = \bigcup_{\beta \in S_\alpha} \mathscr{B}_\beta, \qquad r_\alpha = \bigcup_{\beta \in S_\alpha} r_\beta.$$

In case S_α is not cofinal in α, in which case $\operatorname{cf}(\alpha) = \omega$, of course, pick a sequence $(\alpha_n \mid n < \omega)$ cofinal in α with $\alpha_0 = \sup(S_\alpha)$, let ψ be least such that $G(\psi) > \operatorname{otp}(S_\alpha)$, and pick a monotone sequence $(\varphi_n \mid n < \omega)$ of ordinals such that $\varphi_0 = 0$, $\varphi_1 > \psi$, and $\varphi_n < \mu$, with

$$(\mathscr{B}_{\alpha_i}, r_{\alpha_i}) \propto_{\varphi_{i+1}} (\mathscr{B}_{\alpha_{i+1}}, r_{\alpha_{i+1}})$$

for all $i < \omega$. Then set

$$\mathscr{B}_\alpha = \bigcup_{i < \omega} \mathscr{B}_{\alpha_i}.$$

For $x \in B_\alpha$, let $i(x)$ be the least i such that $x \in B_{\alpha_i}$, and set

$$r_\alpha(x) = \max(\varphi_{i(x)}, r_{\alpha_i}(x)).$$

3 I. Check that the above definitions can indeed be carried out, and that they define a sequence $((\mathscr{B}_\alpha, r_\alpha) \mid \alpha < \kappa^+)$ as stated, to prove the Gap-1 Theorem for singular κ.

Further details of the above results can be found in *Chang-Keisler* (1973), *Devlin* (1973), and *Jensen* (1972).

Finally we consider the Gap-2 Theorem.

3 J. Assume $V = L$. Show that for any infinite cardinal κ, if $\mathscr{A}$ is a K-structure of type (ω_2, ω) there is a K-structure $\mathscr{B}$ of type (κ^{++}, κ) such that $\mathscr{B} \equiv \mathscr{A}$. (Hint. If κ is regular, the proof is a straightforward modification of the proof given in this chapter, using the ideas from exercises 3 A through 3 D above. If κ is singular, a mixture of the methods used in the chapter and those of exercises 3 E through 3 I is required, but in this case the proof is quite tricky. In particular, the $\square_\kappa$-sequence used must be obtained from the morass. (More precisely, from the fine structure construction of the morass.)

4. Morasses and the Combinatorial Principle $\diamondsuit_{\omega_2}$

4 A. Show that it is possible to define transitive structures M_ν, for each $\nu \in S^1$ in an ω_1-morass, such that:

(i) $\nu \in \mathrm{dom}(M_\nu)$, and $\mathrm{dom}(M_\nu) \cap \mathrm{On}$ is less than any element of S_{α_ν} and any element of S^0 above α_ν. (So, in particular, M_ν is countable for all $\nu \in S^1 \cap \omega_1$.)

(ii) For $\nu \dashv \tau$, there is an embedding $\sigma_{\nu\tau}\colon M_\nu \prec_1 M_\tau$ such that all of the following conditions are satisfied:

(iii) $(\sigma_{\nu\tau} \mid \nu \dashv \tau)$ is a commutative system.

(iv) $\sigma_{\nu\tau} \restriction \nu = \pi_{\nu\tau} \restriction \nu$.

(v) if $\bar\tau \dashv \tau$, $\bar\nu \in S_{\alpha_\tau} \cap \bar\tau$, $\nu = \pi_{\bar\tau\tau}(\bar\nu)$, then $\sigma_{\bar\nu\nu} = \sigma_{\bar\tau\tau} \restriction M_{\bar\nu}$.

(vi) if $\tau \in S^1$ is a limit point of $\dashv$, then $M_\tau = \bigcup_{\nu \dashv \tau} \sigma_{\nu\tau}'' M_\nu$.

(vii) $(\mathscr{P}(\nu) \cap M_\nu \mid \nu \in S_{\omega_1})$ is a $\diamondsuit_{\omega_2}$-sequence.

(viii) $(\mathscr{P}(\nu) \cap \Sigma_1(M_\nu) \mid \nu \in S_{\omega_1})$ is a $\diamondsuit^+_{\omega_2}$-sequence.

(Hint. Consider the structures $\langle J_{\varrho(\nu)}, A(\nu)\rangle$ used to construct the morass.)

4 B. Use 4 A to construct an ω_2-Souslin tree by means of a morass-like system of countable trees and embeddings between them.

5. A Coarse Morass

We investigate what kind of morass structure can be constructed using only elementary properties of L.

Let us call an ordinal ν *special* iff:

(i) either $L_\nu \models \mathrm{ZF}^-$ or else $\{\tau \in \nu \mid L_\tau \models \mathrm{ZF}^-\}$ is unbounded in ν;

(ii) $L_\nu =$ "there is exactly one uncountable cardinal".

For example, ω_2 is special. We define

$$S^1 = \{\nu \in \omega_2 \mid \nu \text{ is special}\};$$
$$\alpha_\nu = \omega_1^{L_\nu}, \quad \text{for } \nu \in S^1;$$
$$S^0 = \{\alpha_\nu \mid \nu \in S^1\};$$
$$S_\alpha = \{\nu \in S^1 \mid \alpha_\nu = \alpha\}, \quad \text{for } \alpha \in S^0.$$

5 A. Prove that $S_{\alpha_\nu} \cap \nu$ is uniformly definable in L_ν for $\nu \in S^1$.

5 B. Prove the following. Let $\nu \in S_\alpha$. Then there is an admissible ordinal $\beta = \beta(\nu) > \nu$ such that for some element p of L_β, every member of L_β is definable from parameters in $\alpha \cup \{p\}$ in L_β. Moreover, if τ succeeds ν in S_α, then there is such a $\beta < \tau$.

For each $\nu \in S^1$, let $\beta(\nu)$ be the least ordinal as above, and let $p(\nu)$ be the $<_L$-least such parameter.

5 C. Prove that ν is uniformly definable in $L_{\beta(\nu)}$.

5 D. Let $\nu \in S_\alpha$, and set $\beta = \beta(\nu)$, $p = p(\nu)$. Let $\bar{\alpha} < \alpha$, and suppose that X is the smallest $X \prec L_\beta$ such that $X \cap \alpha = \bar{\alpha}$ and $p \in X$. Let $\pi^{-1}: X \cong L_{\bar{\beta}}$, and set $\bar{\nu} = \pi^{-1}(\nu)$, $\bar{p} = \pi^{-1}(p)$. Show that $\bar{\alpha} \in A$, $\bar{\nu} \in S_{\bar{\alpha}}$, $\bar{\beta} = \beta(\bar{\nu})$, $\bar{p} = p(\bar{\nu})$.

For $\nu, \tau \in S^1$, define $\nu \dashv \tau$ iff $\alpha_\nu < \alpha_\tau$ and there is an embedding $\sigma: L_{\beta(\nu)} \prec L_{\beta(\tau)}$ such that $\sigma \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu$ and $\sigma(p(\nu)) = p(\tau)$.

5 E. Prove that if $\nu \dashv \tau$, the map σ above is unique.

Denote the unique map σ in the above by $\sigma_{\nu\tau}$. Note that by 5 C. above, $\sigma_{\nu\tau}(\nu) = \tau$. Hence $(\sigma_{\nu\tau} \restriction L_\nu): L_\nu \prec L_\tau$. Note also that $\sigma_{\nu\tau} \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu$, and $\sigma_{\nu\tau}(\alpha_\nu) = \alpha_\tau$. Let $\pi_{\nu\tau} = \sigma_{\nu\tau} \restriction (\nu + 1)$.

5 *F*. Prove that $\dashv$ is a tree ordering on S^1.

5 G. Verify that the system just constructed satisfies morass axioms (M 0) through (M 5), and investigate what happens when you try to prove (M 6) and (M 7).

The structure defined above is sometimes referred to as a *coarse morass*.

6. *Morasses and Large Cardinal Axioms*

Prove that if $V = L[A]$, where $A \subseteq \omega_1$, then there is a morass. Deduce that if ω_2 is not inaccessible in L, then there is a morass in the real world.