Chapter V
The Story of $0^{\#}$

In this chapter we investigate the effect upon $V = L$ of the postulated existence of various large cardinals in the universe. This represents a different approach to constructibility from that adopted hitherto. Previously we have been looking at the *internal* structure of the constructible universe. We now step back and regard *L from the outside* as it were.

It is assumed that the reader has a prior acquaintance with large cardinal theory. Admittedly, our account is self-contained (except for the omission of some proofs); but the results we shall obtain cannot really be appreciated without some familiarity with the standard theory of the cardinal properties concerned. The relevant material can be found in *Drake* (1974) and *Jech* (1978).

We shall make considerable use of model-theoretic techniques, usually for models of the languages $\mathscr{L}(A_1, \ldots, A_n)$. It will be convenient to use some of the standard notation of model theory. In particular, we shall write the satisfaction relation as

$$\langle M, \in, A_1, \ldots, A_n \rangle \models \varphi$$

rather than

$$\models_{\langle M, \in, A_1, \ldots, A_n \rangle} \varphi .$$

We shall also not bother to distinguish between an element, x, of a structure and the constant, $\mathring{x}$, of $\mathscr{L}_V$ which denotes it. If $t(\mathring{x}_0, \ldots, \mathring{x}_m)$ is a term of $\mathscr{L}_M(A_1, \ldots, A_n)$ (so $x_0, \ldots, x_m \in M$), we write $t^{\mathscr{A}}(x_0, \ldots, x_m)$ for the interpretation of the term $t(\mathring{x}_0, \ldots, \mathring{x}_m)$ in the structure $\mathscr{A} = \langle M, \ , A_1, \ldots, A_n \rangle$.

We shall also speak of models of ZFC, BS, etc. In each such case we mean these theories formulated in the language $\mathscr{L}$, and not in LST as was originally the case.

1. A Brief Review of Large Cardinals

A cardinal κ is said to be *weakly inaccessible* iff it is an uncountable, regular limit cardinal, and (*strongly*) *inaccessible* iff it is uncountable and regular and has the property that $(\forall \lambda < \kappa)(2^\lambda < \kappa)$. It is clear that all inaccessible cardinals are weakly inaccessible, and that if the GCH be assumed then the two notions of inaccessibility coincide.

If κ is inaccessible, then V_κ (and L_κ) is a model of ZFC. Hence by Gödel's Second Incompleteness Theorem, the existence of inaccessible cardinals is not provable in ZFC.

1.1 Theorem.
(i) *If κ is a cardinal, then $[\kappa$ is a cardinal$]^L$.*
(ii) *If κ is a limit cardinal, then $[\kappa$ is a limit cardinal$]^L$.*
(iii) *If κ is a regular cardinal, then $[\kappa$ is a regular cardinal$]^L$.*
(iv) *If κ is a weakly inaccessible cardinal, then $[\kappa$ is an inaccessible cardinal$]^L$.*

Proof. (i)–(iii). Each of the properties is easily seen to be Π_1, and hence D-absolute.
(iv) By (i)–(iii) and the fact that $[\mathrm{GCH}]^L$. □

A cardinal κ is said to be *Mahlo* iff it is inaccessible and the set

$$\{\lambda \in \kappa \mid \lambda \text{ is inaccessible}\}$$

is stationary in κ.

1.2 Theorem. *If κ is Mahlo, then $[\kappa$ is Mahlo$]^L$.*

Proof. Again, this property is easily seen to be D-absolute. □

A cardinal κ is said to be *weakly compact* iff it is uncountable and satisfies the partition property

$$\kappa \to (\kappa)^2_2.$$

What does this mean? In order to explain we need some notation. If X is a set of ordinals and α is an ordinal, $[X]^\alpha$ denotes the set of all strictly increasing α-sequences of members of X. We set

$$[X]^{<\alpha} = \bigcup_{\beta<\alpha} [X]^\beta$$
$$[X]^{\leqslant\alpha} = \bigcup_{\beta\leqslant\alpha} [X]^\beta.$$

Let X be a set of ordinals, α an ordinal, μ a cardinal. By a *μ-partition* of $[X]^\alpha$ we mean a function

$$f\colon [X]^\alpha \to \mu,$$

which we regard as partitioning $[X]^\alpha$ into μ disjoint classes. A subset Y of X is said to be *homogeneous* for the partition f iff

$$|f''[Y]^\alpha| = 1,$$

i.e. iff all strictly increasing α-sequences of members of Y lie in the same partition class. We write

$$\kappa \to (\lambda)^\alpha_\mu$$

iff every μ-partition of $[\kappa]^\alpha$ has a homogeneous set of cardinality λ. This notation is due to Erdös and Rado. The idea behind it is the easily observed fact that a valid partition relation remains valid if the parameter on the left of the arrow is increased or if any parameter on the right of the arrow is decreased.

The well known *Ramsey's Theorem* states that

$$\omega \to (\omega)^m_n$$

for all $m, n \in \omega$. Generalising this to some extent is the *Erdös-Rado Theorem* that for any cardinal κ and any $n \in \omega$,

$$\mathscr{I}_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa,$$

where $\mathscr{I}_n(\kappa)$ is the n-th iterate of the exponential function 2^λ, starting from κ (i.e. $\mathscr{I}_0(\kappa) = \kappa$, $\mathscr{I}_1(\kappa) = 2^\kappa$, $\mathscr{I}_2(\kappa) = 2^{\mathscr{I}_1(\kappa)}$, etc.).

A weakly compact cardinal, then, is one for which the generalised Ramsey's Theorem

$$\kappa \to (\kappa)^2_2$$

holds. All weakly compact cardinals are Mahlo. The name "weakly compact" stems from the equivalent definition that a weakly compact cardinal is a cardinal κ for which the "κ-compactness property" is valid for any "κ-language". By a *κ-language* we mean a first-order language having κ many basic symbols, whose syntax allows conjunctions and disjunctions of any length less than κ and quantification over any sequence of variables of length less than κ. (In this context, an ordinary first-order language would be called an "ω-language".) The *κ-compactness property* for such a language says that if a set of at most κ sentences of the language is *κ-satisfiable* (i.e. any subset of cardinality less than κ has a model), then the entire set has a model. The whole idea is to generalise to an uncountable cardinal κ, everything connected with the compactness theorem of ordinary logic.

The following theorem lists several standard, equivalent formulations of the notion of weak compactness. Proofs of the various equivalences may be found in *Drake* (1974) or *Jech* (1978).

1.3 Theorem. *Let κ be an uncountable cardinal. The following are equivalent*:

(i) *κ is weakly compact (i.e. $\kappa \to (\kappa)^2_2$)*;
(ii) $(\forall n \in \omega)(\forall \lambda < \kappa)[\kappa \to (\kappa)^n_\lambda]$;
(iii) *the κ-compactness property holds for any κ-language*;
(iv) *κ is Π^1_1-indescribable: i.e. if φ is a sentence of $\mathscr{L}(U, A_1, \ldots, A_n)$ such that*

$$(\forall U \subseteq V_\kappa)[\langle V_\kappa, \in, U, A_1, \ldots, A_n\rangle \models \varphi]$$

for $A_1, \ldots, A_n \subseteq V_\kappa$, then for some $\alpha < \kappa$,

$$(\forall U \subseteq V_\alpha)[\langle V_\alpha, \in, U, A_1 \cap V_\alpha, \ldots, A_n \cap V_\alpha\rangle \models \varphi];$$

(v) *κ has Keisler's Extension Property: every structure of the form $\langle V_\kappa, \varepsilon, U\rangle$ has a transitive elementary extension which contains κ*;

(vi) *If $\mathcal{F}$ is a κ-complete field of sets which is κ-generated by a set of cardinality at most κ, then $\mathcal{F}$ has a κ-complete ultrafilter;*

(vii) *κ is inaccessible and if $\mathcal{F}$ is a κ-complete field of sets of cardinality κ, then $\mathcal{F}$ has a κ-complete ultrafilter;*

(viii) *κ is inaccessible and there is no κ-Aronszajn tree.* □

The following lemma is relevant to our present purposes.

1.4 Lemma. *Let κ be a weakly compact cardinal, and let $A \subseteq \kappa$. If $A \cap \alpha \in L$ for all $\alpha < \kappa$, then $A \in L$.*

Proof. By assumption,

$$\langle V_\kappa, \in, A\rangle \models (\forall \alpha)(A \cap \alpha \in L).$$

By Keisler's Extension Property, let M be a transitive set such that $V_\kappa \cup \{\kappa\} \subseteq M$ and for some set $A' \subseteq M$,

$$\langle V_\kappa, \in, A\rangle \prec \langle M, \in, A'\rangle.$$

We have

$$\langle M, \in, A'\rangle \models (\forall \alpha)(A' \cap \alpha \in L).$$

In particular, since $\kappa \in M$,

$$\langle M, \in, A'\rangle \models A' \cap \kappa \in L.$$

But $A' \cap \kappa = A$. So, noting that set-membership is absolute for M,

$$A \in (L)^M.$$

Now, as κ is inaccessible, V_κ is a model of ZFC. Thus M is a model of ZFC. So by II.2.10, $(L)^M \subseteq L$. Thus $A \in L$, and we are done. □

Utilising 1.4 we have (see also Exercise 1.):

1.5 Theorem. *If κ is weakly compact, then $[\kappa$ is weakly compact$]^L$.*

Proof. By 1.1 we know that $[\kappa$ is inaccessible$]^L$. So by 1.3 it suffices to prove that [there are no κ-Aronszajn trees$]^L$.

Let $T \in L$ be, in L, a κ-tree. We may assume that T has domain κ and that $\alpha <_T \beta$ implies $\alpha < \beta$. It is clear that T is a κ-tree in the real world. Hence as κ is weakly compact, there is (in V) a κ-branch, b, of T. For any $\alpha < \kappa$, let γ be the least ordinal in $b - \alpha$. Then

$$b \cap \alpha = \{\xi \in T \mid \xi <_T \gamma\} \in L.$$

So by 1.4, $b \in L$. But clearly,

$$[b \text{ is a } \kappa\text{-branch of } T]^L.$$

Thus T is not a κ-Aronszajn tree in the sense of L. The proof is complete. □

We shall return to the notion of weak compactness, and a strengthening of it (ineffability) in Chapter VII. In the meantime we consider a much more powerful large cardinal notion.

We write

$$\kappa \to (\lambda)_{\mu}^{<\omega}$$

if, whenever f is a μ-partition of $[\kappa]^{<\omega}$ (i.e. f: $[\kappa]^{<\omega} \to \mu$) there is a set $X \subseteq \kappa$ of cardinality λ such that

$$(\forall n \in \omega)[|f''[X]^n| = 1].$$

Thus, X is simultaneously homogeneous for each of the partitions $f \upharpoonright [X]^n$, $n \in \omega$. We cannot expect X to be "homogeneous" for all of f in the sense that $|f''[X]^{<\omega}| = 1$, since the value of f could depend upon the length of the argument. There is thus no real danger of confusion if we agree to say that a set X for which $(\forall n \in \omega)[|f''[X]^n| = 1]$ is *homogeneous* for f.

Already the existence of a cardinal κ such that

$$\kappa \to (\omega)_2^{<\omega}$$

is a powerful assumption, implying the existence of (many) weakly compact cardinals.

For any cardinal λ, if there is a cardinal κ such that

$$\kappa \to (\lambda)_2^{<\omega},$$

then the least such κ is denoted by $\kappa(\lambda)$. The cardinals $\kappa(\lambda)$ are called the *Erdös cardinals.* They are all inaccessible, and $\kappa(\omega)$ excceeds the first weakly compact cardinal (and the first ineffable cardinal). If $\lambda < \mu$, then $\kappa(\lambda) < \kappa(\mu)$. (See *Drake* (1974) or *Jech* (1978) for all details.)

A cardinal κ such that $\kappa(\kappa) = \kappa$ is called a *Ramsey cardinal.* Thus κ is a Ramsey cardinal iff

$$\kappa \to (\kappa)_1^{<\omega}.$$

Since this is not a simple generalisation of Ramsey's Theorem (the obvious generalisation being provided by the weakly compact cardinals), the name "Ramsey cardinal" is slightly misleading, but is now well established.

The Erdös cardinals have powerful model-theoretic properties, as we show next. We consider structures of the form

$$\mathscr{A} = \langle A, <_A, \dots \rangle,$$

where $<_A$ linearly orders some subset of A, called the *field* of $<_A$. By the *length* of $\mathscr{A}$ we mean the cardinality of the set of all functions, relations and constants of $\mathscr{A}$. If the length of $\mathscr{A}$ is infinite, then this is just the cardinality of the language of $\mathscr{A}$.

An infinite subset, H, of the field of $<_A$ is said to be *$\mathscr{A}$-indiscernible* iff for each $n \in \omega$ and each pair $(a_0, \dots, a_n)$, $(b_0, \dots, b_n) \in [H]^{n+1}$ it is the case that for all formulas $\varphi(v_0, \dots, v_n)$ in the language of $\mathscr{A}$:

$$\mathscr{A} \models \varphi(a_0, \dots, a_n) \quad \text{iff } \mathscr{A} \models \varphi(b_0, \dots, b_n).$$

In other words, as far as first-order properties are concerned, for each $n \in \omega$, all increasing n-tuples from H look the same to $\mathscr{A}$.

1.6 Theorem. *Let λ be an infinite cardinal. The following conditions on κ are equivalent*:

(i) $\kappa \to (\lambda)_2^{<\omega}$;

(ii) $\kappa \to (\lambda)_{2^\omega}^{<\omega}$;

(iii) *for all* $\mu < \kappa(\lambda)$, $\kappa \to (\lambda)_\mu^{<\omega}$;

(iv) *every structure of the form*

$$\mathscr{A} = \langle A, <_A, \dots \rangle,$$

of countable length, such that $\kappa \subseteq \text{field}(<_A)$ *and* $<_A \upharpoonright \kappa$ *is the usual order on* κ, has an *$\mathscr{A}$-indiscernible subset of cardinality* λ;

(v) *as in* (iv) *except that* $\mathscr{A}$ *may have any length less than* $\kappa(\lambda)$.

Proof. The proofs of the equivalence of (i), (ii) and (iii) can be found in *Drake* (1974) and *Jech* (1978), but, since they are not really relevant to us here we shall not give them. We prove the equivalence of (iii) and (v), this being the result that we require. (A similar argument yields the equivalence of (iv) and (ii), as is easily seen.)

Assume (iii). Let $\mathscr{A}$ be a stated in (v). Define a function f on $[\kappa]^{<\omega}$ by letting $f(a_0, \dots, a_n)$ be the set of all formulas $\varphi(v_0, \dots, v_n)$ in the language of $\mathscr{A}$ such that

$$\mathscr{A} \models \varphi(a_0, \dots, a_n).$$

Since *length* $(\mathscr{A}) < \kappa(\lambda)$ and $\kappa(\lambda)$ is inaccessible, the range of f has cardinality less than $\kappa(\lambda)$. So, by (iii), f has a homogeneous set, H, of cardinality λ. Clearly, H is $\mathscr{A}$-indiscernible.

Now assume (v). Given a partition

$$f\colon [\kappa]^{<\omega} \to \mu < \kappa(\lambda),$$

consider the structure

$$\mathscr{A} = \langle \kappa, <, (f \upharpoonright [\kappa]^n)_{n<\omega}, (\xi)_{\xi<\mu} \rangle.$$

The length of $\mathscr{A}$ is less than $\kappa(\lambda)$, so by (v), $\mathscr{A}$ has an $\mathscr{A}$-indiscernible subset, H, of cardinality λ. Clearly, H is homogeneous for f. □

A *measurable* cardinal is an uncountable cardinal κ such that there is a function

$$\mu\colon \mathscr{P}(\kappa) \to 2$$

with the properties:

(i) $\mu(\{\alpha\}) = 0$ for all $\alpha \in \kappa$;
(ii) $\mu(\kappa) = 1$;
(iii) if X_α, $\alpha < \lambda$, are disjoint subsets of κ, where $\lambda < \kappa$, then

$$\mu(\bigcup_{\alpha<\lambda} X_\alpha) = \sum_{\alpha<\lambda} \mu(X_\alpha).$$

(Such a function μ is called a *two-valued measure* on κ.) Measurable cardinals are extremely large. In particular, if κ is measurable, then κ is Ramsey, and indeed is the κ-th Ramsey cardinal. However, as far as L is concerned, cardinals well below the first measurable cardinal (if it exists) already have a highly significant effect. The critical point is the jump from $\kappa(\omega)$ to $\kappa(\omega_1)$, as we show next.

1.7 Theorem. *If* $\kappa \to (\omega)_2^{<\omega}$, *then* $[\kappa \to (\omega)_2^{<\omega}]^L$.

Proof. Let $f \in L$ be such that, in the sense of L, $f\colon [\kappa]^{<\omega} \to 2$. Then, clearly, f is such a function in the real world. Define

$$H = \{\sigma \in [\kappa]^{<\omega} \mid (\forall n \leqslant |\sigma|)[|f \upharpoonright [\sigma]^n| = 1]\}.$$

Notice that the definition of H is absolute for L. We regard H as a poset under the ordering $\supseteq$. Since $\kappa \to (\omega)_2^{<\omega}$, f has an infinite homogeneous set X. Let σ_n consist of the first n elements of X, for each $n \in \omega$. Then $\sigma_n \in H$, and $(\sigma_\nu \mid n < \omega)$ is a $\supseteq$-decreasing chain in the poset H. Thus H is not well-founded. So by I.8.7 and I.8.3,

$$[H \text{ is not well-founded}]^L.$$

So let $(\tau_n \mid n < \omega)$ be a $\supseteq$-decreasing chain from H in L. Then $Y = \bigcup_{n<\omega} \tau_n \in L$ is an infinite homogeneous set for f. This proves the theorem. □

1.8 Theorem. *If there exists a* κ *such that* $\kappa \to (\omega_1)_2^{<\omega}$, *then* $\mathscr{P}^L(\omega)$ *is countable* (*so in particular*, $V \neq L$).

Proof. Let $\kappa = \kappa(\omega_1)$, and consider the structure

$$\mathscr{A} = \langle L_\kappa, \in, \mathscr{P}^L(\omega)\rangle.$$

Let $X \subseteq \kappa$ be an uncountable, $\mathscr{A}$-indiscernible set, and let

$$\mathscr{B} = \langle B, \in, \mathscr{P}^L(\omega) \cap B\rangle$$

be the smallest $\mathscr{B} \prec \mathscr{A}$ such that $X \subseteq B$ (see II.5.3). Then every element of B is of the form $t^{\mathscr{A}}(\vec{x})$ for some term t of set theory and some $(\vec{x}) \in [X]^{<\omega}$. Suppose that $t^{\mathscr{A}}(\vec{x}) \subseteq \omega$. Now, each $n \in \omega$ is definable in $\mathscr{A}$, and the validity of the sentence

$$n \in t(\vec{x})$$

in $\mathscr{A}$ is independent of the exact choice of $(\vec{x})$ from $[X]^{<\omega}$. Hence $t^{\mathscr{A}}(\vec{x})$ does not depend upon $\vec{x}$. But there are only countably many terms t. Thus $\mathscr{P}^L(\omega) \cap B$ must be countable.

Now let

$$\pi\colon \mathscr{B} \cong \mathscr{C} = \langle L_\lambda, \in, U \rangle .$$

Since $\lambda \geqslant \omega_1$, we have $\mathscr{P}^L(\omega) \subseteq L_\lambda$. But

$$\mathscr{A} \models (\forall x \subseteq \omega)(x \in \mathscr{P}^L(\omega)),$$

so

$$\mathscr{C} \models (\forall x \subseteq \omega)(x \in U).$$

Hence $\mathscr{P}^L(\omega) \subseteq U$. But $|U| = |\mathscr{P}^L(\omega) \cap B| = \omega$, so we are done. □

2. *L-Indiscernibles and* $0^{\#}$

In this section we shall obtain a considerable strengthening of 1.8, by proving that if $\kappa(\omega_1)$ exists, then the class of all uncountable cardinals is L-indiscernible. (In particular, this will imply that every uncountable cardinal is inaccessible in the sense of L, giving the conclusion of 1.8 at once.) The existence of $\kappa(\omega_1)$ will also be shown to imply the existence of a truth definition for L, so we may consider the set of all formulas $\varphi(v_0, \ldots, v_n)$ of $\mathscr{L}$ such that $\models_L \varphi(\kappa_0, \ldots, \kappa_n)$ for any strictly increasing sequence $\kappa_0, \ldots, \kappa_n$ of uncountable cardinals. Denoting the set of all Gödel numbers of formulas in this set by the symbol $0^{\#}$, we shall go on to show that the set $0^{\#}$ has an alternative definition, which does not depend upon the existence of L-indiscernibles and a truth definition for L, and that the mere existence of a set of integers satisfying this definition is itself sufficient to ensure that the uncountable cardinals are L-indiscernible.

The techniques which we shall employ are essentially model-theoretic, and originate with some work of Ehrenfeucht and Mostowski concerning models with indiscernibles.

By examining the proof of II.2.9, we see that there is an extension of the $\mathscr{L}$-theory BS, let us call it BSL, which consists of BS together with finitely many instances of the Σ_0-Collection Schema of KP, such that:

(i) $L_\lambda \models \text{BSL}$ for any limit ordinal $\lambda > \omega$; and
(ii) if M is a transitive model of BSL, then for any $\alpha \in M$, $(L_\alpha)^M = L_\alpha$.

(This relates to the proof of II.2.12. We simply require enough instances of Σ_0-Collection to enable us to define the constructible hierarchy.)

Suppose $\mathscr{A} = \langle A, E \rangle$ is a model of the $\mathscr{L}$-theory BSL + $(V = L)$. If $X \subseteq A$, we denote by $\mathscr{A} \upharpoonright X$ the set

$$\{t^{\mathscr{A}}(x_0, \ldots, x_n) \mid t \text{ is a term of } \mathscr{L} \text{ and } x_0, \ldots, x_n \in X\}.$$

It follows from the fact that L has a definable well-ordering that $\mathscr{A} \upharpoonright X \prec \mathscr{A}$. (See II.5.3 in this connection.) We say that $\mathscr{A} \upharpoonright X$ is the elementary substructure of $\mathscr{A}$ *generated by* X.

We shall call a set, Σ, of formulas of $\mathscr{B}$ an *Ehrenfeucht-Mostowski set* (or $E-M$ *set* for short) iff there is a model $\mathscr{A} = \langle A, E\rangle$ of BSL + ($V = L$) and an infinite set $H \subseteq \mathrm{On}^{\mathscr{A}}$ which is $\mathscr{A}$-indiscernible, such that Σ is the set of all $\mathscr{L}$-formulas which are valid in $\mathscr{A}$ on increasing sequences of indiscernibles from H.

Let Σ be an $E-M$ set, and let α be an infinite ordinal. By a (Σ, α)-*model* we mean a pair $(\mathscr{A}, H)$ such that:

(i) $\mathscr{A} = \langle A, E\rangle$ is a model of BSL + ($V = L$);
(ii) $H \subseteq \mathrm{On}^{\mathscr{A}}$ is an $\mathscr{A}$-indiscernible set of order-type α (under $\prec^{\mathscr{A}}_{\mathrm{On}}$);
(iii) $\mathscr{A} = \mathscr{A} \upharpoonright H$ (i.e. H generates $\mathscr{A}$);
(iv) Σ is the set of all $\mathscr{L}$-formulas which are valid in $\mathscr{A}$ on increasing tuples from H.

2.1 Lemma. *Let Σ be an $E-M$ set, and let α, β be infinite ordinals, $\alpha \leqslant \beta$. Let $(\mathscr{A}_\alpha, H_\alpha)$ be a (Σ, α)-model, $(\mathscr{A}_\beta, H_\beta)$ a (Σ, β)-model, and let h: $H_\alpha \to H_\beta$ be order-preserving. Then there is an embedding $\tilde{h}$: $\mathscr{A}_\alpha \prec \mathscr{A}_\beta$ such that $h \subseteq \tilde{h}$. Moreover, if $\beta = \alpha$ and h is onto H_β, then $\tilde{h}$ is an isomorphism of $\mathscr{A}_\alpha$ onto $\mathscr{A}_\beta$; so in particular, the (Σ, α)-model is unique up to isomorphism.*

Proof. Since H_α generates $\mathscr{A}_\alpha$, for any $a \in \mathscr{A}_\alpha$ there is a term t and elements $\vec{x}$ of H_α such that $a = t^{\mathscr{A}_\alpha}(\vec{x})$. Set $\tilde{h}(a) = t^{\mathscr{A}_\beta}(\overrightarrow{hx})$.

We must first of all check that $\tilde{h}$ is well-defined. Suppose that there are terms t_1, t_2 and elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ of H_α such that

$$a = t_1^{\mathscr{A}_\alpha}(x_1, \ldots, x_n) = t_2^{\mathscr{A}_\alpha}(y_1, \ldots, y_m).$$

Let $z_1, \ldots, z_k$ enumerate the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ in increasing order, and let $\varphi(z_1, \ldots, z_k)$ be the formula

$$t_1(x_1, \ldots, x_n) = t_2(y_1, \ldots, y_m).$$

Then $\varphi(z_1, \ldots, z_k) \in \Sigma$, since φ is true in $\mathscr{A}_\alpha$ on the increasing sequence $z_1, \ldots, z_k$ from H_α. Hence φ is true in $\mathscr{A}_\beta$ on any increasing sequence from H_β. But $h(z_1), \ldots, h(z_k)$ is an increasing sequence from H_β. Thus

$$\mathscr{A}_\beta \models \varphi(h(z_1), \ldots, h(z_k)).$$

In other words,

$$t_1^{\mathscr{A}_\beta}(h(x_1), \ldots, h(x_n)) = t_2^{\mathscr{A}_\beta}(h(y_1), \ldots, h(y_m)),$$

so h is well-defined.

Similarly, we can show that $\tilde{h}$ is one-one and preserves the $\in$-relations of the two models. To show that $\tilde{h}$ is elementary, it suffices to show that $\tilde{h}$ preserves the validity of formulas on tuples from H_α only (since H_α generates $\mathscr{A}_\alpha$), which again can be done by passing through Σ as above. The rest of the lemma follows easily now. □

2.2 Lemma. *Let Σ be an $E-M$ set. For each infinite ordinal α there is a unique (up to isomorphism) (Σ, α)-model.*

Proof. Uniqueness was established in 2.1, so we need only concentrate on existence. We introduce new individual constants c_ν, $\nu < \alpha$, to the language $\mathscr{L}$. Let $\mathscr{A}$ be a model of BSL + $(V = L)$, and let $H \subseteq \mathrm{On}^{\mathscr{A}}$ be an $\mathscr{A}$-indiscernible set such that Σ is the set of all $\mathscr{L}$-formulas true in $\mathscr{A}$ on increasing sequences from H. (Such $\mathscr{A}, H$ exist because Σ is an $E-M$ set.) Consider the following theory in the language $\mathscr{L} \cup \{c_\nu \mid \nu < \alpha\}$:

$$T = \{\varphi \mid \varphi \text{ is a sentence of } \mathscr{L} \text{ and } \mathscr{A} \models \varphi\} \cup \{\mathrm{On}(c_\nu) \mid \nu < \alpha\}$$
$$\cup \{c_\nu < c_\tau \mid \nu < \tau < \alpha\} \cup \{\varphi(c_{\nu_0}, \dots, c_{\nu n}) \mid \varphi \in \Sigma \;\&\; \nu_0 < \dots < \nu_n < \alpha\}.$$

It is clear that T is finitely satisfiable in $\mathscr{A}$. So by the compactness theorem, T has a model, say $\mathscr{B}$. Let

$$K = \{c_\nu^{\mathscr{B}} \mid \nu < \alpha\}.$$

Clearly, $\mathscr{B}$ is a model of BSL + $(V = L)$, $K \subseteq \mathrm{On}^{\mathscr{B}}$ is a $\mathscr{B}$-indiscernible set of order-type α, and Σ is the set of formulas of $\mathscr{L}$ which are valid in $\mathscr{B}$ on increasing tuples from K. Thus $(\mathscr{B} \restriction K, K)$ is a (Σ, α)-model. □

So far we have said nothing regarding the existence of an $E-M$ set. In fact the results of this section will depend not just upon the existence of an $E-M$ set, but of an $E-M$ set with some very special properties. We shall describe these properties and their implications for the (Σ, α)-models next, before turning out attention to the construction of an $E-M$ set of the type desired (which will require the existence of large cardinals).

An $E-M$ set Σ is said to be *cofinal* if it contains all formulas of the form:

$$\mathrm{On}(t(v_0, \dots, v_{n-1})) \to t(v_0, \dots, v_{n-1}) < v_n$$

for any $\mathscr{L}$-term t.

2.3 Lemma. *Let Σ be an $E-M$ set. The following are equivalent:*

(i) *Σ is cofinal;*
(ii) *for every limit ordinal α, if $(\mathscr{A}, H)$ is the (Σ, α)-model, then H is cofinal in $\mathrm{On}^{\mathscr{A}}$;*
(iii) *for some limit ordinal α, if $(\mathscr{A}, H)$ is the (Σ, α)-model, then H is cofinal in $\mathrm{On}^{\mathscr{A}}$.*

Proof. (i) → (ii). If $(\mathscr{A}, H)$ is the (Σ, α)-model and $x \in \mathrm{On}^{\mathscr{A}}$, then there is a term t and elements $\vec{h}$ of H such that $x = t^{\mathscr{A}}(\vec{h})$. But then if $k \in H$, $k > \vec{h}$, we have $x < k$ by the requirements on Σ.

(ii) → (iii). Trivial.

(iii) → (i). Let t be any term, and let $\varphi(v_0, \dots, v_n)$ be the formula

$$\mathrm{On}(t(v_0, \dots, v_{n-1})) \to t(v_0, \dots, v_{n-1}) < v_n.$$

We must show that $\varphi \in \Sigma$. It suffices to show that for some increasing sequence $h_0, \ldots, h_n$ from H,

$$\mathscr{A} \models \varphi(h_0, \ldots, h_n).$$

Choose $h_0, \ldots, h_{n-1}$ arbitrary increasing from H. If $t^{\mathscr{A}}(h_0, \ldots, h_{n-1}) \notin \mathrm{On}^{\mathscr{A}}$, we are done already. Otherwise, by our assumption on H we can find $h_n \in H$, $h_n > h_{n-1}$, such that $h_n > t^{\mathscr{A}}(h_0, \ldots, h_{n-1})$, and again we are done. □

An $E-M$ set Σ is said to be *remarkable* if, for every term t of $\mathscr{L}$, if the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) < v_n$$

is in Σ, then so too is the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) = t(v_0, \ldots, v_{n-1}, v_{n+m+1}, \ldots, v_{n+2m+1}).$$

2.4 Lemma. *Let Σ be a remarkable, cofinal, $E-M$ set. Let λ be a limit ordinal, and let $(\mathscr{A}, H)$ be the (Σ, λ)-model. Let $(h_\gamma \mid \gamma < \lambda)$ be the monotone enumeration of H. Let $\alpha < \lambda$ be a limit ordinal, and set $K = \{h_\gamma \mid \gamma < \alpha\}$. Let $\mathscr{B} = \mathscr{A} \upharpoonright K$. Then $(\mathscr{B}, K)$ is the (Σ, α)-model and*

$$\mathrm{On}^{\mathscr{B}} = \{x \in \mathrm{On}^{\mathscr{A}} \mid x < h_\alpha\}.$$

Proof. It is immediate (by uniqueness) that $(\mathscr{B}, K)$ is the (Σ, α)-model. And since Σ is cofinal, 2.3 (ii) tells us that K is cofinal in $\mathrm{On}^{\mathscr{B}}$, so

$$\mathrm{On}^{\mathscr{B}} \subseteq \{x \in \mathrm{On}^{\mathscr{A}} \mid x < h_\alpha\}.$$

Hence the lemma boils down to proving that if $x \in \mathrm{On}^{\mathscr{A}}$ and $x < h_\alpha$, then in fact $x \in \mathrm{On}^{\mathscr{B}}$.

Well, since H generates $\mathscr{A}$, there is a term t and elements $k_0, \ldots, k_{n-1}$ of K, $l_0, \ldots, l_m$ of $H - K$, such that $k_0 < \ldots < k_{n-1} < l_0 < \ldots < l_m$ and $x = t^{\mathscr{A}}(\vec{k}, \vec{l})$. By virtue of our convention concerning the indication of variables present in terms, we may assume that $l_0 = h_\alpha$ here. Now, $x < h_\alpha$, so $t^{\mathscr{A}}(\vec{k}, \vec{l}) < h_\alpha$. Thus the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) < v_n$$

is in Σ. So, by remarkability, the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) = t(v_0, \ldots, v_{n-1}, v_{n+m+1}, \ldots, v_{n+2m+1})$$

is in Σ. Thus for any increasing sequence $l'_0, \ldots, l'_m$ from H with $k_{n-1} < l'_0$, we have $t^{\mathscr{A}}(\vec{k}, \vec{l}) = t^{\mathscr{A}}(\vec{k}, \vec{l}')$. But α is a limit ordinal, so we can find such $l'_0, \ldots, l'_m$ with $l'_m < h_\alpha$. Then, since $\vec{k}, \vec{l}' \in K$, we have

$$x = t^{\mathscr{A}}(\vec{k}, \vec{l}) = t^{\mathscr{A}}(\vec{k}, \vec{l}') \in \mathscr{A} \upharpoonright K = \mathscr{B},$$

as required. □

Thus, if Σ is a remarkable, cofinal, $E-M$ set and $(\mathscr{A}, H)$ is the (Σ, λ)-model for some limit ordinal λ, then if we pick any limit ordinal $\alpha < \lambda$ and let K consist of the first α elements of H, the ordinals of the (Σ, α)-model $(\mathscr{A} \upharpoonright K, K)$ form an initial segment of the ordinals of $\mathscr{A}$. Another consequence of remarkability is that the indiscernibles form a club subset of the ordinals of the model:

2.5 Lemma. *Let Σ be a remarkable, cofinal $E-M$ set. Let λ be a limit ordinal, and let $(\mathscr{A}, H)$ be the (Σ, λ)-model. Then H is closed and unbounded in* $\mathrm{On}^{\mathscr{A}}$.

Proof. Unboundedness was proved in 2.3. We verify closure. Let $(h_\gamma \mid \gamma < \lambda)$ be the monotone enumeration of H. Let $\alpha < \lambda$ be a limit ordinal. We must prove that h_α is the least upper bound of the set $K = \{h_\gamma \mid \gamma < \alpha\}$ in $\mathrm{On}^{\mathscr{A}}$. Well, we know from 2.4 that $(\mathscr{A} \upharpoonright K, K)$ is the (Σ, α)-model. But Σ is cofinal, so K is a cofinal subset of $\mathrm{On}^{\mathscr{A} \upharpoonright K}$. It thus suffices to show that h_α is the least upper bound of $\mathrm{On}^{\mathscr{A} \upharpoonright K}$ in $\mathrm{On}^{\mathscr{A}}$. But by 2.4 again,

$$\mathrm{On}^{\mathscr{A} \upharpoonright K} = \{x \in \mathrm{On}^{\mathscr{A}} \mid x < h_\alpha\},$$

so in particular, h_α is the least upper bound of $\mathrm{On}^{\mathscr{A} \upharpoonright K}$ in $\mathrm{On}^{\mathscr{A}}$. □

We shall be particularly interested in well-founded (Σ, α)-models. For suppose $\mathscr{A}$ is a well-founded (Σ, α)-model. Then $\mathscr{A}$ is a well-founded model of the Axiom of Extensionality, in particular, so by the collapsing lemma there is an isomorphism

$$\pi: \mathscr{A} \cong \langle M, \in \rangle,$$

where M is a transitive set, Now, M is a transitive model of the theory BSL + $(V = L)$. So by virtue of our choice of this theory (see earlier)

$$M = (V)^M = (L)^M = L_\lambda,$$

where $\lambda = \sup(M \cap \mathrm{On})$. Hence $\mathscr{A} \cong L_\lambda$.

The well-foundedness of the (Σ, α)-model will depend upon the $E-M$ set Σ. We shall call an $E-M$ set Σ *well-founded* if, for all infinite ordinals α, the (Σ, α)-model is well-founded.

2.6 Lemma. *Let Σ be an $E-M$ set. The following are equivalent:*

(i) *Σ is well-founded;*

(ii) *for some $\alpha \geqslant \omega_1$, the (Σ, α)-model is well-founded;*

(iii) *for all infinite $\alpha < \omega_1$, the (Σ, α)-model is well-founded.*

Proof. (i) → (ii). Immediate.

(ii) → (iii). Choose $\alpha \geqslant \omega_1$ so that the (Σ, α)-model is well-founded. As we observed earlier, up to isomorphism the (Σ, β)-model is a submodel of the (Σ, α)-model for any infinite $\beta < \omega_1$, which proves (iii).

(iii) → (i). Suppose Σ were not well-founded. Then for some infinite α, the (Σ, α)-model, $(\mathscr{A}, H)$ say, is not well-founded. Let $a_n \in A$, $n < \omega$, be such that

$a_{n+1} E a_n$, where $\mathscr{A} = \langle A, E\rangle$. Each a_n is of the form $t_n^{\mathscr{A}}(\vec{h}_n)$ for some $\mathscr{L}$-term t and some $\vec{h}_n \in H$. Let K be a countably infinite subset of H which contains all $\vec{h}_n$, $n < \omega$. Let $\mathscr{B} = \mathscr{A} \restriction K$. Then $(\mathscr{B}, K)$ is the (Σ, β)-model, where $\beta = \operatorname{otp}(K) < \omega_1$. But $\alpha_n \in B$ for all n, where $\mathscr{B} = \langle B, E\rangle$, so $\mathscr{B}$ is not well-founded. This contradicts (iii). □

If Σ is a well-founded $E-M$ set, then for any infinite ordinal α, there is a unique transitive (Σ, α)-model. We denote this model by $M(\Sigma, \alpha)$. We observed above that $M(\Sigma, \alpha)$ has the form $(\langle L_\lambda, \in\rangle, H)$, where λ is a limit ordinal greater than ω, and where $H \subseteq \lambda$. In case α is an uncountable cardinal, we can say even more, namely:

2.7 Lemma. *Let Σ be a well-founded, remarkable, cofinal, $E-M$ set. If κ is an uncountable cardinal, then the universe of $M(\Sigma, \kappa)$ is L_κ.*

Proof. Let $M(\Sigma, \kappa)$ be (L_γ, H). Since $H \subseteq \gamma$ and $|H| = \kappa$, we know that $\gamma \geqslant \kappa$. Suppose that $\gamma > \kappa$. Since $H = \{h_\alpha \mid \alpha < \kappa\}$ is cofinal in γ, we can find a limit ordinal $\alpha < \kappa$ such that $h_\alpha > \kappa$. Let $K = \{h_\beta \mid \beta < \alpha\}$ and set $N = L_\gamma \restriction K$. By 2.4,

$$\mathrm{On}^N = \{x \in \gamma \mid x < h_\alpha\} = h_\alpha.$$

Thus $\kappa \subseteq \mathrm{On}^N$. But $|N| = |K| = |\alpha| < \kappa$, so this is absurd. Hence $\gamma = \kappa$ and we are done. □

For each uncountable cardinal κ, let H_κ denote the unique subset of κ (if it exists) such that (L_κ, H_κ) is the (Σ, κ)-model $M(\Sigma, \kappa)$. By 2.5, we know that H_κ is a club subset of κ.

2.8 Lemma. *If $\kappa < \lambda$ are uncountable cardinals, then $H_\kappa = H_\lambda \cap \kappa$ and $L_\kappa = L_\lambda \restriction H_\kappa$.*

Proof. Let $(h_\nu \mid \nu < \lambda)$ enumerate H_λ in increasing order. Set $K = \{h_\nu \mid \nu < \kappa\}$, and let $N = L_\lambda \restriction K$. Then (N, K) is a (Σ, k)-model, so $N \cong L_\kappa$. But On^N is an initial segment of λ. Hence N must be transitive. But then we must have $N = L_\kappa$, and moreover $K = H_\kappa$, $h_\kappa = \kappa$, and $H_\kappa = K = H_\lambda \cap \kappa$. □

2.9 Corollary. *If λ is an uncountable cardinal, then H_λ contains all uncountable cardinals below λ.*

Proof. Let $\kappa < \lambda$ be an uncountable cardinal. Then, as we saw above,

$$\kappa = h_\kappa \in H_\lambda. \quad \square$$

Of course, we have still said nothing concerning the existence of $E-M$ sets. We are now about to rectify this omission. We show first that if there is a well-founded, remarkable, cofinal $E-M$ set, then it must be unique.

2.10 Lemma. *If there is a well-founded, remarkable, cofinal, $E-M$ set, then it is unique.*

Proof. Let Σ be a well-founded, remarkable, cofinal $E-M$ set. Now, $(L_{\omega_\omega}, H_{\omega_\omega})$ is the transitive (Σ, ω_ω)-model, and by 2.9, $\omega_n \in H_{\omega_\omega}$ for all $n < \omega$. Thus for any $\mathscr{L}$-formula φ,

$$\varphi(v_1, \ldots, v_n) \in \Sigma \quad \text{iff } L_{\omega_\omega} \models \varphi(\omega_1, \ldots, \omega_n).$$

This determines Σ uniquely. □

The unique well-founded, remarkable, cofinal $E-M$ set, if it exists, is denoted by the symbol $0^\#$ ("zero sharp"). It is possible to carry out a similar development for the relativised universe $L[a]$ for any set $a \subseteq \omega$, in which case the corresponding $E-M$ set is denoted by $a^\#$. (This is considered in Exercise 2.) Summarising our previous results, we have:

2.11 Theorem. *Assume $0^\#$ exists. Then there is a club class H of ordinals such that:*

(i) *H contains all uncountable cardinals;*

and for any uncountable cardinal κ, if we set $H_\kappa = H \cap \kappa$, then:

(ii) *H_κ has order-type κ and is club in κ;*
(iii) *H_κ is L_κ-indiscernible;*
(iv) $L_\kappa = L_\kappa \upharpoonright H_\kappa$.

Proof. We set $H = \bigcup_\kappa H_\kappa$, where H_κ is a described earlier. The theorem is immediate now. □

2.12 Theorem. *Assume $0^\#$ exists. If $\kappa < \lambda$ are uncountable cardinals, then $L_\kappa \prec L_\lambda$.*

Proof. We know that

$$M(0^\#, \kappa) = (L_\kappa, H_\kappa), \qquad M(0^\#, \lambda) = (L_\lambda, H_\lambda).$$

So, by 2.8, we have

$$L_\kappa = L_\lambda \upharpoonright H_\kappa \prec L_\lambda. \quad \square$$

The existence of $0^\#$ also provides us with a truth definition for L:

2.13 Theorem (Metatheorem). *There is a formula $\Theta(x)$ of* LST *such that, for any* LST *formula $\Phi(v_0, \ldots, v_n)$, if φ is the $\mathscr{L}$-formula corresponding to Φ* (as in I.9.11), ZF $\vdash$ "*if $0^\#$ exists, then* $(\forall a_0, \ldots, a_n \in \mathrm{L})[\Phi^L(a_0, \ldots, a_n) \leftrightarrow \Theta(\varphi(\mathring{a}_0, \ldots, \mathring{a}_n))]$".

Proof. Given any formula $\Phi(v_0, \ldots, v_n)$ of LST, the reflection principle (I.8.2) provides us with an uncountable cardinal κ such that

$$(\forall \vec{a} \in L_\kappa)[\Phi^L(\vec{a}) \leftrightarrow \Phi^{L_\kappa}(\vec{a})].$$

But by 2.12, together with I.9.11, the actual choice of κ here is irrelevant in the case that $0^\#$ exists. Thus, given any $\vec{a} \in L$, if κ is any uncountable cardinal such that $\vec{a} \in L_\kappa$, then providing that $0^\#$ exists, we have (using I.9.11)

$$\Phi^L(\vec{a}) \leftrightarrow \Phi^{L_\kappa}(\vec{\alpha}) \leftrightarrow \models_{L_\kappa} \varphi(\vec{\mathring{a}}).$$

Thus $\Theta(x)$ is the LST formula which says:

"x is a sentence of $\mathscr{L}_L$, and if κ is the least uncountable cardinal such that $x \in L_\kappa$, then $\models_{L_\kappa} x$".

By virtue of 2.13, we may speak about "elementary submodels of L" quite openly, and indeed may state the following theorem:

2.14 Theorem. *If $0^{\#}$ exists, then for any uncountable cardinal κ, $L_\kappa \prec L$.*

Proof. Since

$$L = \bigcup \{L_\kappa \mid \kappa \text{ is an uncountable cardinal}\},$$

this follows easily from 2.12. □

Before we turn to an existence proof for $0^{\#}$, we give one more consequence of its existence.

2.15 Theorem. *Assume that $0^{\#}$ exists, and let κ be any uncountable cardinal. Then:*

(i) $[\kappa \text{ is inaccessible}]^L$;
(ii) $[\kappa \to (\omega)_2^{<\omega}]^L$;
(iii) $|\mathscr{P}^L(\kappa)| = \kappa$.

Proof. (i) If $\lambda = \omega_1$, then

$$[\lambda \text{ is regular}]^L,$$

and if $\mu = \omega_\omega$, then

$$[\mu \text{ is a limit cardinal}]^L.$$

So by the L-indiscernibility of the cardinals,

$$[\kappa \text{ is a regular limit cardinal}]^L.$$

Since $[\mathrm{GCH}]^L$, this proves (i).

(ii) Suppose not, and let $f \in L$, $f\colon [\kappa]^{<\omega} \to 2$ be the $<_L$-least partition with no infinite homogeneous set (in the sense of L). In the real world, $f\colon [\kappa]^{<\omega} \to 2$, of course. Now, f is definable from κ in L_{κ^+} (by the above definition, which is clearly absolute for L_{κ^+}). It follows that, in the real world, H_κ is homogeneous for f. For if t is a term such that

$$f(\sigma) = t^{L_{\kappa^+}}(\sigma, \kappa)$$

for all $\sigma \in [\kappa]^{<\omega}$, then for any $\alpha_1 < \ldots < \alpha_n$, $\beta_1 < \ldots < \beta_n$ from H_κ, if $i = 0, 1$, then

$$\begin{aligned} f(\alpha_1, \ldots, \alpha_n) = i \quad & \text{iff } L_{\kappa^+} \models t(\alpha_1, \ldots, \alpha_n, \kappa) = i \\ & \text{iff } L_{\kappa^+} \models t(\beta_1, \ldots, \beta_n, \kappa) = i \\ & \text{iff } f(\beta_1, \ldots, \beta_n) = i. \end{aligned}$$

But then, exactly as in 1.7 it follows that there is, in L, an infinite set which is homogeneous for f (in the sense of L), contradicting the choice of f. This proves (ii).

(iii) By (i) $[\lambda \text{ is inaccessible}]^L$, where $\lambda = \kappa^+$. This implies (iii) at once. □

Note that a particular consequence of the above (and previous) results is that the existence of $0^{\#}$ cannot be established in ZFC alone. We shall now look into this question of existence of $0^{\#}$.

2.16 Theorem. *The following are equivalent*:

(i) $0^{\#}$ *exists*;

(ii) *for every uncountable cardinal* κ, L_κ *has an uncountable set of indiscernibles*;

(iii) *for some uncountable cardinal* κ, L_κ *has an uncountable set of indiscernibles.*

Proof. (i) → (ii). By 2.11 (iii).

(ii) → (iii). Trivial.

(iii) → (i). Let λ be the least limit ordinal such that L_λ has an uncountable set of indiscernibles. Let $H \subseteq \lambda$ be an L_λ-indiscernible set of order-type ω_1, chosen so that h_ω is as small as possible, where $(h_\nu \mid \nu < \omega_1)$ is the monotone enumeration of H. Let Σ be the $E-M$ set determined by the indiscernible set H in L_λ. We show that Σ is well-founded, remarkable, and cofinal.

(a) Σ is well-founded. Well, clearly, $L_\lambda \upharpoonright H$ is well-founded. But $(L_\lambda \upharpoonright H, H)$ is the (Σ, ω_1)-model. So by 2.6, Σ is well-founded.

(b) Σ is cofinal. For suppose not. Then by 2.3, H is not cofional in (the ordinals of) $L_\lambda \upharpoonright H$. So for some $\mathscr{L}$-term t and some $\nu_1 < \ldots < \nu_n < \omega_1$,

$$\gamma = t^{L_\lambda}(h_{\nu_1}, \ldots, h_{\nu_n}) \geqslant \sup(H).$$

We may assume that γ is a limit ordinal here. (For otherwise, if $\gamma = \delta + m$, we may replace t by the term

$$t'(h_{\nu_1}, \ldots, h_{\nu_n}) = t(h_{\nu_1}, \ldots, h_{\nu_n}) - m.)$$

Let

$$K = \{h_\nu \mid \nu_n < \nu < \omega_1\}.$$

Clearly, K is a set of indiscernibles for L_γ. But $\gamma < \lambda$, so this contradicts the choice of λ. Hence Σ must be cofinal.

(c) Σ is remarkable. To see this, suppose that the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) < v_n$$

is in Σ, for some $\mathscr{L}$-term t. Partition H into increasing, finite pieces

$$\vec{c}, \vec{d}_0, \vec{d}_1, \ldots, \vec{d}_\nu, \ldots \quad (\nu < \omega_1),$$

where $\vec{c}$ has length n and each $\vec{d}_\nu$ has length $m + 1$, and where

$$\max(\vec{c}) < \min(\vec{d}_0) < \max(\vec{d}_0) < \min(\vec{d}_1) < \max(\vec{d}_1) < \min(\vec{d}_2) < \ldots.$$

Notice that, in particular,

$$\vec{d}_\omega = h_\omega, h_{\omega+1}, \ldots, h_{\omega+m}.$$

By indiscernibility, one of the following must occur:

(A) $t^{L_\lambda}(\vec{c}, \vec{d}_\nu) = t^{L_\lambda}(\vec{c}, \vec{d}_\tau)$ for all $\nu < \tau < \omega_1$;

(B) $t^{L_\lambda}(\vec{c}, \vec{d}_\nu) < t^{L_\lambda}(\vec{c}, \vec{d}_\tau)$ for all $\nu < \tau < \omega_1$;

(C) $t^{L_\lambda}(\vec{c}, \vec{d}_\nu) > t^{L_\lambda}(\vec{c}, \vec{d}_\tau)$ for all $\nu < \tau < \omega_1$.

Since Σ is determined by H in L_λ, if we can prove that (A) must occur, we shall be done, since this will imply that Σ contains the formula

$$t(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m}) = t(v_0, \ldots, v_{n-1}, v_{n+m+1}, \ldots, v_{n+2m+1}).$$

Well, (C) is clearly impossible, since that would give us a decreasing ω_1-sequence of ordinals. So let us assume (B) and work for a contradiction. Set

$$h'_\nu = t^{L_\lambda}(\vec{c}, \vec{d}_\nu), \qquad \nu < \omega_1.$$

By (B), the sequence $(h'_\nu \mid \nu < \omega_1)$ is strictly increasing. And it is easily checked that $\{h_\nu \mid \nu < \omega_1\}$ is L_λ-indiscernible. But by choice of t, $h'_\omega < h_\omega$, so this contradicts our choice of H, h_ω, and we are done. □

2.17 Corollary. *If $\kappa(\omega_1)$ exists, then $0^\#$ exists. Hence if there is a measurable cardinal, then $0^\#$ exists.*

Proof. By 1.6. □

3. *Definability of* $0^\#$

We have already seen that the existence of $0^\#$ has a profound effect upon the constructible universe. In this section we investigate the logical complexity of the set $0^\#$ as a subset of the set of all formulas of $\mathscr{L}$. In particular we shall show that $0^\#$ has strong absoluteness properties.

3.1 Lemma. *There is a Π_1 formula $\Phi(x)$ of* LST *such that*

$$\Phi(x) \leftrightarrow x = 0^\#.$$

Proof. By 2.10, $0^\#$ is unique, if it exists, and what we must show is that the predicate

"x is a well-founded, remarkable, cofinal $E-M$ set"

can be expressed in a Π_1 fashion.

We commence by examining the predicate

"Σ is an $E-M$ set".

Let $\mathscr{L}^+$ be the language $\mathscr{L}$ together with the extra constant symbols c_n, $n < \omega$. For any set, Σ, of $\mathscr{L}$-formulas, let Σ^+ be the set of $\mathscr{L}^+$-sentences which consists of:

(i) the axioms of BSL + $(V = L)$;
(ii) $\varphi(c_0, \ldots, c_n)$, for each $\varphi(v_0, \ldots, v_n) \in \Sigma$;
(iii) $\mathrm{On}(c_n)$, for all $n < \omega$;
(iv) $(c_n < c_m)$, for all $n < m < \omega$;
(v) $\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n})$, for each $\varphi(v_0, \ldots, v_{n-1}) \in \Sigma$ and each $i_1 < \ldots < i_n < \omega$, $j_1 < \ldots < j_n < \omega$.

Claim. Σ is an $E-M$ set iff Σ^+ is consistent.

Proof of claim: Suppose Σ is an $E-M$ set. Then Σ is the set of all $\mathscr{L}$-formulas which are true on increasing tuples from an $\mathscr{A}$-indiscernible set $\{a_n | n < \omega\}$ in some model $\mathscr{A}$ of BSL + $(V = L)$. Clearly, $\langle \mathscr{A}, (a_n)_{n<\omega} \rangle$ is a model of Σ^+, so Σ^+ is consistent.

Conversely, suppose Σ^+ is consistent, and let $\langle \mathscr{A}, (a_n)_{n<\omega} \rangle$ be a model of Σ^+. Clearly, $\{a_n | n < \omega\}$ is $\mathscr{A}$-indiscernible and $(\mathscr{A}, \{a_n | n < \omega\})$ is a (Σ, ω)-model, so Σ is an $E-M$ set. The claim is proved.

By the claim we have:

Σ is an $E-M$ set iff there does not exist a proof of the sentence $(0 = 1)$ from the sentences in Σ^+.

More precisely:

Σ is an $E-M$ set iff there does not exist a finite sequence of $\mathscr{L}^+$-formulas such that the last formula in the sequence is $(0 = 1)$ and each formula of the sequence is either a consequence of previous formulas by modus ponens or else is an axiom of logic or else an axiom of BSL + $(V=L)$ or else is of the form $\varphi(c_0, \ldots, c_n)$ for some $\varphi(v_0, \ldots, v_n) \in \Sigma$ or else of the form $\mathrm{On}(c_n)$ for some $n < \omega$ or else of the form $(c_n < c_m)$ for some $n < m < \omega$ or else of the form $(\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n}))$ for some $\varphi(v_0, \ldots, v_{n-1})$ in Σ and some $i_1 < \ldots < i_n < \omega$, $j_1 < \ldots < j_n < \omega$.

Now, provided that the constants c_n are suitably chosen (e.g. take $\mathscr{L}_\omega$ as the language $\mathscr{L}^+$ and use the constant symbol $\mathring{n}$ for c_n), all quantifiers in the above definition can be bound (without loss of generality) by V_ω. Thus the above characterisation of the predicate "Σ is an $E-M$ set" is Σ_0 in the parameter V_ω.

It is easily seen that the predicates "Σ is cofinal" and "Σ is remarkable" are also Σ_0 in the parameter V_ω. Thus there is a Σ_0 formula $\Psi(x, y)$ of LST such that

$$\Sigma \text{ is a remarkable, cofinal } E-M \text{ set} \leftrightarrow \Psi(\Sigma, V_\omega).$$

But $V_\omega = L_\omega$. Hence

$$\begin{aligned}&\Sigma \text{ is a remarkable, cofinal } E\!-\!M \text{ set}\\ &\leftrightarrow \forall\alpha\,\forall u\,[[\mathrm{On}(\alpha) \wedge \lim(\alpha) \wedge (\forall\beta\in\alpha)(\beta = 0 \vee \mathrm{succ}(\beta)) \wedge u = L_\alpha]\\ &\qquad \to \Psi(\Sigma, u)].\end{aligned}$$

The formula on the right here is Π_1. (In fact there is an equivalent Σ_1 formula, as the reader may readily verify, but we do not require this fact.) So we are left with proving that the predicate

"Σ is well-founded"

(for a remarkable, cofinal $E\!-\!M$ set Σ) is Π_1.

By definition,

Σ is well-founded iff for all α, the (Σ, α)-model is well-founded.

Now, if Σ is a remarkable, cofinal $E\!-\!M$ set, then for every limit ordinal α there is a unique (up to isomorphism) (Σ, α)-model, and we can find one of the form $(\langle A, E\rangle, \alpha)$, where $E \cap (\alpha\times\alpha) = \in \cap (\alpha\times\alpha)$. Let us call such a model a *standardised* (Σ, α)-model. Then:

$(\langle A, E\rangle, \alpha)$ is a standardised (Σ, α)-model iff

(i) $\langle A, E\rangle$ is a model of BSL + $(V = L)$ $\wedge$

(ii) $\alpha \subseteq \mathrm{On}^{\langle A, E\rangle} \wedge E \cap (\alpha\times\alpha) = \in \cap (\alpha\times\alpha) \wedge$

(iii) α is $\langle A, E\rangle$-indiscernible $\wedge$

(iv) α generates $\langle A, E\rangle$ $\wedge$

(v) Σ is the set of all $\mathscr{L}$-formulas valid in $\langle A, E\rangle$ on increasing tuples from α.

Now, in each of the clauses (i)–(v) above, all necessary quantifiers may be bound either by A or by α or by V_ω. (This is a routine matter which we leave to the reader to check.) Thus there is a Σ_0 formula $\Theta(w, x, y, z)$ of LST such that

$(\langle A, E\rangle, \alpha)$ is a standardised (Σ, α)-model iff $\Theta(\Sigma, \langle A, E\rangle, \alpha, V_\omega)$.

But, clearly,

$$\begin{aligned}\Theta(\Sigma, \langle A, E\rangle, \alpha, V_\omega) \quad \text{iff } &\exists\gamma\,\exists u\,[\mathrm{On}(\gamma) \wedge \lim(\gamma)\\ &\wedge (\forall\beta\in\gamma)(\beta = 0 \vee \mathrm{succ}(\beta)) \wedge (u = L_\gamma)\\ &\wedge \Theta(\Sigma, \langle A, E\rangle, \alpha, u)].\end{aligned}$$

Thus the predicate "$(\langle A, E\rangle, \alpha)$ is a standardised (Σ, α)-model" (as a predicate on $\langle A, E\rangle, \alpha, \Sigma$) is Σ_1. But (for a remarkable, cofinal $E\!-\!M$ set Σ):

$$\begin{aligned}\Sigma \text{ is well-founded} \leftrightarrow \forall\alpha\,\forall\langle A, E\rangle\,&[\text{if } (\langle A, E\rangle, \alpha) \text{ is a standardised}\\ &(\Sigma, \alpha)\text{-model, then } E \text{ is well-founded on } A].\end{aligned}$$

This is easily seen to be Π_1, so we are done. □

3.2 Corollary. $0^{\#} \notin L$.

Proof. If $0^{\#} \in L$, then since Π_1 properties are D-absolute, $\Phi(0^{\#})$ implies $\Phi^L(0^{\#})$, where Φ is the Π_1 formula from 3.1. But then we can prove all of the results of section 2 *inside* L, which is absurd. □

4. $0^{\#}$ *and Elementary Embeddings*

The existence of $0^{\#}$ is closely connected with the existence of elementary embeddings of the form j: $L_\kappa \prec L_\kappa$, where κ is a cardinal. The simplest such result is the following:

4.1 Theorem. *If* $0^{\#}$ *exists, then for any uncountable cardinal* κ *there is a non-trivial embedding* j: $L_\kappa \prec L_\kappa$.

Proof. Let $(h_\alpha \mid \alpha < \kappa)$ be the monotone enumeration of H_κ. Define j: $H_\kappa \to H_\kappa$ by $j(h_\alpha) = h_{\alpha+1}$. By 2.1, j extends to an embedding $\tilde{j}$: $L_\kappa \prec L_\kappa$. □

The main effort in this section is directed towards proving the converse to 4.1. In fact we shall prove a stronger result. In order to imply the existence of $0^{\#}$ it is enough to have an embedding j: $L_\alpha \prec L_\beta$ for some limit ordinals α, β such that $j(\gamma) \neq \gamma$ for some $\gamma < |\alpha|$. In order to do this we shall first of all prove a converse to 4.1 under some additional assumptions. We require some prior definitions.

Say that a cardinal κ is of γ-*type* 0 if it is a limit cardinal and $\mathrm{cf}(\kappa) > \gamma$. Notice that there are arbitrarily large cardinals of γ-type 0, for any given ordinal γ. Moreover, if $(\kappa_\nu \mid \nu < \theta)$ is an increasing sequence of cardinals of γ-type 0 such that $\mathrm{cf}(\theta) > \gamma$, then $\sup_{\nu<\theta} \kappa_\nu$ is of γ-type 0.

A cardinal κ is said to be of γ-*type* 1 if it is of γ-type 0 and

$$|\{\lambda \in \kappa \mid \lambda \text{ is of } \gamma\text{-type } 0\}| = \kappa.$$

Since the γ-type 0 cardinals are closed under limits of η-sequences whenever $\mathrm{cf}(\eta) > \gamma$, it is easily proved that there are arbitrarily large cardinals of γ-type 1. Moreover, it is clear that the γ-type 1 cardinals are closed under limits of η-sequences whenever $\mathrm{cf}(\eta) > \gamma$.

Proceeding in a recursive fashion now, say that a cardinal κ is of γ-*type* $\nu + 1$ if it is of γ-type ν and

$$|\{\lambda \in \kappa \mid \lambda \text{ is of } \gamma\text{-type } \nu\}| = \kappa.$$

Provided the γ-type ν cardinals are unbounded and closed under limits of η-sequences whenever $\mathrm{cf}(\eta) > \delta$ for some $\delta \geqslant \gamma$, the same will be true of the γ-type $\nu + 1$ cardinals.

If τ is a limit ordinal, we say that a cardinal κ is of *γ-type* τ iff it is of γ-type ν for every $\nu < \tau$. If, for each $\nu < \tau$, the γ-type ν cardinals are unbounded and closed under limits of η-sequences whenever $\mathrm{cf}(\eta) > \delta$ for some $\delta \geqslant \gamma$, then, provided $\tau < \delta$, the same will be true of the γ-type τ cardinals.

4.2 Theorem. *Let κ be a cardinal. Suppose that there is an embedding*

$$e: L_\kappa \prec L_\kappa$$

such that for some ordinal $\gamma < \kappa$:

(i) $e \upharpoonright \gamma = \mathrm{id} \upharpoonright \gamma$;
(ii) $e(\gamma) > \gamma$;
(iii) *if $\lambda > \kappa$ is of γ-type* 0, *then* $e(\lambda) = \lambda$.

Suppose further that κ is of γ-type ω_1. Then $0^{\#}$ exists.

Proof. By 2.16, it suffices to show that L_κ has an uncountable, indiscernible subset.

For each $\nu < \omega_1$, let

$$U_\nu = \{\lambda \in \kappa \mid \lambda \text{ is a } \gamma\text{-type } \nu \text{ cardinal}\}.$$

Since κ has γ-type ω_1, $|U_\nu| = \kappa$ for each $\nu < \omega_1$. Moreover,

$$U_0 \supseteq U_1 \supseteq \ldots \supseteq U_\nu \supseteq \ldots \quad (\nu < \omega_1),$$

and for each $\nu < \omega_1$,

$$U_{\nu+1} = \{\lambda \in U_\nu \mid |U_\nu \cap \lambda| = \lambda\},$$

with

$$U_\delta = \bigcap_{\nu<\delta} U_\nu, \quad \text{if } \lim(\delta),\ \delta < \omega_1.$$

For each $\nu < \omega_1$, let

$$M_\nu = L_\kappa \upharpoonright (\gamma \cup U_\nu).$$

Thus,

$$M_\nu \prec L_\kappa \quad \text{and} \quad |M_\nu| = \kappa.$$

In particular, the transitive collapse of M_ν is L_κ. Let

$$i_\nu: L_\kappa \cong M_\nu.$$

Thus

$$i_\nu: L_\kappa \prec L_\kappa.$$

Set

$$\gamma_\nu = i_\nu(\gamma).$$

Claim 1. Let $\nu, \tau < \omega_1$. Then:

(i) γ_ν is the least ordinal in $M_\nu - (\gamma + 1)$;
(ii) if $\nu < \tau$ and $x \in M_\tau$, then $i_\nu(x) = x$;
(iii) if $\nu < \tau$, then $i_\nu(\gamma_\tau) = \gamma_\tau$;
(iv) if $\nu < \tau$, then $\gamma_\nu < \gamma_\tau$.

Proof. (i) Since $\gamma \in M_\nu$, $i_\nu \upharpoonright \gamma = \mathrm{id} \upharpoonright \gamma$ and $i_\nu(\gamma)$ is the least element of $M_\nu - \gamma$. So it suffices to prove that $\gamma \notin M$. Since $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_\nu \supseteq \ldots$ $(\nu < \omega_1)$, it is enough to prove that $\gamma \notin M_0$. Consider any $x \in M_0$. Then $x = t^{L_\kappa}(\eta_1, \ldots, \eta_n)$ for some $\mathscr{L}$-term t and some $\eta_1, \ldots, \eta_n \in \gamma \cup U_0$. By the assumptions (i) and (iii) of the lemma, $e(\eta_1) = \eta_1, \ldots, e(\eta_n) = \eta_n$. Thus

$$e(x) = e(t^{L_\kappa}(\eta_1, \ldots, \eta_n)) = t^{L_\kappa}(e(\eta_1), \ldots, e(\eta_n)) = t^{L_\kappa}(\eta_1, \ldots, \eta_n) = x.$$

So by assumption (ii) of the lemma, $x \neq \gamma$. Thus $\gamma \notin M_0$.

(ii) Let $x \in M_\tau$. Then for some $\mathscr{L}$-term t and some $\eta_1, \ldots, \eta_n \in \gamma \cup U_\tau$, $x = t^{L_\kappa}(\eta_1, \ldots, \eta_n)$. If $\eta \in \gamma$, then since $\gamma \subseteq M_\nu$, $i_\nu(\eta) = \eta$. If $\eta \in U_\tau$, then since $\nu < \tau$, $|U_\nu \cap \eta| = \eta$, so $i_\nu^{-1}(\eta) = \eta$, so $i_\nu(\eta) = \eta$. Thus

$$i_\nu(x) = i_\nu(t^{L_\kappa}(\eta_1, \ldots, \eta_n)) = t^{L_\kappa}(i_\nu(\eta_1), \ldots, i_\nu(\eta_n)) = t^{L_\kappa}(\eta_1, \ldots, \eta_n) = x.$$

(iii) An immediate consequence of (ii).

(iv) If $\nu < \tau$, then $M_\nu \subseteq M_\tau$, so $\gamma_\nu \leqslant \gamma_\tau$. Now by result (i) of this claim, $\gamma_\nu > \gamma$, so applying i_ν, we get $i_\nu(\gamma_\nu) > i_\nu(\gamma) = \gamma_\nu$. But by result (iii), $i_\nu(\gamma_\tau) = \gamma_\tau$. Hence $\gamma_\nu \neq \gamma_\tau$. Thus $\gamma_\nu < \gamma_\tau$.

The claim is proved.

For $\nu < \tau < \omega_1$, set

$$M_{\nu t} = L_\kappa \upharpoonright (\gamma_\nu \cup U_\tau).$$

Let

$$i_{\nu\tau}\colon L_\kappa \cong M_{\nu\tau}.$$

Thus

$$i_{\nu\tau}\colon L_\kappa \prec L_\kappa.$$

Claim 2. Let $\nu < \tau$. Then:

(i) if $\xi < \nu$, then $i_{\nu\tau}(\gamma_\xi) = \gamma_\xi$;
(ii) $i_{\nu\tau}(\gamma_\nu) = \gamma_\tau$;
(iii) if $\xi > \tau$, then $i_{\nu\tau}(\gamma_\xi) = \gamma_\xi$.

Proof. (i) Since $\gamma_\nu \subseteq M_{\nu\tau}$, we have $i_{\nu\tau} \upharpoonright \gamma_\nu = \mathrm{id} \upharpoonright \gamma_\nu$, so this is immediate.

(ii) Since $\gamma_\nu > \gamma$, we have $M_\tau \subseteq M_{\nu\tau}$, so $\gamma_\tau \in M_{\nu\tau}$. But $i_{\nu\tau} \upharpoonright \gamma_\nu = \mathrm{id} \upharpoonright \gamma_\nu$, so $i_{\nu\tau}(\gamma_\nu)$ is the least ordinal in $M_{\nu\tau}$ greater than or equal to γ_ν. Hence $\gamma_\nu \leqslant i_{\nu\tau}(\gamma_\nu) \leqslant \gamma_\tau$. It therefore suffices to show that there is no ordinal $\delta \in M_{\nu\tau}$ such that $\gamma_\nu \leqslant \delta < \gamma_\tau$. Suppose that there were such a δ. Then for some $\mathscr{L}$-term t, $\delta = t^{L_\kappa}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$, where $\xi_1, \ldots, \xi_n \in \gamma_\nu$ and $\eta_1, \ldots, \eta_k \in U_\tau$. Thus

$$L_\kappa \models (\exists\, \xi_1, \ldots, \xi_n < \gamma_\nu)\,[\gamma_\nu \leqslant t\,(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) < \gamma_\tau].$$

Applying i_ν^{-1}, we get (since $i_\nu(\gamma_\tau) = \gamma_\tau$, $i_\nu \restriction U_\tau = \mathrm{id} \restriction U_\tau$, and $i_\nu(\gamma) = \gamma_\nu$)

$$L_\kappa \models (\exists\, \xi_1, \ldots, \xi_n < \gamma)\,[\gamma \leqslant t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) < \gamma_\tau].$$

So for some $\xi_1, \ldots, \xi_n < \gamma$ we have

$$\gamma \leqslant t^{L_\kappa}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) < \gamma_\tau.$$

But $t^{L_\kappa}(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) \in M_\tau$, so this contradicts (i) of Claim 1.

(iii) If $x \in M_{\tau+1}$, then $x = t^{L_\kappa}(\eta_1, \ldots, \eta_n)$ for some $\mathscr{L}$-term t and some $\eta_1, \ldots, \eta_n \in \gamma \cup U_{\tau+1}$. Now, $i_{\nu\tau} \restriction \gamma_\nu = \mathrm{id} \restriction \gamma_\nu$, so $i_{\nu\tau} \restriction \gamma = \mathrm{id} \restriction \gamma$. And if $\eta \in U_{\tau+1}$, then $|U_\tau \cap \eta| = \eta$, so $i_{\nu\tau}^{-1}(\eta) = \eta$, giving $i_{\nu\tau}(\eta) = \eta$. Thus $i_{\nu\tau}(x) = x$. In particular, $i_{\nu\tau}(\gamma_\xi) = \gamma_\xi$ for all $\xi > \tau$.

The claim is proved.

Claim 3. The set $\{\gamma_\nu \mid \nu < \omega_1\}$ is L_κ-indiscernible.

Proof. Let $\varphi(v_1, \ldots, v_n)$ be any $\mathscr{L}$-formula, and let $\nu_1 < \ldots < \nu_n < \omega_1$, $\tau_1 < \ldots < \tau_n < \omega_1$. We show that

$$L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_1}) \quad \text{iff } L_\kappa \models \varphi(\gamma_{\tau_1}, \ldots, \gamma_{\tau_n}).$$

Pick $\delta_1 < \ldots < \delta_n < \omega_1$ so that $\nu_n, \tau_n < \delta_1$. Applying $i_{\nu_n \delta_n}$ we get, using Claim 2,

$$L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_{n-1}}, \gamma_{\nu_n}) \quad \text{iff } L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_{n-1}}, \gamma_{\delta_n}).$$

Applying $i_{\nu_{n-1} \delta_{n-1}}$ now gives

$$L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_{n-2}}, \gamma_{\nu_{n-1}}, \gamma_{\delta_n}) \quad \text{iff } L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_{n-2}}, \gamma_{\delta_{n-1}}, \gamma_{\delta_n}).$$

Successively applying $i_{\nu_{n-2} \delta_{n-2}}, \ldots, i_{\nu_1 \delta_1}$ now gives, in the end, the equivalence

$$L_\kappa \models \varphi(\gamma_{\nu_1}, \ldots, \gamma_{\nu_n}) \quad \text{iff } L_\kappa \models \varphi(\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}).$$

Repeating the above procedure with $\tau_1, \ldots, \tau_n$ in place of $\nu_1, \ldots, \nu_n$, we get

$$L_\kappa \models \varphi(\gamma_{\tau_1}, \ldots, \gamma_{\tau_n}) \quad \text{iff } L_\kappa \models \varphi(\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}).$$

The above two equivalences combine to give the desired result. That proves the claim, and with it the theorem. □

We shall make use of 4.2 in our proof of the next result, the (strong) converse to 4.1.

4.3 Theorem. *Assume there is an embedding $j: L_\alpha \prec L_\beta$, where α, β are limit ordinals, and that $j(\gamma) \neq \gamma$ for some $\gamma < |\alpha|$. Then $0^\#$ exists.* □

The proof of 4.3 will take some time. Fix j, α, β as above, and let γ be the least ordinal such that $j(\gamma) \neq \gamma$. Thus $j \upharpoonright \gamma = \mathrm{id} \upharpoonright \gamma$ and $j(\gamma) > \gamma$. Let κ be a cardinal of γ-type ω_1. We prove 4.3 by using j to construct an embedding $e: L_\kappa \prec L_\kappa$ to satisfy the hypotheses of 4.2.

Let $\lambda = \kappa^+$. Notice that since $\gamma < |\alpha|$, $\mathscr{P}(\gamma) \cap L \subseteq L_\alpha$. Hence we may define

$$D = \{X \subseteq \gamma \mid X \in L \ \&\ \gamma \in j(X)\}.$$

Since $j: L_\alpha \prec L_\beta$ is elementary, the following lemma is easily proved.

4.4 Lemma.

(i) *$\gamma \in D$ and $\emptyset \notin D$;*
(ii) *if $X \in D$ and $Y \in D$, then $X \cap Y \in D$;*
(iii) *if $X \in D$ and $X \subseteq Y \subseteq \gamma$, where $Y \in L$, then $Y \in D$;*
(iv) *if $X \subseteq \gamma$, $X \in L$, then either $X \in D$ or else $\gamma - X \in D$;*
(v) *if $\bar{\gamma} < \gamma$ and $\{X_\xi \mid \xi < \bar{\gamma}\} \subseteq D$ and $(X_\xi \mid \xi < \bar{\gamma}) \in L$, then $\bigcap_{\xi < \bar{\gamma}} X_\xi \in D$.* □

Thus D is an ultrafilter in the field of sets $\mathscr{P}^L(\gamma)$ which is γ-complete with regards to families of sets in L. We do not necessarily have $D \in L$; indeed, it is a consequence of our ensuing results that $D \notin L$.

We use D to construct a kind of "ultrapower" of L_λ. Set

$$F = \{f \in L \mid f: \gamma \to L_\lambda\}.$$

Notice that as $\mathrm{cf}(\lambda) = \lambda > \gamma$, if $f \in F$ then in fact $f \in L_\lambda$. This fact will be relevant later on. Define an equivalence relation on F by

$$f \sim g \quad \text{iff } \{\nu \in \gamma \mid f(\nu) = g(\nu)\} \in D.$$

(Since $\{\nu \in \gamma \mid f(\nu) = g(\nu)\} \in L$ whenever $f, g \in F$, this definition makes sense. And, using the results of 4.4, it is easily checked that $\sim$ is an equivalence relation.) Let $[f]$ denote the equivalence class of f, and set

$$M = \{[f] \mid f \in F\}.$$

Define a binary relation, E, on M by

$$[f]\, E\, [g] \quad \text{iff } \{\nu \in \gamma \mid f(\nu) \in g(\nu)\} \in D.$$

(Again, for $f, g \in F$, $\{\nu \in \gamma \mid f(\nu) \in g(\nu)\} \in L$. And using 4.4 it is easily seen that E is well-defined on M.)

4.5 Lemma. *Let $\varphi(v_0, \ldots, v_n)$ be any $\mathscr{L}$-formula, and let $[f_0], \ldots, [f_n] \in M$. Then*

$$\langle M, E \rangle \models \varphi([f_0], \ldots, [f_0]) \quad \textit{iff} \ \{v \in \gamma \mid L_\lambda \models \varphi(f_0(v), \ldots, f_n(v))\} \in D.$$

Proof. Notice first that, since $f_0, \ldots, f_n \in L_\lambda$, $\{v \in \gamma \mid L_\lambda \models \varphi(f_0(v), \ldots, f_n(v))\} \in L$. The lemma is proved by induction on the length of φ.

If φ is primitive, the result is true by definition of $\langle M, E \rangle$.

If φ is of the form $\neg \psi$ or else $\psi_1 \wedge \psi_2$, the induction step is trivial, using the results of 4.4.

Suppose finally that φ has the form $\exists y \psi(y, v_0, \ldots, v_n)$ and that the result holds for ψ. If

$$\langle M, E \rangle \models \exists y \psi(y, [f_0], \ldots, [f_n]),$$

then for some $[g] \in M$,

$$\langle M, E \rangle \models \psi([g], [f_0], \ldots, [f_n]),$$

so by induction hypothesis

$$X = \{v \in \gamma \mid L_\lambda \models \psi(g(v), f_0(v), \ldots, f_n(v))\} \in D.$$

But clearly,

$$X \subseteq \{v \in \gamma \mid L_\lambda \models \exists y \psi(y, f_0(v), \ldots, f_n(v))\} \in L.$$

Hence

$$\{v \in \gamma \mid L_\lambda \models \exists y \psi(y, f_0(v), \ldots, f_n(v))\} \in D.$$

Conversely, suppose that

$$Y = \{v \in \gamma \mid L_\lambda \models \exists y \psi(y, f_0(v), \ldots, f_n(v))\} \in D.$$

In particular, $Y \in L$. Define $g: \gamma \to L_\lambda$ by

$$g(v) = \begin{cases} \text{the } <_L\text{-least } y \text{ such that } L_\lambda \models \psi(y, f_0(v), \ldots, f_n(v)), & \text{if } v \in Y, \\ \emptyset, \quad \text{if } v \notin Y. \end{cases}$$

Clearly, $g \in L$. Hence $[g] \in M$. But

$$\{v \in \gamma \mid L_\lambda \models \psi(g(v), f_0(v), \ldots, f_n(v))\} = Y \in D.$$

So by induction hypothesis,

$$\langle M, E \rangle \models \psi([g], [f_0], \ldots, [f_n]).$$

Hence

$$\langle M, E \rangle \models \exists y \psi(y, [f_0], \ldots, [f_n]).$$

The proof is complete. □

4.6 Lemma. *$\langle M, E\rangle$ is well-founded.*

Proof. Suppose not, and let $[g_{n+1}] \ni [g_n]$ for all $n < \omega$. Now, $g_n \in L_\lambda$ for all $n < \omega$, so pick $X \prec L_\lambda$ such that $(\gamma + 1) \cup \{g_n | n < \omega\} \subseteq X$ and $|X| = |\gamma|$. Let $\sigma\colon X \cong L_\delta$. Then $|\delta| = |\gamma| < |\alpha|$, and hence $\delta < \alpha$. Thus $\bar{g}_n \in L_\alpha$, where we set $\bar{g}_n = \sigma(g_n)$. Now, $\sigma^{-1}\colon L_\delta \prec L_\lambda$ and $\sigma \restriction \gamma = \mathrm{id} \restriction \gamma$, so for each $n < \omega$,

$$\{\nu \in \gamma \,|\, \bar{g}_{n+1}(\nu) \in \bar{g}_n(\nu)\} = \{\nu \in \gamma \,|\, g_{n+1}(\nu) \in g_n(\nu)\} \in D.$$

Thus for each $n < \omega$,

$$\gamma \in j(\{\nu \in \gamma \,|\, \bar{g}_{n+1}(\nu) \in \bar{g}_n(\nu)\}) = \{\nu \in j(\gamma) \,|\, [j(\bar{g}_{n+1})](\nu) \in [j(\bar{g}_n)](\nu)\}.$$

In other words, for all $n < \omega$, we have

$$[j(\bar{g}_{n+1})](\gamma) \in [j(\bar{g}_n)](\gamma).$$

But this is absurd. The lemma is proved. □

We can define a map $k\colon L_\lambda \to M$ by

$$k(x) = [c_x],$$

where c_x is the constant function $(x \,|\, \nu < \gamma)$. Using 4.5, it is easily seen that

$$k\colon \langle L_\lambda, \in\rangle \prec \langle M, E\rangle,$$

so by 4.6 there is an isomorphism

$$\varrho\colon \langle M, E\rangle \cong \langle L_\mu, \in\rangle,$$

for some $\mu \geqslant \lambda$. Let $\pi = \varrho \circ k$. Thus

$$\pi\colon L_\lambda \prec L_\mu.$$

4.7 Lemma.

(i) $\pi \restriction \gamma = \mathrm{id} \restriction \gamma$;
(ii) $\pi(\gamma) > \gamma$;
(iii) *if $\theta < \lambda$ is a cardinal of γ-type* 0, *then* $\pi(\theta) = \theta$.

Proof. (i) Let $\nu < \gamma$. Then

$$\pi(\nu) = \varrho \circ k(\nu) = \varrho([c_\nu]).$$

So, as ϱ is the collapsing isomorphism for $\langle M, E\rangle$,

$$\pi(\nu) = \mathrm{otp}(\langle A, E\rangle),$$

where

$$A = \{[f] \in M \,|\, [f]\, E\, [c_\nu]\}.$$

Now,

$$[f]\,E\,[c_\nu] \quad \text{iff } \{\xi \in \gamma \mid f(\xi) \in c_\nu(\xi)\} \in D$$
$$\text{iff } \{\xi \in \gamma \mid f(\xi) \in \nu\} \in D$$
$$\text{iff } \bigcup_{\zeta < \nu} \{\xi \in \gamma \mid f(\xi) = \zeta\} \in D.$$

Using 4.4(v), we get

$$[f]\,E\,[c_\nu] \quad \text{iff } (\exists \zeta < \nu)[\{\xi \in \gamma \mid f(\xi) = \zeta\} \in D]$$
$$\text{iff } (\exists \zeta < \nu)[[f] = [c_\zeta]].$$

Thus

$$A = \{[c_\zeta] \mid \zeta < \nu\}.$$

But

$$\xi < \zeta \rightarrow [c_\xi]\,E\,[c_\zeta].$$

Thus

$$\pi(\nu) = \operatorname{otp}(\langle A, E\rangle) = \nu.$$

(ii) For all $\nu < \gamma$, $\gamma - \nu \in D$, so

$$\nu < \gamma \rightarrow [c_\nu]\,E\,[\mathrm{id} \restriction \gamma]\,E\,[c_\gamma].$$

Thus $\pi(\gamma) \geqslant \gamma + 1$.

(iii) Suppose that $[g]\,E\,[c_\theta]$. Thus

$$\{\nu \in \gamma \mid g(\nu) \in \theta\} \in D.$$

Define $f\colon \gamma \rightarrow \theta$ by

$$f(\nu) = \begin{cases} g(\nu), & \text{if } g(\nu) \in \theta, \\ 0\;\;\;\;\;, & \text{if } g(\nu) \notin \theta. \end{cases}$$

Then $f \in L$, so $f \in F$, and $[f] = [g]$. But $\operatorname{cf}(\theta) > \gamma$, so $f''\gamma \subseteq \nu$ for some $\nu < \theta$. Thus $[f]\,E\,[c_\nu]$, i.e. $[g]\,E\,[c_\nu]$. We have therefore shown that the set $\{[c_\nu] \mid \nu < \theta\}$ is E-cofinal in $[c_\theta]$, i.e. that $\{k(\nu) \mid \nu < \theta\}$ is E-cofinal in $k(\theta)$. But ϱ is the collapsing isomorphism for $\langle M, E\rangle$. Thus

$$\varrho(k(\theta)) = \sup_{\nu<\theta} \varrho(k(\nu)),$$

i.e.

$$\pi(\theta) = \sup_{\nu<\theta} \pi(\nu).$$

But for $\nu < \theta$, if $[g]\,E\,[c_\nu]$, then as above we have $[g] = [f]$ for some $f \in ({}^{\gamma}\nu) \cap L$, so, noting that θ is a limit cardinal and that $[\mathrm{GCH}]^L$, we have

$$|\pi(\nu)| = |\varrho \circ k(\nu)| = |\varrho([c_\nu])| = |\{[g] \mid [g]\,E\,[c_\nu]\}| \leqslant |({}^{\gamma}\nu) \cap L| < \theta.$$

Thus $\pi(\theta) \leqslant \theta$, and so, in fact, $\pi(\theta) = \theta$. □

Since $\kappa < \lambda$ is of γ-type ω_1, it follows from 4.7 (iii) that $\pi(\kappa) = \kappa$. Hence

$$(\pi \upharpoonright L_\kappa)\colon L_\kappa \prec L_\kappa .$$

Setting $e = \pi \upharpoonright L_\kappa$, 4.7 implies that e is as in 4.2. That completes the proof of 4.3.

5. The Covering Lemma

It is the very essence of $0^\#$ that its existence implies that V is very different from L. In this section we show that if $0^\#$ does not exist, then V is very similar to L. More precisely, we shall prove the following result.

5.1 Theorem (The Covering Lemma). *Assume $0^\#$ does not exist. If X is an uncountable set of ordinals, then there is a constructible set, Y, of ordinals such that $X \subseteq Y$ and $|Y| = |X|$.* (Thus, every uncountable set of ordinals is *covered* by a constructible set of ordinals of the same (real) cardinality.)

The proof will take some time. Before we commence, let us notice that if $0^\#$ does exist, then the conclusion of 5.1 fails badly. For example, if $0^\#$ exists, then ω_ω is inaccessible in L, so the countable set $\{\omega_n \mid n < \omega\}$, being cofinal in ω_ω, can only be covered by a constructible set of cardinality at least ω_ω.

It is also instructive to give some examples of how the covering lemma effects the set theory of V, making it resemble L to some extent.

5.2 Theorem. *Assume $0^\#$ does not exist. Let κ be a singular cardinal. If $2^{\mathrm{cf}(\kappa)+\omega_1} \leqslant \kappa^+$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. In particular, if κ is a singular cardinal such that $(\forall \lambda < \kappa)(2^\lambda < \kappa)$, then $2^\kappa = \kappa^+$.*

Proof. Let κ be a singular cardinal such that $2^{\mathrm{cf}(\kappa)+\omega_1} \leqslant \kappa^+$. Let A be the set of all subsets of κ of cardinality $\mathrm{cf}(\kappa)$. We know (see I.5.8) that $|A| > \kappa$, so we must prove here that $|A| \leqslant \kappa^+$ in order to obtain the first part of the theorem. (The second part follows easily by cardinal arithmetic.)

Let $X \in A$. By the covering lemma there is a set $Y \in L$, $Y \subseteq \kappa$, such that $X \subseteq Y$ and $|Y| = \mathrm{cf}(\kappa) + \omega_1$. Given such a set Y, how many subsets can it have (in V)? It has $2^{|Y|}$ many, of course. So, by hypothesis, Y has at most κ^+ subsets (in V). Now we ask ourselves how many such sets Y there are? Clearly, there are at most $|(2^\kappa)^L|$. But GCH is valid in L. So the number of possible sets Y is at most $|(\kappa^+)^L| \leqslant \kappa^+$. So the set X is one of at most κ^+ subsets of one of at most κ^+ constructible sets. There are thus at most κ^+ sets $X \in A$. □

Further consequences of the covering lemma for cardinal arithmetic are considered in Exercise 3.

5.3 Theorem. *Assume $0^\#$ does not exist. Let κ be a singular cardinal. Then $[\kappa$ is singular$]^L$.*

Proof. Let $X \subseteq \kappa$ be cofinal in κ, $|X| = \mathrm{cf}(\kappa)$. Let $X \subseteq Y \subseteq \kappa$, $Y \in L$, $|Y| = |X| + \omega_1$. Since $Y \in L$, $|Y| < \kappa$, and $\sup(Y) = \kappa$, we must have $[\kappa$ is singular$]^L$. □

Notice that as an immediate consequence of 5.3 we have:

$0^{\#}$ exists iff ω_ω is regular in L.

5.4 Theorem. *Assume $0^{\#}$ does not exist. Let κ be a singular cardinal. If $(\forall \alpha < \kappa)[\mathscr{P}(\alpha) \subseteq L]$, then $\mathscr{P}(\kappa) \subseteq L$.*

Proof. Let $A \subseteq \kappa$. We show that $A \in L$. Let $\lambda = \mathrm{cf}^L(\kappa)$, and let $(\kappa_\nu \mid \nu < \lambda) \in L$ be cofinal in κ. By 5.3, $\lambda < \kappa$. Let $f \in L, f\colon \kappa \leftrightarrow L_\kappa$. For each $\nu < \lambda$, $A \cap \kappa_\nu \in L$, so $A \cap \kappa_\nu \in L_\kappa$, and we can find an $\alpha_\nu < \kappa$ so that $A \cap \kappa_\nu = f(\alpha_\nu)$. Let $X = \{\alpha_\nu \mid \nu < \lambda\}$. Pick $Y \in L$, $Y \subseteq \kappa$, so that $X \subseteq Y$ and $|Y| = |X| + \omega_1 < \kappa$. Then $\mu = |Y|^L < \kappa$. Let $j \in L, j\colon \mu \leftrightarrow Y$. Since $j^{-1}{}''X \subseteq \mu < \kappa$, we have $j^{-1}{}''X \in L$. So, as $j \in L$, we have $X \in L$. But $f \in L$, so it follows that $A = \bigcup\{f(\alpha) \mid \alpha \in X\} \in L$. □

5.5 Theorem. *Assume $0^{\#}$ does not exist. If κ is a singular cardinal, then $(\kappa^+)^L = \kappa^+$.*

Proof. Let $\lambda = (\kappa^+)^L$. Suppose that $\lambda < \kappa^+$. Thus $|\lambda| = \kappa$, and so $\mathrm{cf}(\lambda) < \kappa$. Let $X \subseteq \lambda$ be cofinal in λ, $|X| = \mathrm{cf}(\lambda)$. Let $Y \in L$, $X \subseteq Y \subseteq \lambda$, $|Y| = |X| + \omega_1 < \kappa$. Then $|Y|^L < \kappa$. So as Y is cofinal in λ, $[\mathrm{cf}(\lambda)]^L < \kappa < \lambda$. But $[\lambda \text{ is regular}]^L$. Contradiction. □

5.6 Theorem. *Assume $0^{\#}$ does not exist. Let κ be a singular cardinal. Then $\square_\kappa$ holds.*

Proof. In L, $\square_\kappa$ is valid, so let $(C_\alpha \mid \alpha < (\kappa^+)^L \wedge \lim(\alpha)) \in L$ be a $\square_\kappa$-sequence *in the sense of* L. By 5.5, this sequence is clearly a $\square_\kappa$-sequence in the real world. □

5.7 Theorem. *Assume $0^{\#}$ does not exist. If* GCH *holds, then for every singular cardinal κ there is a κ^+-Souslin tree.*

Proof. By 5.6 and IV.2.11. □

A slight strengthening of 5.7 is considered in Exercise ID.

We turn now to the proof of the Covering Lemma. It turns out to be a little more convenient to work with the Jensen hierarchy of constructible sets, $(J_\alpha \mid \alpha \in \mathrm{On})$, rather than the hierarchy $(L_\alpha \mid \alpha \in \mathrm{On})$. The Jensen hierarchy was introduced briefly in IV.4, and is studied in detail in Chapter VI. In the meantime, we summarise the facts we need concerning this hierarchy. Note that although the Covering Lemma can be proved using the Fine Structure Theory outlined in IV.4, we shall give here a proof which is free of Fine Structure. Consequently, this section may be read independently of IV.4.

The *rudimentary functions* were defined in IV.4, so, even though you are not require to have read IV.4, there seems little point in repeating the definition here. For any set U, $\mathrm{rud}(U)$ denotes the closure of $U \cup \{U\}$ under the rudimentary functions. If U is transitive, so is $\mathrm{rud}(U)$. The Jensen hierarchy is defined by the recursion

$$J_0 = \emptyset;$$
$$J_{\alpha+1} = \mathrm{rud}(J_\alpha);$$
$$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha, \quad \text{if } \lim(\lambda).$$

Each J_α is transitive, $\alpha < \beta$ implies $J_\alpha \cup \{J_\alpha\} \subseteq J_\beta$, and $J_\alpha \cap \mathrm{On} = \omega\alpha$. We have $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$, so $J_\alpha = L_\alpha$ iff $\omega\alpha = \alpha$. Each J_α is an amenable set, and for all α,

$$J_{\alpha+1} \cap \mathscr{P}(J_\alpha) = \mathrm{Def}(J_\alpha).$$

The Jensen hierarchy thus resembles the usual L_α-hierarchy to a great extent, the main difference being that the slightly more rapid growth of the Jensen hierarchy makes *each level* amenable, not just the limit levels as is the case with the L_α-hierarchy.

There is a single rudimentary function $\mathbf{S}$ such that $U \cup \{U\} \subseteq \mathbf{S}(U)$, and in case U is transitive, $\mathrm{rud}(U) = \bigcup_{n<\omega} \mathbf{S}^n(U)$, where $\mathbf{S}^n$ denotes the n'th iterate of $\mathbf{S}$. We define a refinement of the Jensen hierarchy by the recursion

$$\begin{aligned} S_0 &= \emptyset; \\ S_{\alpha+1} &= \mathbf{S}(S_\alpha); \\ S_\lambda &= \bigcup_{\alpha<\lambda} S_\alpha, \quad \text{if } \lim(\lambda). \end{aligned}$$

Then $\alpha < \beta$ implies $S_\alpha \cup \{S_\alpha\} \subseteq S_\beta$, $S_\alpha \cap \mathrm{On} = \alpha$, and $J_\alpha = S_{\omega\alpha}$.

Every rudimentary function is Σ_0 and uniformly $\Sigma_0^{J_\alpha}$ for all $\alpha > 0$. Consequently, both $(J_\alpha \mid \alpha \in \mathrm{On})$ and $(S_\alpha \mid \alpha \in \mathrm{On})$ are Σ_1, and if $\alpha > 0$, then $(J_\nu \mid \nu < \alpha)$ and $(S_\nu \mid \nu < \omega\alpha)$ are uniformly $\Sigma_1^{J_\alpha}$.

There is a well-ordering $<_J$ of L, which is Σ_1, such that $<_J \cap (J_\alpha \times J_\alpha)$ is an initial segment of $<_J \cap (J_\beta \times J_\beta)$ whenever $\alpha < \beta$. If $\alpha > 0$, $x <_J y \in J_\alpha$ implies $x \in J_\alpha$. Moreover, $<_J \cap (J_\alpha \times J_\alpha)$ is uniformly $\Sigma_1^{J_\alpha}$ for $\alpha > 0$.

The Condensation Lemma is valid for the Jensen hierarchy: if $\alpha > 0$ and $X \prec_1 J_\alpha$, then $X \cong J_\beta$ for some unique $\beta \leqslant \alpha$.

We have already mentioned that we shall give here a proof of the Covering Lemma which does not require any of the Fine Structure Theory. For those familiar with that theory (from IV.4, perhaps) we mention that it is the following, relatively crude notion which suffices here in place of the full Fine Structure apparatus.

Let $\varphi(v_0, v_1, \ldots, v_m)$ be any $\mathscr{L}$-formula. Let $\alpha > 0$. The *J_α-skolem function for*
h_α^φ φ is the function $h_\alpha^\varphi \colon (J_\alpha)^m \to J_\alpha$ defined by

$$h_\alpha^\varphi(x_1, \ldots, x_m) = \begin{cases} \text{the } <_J\text{-least } y \in J_\alpha \text{ such that } \models_{J_\alpha} \varphi(y, x_1, \ldots, x_m), \\ \qquad \text{if such a } y \text{ exists}, \\ \emptyset, \qquad \text{if no such } y \text{ exists}. \end{cases}$$

$H_\alpha^n(A)$ Let $\alpha > 0$, $n < \omega$, $A \subseteq J_\alpha$. We denote by $H_\alpha^n(A)$ the closure of A under all J_α-skolem functions h_α^φ for which φ is Σ_n. It is easily seen that if $n > 0$,
$H_\alpha^\omega(A)$ $A \subseteq H_\alpha^n(A) \prec_n J_\alpha$. Similarly, if we denote by $H_\alpha^\omega(A)$ the closure of A under all J_α-skolem functions, then $A \subseteq H_\alpha^\omega(A) \prec J_\alpha$. (We sometimes write $\prec_\omega$ to mean $\prec$ in such contexts.) We also have $A \subseteq H_\alpha^0(A) \prec_1 J_\alpha$. To see this, suppose $\psi(v_0, \ldots, v_n)$ is Σ_1 and that $x_1, \ldots, x_n \in H_\alpha^0(A)$ are such that

$$\models_{J_\alpha} \exists y \varphi(y, x_1, \ldots, x_n)$$

Let φ be a Σ_0 formula such that $\psi(y, \vec{x})$ is equivalent to $\exists z\varphi(z, y, \vec{x})$. Then

$$\models_{J_\alpha} \exists w\varphi((w)_0, (w)_1, \vec{x}).$$

By definition,

$$\models_{J_\alpha} \varphi((h_\alpha^\varphi(\vec{x}))_0, (h_\alpha^\varphi(\vec{x}))_1, \vec{x}).$$

So

$$\models_{J_\alpha} \exists z\varphi(z, (h_\alpha^\varphi(\vec{x}))_1, \vec{x}),$$

i.e.

$$\models_{J_\alpha} \varphi((h_\alpha^\varphi(\vec{x}))_1, \vec{x}).$$

So we shall be done if we can show that $(h_\alpha^\varphi(\vec{x}))_1 \in H_\alpha^0(A)$. Well, for any ordered pair $z \in H_\alpha^0(A)$,

$$(z)_1 = \text{the } <_J\text{-least } y \in J_\alpha \text{ such that } \models_{J_\alpha} \theta(y, z),$$

where θ is the Σ_0-formula $(\exists x \in z)[z = (x, y)]$. Hence $(z)_1 = h_\alpha^\theta(z) \in H_\alpha^0(A)$, and we are done.

5.8 Lemma. *Let $\alpha > 0$, $1 \leqslant n \leqslant \omega$. Let j: $J_\alpha \prec_n J_\beta$. Let $\varphi(v_0, \ldots, v_k)$ be any Σ_n-formula of $\mathscr{L}$. Then for all $x_1, \ldots, x_k \in J_\alpha$,*

$$j(h_\alpha^\varphi(x_1, \ldots, x_k)) = h_\beta^\varphi(j(x_1), \ldots, j(x_k)).$$

Proof. Suppose first that there is no $y \in J_\alpha$ such that $\models_{J_\alpha} \varphi(y, x_1, \ldots, x_k)$. Thus

$$\models_{J_\alpha} \neg \exists y\varphi(y, x_1, \ldots, x_k).$$

Applying j, we get

$$\models_{J_\beta} \neg \exists y\varphi(y, j(x_1), \ldots, j(x_k)).$$

Hence in this case, we have

$$h_\alpha^\varphi(x_1, \ldots, x_k) = \emptyset \quad \text{and} \quad h_\beta^\varphi(j(x_1), \ldots, j(x_k)) = \emptyset,$$

and the lemma is immediate.

Now suppose there is a $y \in J_\alpha$ such that $\models_{J_\alpha} \varphi(y, x_1, \ldots, x_k)$. Set

$$y_0 = h_\alpha^\varphi(x_1, \ldots, x_k).$$

Then

$$\models_{J_\alpha} \varphi(y_0, x_1, \ldots, x_k).$$

So, applying j,

$$\models_{J_\beta} \varphi(j(y_0), j(x_1), \ldots, j(x_k)).$$

Thus if $j(y_0) \neq h_\beta^\varphi(j(x_1), \ldots, j(x_k))$, we must have

$$\models_{J_\beta} \exists z [z <_J j(y_0) \wedge \varphi(z, j(x_1), \ldots, j(x_k))].$$

Applying j^{-1}, we get

$$\models_{J_\alpha} \exists z [z <_J y_0 \wedge \varphi(z, x_1, \ldots, x_k)],$$

contrary to the choice of y_0. □

The main technique involved in the proof of the Covering Lemma is that of constructing limits of directed systems of embeddings. For the benefit of readers not familiar with this technique, we give here a brief outline of what is involved.

A *directed set* is a poset $(I, \leqslant)$ such that whenever $i, j \in I$ there is a $k \in I$ such that $i, j \leqslant k$. A simple example of such is the set $([X]^{<\omega}, \subseteq)$ of all finite subsets of a set X, ordered by inclusion.

Let $n \leqslant \omega$. A *directed Σ_n-elementary system* consists of a family $(\mathscr{A}_i \mid i \in I)$ of structures (of the same kind), indexed by members of a directed set I, together with embeddings $\sigma_{ij}: \mathscr{A}_i \prec_n \mathscr{A}_j$ for each $i, j \in I$, $i \leqslant j$, satisfying the *commutativity* condition $\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}$ for $i \leqslant j \leqslant k$. In case $n = \omega$, here, we speak simply of a *directed elementary system.*

A *direct limit* of a directed Σ_n-elementary system $\langle (\mathscr{A}_i)_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle$ consists of a structure $\mathscr{A}$ (of the same kind as all the $\mathscr{A}_i$), together with embeddings σ_i: $\mathscr{A}_i \prec_n \mathscr{A}$ such that $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i$ for $i \leqslant j$, satisfying the condition that if $x \in \mathscr{A}$ then $x \in \operatorname{ran}(\sigma_i)$ for some $i \in I$. If $\langle \mathscr{A}, (\sigma_i)_{i \in I} \rangle$, $\langle \mathscr{B}, (\tau_i)_{i \in I} \rangle$ are direct limits of the same system $\langle (\mathscr{A}_i)_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle$ we may define an isomorphism π: $\mathscr{A} \cong \mathscr{B}$ as follows: let $x \in \mathscr{A}$. Pick $i \in I$ so that $x = \sigma_i(\bar{x})$ for some $\bar{x} \in \mathscr{A}_i$. Let $\pi(x) = \tau_i(\bar{x})$. It is easily checked (using the commutativity condition) that the choice of i is unimportant here, and that π is a well-defined isomorphism. Since any two direct limits are isomorphic, we often speak of *the* direct limit of a directed elementary system. That there always is a direct limit may be demonstrated as follows.

Let $\langle (\mathscr{A}_i)_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle$ be a directed Σ_n-elementary system. For simplicity, suppose that $\mathscr{A}_i = \langle A_i, R_i \rangle$, where $R_i \subseteq A_i^n$. Set

$$U = \bigcup_{i \in I} A_i.$$

Define an equivalence relation $\sim$ on U as follows. Let $x, y \in U$. Pick $i, j \in I$ so that $x \in A_i$, $y \in A_j$. Say $x \sim y$ iff there is a $k \geqslant i, j$ such that $\sigma_{ik}(x) = \sigma_{jk}(y)$. (We leave it to the reader to check that this is an equivalence relation.) Let A be the set of equivalence classes of elements of U under $\sim$. Define a relation $R \subseteq A^n$ as follows. Let $X_1, \ldots, X_n \in A$. Since I is directed we can find an $i \in I$ such that there are elements $x_1 \in X_1 \cap A_i, \ldots, x_n \in X_n \cap A_i$. Set $R(X_1, \ldots, X_n)$ iff $R_i(x_1, \ldots, x_n)$. (We leave it to the reader to check that R is well-defined here.) Let $\mathscr{A} = \langle A, R \rangle$. For $i \in I$, define σ_i: $A_i \to A$ by letting $\sigma_i(x)$ be the equivalence class of x. It is routine to show that σ_i: $\mathscr{A}_i \prec_n \mathscr{A}$.

In cases where the direct limit of a system is well-founded, we usually take the transitive collapse of the limit as "the direct limit" to work with. In this connection, the following result is sometimes useful.

5.9 Lemma. *If* $e\colon J_\alpha \prec_0 M$, *where* M *is transitive, then* $e(S_\nu) = S_{e(\nu)}$ *for all* $\nu < \omega\alpha$.

Proof. Let φ be the canonical Σ_0 formula which defines the S_ν-hierarchy: that is, for any γ and any $\nu < \omega\gamma$, if N is a transitive set such that $J_\gamma \subseteq N$, then

$$x = S_\nu \quad \text{iff } \models_N \exists y \varphi(y, x, \nu).$$

Let $\nu < \omega\alpha$ be given. Set $x = S_\nu$. Then

$$\models_{J_\alpha} \exists y \varphi(y, x, \nu).$$

So for some $y \in J_\alpha$,

$$\models_{J_\alpha} \varphi(y, x, \nu).$$

Applying $e\colon J_\alpha \prec_0 M$,

$$\models_M \varphi(e(y), e(x), e(\nu)).$$

Thus

$$\models_M \exists y \varphi(y, e(x), e(\nu)).$$

Thus $e(x) = S_{e(\nu)}$, as required. □

There are various ways of obtaining a structure J_δ as a direct limit of a directed system of smaller structues. We describe below three methods that will be of use to us.

Let $\delta > \omega$ be given. For each integer $n > 0$ and each infinite ordinal $\eta < \omega\delta$, we define a directed Σ_n-elementary system $S^n_\delta(\eta)$ whose limit is J_δ, as follows. Let $I = I^n_\delta(\eta)$ be the set of pairs (α, p) such that $\alpha < \eta$ and p is a finite subset of J_δ. Partially order I by setting $S^n_\delta(\eta)$ $I^n_\delta(\eta)$

$$(\alpha, p) \leqslant (\beta, q) \quad \text{iff } \alpha \leqslant \beta \text{ and } p \subseteq q.$$

Under this ordering, I is a directed set. We use I to index the system $S^n_\delta(\eta)$.

Let $i = (\alpha, p) \in I$. Then

$$\alpha \cup p \subseteq H^n_\delta(\alpha \cup p) \prec_n J_\delta.$$

By the Condensation Lemma, let

$$\sigma_i\colon J_{\varrho(i)} \cong H^n_\delta(\alpha \cup p).$$

For $i, j \in I$, $i \leqslant j$, set

$$\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i.$$

(This is clearly well-defined.) Thus

$$\sigma_{ij}\colon J_{\varrho(i)} \prec_n J_{\varrho(j)},$$

and

$$\sigma_i\colon J_{\varrho(i)} \prec_n J_\delta.$$

Then $S^n_\delta(\eta) = \langle (J_{\varrho(i)})_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle$ is a directed Σ_n-elementary system whose direct limit is $\langle J_\delta, (\sigma_i)_{i \in I} \rangle$.

$S^{\omega}_{\delta}(\eta)$ $I^{\omega}_{\delta}(\eta)$ We may also represent J_δ as the limit of a directed Σ_1-elementary system $S^{\omega}_{\delta}(\eta)$, described next. As above, let $\omega \leqslant \eta < \omega\delta$. Let $I = I^{\omega}_{\delta}(\eta)$ be the set of all triples (k, α, p) such that $0 < k < \omega$, $\alpha < \eta$, and p is a finite subset of J_δ. Partially order I by setting

$$(k, \alpha, p) \leqslant (l, \beta, q) \quad \text{iff } k \leqslant l \text{ and } \alpha \leqslant \beta \text{ and } p \subseteq q .$$

The directed set I is used to index the system $S^{\omega}_{\delta}(\eta)$.

Let $i = (k, \alpha, p) \in I$. Then

$$\alpha \cup p \subseteq H^k_\delta(\alpha \cup p) \prec_k J_\delta .$$

Since $k > 0$, by the Condensation Lemma we may let

$$\sigma_i \colon J_{\varrho(i)} \cong H^k_\delta(\alpha \cup p) .$$

Then $\sigma_i \colon J_{\varrho(i)} \prec_1 J_\delta$, and in fact $\sigma_i \colon J_{\varrho(i)} \prec_k J_\delta$. For $i \leqslant j$, we may clearly define

$$\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i .$$

Then

$$\sigma_{ij} \colon J_{\varrho(i)} \prec_1 J_{\varrho(j)} .$$

$S^{\omega}_{\delta}(\eta)$ is the directed Σ_1-elementary system

$$\langle (J_{\varrho(i)})_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle .$$

Its direct limit is, of course,

$$\langle J_\delta, (\sigma_i)_{i \in I} \rangle .$$

Our third directed system to give J_δ as its limit will apply only in the case when δ is a limit ordinal. For $\omega \leqslant \eta < \omega\delta$, we define the directed Σ_1-elementary system $S^0_\delta(\eta)$ as follows. Let $I = I^0_\delta(\eta)$ be the set of all triples (ν, α, p) such that $0 < \nu < \delta$, $\alpha < \eta$, $\alpha \leqslant \nu$, and p is a finite subset of J_ν. Partially order I by setting

$S^0_\delta(\eta)$, $I^0_\delta(\eta)$

$$(\nu, \alpha, p) \leqslant (\mu, \beta, q) \quad \text{iff } [\nu = \mu \text{ or } J_\nu \in q] \text{ and } \alpha \leqslant \beta \text{ and } p \subseteq q .$$

The directed set I will be used to index the system $S^0_\delta(\eta)$.

Let $i = (\nu, \alpha, p) \in I$. Then

$$\alpha \cup p \subseteq H^0_\nu(\alpha \cup p) \prec_1 J_\nu .$$

By the Condensation Lemma,

$$\sigma_i \colon J_{\varrho(i)} \cong H^0_\nu(\alpha \cup p) .$$

Thus

$$\sigma_i \colon J_{\varrho(i)} \prec_1 J_\nu .$$

Suppose that $i = (\nu, \alpha, p)$, $j = (\mu, \beta, q)$ are elements of I, $i \leqslant j$. Let $x \in H^0_\nu(\alpha \cup p)$. If $\nu = \mu$, then $x \in H^0_\mu(\alpha \cup p) \subseteq H^0_\mu(\beta \cup q)$. If $\nu \neq \mu$, then since $i \leqslant j$ we must have $J_\nu \in q$. Now, x is Σ_1-definable from elements of $\alpha \cup p$ in J_ν. So there is a Σ_0-formula φ of $\mathscr{L}$ such that x is the unique element of J_ν such that

$$\models_{J_\mu} \varphi(x, \eta_1, \ldots, \eta_n, y_1, \ldots, y_m, J_\nu),$$

where $\eta_1, \ldots, \eta_n \in \alpha$ and $y_1, \ldots, y_m \in p$. [To obtain φ, take a formula which defines x from elements of $\alpha \cup p$ in J_ν and bind all quantifiers by J_ν.] Thus $x \in H^0_\mu(\beta \cup q)$. So we have proved that $H^0_\nu(\alpha \cup p) \subseteq H^0_\mu(\beta \cup q)$. Hence we may define

$$\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i.$$

Since Σ_0 formulas are absolute for transitive sets (I.9.14), $J_\nu \prec_0 J_\mu$. Hence

$$\sigma_{ij}\colon J_{\varrho(i)} \prec_0 J_{\varrho(j)}.$$

$S^0_\delta(\eta)$ is the directed Σ_0-elementary system

$$\langle (J_{\varrho(i)})_{i \in I}, (\sigma_{ij})_{i \leqslant j} \rangle.$$

Its direct limit is

$$\langle J_\delta, (\sigma_i)_{i \in I} \rangle.$$

(Again, by Σ_0-absoluteness, $J_\nu \prec_0 J_\delta$ for all $\nu < \delta$, so $\sigma_i\colon J_{\varrho(i)} \prec_0 J_\delta$.)

The relevance of the above directed system "representations" of structures J_δ lies in the fact that they enable us to represent a possibly large J_δ in terms of small structures $J_{\varrho(i)}$. For, although the directed system will have to be large, in the sense that the index set I must be large, the individual structures $J_{\varrho(i)}$ may all be relatively small. We investigate this phenomenon next.

Consider any of the systems $S^n_\delta(\eta)$ just defined, where $\delta > \omega$, $0 \leqslant n \leqslant \omega$, $\omega \leqslant \eta < \delta$, with $\lim(\delta)$ in case $n = 0$. Let γ be any admissible ordinal. We shall say that $S^n_\delta(\eta)$ is *below* γ if $\varrho(i) < \gamma$ for all $i \in I^n_\delta(\eta)$.

5.10 Lemma. *If $S^n_\delta(\eta)$ is below γ, then $\sigma_{ij} \in J_\gamma$ for all $i, j \in I^n_\delta(\eta)$, $i \leqslant j$.*

Proof. Consider first the case $0 < n < \omega$. Let $i, j \in I^n_\delta(\eta)$, $i \leqslant j$, $i = (\alpha, p)$, $j = (\beta, q)$. Then $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i$, where

$$\sigma_i\colon J_{\varrho(i)} \cong H^n_\delta(\alpha \cup p), \qquad \sigma_j\colon J_{\varrho(j)} \cong H^n_\delta(\beta \cup q).$$

Now, $H^n_\delta(\alpha \cup p)$ is the closure of $\alpha \cup p$ under the Σ_n skolem functions h^φ_δ. Since $\alpha \leqslant \beta$ and $p \subseteq q$, $H^n_\delta(\alpha \cup p) \subseteq H^n_\delta(\beta \cup q)$. Using 5.8 and applying

$$\sigma_j\colon J_{\varrho(j)} \cong H^n_\delta(\beta \cup q) \prec_n J_\delta$$

"backwards", we see that the set

$$\operatorname{ran}(\sigma_j^{-1} \circ \sigma_i) = \sigma_j^{-1} {}'' H^n_\delta(\alpha \cup p)$$

is the closure of $\alpha \cup \sigma_j^{-1}(p)$ under the Σ_n skolem functions $h^{\varphi}_{\varrho(j)}$, i.e.

$$\operatorname{ran}(\sigma_{ij}) = H^n_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p)).$$

Consider the definition of $H^n_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p))$ from $\varrho(j)$, n, α, $\sigma_j^{-1}(p)$. It is the subset of $J_{\varrho(j)}$ consisting of all those elements x of $J_{\varrho(j)}$ which may be obtained from elements of $\alpha \cup \sigma_j^{-1}(p)$ by finitely many applications of functions of the form $h^{\varphi}_{\varrho(j)}$ where φ is a Σ_n formula of $\mathscr{L}$. Thus, it is easily seen that $H^n_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p))$ is a $\Delta_1(J_\gamma)$ subset of $J_{\varrho(j)}$. But J_γ is admissible. Hence by Δ_1-Comprehension (I.11.1), $H^n_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p)) \in J_\gamma$.

Again, σ_{ij}^{-1} is the collapsing isomorphism for the set $H^n_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p))$, so σ_{ij}^{-1} is a $\Delta_1(J_\gamma)$ subset of $J_{\varrho(j)}$, whence $\sigma_{ij}^{-1} \in J_\gamma$. Thus $\sigma_{ij} \in J_\gamma$, as required.

Consider next the case $n = \omega$. If $i, j \in I^{\omega}_{\delta}(\eta)$, $i \leqslant j$, $i = (k, \alpha, p)$, $j = (l, \beta, q)$, then, much as above, we have

$$\sigma_{ij}\colon J_{\varrho(i)} \cong H^k_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p)),$$

and again as before this implies that $\sigma_{ij} \in J_\gamma$.

Finally suppose $n = 0$. Let $i, j \in I^0_\delta(\eta)$, $i \leqslant j$, $i = (\nu, \alpha, p)$, $j = (\mu, \beta, q)$. If $\nu = \mu$, then

$$\sigma_i\colon J_{\varrho(i)} \cong H^0_\nu(\alpha \cup p), \qquad \sigma_j\colon J_{\varrho(j)} \cong H^0_\nu(\beta \cup q),$$

so as in the above cases

$$\sigma_{ij}\colon J_{\varrho(i)} \cong H^0_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p)),$$

and as before we can conclude that $\sigma_{ij} \in J_\gamma$.

Now suppose $J_\nu \in q$. Then

$$\sigma_i\colon J_{\varrho(i)} \cong H^0_\nu(\alpha \cup p), \qquad \sigma_j\colon J_{\varrho(j)} \cong H^0_\mu(\beta \cup q).$$

Now, if φ is Σ_0 and $x_1, \ldots, x_m, y \in J_\nu$, then, as is easily seen,

$$y = h^{\varphi}_{\nu}(x_1, \ldots, x_m) \quad \text{iff } y = h^{\varphi}_{\mu}(x_1, \ldots, x_m).$$

Thus we may in fact apply the same argument as before to obtain

$$\sigma_{ij}\colon J_{\varrho(i)} \cong H^0_{\varrho(j)}(\alpha \cup \sigma_j^{-1}(p)).$$

Again this implies that $\sigma_{ij} \in J_\gamma$, so we are done in all cases. □

We are now ready to begin our proof of the Covering Lemma. We shall
τ assume from now on that the Covering Lemma is false. We fix τ the least ordinal
X such that there is an uncountable set $X \subseteq \tau$ which is not a subset of any constructible set of cardinality $|X|$. This choice of τ has two immediate consequences.

5.11 Lemma. $[\tau$ *is a cardinal*$]^L$.

Proof. Suppose otherwise. Let $\lambda = |\tau|^L$, and let $j \in L$, $j\colon \lambda \leftrightarrow \tau$. Let $\bar{X} = j^{-1}{}''X$. Since $\bar{X} \subseteq \lambda < \tau$, the minimality of τ guarantees the existence of a set $\bar{Y} \in L$, $\bar{X} \subseteq \bar{Y} \subseteq \lambda$, $|\bar{Y}| = |\bar{X}|$. Let $Y = j''\bar{Y}$. Then $Y \in L$, $X \subseteq Y \subseteq \tau$, and $|Y| = |X|$. Contradiction. □

5.12 Lemma. *If $Y \in L$ and $|Y|^L < \tau$, then Y cannot cover X.*

Proof. Let $\lambda = |Y|^L$, and let $j \in L$, $j\colon \lambda \leftrightarrow Y$. Suppose $X \subseteq Y$. Let $\bar{X} = j^{-1}{}''X$. Then $\bar{X} \subseteq \lambda < \tau$, so by choice of τ there is a set $\bar{Z} \in L$, $\bar{X} \subseteq \bar{Z} \subseteq \lambda$, $|\bar{Z}| = |\bar{X}|$. Let $Z = j''\bar{Z}$. Then $Z \in L$, $X \subseteq Z \subseteq \tau$, $|Z| = |X|$, contrary to the choice of X. □

The overall strategy behind our proof of the Covering Lemma is as follows. Let $M \prec J_\tau$, $|M| = |X|$, $X \subseteq M$, and let $\pi\colon J_\gamma \cong M$. Since $X \subseteq \tau$, we know that (by choice of X) $|X| < |\tau|$ (for otherwise $\tau \in L$ is a cover of X of cardinality $|X|$). Thus $|J_\gamma| < |\tau|$. But X is cofinal in τ and $X \subseteq \operatorname{ran}(\pi)$. Hence $\pi\colon J_\gamma \prec J_\tau$ is non-trivial. Let β be least such that $\pi(\beta) > \beta$. If $\beta < |\gamma|$, then by 4.3, $0^\#$ exists, and we have our sought-after contradiction. What if $\beta \geqslant |\gamma|$? Then we try to find a $\delta \geqslant \gamma$, $|\delta| > \beta$, such that it is possible to find an embedding $\tilde{\pi}\colon J_\delta \prec J_\nu$ which extends π (so $\tilde{\pi}(\beta) > \beta$), in which case 4.3 may again be applied. The question is, how might we extend π as desired? Well, by choosing M carefully in the first place, we find a δ such that J_δ is the direct limit of a system which is below γ. (Note that as $J_\gamma \equiv J_\tau$, γ is admissible, by virtue of 5.11.) Thus the map π sends the members of this system to a directed system inside J_τ. If the direct limit of this system is well-founded, and thus of the form J_ν for some ν, then it will be easy to construct an embedding $\tilde{\pi}\colon J_\delta \prec J_\nu$ which extends π, as we shall see. However, as we shall discover, the choice of M, in particular, must be made very carefully indeed, making use of the special properties of τ and X.

We defer until later the actual choice of the submodel $M \prec J_\tau$. We assume simply that we have found some embedding $\pi\colon J_\gamma \prec J_\tau$. Note that by 5.11, γ will be an admissible ordinal. Let us further assume that δ, n, η are such that $\delta > \omega$, $0 \leqslant n \leqslant \omega$, $\omega \leqslant \eta < \delta$ and that $S^n_\delta(\eta)$ is below γ. Then we may define a directed system $\pi^* S^n_\delta(\eta)$, of the same degree of elementarity as $S^n_\delta(\eta)$, as follows. As index set we take the set $I^n_\delta(\eta)$. Associated with $i \in I^n_\delta(\eta)$ will be the structure $J_{\pi(\varrho(i))}$. For $i, j \in I^n_\delta(\eta)$, $i \leqslant j$, the embedding associated with i, j will be $\pi(\sigma_{ij})$. Since π is elementary, if $\sigma_{ij}\colon J_{\varrho(i)} \prec_k J_{\varrho(j)}$, then $\pi(\sigma_{ij})\colon J_{\pi(\varrho(i))} \prec_k J_{\pi(\varrho(j))}$, so this makes sense, and moreover, $\pi^* S^n_\delta(\eta)$ so defined is a directed system. The lemma below shows how, under these circumstances, it is possible to extend π from J_γ to J_δ. (Actually, in the form stated, all that we get is that $\tilde{\pi}$ extends $\pi \upharpoonright \eta$, but in our main application of the lemma we shall have $\eta = \gamma$, in which case we really will have $\pi \subseteq \tilde{\pi}$.)

π, γ
δ, n, η
$\pi^* S^n_\delta(\eta)$

5.13 Lemma. *Let $\langle\langle U, E\rangle, (\theta_i)_{i \in I}\rangle$ be the direct limit of the system $\pi^* S^n_\delta(\eta)$. Then there is an embedding $\tilde{\pi}\colon \langle J_\delta, \in\rangle \prec_{1+n} \langle U, E\rangle$. Moreover, if $\langle U, E\rangle$ is well-founded, we may take $\langle U, E\rangle$ to be of the form $\langle J_\mu, \in\rangle$ for some μ, in which case $\tilde{\pi} \upharpoonright \eta = \pi \upharpoonright \eta$.*

Proof. Let $x \in J_\delta$. For some $i \in I^n_\delta(\eta)$, $x \in \operatorname{ran}(\sigma_i)$, say $x = \sigma_i(\bar{x})$, where $\bar{x} \in J_{\varrho(i)}$. Let $\bar{y} = \pi(\bar{x})$, and set $\tilde{\pi}(x) = \theta_i(\bar{y})$. (Thus $\tilde{\pi}(x) = \theta_i \circ \pi \circ \sigma_i^{-1}(x)$.) It is routine to verify that $\tilde{\pi}$ is well-defined. And in the cases $n < \omega$, it is immediate that $\tilde{\pi}$ is Σ_n-elementary. To show that in these cases $\tilde{\pi}$ is in fact Σ_{n+1}-elementary, we argue as follows.

Suppose that φ is Π_n and that

$$\langle U, E\rangle \models \exists y \varphi(y, \tilde{\pi}(x)).$$

Then for some $y \in U$,

$$\langle U, E\rangle \models \varphi(y, \tilde{\pi}(x)).$$

Pick i so that $\tilde{\pi}(x)$, $y \in \operatorname{ran}(\theta_i)$, say $\tilde{\pi}(x) = \theta_i(\bar{x})$, $y = \theta_i(\bar{y})$. Since

$$\theta_i \colon \langle J_{\pi(\varrho(i))}, \in\rangle \prec_n \langle U, E\rangle,$$

we have

$$J_{\pi(\varrho(i))} \models \varphi(\bar{y}, \bar{x}).$$

But $\bar{x} = \theta_i^{-1} \circ \tilde{\pi}(x) = \pi \circ \sigma_i^{-1}(x)$. So we may rewrite the above as

$$J_{\pi(\varrho(i))} \models \varphi(\bar{y}, \pi \circ \sigma_i^{-1}(x)).$$

Thus

$$J_{\pi(\varrho(i))} \models \exists z \varphi(z, \pi \circ \sigma_i^{-1}(x)).$$

Then, since $\pi \colon J_\gamma \prec J_\tau$, we deduce that

$$J_{\varrho(i)} \models \exists z \varphi(z, \sigma_i^{-1}(x)).$$

So for some $z \in J_{\varrho(i)}$,

$$J_{\varrho(i)} \models \varphi(z, \sigma_i^{-1}(x)).$$

But $\sigma_i \colon J_{\varrho(i)} \prec_n J_\delta$. So

$$J_\delta \models \varphi(\sigma_i(z), x).$$

Thus

$$J_\delta \models \exists y \varphi(y, x).$$

The argument in the other direction is similar, and we leave it to the reader to supply.

In the case $n = \omega$, to prove that $\tilde{\pi}$ is elementary, we argue as follows. Let φ be a formula which we wish to show is preserved by $\tilde{\pi}$. Suppose that φ is Σ_m. Pick $i = (k, \alpha, p)$ in $I_\delta^\omega(\eta)$ "large" enough so that $\operatorname{ran}(\sigma_i)$ contains all parameters involved and so that $k \geqslant m$. Then use the fact that σ_i and θ_i are Σ_k-elementary. (We leave the details to the reader.)

Now suppose that $\langle U, E\rangle$ is well-founded. Then we may assume that U is transitive and that $E = \in \cap U^2$. Let $U \cap \mathrm{On} = \omega\mu$. (It is clear that $U \cap \mathrm{On}$ must be a limit ordinal.) We prove that $U = J_\mu$. First of all set $x \in U$. Pick i so that

$x = \theta_i(\bar{x})$ for some $\bar{x} \in J_{\pi(\varrho(i))}$. For some $\nu < \omega \cdot \pi(\varrho(i))$, $\bar{x} \in S_\nu$. Applying θ_i: $J_{\pi(\varrho(i))} \prec_0 U$ and using 5.9, we have

$$x = \theta_i(\bar{x}) \in \theta_i(S_\nu) = S_{\theta_i(\nu)} \subseteq \bigcup_{\xi < \omega\mu} S_\xi = J_\mu.$$

Now let $x \in J_\mu$. For some $\nu < \omega\mu$, $x \in S_\nu$. Since $\nu \in U$ we can find an i such that $\nu = \theta_i(\bar{\nu})$, where $\bar{\nu} < \pi(\varrho(i))$. Then by 5.9 again, $\theta_i(S_{\bar{\nu}}) = S_{\theta_i(\bar{\nu})} = S_\nu$, so $S_\nu \in \operatorname{ran}(\theta_i)$. Thus $S_\nu \in U$. But U is transitive. Hence $x \in U$.

Finally, assume now that $U = J_\mu$. We show that $\tilde{\pi} \upharpoonright \eta = \pi \upharpoonright \eta$. Let $\xi < \eta$ be given. Pick $i \in I^n_\delta(\eta)$ so that $\xi \in \alpha$, where $i = (\alpha, p)$ if $0 < n < \omega$, $i = (\nu, \alpha, p)$ if $n = 0$, and $i = (k, \alpha, p)$ if $n = \omega$. Then $\sigma_{ij}(\xi) = \xi$ for all $j \geqslant i$, so $\sigma_i(\xi) = \xi$. Again, since $\sigma_{ij}(\xi) = \xi$ for all $j \geqq i$, applying π: $J_\gamma \prec J_\tau$ we have $[\pi(\sigma_{ij})](\pi(\xi)) = \pi(\xi)$ for all $j \geqslant i$. Hence as U is transitive, $\theta_i(\pi(\xi)) = \pi(\xi)$. Thus $\tilde{\pi}(\xi) = \pi(\xi)$. □

The proof of the following lemma is very complicated, and is deferred until later.

5.14 Lemma. *There is an admissible ordinal γ and an embedding π: $J_\gamma \prec J_\tau$ such that $|\gamma| = |X|$, $X \subseteq \operatorname{ran}(\pi)$, and whenever $\delta \geqslant \gamma$, then $S^\omega_\delta(\gamma)$ is below γ and the direct limit of $\pi^* S^\omega_\delta(\gamma)$ is well-founded.* □

Using 5.14, it is very easy to obtain the contradiction which proves the Covering Lemma. Namely:

5.15 Lemma. $0^\#$ *exists.*

Proof. Since $X \subseteq \operatorname{ran}(\pi)$ and $|\gamma| = |X| < |\tau|$ we can find a β such that $\pi(\beta) \neq \beta$. Pick $\delta \geqslant \gamma$, $|\delta| > \beta$. By 5.14, $S^\omega_\delta(\gamma)$ is below γ and the direct limit of $\pi^* S^\omega_\delta(\gamma)$ is well-founded. So by 5.13 we may take this limit to be J_μ for some μ, and there is an embedding $\tilde{\pi}$: $J_\delta \prec J_\mu$ such that $\tilde{\pi} \upharpoonright \gamma = \pi \upharpoonright \gamma$. But $\tilde{\pi}(\beta) \neq \beta$ and $\beta < |\delta|$. So by 4.3, $0^\#$ exists. □

Now let us begin our attack on 5.14. The part that makes use of the fact that X cannot be covered by a constructible set Y such that $|Y|^L < \tau$ (see 5.12) is the proof that $S^\omega_\delta(\gamma)$ is below γ for any $\delta \geqslant \gamma$. In fact, we shall prove, by induction on δ, n, that if $\delta \geqslant \gamma$ and $0 < n \leqq \omega$, then $S^n_\delta(\gamma)$ is below γ. (This is why we need to consider three types of directed system, not just $S^\omega_\delta(\gamma)$.) This in turn means that we must be even more careful in our original choice of γ, π. More precisely, instead of simply proving 5.14 as stated, we prove the following two results, which together imply 5.14 at once.

5.16 Lemma. *There is an admissible ordinal γ and an embedding π: $J_\gamma \prec J_\tau$ such that:*

(i) $|\gamma| = |X|$ *and* $X \subseteq \operatorname{ran}(\pi)$;

(ii) *if $\delta \geqslant \gamma$, $n \leqslant \omega$, and if* $\lim(\delta)$ *in case $n = 0$, then,* IF *$S^n_\delta(\gamma)$ is below γ, then the direct limit of $\pi^* S^n_\delta(\gamma)$ is well-founded.* □

5.17 Lemma. *Let $\delta \geqslant \gamma$, $0 < n \leqslant \omega$. Then $S^n_\delta(\gamma)$ is below γ.* □

We prove 5.17 first, since 5.16 is the more complex of the two. It is clear that 5.17 follows directly from the following lemma (which is in fact only a reformulation of 5.17 in the cases $n < \omega$, being stronger only in the case $n = \omega$).

5.18 Lemma. *Let $\delta \geqslant \gamma$, $0 < n \leqslant \omega$. For every $\alpha < \gamma$ and every finite set $p \subseteq J_\delta$,*

$$\operatorname{otp}[H_\delta^n(\alpha \cup p) \cap \mathrm{On}] < \gamma .$$

Proof. Suppose the lemma is false. Let $\delta \geqslant \gamma$ be the least ordinal for which it fails (for some n), and let n be least such that $0 < n \leqslant \omega$ and the lemma fails for δ, n. We wish to apply 5.16(ii) to $\delta, n-1$. In order to do this we must know that if $n = 1$, then $\lim(\delta)$. This is in fact the case, but we shall defer the proof for a moment, and simply assume it.

Claim 1. $S_\delta^{n-1}(\gamma)$ is below γ. (If $n = \omega$, then of course $n - 1 = \omega$.)

Proof. Suppose first that $n > 1$. Let $i \in I_\delta^{n-1}(\gamma)$. Then

$$J_{\varrho(i)} \cong H_\delta^k(\alpha \cup p)$$

for some k, $0 < k < n$ and some finite $p \subseteq J_\delta$. (If $n < \omega$, then in fact $k = n - 1$.) By the minimality of n,

$$\operatorname{otp}[H_\delta^k(\alpha \cup p) \cap \mathrm{On}] < \gamma .$$

Hence $\varrho(i) < \gamma$.

Now consider the case $n = 1$. Let $i \in I_\delta^0(\gamma)$. Then

$$J_{\varrho(i)} \cong H_\mu^0(\alpha \cup p)$$

for some $\mu < \delta$, $\alpha < \gamma$, and some finite $p \subseteq J_\mu$. If $\mu \geqslant \gamma$, then by the minimality of δ,

$$\operatorname{otp}[H_\mu^0(\alpha \cup p) \cap \mathrm{On}] < \gamma ,$$

whilst if $\mu < \gamma$, then trivially

$$\operatorname{otp}[H_\mu^0(\alpha \cup p) \cap \mathrm{On}] \leqslant \omega\mu < \gamma ,$$

so again we have $\varrho(i) < \gamma$. The claim is proved.

Claim 2. There are $\alpha_0 < \gamma$, $p_0 \subseteq J_\delta$, p_0 finite, such that

$$J_\delta = H_\delta^n(\alpha_0 \cup p_0) .$$

Proof. Pick $\alpha < \gamma$, $p \subseteq J_\delta$, p finite, such that

$$\operatorname{otp}[H_\delta^n(\alpha \cup p) \cap \mathrm{On}] \geqslant \gamma .$$

Let

$$j \colon J_{\bar\delta} \cong H_\delta^n(\alpha \cup p) .$$

Set $\alpha_0 = \alpha$, $p_0 = j^{-1}(p)$. By 5.8,

$$J_{\bar\delta} = H_{\bar\delta}^n(\alpha_0 \cup p_0) .$$

But $\bar\delta \geqslant \gamma$. So by the minimality of δ we have $\bar\delta = \delta$. The claim is proved.

By Claim 1 and 5.16 (ii), the direct limit of $\pi^* S_\delta^{n-1}(\gamma)$ is well-founded. Thus by 5.13 we may take this limit to be of the form J_ν, and there is an embedding $\tilde{\pi}$: $J_\delta \prec_n J_\nu$ such that $\tilde{\pi} \restriction \gamma = \pi \restriction \gamma$. Let $\beta_0 = \tilde{\pi}(\alpha_0)$, $q_0 = \tilde{\pi}(p_0)$. By Claim 2 and 5.8, we have, applying $\tilde{\pi}$,

$$\operatorname{ran}(\tilde{\pi}) = \tilde{\pi}'' J_\delta = \tilde{\pi}'' H_\delta^n(\alpha_0 \cup p_0) \subseteq H_\nu^n(\beta_0 \cup q_0).$$

But $X \subseteq \operatorname{ran}(\pi) \subseteq \operatorname{ran}(\tilde{\pi})$. Hence

$$X \subseteq H_\nu^n(\beta_0 \cup q_0).$$

Now, clearly, $Y = H_\nu^n(\beta_0 \cup q_0) \in L$. Moreover $|Y|^L = |\beta_0|^L + \omega$. But

$$\beta_0 = \tilde{\pi}(\alpha_0) = \pi(\alpha_0) \in J_\tau,$$

so $\beta_0 < \tau$. Thus by 5.11, $|\beta_0|^L < \tau$. Thus Y contradicts 5.12, and we are done.

We are left with the proof that if $n = 1$, then $\lim(\delta)$.[5] Suppose, on the contrary, that we had $n = 1$ and $\delta = \beta + 1$. Note that as $\delta \geqslant \gamma$ and γ is a limit ordinal, we must have $\beta \geqslant \gamma$. Choose $\alpha < \gamma$, $p \subseteq J_\delta$ finite, so that

$$\operatorname{otp}[H_\delta^1(\alpha \cup p) \cap \mathrm{On}] \geqslant \gamma.$$

Now,

$$H_\delta^1(\alpha \cup p) \cap \mathrm{On} = H_\delta^1(\alpha \cup p) \cap \omega\delta.$$

Since $\delta = \beta + 1$, if we intersect $H_\delta^1(\alpha \cup p) \cap \omega\delta$ with $\omega\beta$ we lose at most ω elements. But

$$\operatorname{otp}[H_\delta^1(\alpha \cup p) \cap \omega\delta] \geqslant \gamma = \omega\gamma.$$

Thus we must have

$$(*) \qquad \operatorname{otp}[H_\delta^1(\alpha \cup p) \cap \omega\beta] \geqslant \gamma.$$

Let $p = \{a_1, \ldots, a_l\}$. Since $a_1, \ldots, a_l \in J_\delta = \operatorname{rud}(J_\beta)$, there are rudimentary functions $f_1, \ldots, f_l$ and elements $b_1, \ldots, b_l$ of J_β such that

$$a_1 = f_1(J_\beta, b_1), \ldots, a_1 = f_l(J_\beta, b_l).$$

Let $q = \{b_1, \ldots, b_l\}$. We prove that

$$(**) \qquad H_\delta^1(\alpha \cup p) \cap \omega\beta \subseteq H_\beta^\omega(\alpha \cup q) \cap \omega\beta.$$

5 This part of the proof makes use of some technical facts concerning the Jensen hierarchy of constructible sets and the properties of rudimentary functions. These facts are proved in Chapter VI, and we simply quote them in the present account. Consequently, the reader not already familiar with the Jensen hierarchy may prefer to simply take the result $n = 1 \rightarrow \lim(\delta)$ on trust, or else to merely skip through the account given. In any event, it hardly seems worth postponing a proof of the Covering Lemma until after Chapter VI, when this one technical detail in the proof is the only point where such knowledge is required.

By (*), this implies that $\operatorname{otp}[H^\omega_\beta(\alpha \cup q) \cap \mathrm{On}] \geqslant \gamma$, contrary to the choice of δ, which completes the proof.

So let $\xi \in H^1_\delta(\alpha \cup p) \cap \omega\beta$. We must prove that $\xi \in H^\omega_\beta(\alpha \cup q)$. Let φ be a Σ_0-formula, and let $\xi_1, \ldots, \xi_k < \alpha$ be such that ξ is the least (which for ordinals is the same as the $<_J$-least) ordinal in $\omega\delta$ such that

$$(1) \qquad \models_{J_\delta} \exists x\, \varphi(x, \xi, \xi_1, \ldots, \xi_k, a_1, \ldots, a_l).$$

Pick $x \in J_\delta$ such that

$$(2) \qquad \models_{J_\delta} \varphi(x, \xi, \xi_1, \ldots, \xi_k, a_1, \ldots, a_l).$$

Then we can find a rudimentary function f and an element y of J_β such that $x = f(J_\beta, y)$. So,

$$(3) \qquad \models_{J_\delta} \varphi(f(J_\beta, y), \xi, \xi_1, \ldots, \xi_k, f_1(J_\beta, b_1), \ldots, f_l(J_\beta, b_l)).$$

Since φ is Σ_0 and $f, f_1, \ldots, f_l$ are rudimentary, the formula

$$\varphi(f(x, y), \xi, \xi_1, \ldots, \xi_k, f_1(x, b_1), \ldots, f_l(x, b_l))$$

is Σ_0 in the variables $x, y, \xi, \xi_1, \ldots, \xi_k, b_1, \ldots, b_l$. This depends upon a property of rudimentary functions that we have not mentioned before, that if $R(x)$ is a Σ_0 predicate and f is rudimentary, then $R(f(\vec{x}))$ is a Σ_0 predicate. For a proof of this fact we refer the reader to VI.1.3. It follows that there is a formula ψ of $\mathscr{L}$ such that (3) is equivalent to

$$(4) \qquad \models_{J_\beta} \psi(y, \xi, \xi_1, \ldots, \xi_k, b_1, \ldots, b_l).$$

This requires another result not yet proved, which says that Σ_0-definability over $\operatorname{rud}(U)$ for elements of a transitive rud closed set U, using parameters $U, \vec{a}$, where $\vec{a} \in U$, is equivalent to definability over U using parameters $\vec{a}$. This is proved in VI.1.18. By (4) we have

$$(5) \qquad \models_{J_\beta} \exists y \psi(y, \xi, \xi_1, \ldots, \xi_k, b_1, \ldots, b_l).$$

Moreover, ξ is the least such. For suppose, on the contrary, that $\xi' < \xi$ is such that

$$(6) \qquad \models_{J_\beta} \exists y \psi(y, \xi', \xi_1, \ldots, \xi_k, b_1, \ldots, b_l).$$

Then, using the equivalence of (3) and (4) we can find a $y' \in J_\beta$ such that

$$(7) \qquad \models_{J_\delta} \varphi(f(J_\beta, y'), \xi', \xi_1, \ldots, \xi_k, f_1(J_\beta, b_1), \ldots, f_l(J_\beta, b_l)).$$

So, setting $x' = f(J_\beta, y')$, we have

$$(8) \qquad \models_{J_\delta} \varphi(x', \xi', \xi_1, \ldots, \xi_k, a_1, \ldots, a_l).$$

This contradicts the choice of ξ. Since ξ is the least ordinal satisfying (5), we have $\xi \in H^\omega_\beta(\alpha \cup q)$. This proves (**), and completes our proof of the lemma. $\square$

This leaves us with the proof of 5.16. We shall define, by recursion, a chain of submodels

$$M_0 \prec M_1 \prec \ldots \prec M_\theta \prec \ldots \prec J_\tau \quad (\theta \leqslant \omega_1)$$

such that $X \subseteq M_0$. Setting

$$\pi_\theta \colon J_{\gamma(\theta)} \cong M_\theta$$ π_θ, $\gamma(\theta)$

for each $\theta \leqslant \omega_1$, we shall let $\gamma = \gamma(\omega_1)$, $\pi = \pi_{\omega_1}$ to obtain the lemma. That is, we shall have $|\gamma| = |X|$, and whenever $\delta \geqslant \gamma$ and $n \leqslant \omega$, with $\lim(\delta)$ in case $n = 0$, if $S^n_\delta(\gamma)$ is below γ, then the direct limit of $\pi^* S^n_\delta(\gamma)$ is well-founded. The idea is to include in the models M_θ, $\theta < \omega_1$, withnesses to any possible failure of well-foundedness of any eventual $\pi^* S^n_\delta(\gamma)$, so that the well-foundedness can be established by a proof by contradiction. γ, π

To commence, we set

$$M_0 = H^\omega_\tau(X).$$ M_θ

And a limit stages $\theta < \omega_1$, we set

$$M_\theta = \bigcup_{\psi < \theta} M_\varphi .$$

This leaves us with the case where $\theta < \omega_1$ and M_θ has been defined.

Consider a pair (n, η) such that $n \leqslant \omega$ and $\omega < \eta \leqslant \gamma(\theta)$. Suppose that there is a $\delta \geqslant \eta$ such that $S^n_\delta(\eta)$ is below $\gamma(\theta)$ and the direct limit of $\pi^*_\theta S^n_\delta(\eta)$ is not well-founded. Let δ_0 be the least such δ. Since the limit of $\pi^*_\theta S^n_{\delta_0}(\eta)$ is not well-founded, we can find a sequence $(a_k \mid k < \omega)$ and elements $j_k \in I^n_{\delta_0}(\eta)$, $j_k \leqslant j_{k+1}$, such that $a_k \in J_{\pi_\theta(\varrho(j_k))}$ and $a_{k+1} \in [\pi_\theta(\sigma_{j_k, j_{k+1}})](a_k)$, where $\varrho(i)$, σ_{ij} relate to the system $S^n_{\delta_0}(\eta)$ here.[6]

For each pair (n, η) as above, we pick one such sequence $(a_k \mid k < \omega)$. We let N_θ be the set of all elements a_k, $k < \omega$, chosen in this way. (Of course, it is possible that $N_\theta = \emptyset$.) We set

$$M_{\theta+1} = H^\omega_\tau(M_\theta \cup N_\theta).$$

By induction on $\theta < \omega_1$ we easily see that $|N_\theta| \leqslant |X|$ for all $\theta < \omega_1$. Thus $|M_\theta| = |X|$ for all $\theta \leqslant \omega_1$. In particular, if $\gamma = \gamma(\omega_1)$, then $|\gamma| = |X|$. Setting $\pi = \pi_{\omega_1}$, we have $\pi \colon J_\gamma \prec J_\tau$ and $X \subseteq \operatorname{ran}(\pi)$. So what we must show is that if

6 To avoid the necessity of extra notation in a situation where the notational complexity is already at the limit of human tolerance, we shall frequently use the symbols $\varrho(i)$, σ_{ij}, etc. to refer to various dirceted systems of any of the three basic types described earlier, and merely observe which system is referred to each time. In each case, $\varrho(i)$, σ_{ij}, etc. will have the meaning originally defined, *but for the system under consideration at the time.* This desire for notational "simplicity" is also the reason why we made no notational distinction between the three types of directed system which we introduced; with $\varrho(i)$, ϱ_{ij}, etc. having the same meaning in each case.

δ, n $\delta \geqslant \gamma$ and $n \leqslant \omega$, with $\lim(\delta)$ in case $n = 0$, and if $S^n_\delta(\gamma)$ is below γ, then the limit of $\pi^* S^n_\delta(\gamma)$ is well-founded. We assume otherwise and work for a contradiction.

We consider first the case $0 < n < \omega$. Pick sequences $(b_m | m < \omega)$, $(i_m | m < \omega)$
b_m, i_m so that $i_m \in I^n_\delta(\gamma)$, $i_m \leqslant i_{m+1}$, $i_m = (\alpha_m, p_m)$, $\alpha_m < \alpha_{m+1}$, $b_m \in J_{\pi(\varrho(i_m))}$, $b_{m+1} \in$
α_m, p_m $[\pi(\sigma_{i_m, i_{m+1}})](b_m)$, where $\varrho(i)$, σ_{ij} refer to the system $S^n_\delta(\gamma)$.

Now, in order to obtain a contradiction with the construction of M, what we require is that such a sequence $(b_m | m < \omega)$ exists for a system which is below $\gamma(\theta)$ for some $\theta < \omega_1$. But all that we know about $S^n_\delta(\gamma)$ is that it is below γ. (Indeed, for the system $S^n_\delta(\gamma)$ itself, the ordinal γ is clearly the least ordinal such that the system is below γ.) However, the subsystem $\langle (J_{\varrho(i_m)})_{m<\omega}, (\sigma_{i_m, i_s})_{m \leqslant s}\rangle$ is such that the limit of $\langle (J_{\pi(\varrho(i_m))})_{m<\omega}, (\pi(\sigma_{i_m, i_s}))_{m \leqslant s}\rangle$ is not well-founded. The idea now is to use this countable system to construct a system $S^n_{\bar\delta}(\bar\eta)$ which is below $\gamma(\theta)$ for some $\theta < \omega_1$ and for which the direct limit of $\pi^*_\theta S^n_{\bar\delta}(\bar\eta)$ is not well-founded.

Now, $S^n_\delta(\gamma)$ is below γ, so for each m, $\varrho(i_m), \alpha_m, \sigma^{-1}_{i_m}(p_m) \in J_\gamma$. Thus for each m,
θ $\pi(\varrho(i_m)), \pi(\alpha_m), \pi(\sigma^{-1}_{i_m}(p_m)) \in M_{\omega_1}$. Since $M_{\omega_1} = \bigcup_{\theta<\omega_1} M_\theta$, we can find a $\theta < \omega_1$
j such that $\pi(\varrho(i_m)), \pi(\alpha_m), \pi(\sigma^{-1}_{i_m}(p_m)) \in M_\theta$ for all $m < \omega$. Let $j = \pi^{-1} \circ \pi_\theta$. Thus

$$j\colon J_{\gamma(\theta)} \prec J_\gamma.$$

Our next move is to use j in order to "pull back" from J_γ to $J_{\gamma(\theta)}$ the system $\langle (J_{\varrho(i_m)})_{m<\omega}, (\sigma_{i_m, i_k})_{m \leqslant k}\rangle$. Since $\operatorname{ran}(j) = \operatorname{ran}(\pi^{-1} \restriction M_\theta)$, we have $\varrho(i_m)$, α_m, $\sigma^{-1}_{i_m}(p_m) \in \operatorname{ran}(j)$ for all $m < \omega$. For each $m < \omega$, let $\bar\varrho_m < \gamma(\theta)$ be such that
$\bar\varrho_m, \bar\alpha_m$ $j(\bar\varrho_m) = \varrho(i_m)$, let $\bar\alpha_m \leqslant \omega\varrho_m$ be such that $j(\bar\alpha_m) = \alpha_m$, and let $\bar p_m \subseteq J_{\bar\varrho_m}$ be such that
$\bar p_m$ $j(\bar p_m) = \sigma^{-1}_{i_m}(p_m)$.

Now, by definition,

$$\sigma_{i_m}\colon J_{\varrho(i_m)} \cong H^n_\delta(\alpha_m \cup p_m) \prec_n J_\delta.$$

So, using 5.8,

$$J_{\varrho(i_m)} = H^n_{\varrho(i_m)}(\alpha_m \cup \sigma^{-1}_{i_m}(p_m)).$$

But $j\colon J_{\gamma(\theta)} \prec J_\gamma$, so applying j^{-1} we have

$$J_{\bar\varrho_m} = H^n_{\bar\varrho_m}(\bar\alpha_m \cup \bar p_m).$$

Let $m \leqslant s$. Now, if $x \in \bar\alpha_m$, then $j(x) \in j(\bar\alpha_m) = \alpha_m$. So as $\sigma_{i_m, i_s} \restriction \alpha_m = \mathrm{id} \restriction \alpha_m$, we have $\sigma_{i_m, i_s}(j(x)) = j(x)$, and hence $j^{-1}(\sigma_{i_m, i_s}(j(x))) = x$ is defined. Again, suppose $x \in \bar p_m$. Then $j(x) \in j(\bar p_m) = \sigma^{-1}_{i_m}(p_m)$. So $\sigma_{i_m, i_s}(j(x)) \in \sigma^{-1}_{i_s}(p_m)$. But $\sigma^{-1}_{i_s}(p_s) = j(\bar p_s) \in \operatorname{ran}(j) \prec J_\gamma$. So as p_m is a finite subset of the finite set p_s, $\sigma^{-1}_{i_s}(p_m) \in \operatorname{ran}(j)$. Thus $j^{-1}(\sigma_{i_m, i_s}(j(x))) = j^{-1}(\sigma^{-1}_{i_s}(p_m))$ is defined.

Thus we can define an embedding

$\bar\sigma_{ms}$
$$\bar\sigma_{ms}\colon J_{\bar\varrho_m} \prec_n J_{\bar\varrho_s}$$

by setting

$$\bar\sigma_{ms}(h^\varphi_{\bar\varrho_m}(x_1, \ldots, x_k)) = h^\varphi_{\bar\varrho_s}(j^{-1}(\sigma_{i_m, i_s}(j(x_1))), \ldots, j^{-1}(\sigma_{i_m, i_s}(j(x_k)))),$$

for any Σ_n formula $\varphi(v_1, \ldots, v_k)$ and any $x_1, \ldots, x_k \in \bar{\alpha}_m \cup \bar{p}_m$. (By 5.8, this does define a Σ_n-elementary embedding.) Then

$$\langle (J_{\bar{\varrho}_m})_{m<\omega}, (\bar{\sigma}_{ms})_{m \leqslant s} \rangle$$

is a directed Σ_n-elementary system, and moreover the following infinite diagram commutes:

$$\begin{array}{ccc}
J_{\bar{\varrho}_0} & \xrightarrow[j \upharpoonright J_{\bar{\varrho}_0}]{} & J_{\varrho(i_0)} \\
\bar{\sigma}_{01} \downarrow & & \downarrow \sigma_{i_0, i_1} \\
J_{\bar{\varrho}_1} & \xrightarrow[j \upharpoonright J_{\bar{\varrho}_1}]{} & J_{\varrho(i_1)} \\
\bar{\sigma}_{12} \downarrow & & \downarrow \sigma_{i_1, i_2} \\
J_{\bar{\varrho}_2} & \xrightarrow[j \upharpoonright J_{\bar{\varrho}_2}]{} & J_{\varrho(i_2)} \\
\downarrow & & \downarrow \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array}$$

Let $\langle \langle U, E \rangle, (\bar{\sigma}_m)_{m<\omega} \rangle$ be the direct limit of the system $\bar{\sigma}_m$

$$\langle (J_{\bar{\varrho}_m})_{m<\omega}, (\bar{\sigma}_{ms})_{m \leqslant s} \rangle .$$

We have $\bar{\sigma}_m \colon J_{\bar{\varrho}_m} \prec_n \langle U, E \rangle$, $\sigma_{i_m} \colon J_{\varrho(i_m)} \prec_n J_\delta$, and $\sigma_{i_m, i_s} = \sigma_{i_s}^{-1} \circ \sigma_{i_m}$ for $m \leqslant s$. So we can define an embedding

$$e \colon \langle U, E \rangle \prec_n \langle J_\delta, \in \rangle$$

as follows. Let $u \in U$. Pick $m < \omega$ so that $u = \bar{\sigma}_m(x)$ for some $x \in J_{\bar{\varrho}_m}$. Set $e(u) = \sigma_{i_m}(j(x))$. (It is routine to check that e is well-defined and Σ_n-elementary.) In particular, $\langle U, E \rangle$ is well-founded, and we may assume that $\langle U, E \rangle = \langle J_{\bar{\delta}}, \in \rangle$ for some $\bar{\delta}$. $\bar{\delta}$

So, starting with a system

$$\langle (J_{\pi(\varrho(i_m))})_{m<\omega}, (\pi(\sigma_{i_m, i_s}))_{m \leqslant s} \rangle$$

which has a non-well-founded limit (witnessed by the elements b_m), we picked a $\theta < \omega_1$ sufficiently large for us to be able to use $j = \pi^{-1} \circ \pi_\theta$ (so $j \colon J_{\gamma(\theta)} \prec J_\gamma$) in order to "pull-back" from J_γ the system

$$\langle (J_{\varrho(i_m)})_{m<\omega}, (\sigma_{i_m, i_s})_{m \leqslant s} \rangle$$

to a system

$$\langle (J_{\bar{\varrho}_m})_{m<\omega}, (\bar{\sigma}_{ms})_{m \leqslant s} \rangle$$

with limit

$$\langle J_{\bar\delta}, (\bar\sigma_m)_{m<\omega}\rangle .$$

$\bar\eta$ We shall show that for $\bar\eta = \sup_{m<\omega}\bar\alpha_m$, the direct limit of $\pi_\theta^* S_{\bar\delta}^n(\bar\eta)$ is not well-founded. Indeed, we shall show that $\pi_\theta(\bar\sigma_{ms}) = \pi(\sigma_{i_m, i_s})$ for $m \leqslant s$, so that the same elements b_m witness this non-well-foundedness, just as they did for the original system.

Let $\bar\eta = \sup_{m<\omega}\bar\alpha_m$. Since $(\alpha_m \mid m<\omega)$ is strictly increasing and $j(\bar\alpha_m) = \alpha_m$, $(\bar\alpha_m \mid m<\omega)$ is strictly increasing. Hence $\bar\alpha_m < \bar\eta$ for all $m<\omega$. Since $\bar\sigma_m\colon J_{\bar\varrho_m} \prec_n J_{\bar\delta}$, we have $\bar\alpha_m \leqslant \omega\,.\,\bar\varrho_m \leqslant \omega\,.\,\bar\delta$ for all $m<\omega$. Hence $\bar\eta \leqslant \omega\bar\delta$. So we may consider the directed Σ_n-elementary system $S_{\bar\delta}^n(\bar\eta)$.

$q_m, \bar i_m$ Set $q_m = \bar\sigma_m(\bar p_m)$, $\bar i_m = (\bar\alpha_m, \bar q_m)$. Then $\bar i_m \in I_{\bar\delta}^n(\bar\eta)$ and $m \leqslant s$ implies $\bar i_m \leqslant \bar i_s$. And for the system $S_{\bar\delta}^n(\bar\eta)$ we have $\bar\varrho_m = \varrho(\bar i_m)$, $\bar\sigma_m = \sigma_{\bar i_m}$, $\bar\sigma_{ms} = \sigma_{\bar i_m, \bar i_s}$. (This is not a fact that requires any *proof*. We have simply started with a system $\langle (J_{\bar\varrho_m})_{m<\omega}, (\bar\sigma_{ms})_{m\leqslant s}\rangle$ and then defined $\bar\delta$, $\bar\eta$, $\bar i_m$ so that the above equalities are true by definition.)

5.19 Lemma. *In the system $S_{\bar\delta}^n(\bar\eta)$, for each $i \in I_{\bar\delta}^n(\bar\eta)$ there is an $m<\omega$ such that*

$$\sigma_i{}''J_{\varrho(i)} \subseteq \sigma_{\bar i_m}{}''J_{\varrho(\bar i_m)} .$$

Proof. Let $i = (\alpha, p) \in I_{\bar\delta}^n(\bar\eta)$. Since $p \subseteq J_{\bar\delta}$ is finite and $J_{\bar\delta}$ is the direct limit of the system

$$\langle (J_{\bar\varrho_m})_{m<\omega}, (\bar\sigma_{ms})_{m\leqslant s}\rangle,$$

there is an $m<\omega$ such that $p \subseteq \bar\sigma_m{}''J_{\bar\varrho_m}$. Moreover, since $\alpha < \bar\eta$ we can choose m here so that $\bar\alpha_m \geqslant \alpha$. But $\bar\sigma_m = \sigma_{\bar i_m}$ and $\sigma_{\bar i_m}\colon J_{\varrho(\bar i_m)} \cong H_{\bar\delta}^n(\bar\alpha_m \cup \bar q_m)$, so $\bar\sigma_m \restriction \bar\alpha_m = \mathrm{id} \restriction \bar\alpha_m$. Thus

$$\alpha \cup p \subseteq \bar\sigma_m{}''J_{\bar\varrho_m} \prec_n J_{\bar\delta} .$$

It follows that $H_{\bar\delta}^n(\alpha \cup p) \subseteq \bar\sigma_m{}''J_{\bar\varrho_m}$. But in $S_{\bar\delta}^n(\bar\eta)$, by definition,

$$\sigma_i\colon J_{\varrho(i)} \cong H_{\bar\delta}^n(\alpha \cup p) .$$

Thus $\sigma_i{}''J_{\varrho(i)} \subseteq \bar\sigma_m{}''J_{\bar\varrho_m}$. Since $\bar\sigma_m = \sigma_{\bar i_m}$, $\varrho_m = \varrho(\bar i_m)$ we are done. □

It follows from 5.19 that $S_{\bar\delta}^n(\bar\eta)$ is below $\gamma(\theta)$. To see this, let $i \in I_{\bar\delta}^n(\bar\eta)$ be given. Pick $m<\omega$ so that $\sigma_i{}''J_{\varrho(i)} \subseteq \sigma_{\bar i_m}{}''J_{\varrho(\bar i_m)}$. Since $\sigma_i, \sigma_{\bar i_m}$ are one-one and $\in$-preserving, it follows that $\varrho(i) \leqslant \varrho(\bar i_m) = \bar\varrho_m < \gamma(\theta)$, as required.

Since $S_{\bar\delta}^n(\bar\eta)$ is below $\gamma(\theta)$, $\pi_\theta^* S_{\bar\delta}^n(\bar\eta)$ is defined. Now,

$$\sigma_{i_m}\colon J_{\varrho(i_m)} \cong H_\delta^n(\alpha_m \cup p_m),$$
$$\sigma_{i_s}\colon J_{\varrho(i_s)} \cong H_\delta^n(\alpha_s \cup p_s).$$

Thus

$$(*) \qquad \sigma_{i_m, i_s} = \sigma_{i_s}^{-1} \circ \sigma_{i_m}\colon J_{\varrho(i_m)} \cong H_{\varrho(i_s)}^n(\alpha_m \cup \sigma_{i_s}^{-1}(p_m)) .$$

Now, by choice of θ, $\varrho(i_m)$, $\varrho(i_s)$, $\alpha_m \in \text{ran}(j)$. Moreover, the choice of θ ensures that $\sigma_{i_s}^{-1}(p_s) \in \text{ran}(j)$, so as p_m is a finite subset of the finite set p_s, we have $\sigma_{i_s}^{-1}(p_m) \in \text{ran}(j)$. Thus as $j\colon J_{\gamma(\theta)} \prec J_\gamma$, we have $\sigma_{i_m, i_s} \in \text{ran}(j)$. But $j(\bar\varrho_m) = \varrho(i_m)$, $j(\bar\varrho_s) = \varrho(i_s)$, $j(\bar\alpha_m) = \alpha_m$. Thus from (*), applying j^{-1}, we get

$$(**) \qquad j^{-1}(\sigma_{i_m, i_s})\colon J_{\bar\varrho_m} \cong H^n_{\bar\varrho_s}(\bar\alpha_m \cup j^{-1} \circ \sigma_{i_s}^{-1}(p_m)).$$

Now,

$$\begin{aligned} j^{-1} \circ \sigma_{i_s}^{-1}(p_m) &= j^{-1} \circ \sigma_{i_s}^{-1} \circ \sigma_{i_m} \circ j(\bar p_m) && \text{(by choice of } \bar p_m) \\ &= j^{-1} \circ \sigma_{i_m, i_s} \circ j(\bar p_m) && \text{(by definition of } \sigma_{i_m, i_s}) \\ &= \bar\sigma_{ms}(\bar p_m) && \text{(by commutativity of the diagram above)} \\ &= \bar\sigma_s^{-1} \circ \bar\sigma_m(\bar p_m) && \text{(by definition of } \bar\sigma_m, \bar\sigma_s) \\ &= \bar\sigma_s^{-1}(\bar q_m) && \text{(by definition of } \bar q_m). \end{aligned}$$

Thus by (**),

$$j^{-1}(\sigma_{i_m, i_s})\colon J_{\bar\varrho_m} \cong H^n_{\bar\varrho_s}(\bar\alpha_m \cup \bar\sigma_s^{-1}(\bar q_m)).$$

In other words, since $\bar\varrho_m = \varrho(\bar i_m)$, $\bar\varrho_s = \varrho(\bar i_s)$, $\bar\sigma_s = \sigma_{\bar i_s}$,

$$j^{-1}(\sigma_{i_m, i_s})\colon J_{\varrho(\bar i_m)} \cong H^n_{\varrho(\bar i_s)}(\bar\alpha_m \cup \sigma_{\bar i_s}^{-1}(\bar q_m)).$$

But $\bar i_m = (\bar\alpha_m, \bar q_m)$, $\bar i_s = (\bar\alpha_s, \bar q_s)$. Thus, in the same way that we deduced (*), we may obtain

$$\sigma_{\bar i_m, \bar i_s}\colon J_{\varrho(\bar i_m)} \cong H^n_{\varrho(\bar i_s)}(\bar\alpha_m \cup \sigma_{\bar i_s}^{-1}(\bar q_m)).$$

Hence

$$j^{-1}(\sigma_{i_m, i_s}) = \sigma_{\bar i_m, \bar i_s},$$

i.e.

$$j(\sigma_{\bar i_m, \bar i_s}) = \sigma_{i_m, i_s}.$$

Therefore, applying π,

$$\pi \circ j(\sigma_{\bar i_m, \bar i_s}) = \pi(\sigma_{i_m, i_s}).$$

But $j = \pi^{-1} \circ \pi_\theta$. So,

$$\pi_\theta(\sigma_{\bar i_m, \bar i_s}) = \pi(\sigma_{i_m, i_s}).$$

Thus by choice of the elements b_m,

$$b_{m+1} \in [\pi_\theta(\sigma_{\bar i_m, \bar i_{m+1}})](b_m)$$

for all $m < \omega$. Hence the direct limit of the system $\pi_\theta^* S^n_{\bar\delta}(\bar\eta)$ is not well-founded.

We have now arrived at the following situation. We started with a $\delta \geqslant \gamma$ and a $0 < n < \omega$, such that $S_{\delta}^{n}(\gamma)$ is below γ and the limit of $\pi^{*} S_{\delta}^{n}(\gamma)$ is not well-founded. By choosing a suitable embedding $j: J_{\gamma(\theta)} \prec J_{\gamma}$, we were able to "pull back" $S_{\delta}^{n}(\gamma)$ (or at least a subsystem of this large enough to give a non-well-founded π-image) to a system $S_{\bar{\delta}}^{n}(\bar{\eta})$ which is below $\gamma(\theta)$, such that the direct limit of $\pi_{\theta}^{*} S_{\bar{\delta}}^{n}(\bar{\eta})$ is not well-founded. (Remember also that $\pi = \pi_{\omega_1}$.)

Consider now the definition of $M_{\theta+1}$. When the pair $(n, \bar{\eta})$ was considered, $\bar{\delta}$ was, by the above, a candidate in the choice of what we then called δ_0. Hence as δ_0 was chosen minimally, $\delta_0 \leqslant \bar{\delta}$. Let $(a_k \mid k < \omega)$, $(j_k \mid k < \omega)$ be the sequences
a_k, j_k chosen for δ_0, n as described: that is, $j_k \in I_{\delta_0}^{n}(\bar{\eta})$, $j_k \leqslant j_{k+1}$, $a_k \in J_{\pi_\theta(\varrho(j_k))}$, a_{k+1}
β_k, q_k $\in [\pi_\theta(\sigma_{j_k, j_{k+1}})](a_k)$. Let $j_k = (\beta_k, q_k)$.

It is easy to construct an increasing sequence $(m_k \mid k < \omega)$ of integers such that $\beta_k \leqslant \bar{\alpha}_{m_k}$, $q_k \subseteq \sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}$, and in case $\delta_0 < \bar{\delta}$, such that $J_{\delta_0} \in \sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}$, where, as before, these relate to the system $S_{\bar{\delta}}^{n}(\bar{\eta})$. (To get $\beta_k \leqslant \bar{\alpha}_{m_k}$ we use the fact that $\beta_k < \bar{\eta} = \sup_{i<\omega} \bar{\alpha}_i$. To get $q_k \subseteq \sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}$ we use the facts that $\delta_0 \leqslant \bar{\delta}$ and $J_{\bar{\delta}}$ is the direct limit of $S_{\bar{\delta}}^{n}(\bar{\eta})$, together with 5.19. Likewise to obtain $J_{\delta_0} \in \sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}$ in case $\delta_0 < \bar{\delta}$.)

For each $k < \omega$,

$$\beta_k \cup q_k \subseteq [\sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}] \cap J_{\delta_0} \prec_n J_{\delta_0}.$$

(For if $\delta_0 = \bar{\delta}$, this just says that

$$\sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})} \prec_n J_{\bar{\delta}},$$

which we know already. Whilst if $\delta_0 < \bar{\delta}$, then from the fact that

$$\sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})} \prec_n J_{\bar{\delta}}$$

and

$$J_{\delta_0} \in \sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})},$$

we deduce easily that

$$[\sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}] \cap J_{\delta_0} \prec J_{\delta_0},$$

i.e. full elementarity.) It follows that, if σ_{j_k}, $\varrho(j_k)$ refer to the system $S_{\delta_0}^{n}(\bar{\eta})$,

$$\sigma_{j_k}{}'' J_{\varrho(j_k)} = H_{\delta_0}^{n}(\beta_k \cup q_k) \subseteq [\sigma_{\bar{\imath}_{m_k}}{}'' J_{\varrho(\bar{\imath}_{m_k})}] \cap J_{\delta_0}.$$

Thus we can define embeddings

$$e_k \colon J_{\varrho(i_k)} \prec_0 J_{\varrho(\bar{\imath}_{m_k})}$$

by

e_k $$e_k \colon \sigma_{\bar{\imath}_{m_k}}^{-1} \circ \sigma_{j_k}.$$

Now,

$$\sigma_{j_k}\colon J_{\varrho(j_k)} \cong H^n_{\delta_0}(b_k \cup q_k),$$

and

$$\sigma_{\bar{i}_{m_k}}\colon J_{\bar{\varrho}_{m_k}} \cong H^n_{\bar{\delta}}(\bar{\alpha}_{m_k} \cup \bar{q}_{m_k}),$$

so

$$e_k\colon J_{\varrho(j_k)} \cong H^n_{\psi}(\beta_k \cup \sigma^{-1}_{\bar{i}_{m_k}}(q_k)),$$

where

$$\psi = \begin{cases} \sigma^{-1}_{\bar{i}_{m_k}}, & \text{if } \delta_0 < \bar{\delta} \\ \bar{\varrho}_{m_k}, & \text{if } \delta_0 = \bar{\delta}. \end{cases}$$

Thus $e_k \in J_{\gamma(\theta)}$, and so $\pi_\theta(e_k)$ is defined. Moreover, the following diagram clearly commutes:

$$\begin{array}{ccc}
J_{\varrho(j_0)} & \xrightarrow{\;e_0\;} & J_{\varrho(\bar{i}_{m_0})} \\
{\scriptstyle \sigma_{j_0, j_1}}\downarrow & & \downarrow{\scriptstyle \sigma_{\bar{i}_{m_0}, \bar{i}_{m_1}}} \\
J_{\varrho(j_1)} & \xrightarrow{\;e_1\;} & J_{\varrho(\bar{i}_{m_1})} \\
{\scriptstyle \sigma_{j_1, j_2}}\downarrow & & \downarrow{\scriptstyle \sigma_{\bar{i}_{m_1}, \bar{i}_{m_2}}} \\
J_{\varrho(j_2)} & \xrightarrow{\;e_2\;} & J_{\varrho(\bar{i}_{m_2})} \\
\downarrow & & \downarrow \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array}$$

Applying π_θ we obtain the commutative diagram:

$$\begin{array}{ccc}
J_{\pi_\theta(\varrho(j_0))} & \xrightarrow{\;\pi_\theta(e_0)\;} & J_{\pi_\theta(\varrho(\bar{i}_{m_0}))} \\
{\scriptstyle \pi_\theta(\sigma_{j_0, j_1})}\downarrow & & \downarrow{\scriptstyle \pi_\theta(\sigma_{\bar{i}_{m_0}, \bar{i}_{m_1}})} \\
J_{\pi_\theta(\varrho(j_1))} & \xrightarrow{\;\pi_\theta(e_1)\;} & J_{\pi_\theta(\varrho(\bar{i}_{m_1}))} \\
{\scriptstyle \pi_\theta(\sigma_{j_1, j_2})}\downarrow & & \downarrow{\scriptstyle \pi_\theta(\sigma_{\bar{i}_{m_1}, \bar{i}_{m_2}})} \\
J_{\pi_\theta(\varrho(j_2))} & \xrightarrow{\;\pi_\theta(e_2)\;} & J_{\pi_\theta(\varrho(\bar{i}_{m_2}))} \\
\downarrow & & \downarrow \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array}$$

c_k Let $c_k = [\pi_\theta(e_k)](a_k)$. We know that $a_k \in M_{\omega_1}$ and that $\pi_\theta(e_k) \in M_{\omega_1}$. So, as
$\bar{c}_k$ $M_{\omega_1} \prec J_\tau$, we have $c_k \in M_{\omega_1}$. Let $\bar{c}_k = \pi^{-1}(c_k)$.

By its definition, $c_k \in J_{\pi_\theta(\varrho(\bar{i}_{m_k}))}$. Also, $a_{k+1} \in [\pi_\theta(\sigma_{j_k, j_{k+1}})](a_k)$. So, referring to the above diagram, we have (by commutativity)

$$\begin{aligned} c_{k+1} &= [\pi_\theta(e_k)](a_{k+1}) \in [\pi_\theta(e_k) \circ \pi_\theta(\sigma_{j_k, j_{k+1}})](a_k) \\ &= [\pi_\theta(\sigma_{\bar{i}_{m_k}, \bar{i}_{m_{k+1}}}) \circ \pi_\theta(e_k)](a_k) = [\pi_\theta(\sigma_{\bar{i}_{m_k}, \bar{i}_{m_{k+1}}})](c_k). \end{aligned}$$

But $\pi_\theta(\varrho(\bar{i}_m)) = \pi(\varrho(i_m))$ and $\pi_\theta(\sigma_{\bar{i}_m, \bar{i}_s}) = \pi(\sigma_{i_m, i_s})$. (The first of these equalities is easily seen, the second was proved earlier.) So, applying π^{-1} to the above results, we get

$$\bar{c}_k \in J_{\varrho(i_k)} \quad \text{and} \quad \bar{c}_{k+1} \in \sigma_{i_{m_k}, i_{m_{k+1}}}(\bar{c}_k).$$

But then

$$\sigma_{i_{m_{k+1}}}(\bar{c}_{k+1}) \in \sigma_{i_{m_k}}(\bar{c}_k)$$

for all $k < \omega$, which is absurd. That completes the proof in the case $0 < n < \omega$.

The case $n = \omega$ is handled in an entirely similar fashion. The only difference is that we must ensure that the sequences $(b_m | m < \omega)$, $(i_m | m < \omega)$ are chosen so that, if $i_m = (k_m, \alpha_m, p_m)$, then $k_m < k_{m+1}$. We may then proceed as for $0 < n < \omega$. (We leave it to the reader to check all the details. Note that we dealt with the proof of 5.10 in this fashion, giving full details for the case $0 < n < \omega$ and simply indicating the modifications required for the case $n = \omega$. With this as a model, there should be no difficulty for the reader in handling the case $n = \omega$ here as well.)

The case $n = 0$ is also similar. We start with sequences $(b_m | m < \omega)$, $(i_m | m < \omega)$ chosen so that $\mu_m < \mu_{m+1}$, where $i_m = (\mu_m, \alpha_m, p_m)$, so that, in particular, $J_{\mu_m} \in p_{m+1}$ for all m. It is then easy to modify the proof for the case $0 < n < \omega$ to work in this case. At various points we need to rely upon Σ_0-absoluteness between the structures J_δ, J_η involved. Again, the proof of 5.10 indicates the type of modification required, so once again we leave it to the reader to supply the missing details.

That completes our proof of 5.16, and with it the Covering Lemma.

Exercises

1. *The Tree Property* (Section 1)

An uncountable regular cardinal κ is said to have the *tree property* iff there is no κ-Aronszajn tree. By Theorem 1.3 (viii), if κ is weakly compact then κ has the tree property. It follows from Theorems IV.2.4 and VII.1.3 that if $V = L$, the tree property is equivalent to weak compactness. On the other hand, Silver has proved (see *Mitchell* (1972)) that if ZFC + "there is a weakly compact cardinal" is consistent, so too is ZFC + "ω_2 has the tree property". The results below show that the assumption concerning weak compactness here is essential. It is shown that if κ has the tree property, then κ is a weakly compact cardinal in the sense of L.

1 A. Show that if κ has the tree property and [κ is inaccessible]^L, then [κ is weakly compact]^L.

(Outline: Use Theorem 1.3 (vii). Let $\mathscr{F} \in L$ be, in the sense of L, a κ-complete field of subsets of $\mathscr{F}$ of cardinality κ. Pick $\lambda < (\kappa^+)^L$ admissible such that $|\mathscr{F}|^{L_\lambda} = \kappa$. Let $c \in L$, $c: \kappa \leftrightarrow \mathscr{P}(\kappa) \cap L_\lambda$. For each $\alpha < \kappa$, let

$$T_\alpha = \{f \in {}^\alpha 2 \cap L \mid |\bigcap \{c(v) \mid f(v) = 1\} \cap \bigcap \{\kappa - c(v) \mid f(v) = 0\}| = \kappa\},$$

and set

$$T = \bigcup_{\alpha < \kappa} T_\alpha.$$

Show that, under inclusion, T is a tree of height κ and width κ. By the tree property, let $f: \kappa \to 2$ define a κ-branch of T, and set

$$D = \{x \in \mathscr{P}(\kappa) \cap L \mid f(c^{-1}(x)) = 1\}.$$

Show that D is a κ-complete non-principal ultrafilter on $\mathscr{P}(\kappa) \cap L_\lambda$. Let

$$M = \{f \in L_\lambda \mid f: \kappa \to L_\lambda\},$$

and form the ultrapower M/D. Let $i: L \prec M/D$ be the canonical embedding. M/D is well-founded, so let $J: M/D \cong L_\gamma$ be the collapsing isomorphism. Let $g \in L_\lambda$, $g: \kappa \leftrightarrow \mathscr{F}$. Then $j \circ i(g) \in L$ and $j \circ i(g) \restriction \kappa = (j \circ i(g(v)) \mid v < \kappa)$, so $U = \{g(v) \mid \kappa \in j \circ i(g(v))\} \in L$. Since U is, in the sense of L, a κ-complete ultrafilter on $\mathscr{F}$, the proof is complete.)

1 B. Show that if κ has the tree property, then [κ is weakly compact]^L.

(Hint: By 1 A it suffices to show that [κ is inaccessible]^L. Suppose not. Then for some $\mu < \kappa$, $\kappa = (\mu^+)^L$. By Exercise IV.1, in L let T be a special μ^+-Aronszajn tree. In V, T is a κ-tree. Since $T_\alpha \subseteq \{f \mid f: \alpha \xrightarrow{1-1} \mu\}$ for all $\alpha < \kappa$ and the ordering on T is inclusion, T is κ-Aronszajn. Contradiction.)

The following exercise provides an alternative solution to 1 B.

1 C. Let κ be a regular cardinal in L, not weakly compact in L. Show that there is a tree T on κ in L such that if, in the real world, there is a κ-branch through T, then $\mathrm{cf}(\kappa) = \omega$.

(Hint: Let T_0 be, in L, a κ-Aronszajn tree. We may assume that T_0 is an initial part of $2^{<\kappa}$. Define T by putting a triple (α, M, b) into T iff $\alpha < \kappa$, $M = L_\beta$ for some limit ordinal β, $b \in M$, $\alpha \subseteq M$, M is the smallest $M \prec M$ such that $\alpha \cup \{b\} \subseteq M$, b is a function with domain containing α as a subset, and $b \restriction \alpha \in T_0$. We have $(\alpha, M, b) <_T (\alpha', M', b')$ iff $\alpha \leqslant \alpha'$, M is the transitive collapse of the skolem hull of $\alpha \cup \{b'\}$ in M', and b' collapses to b. Show that T is a tree, $T \in L$, and that (α, M, b) has height α in T. Show further that if $((\alpha, M_\alpha, b_\alpha) \mid \alpha < \kappa)$ is a branch through T, and $\langle M, E, b\rangle$ is the limit of the elementary system $\langle M_\alpha, \in, b_\alpha\rangle$, $\alpha < \kappa$, then $\langle M, E\rangle$ is a model of BS + $V = L$ + "b is a function", $\kappa \subseteq M$, and for each $\alpha < \kappa$, $b \restriction \alpha \in T_0$. Thus $b \restriction \kappa$ is a branch through T_0. Thus $b \notin L$, which implies that $\langle M, E\rangle$ cannot be well-founded. This implies that $\mathrm{cf}(\kappa) = \omega$.)

1 D. The following result extends Theorem 5.7. Assume $0^\#$ does not exist. Then for every strong limit cardinal κ, there is a Souslin κ^+-tree.

(Hint: Use Exercise IV.8 to strengthen IV.2.11 appropriately, and combine this with 5.2 and 5.6.)

2. *The Sharp Operation* (Section 2)

Show that for any set $a \subseteq \omega$ there is a set $a^\# \subseteq \omega$ which has the same effect upon $L[a]$ as does $0^\#$ upon L. (i.e. Show that the development of section 2 goes through for $L[a]$ whenever $a \subseteq \omega$.) Is it the case that if $a, b \subseteq \omega$ are such that $a \in L[b]$, then $a^\# \in L[b^\#]$? Investigate the relationship between the various sets $a^\#$, $a \subseteq \omega$.

3. *On the Existence of* $0^\#$ (Section 2)

Show that $0^\#$ exists iff for some (all) uncountable regular cardinal κ, every constructible set $X \subseteq \kappa$ either contains or is disjoint from a club subset of κ.

(Hint: If $0^\#$ exists, show that if $X \subseteq \kappa$, $X \in L$, then either X or else $\kappa - X$ contains $H_\kappa - \gamma$ for some $\gamma < \kappa$. For the converse, let

$$D = \{X \in \mathscr{P}(\kappa) \cap L \mid X \text{ contains a club}\},$$

show that D is an ultrafilter on $\mathscr{P}(\kappa) \cap L$ which is κ-complete for families in L, and use D to construct an ultrapower which allows the use of Theorem 4.3.).

4. *The Covering Lemma and Cardinal Arithmetic* (Section 5)

By the *Singular Cardinals Hypothesis* (SCH) we mean the assertion that for all singular cardinals κ,

$$2^{\operatorname{cf}(\kappa)} < \kappa \quad \text{implies } \kappa^{\operatorname{cf}(\kappa)} = \kappa^+ .$$

Clearly, GCH implies SCH. As is shown in the following exercises, SCH completely determines the cardinal exponentiation of singular cardinals.

4 A. Show that (in ZFC) if κ is a singular cardinal, then

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}(\kappa)}.$$

4 B. Show that SCH implies that for any singular cardinal κ,

$$2^\kappa = \begin{cases} 2^{<\kappa}, & \text{if } (\exists \lambda < \kappa)(2^{<\kappa} = 2^\lambda) \\ (2^{<\kappa})^+, & \text{otherwise.} \end{cases}$$

4 C. Show that SCH implies that for any cardinals κ, λ, singular or regular,

$$\kappa^\lambda = \begin{cases} 2^\lambda, & \text{if } 2^\lambda \geqslant \kappa \\ \kappa, & \text{if } \lambda < \operatorname{cf}(\kappa) \text{ and } 2^\lambda < \kappa \\ \kappa^+, & \text{otherwise.} \end{cases}$$

4 D. Use the Covering Lemma to show that if $0^{\#}$ does not exist, then SCH is valid.

5. *An Application of the Covering Lemma*

Prove that if $0^{\#}$ does not exist, and $\kappa \geqslant \omega_2$ is any cardinal such that $2^{<\kappa} = \kappa$, then there is a set $A \subseteq \kappa$ such that $X \in L[A]$ for every set $X \subseteq \mathrm{On}$ such that $|X| < \kappa$.