# Chapter II
# The Constructible Universe

In Zermelo-Fraenkel set theory, the notion of what consitutes a set is not really defined, but rather is taken as a basic concept. The Zermelo-Fraenkel axioms describe the *properties* of sets and the set-theoretic universe. For instance, if $X$ is an infinite set, the Power Set Axiom tells us that there is a set, $\mathscr{P}(X)$, which consists of *all* subsets of $X$. But the other axioms do not tell us very much about the members of $\mathscr{P}(X)$, or give any indication as to how big a set this is. The Axiom of Comprehension says that $\mathscr{P}(X)$ will contain all sets which are *describable* in a certain, well-defined sense, and AC will provide various choice sets and well-orderings. But the word "all" in the phrase "all subsets of $X$" is not really explained. Of course, as mathematicians we are (are we not?) quite happy with the notion of $\mathscr{P}(X)$, and so long as there are no problems, Zermelo-Fraenkel set theory can be taken as a perfectly reasonable theory. But as we know, ZFC set theory does have a major drawback: there are several easily formulated questions which cannot be answered on the basis of the ZFC axioms alone. A classic example is the status of the *continuum hypothesis*, $2^\omega = \omega_1$. It can be argued that this cannot be decided in ZFC because the ZFC axioms do not say just what constitutes a subset of $\omega$; hence we cannot relate the size of $\mathscr{P}(\omega)$ to the infinite cardinal numbers $\omega_\alpha$, $\alpha \in \mathrm{On}$. (The formal *proof* of the undecidability of CH is rather different from the above "plausibility argument".)

One way of overcoming the difficulty of undecidable questions is to extend the theory ZFC, to obtain a richer theory which provides more information about sets. (An alternative solution is simply to accept as a fact of life that some questions have no answer.) One highly successful extension of ZFC is the *constructible set theory* of Gödel. In this theory the notion of a "set" is made precise (at least relative to the ordinals). The idea is as follows.

The fundamental picture of the set-theoretic universe which the Zermelo-Fraenkel axioms supply is embodied in the cumulative hierarchy of sets. We commence with the null set, $\emptyset$, and obtain all other sets by iteratively applying the (undescribed) power set operation, $\mathscr{P}$. Thus:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \mathscr{P}(V_\alpha); \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \qquad \text{if } \lim(\lambda).$$

Then

$$V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha .$$

In *constructible set theory* we do not take as basic the (so called "unrestricted") power set operator, $\mathscr{P}$. Rather we say that a set can only be said to exist if it is definable over existing sets in much the same way that classes are obtained. Recall from I.10 that a subset $y$ of a set $x$ is said to be *x-definable* iff there is a formula $\varphi(v_0)$ of $\mathscr{L}_x$ such that

$$y = \{a \in x \mid \models_x \varphi(\mathring{a})\}.$$

To obtain the *constructible universe* of sets, we start with the empty set and iterate the operation of taking all the definable subsets at each stage. This provides us with a universe of sets in which the notion of what constitutes a set is very precisely defined (relative to the ordinals).

Now, although we can regard constructible set theory as an *alternative* to Zermelo-Fraenkel set theory, as axiomatic theories the former is an extension of the latter: in fact constructible set theory is just ZFC together with one additional axiom – the *Axiom of Constructibility*. In this volume we are taking ZFC as our basic set theory, and we shall study the notion of constructibility in its own right. Indeed, many mathematicians feel that constructible set theory is *not* a reasonable *fundamental* set theory in the sense that ZFC is, and that constructibility should *only* be studied as an interesting notion within the ZFC framework. In any event, the notion is an interesting and fruitful one, as we hope to demonstrate in the ensuing pages.

In this chapter we define the constructible universe and develop its elementary theory.

# *1. Definition of the Constructible Universe*

Let $X$ be any set. By

$$\mathrm{Def}(X)$$

we mean the set of all subsets of $X$ which are $X$-definable (in the sense of I.10). That is, $\mathrm{Def}(X)$ consists of all sets, $a$, such that for some formula $\varphi(v_0)$ of $\mathscr{L}_X$,

$$a = \{x \in X \mid \models_X \varphi(\mathring{x})\}.$$

The function Def is a well-defined set-theoretic function, and indeed has the definition:

$$\begin{aligned} v = \mathrm{Def}(u) \leftrightarrow (\forall x \in v)(\exists \varphi)&[\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\}) \\ &\wedge (x = \{z \in u \mid \exists \psi(\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi))\})] \\ &\wedge (\forall \varphi)[(\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\})) \\ &\rightarrow (\exists x \in v)(x = \{z \in u \mid \exists \psi(\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi))\})]. \end{aligned}$$

(We shall presently examine the logical complexity of this definition.)

By recursion on $\alpha \in \mathrm{On}$ we define

$$L_0 = \emptyset; \qquad L_{\alpha+1} = \mathrm{Def}(L_\alpha); \qquad L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha, \qquad \text{if } \lim(\lambda).$$

$(L_\alpha \mid \alpha \in \mathrm{On})$ is the *constructible hierarchy*, and is clearly a well-defined function (in the class sense) of ZF set theory (see later for more details). Hence $L$ is a well-defined class (again, more details later), where we set:

$$L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha.$$

$L$ is the *constructible universe.* A set $x$ is said to be *constructible* iff $x \in L$.

Our first lemma below establishes various simple and basic results about the constructible hierarchy.

**1.1 Lemma.**

(i) $\alpha \leqslant \beta$ *implies* $L_\alpha \subseteq L_\beta$.
(ii) *Each* $L_\alpha$ *is transitive.* (*Hence* $L$ *is transitive.*)
(iii) $L_\alpha \subseteq V_\alpha$ *for all* $\alpha$.
(iv) $\alpha < \beta$ *implies* $\alpha, L_\alpha \in L_\beta$. (*Hence* $\mathrm{On} \subseteq L$.)
(v) *For all* $\alpha$, $L \cap \alpha = L_\alpha \cap \mathrm{On} = \alpha$.
(vi) *For* $\alpha \leqslant \omega$, $L_\alpha = V_\alpha$.
(vii) *For* $\alpha \geqslant \omega$, $|L_\alpha| = |\alpha|$.

*Proof.* (i) and (ii). We prove by simultaneous induction on $\alpha$ that;

(a) $\qquad \gamma < \alpha \rightarrow L_\gamma \subseteq L_\alpha$;

(b) $\qquad L_\alpha$ is transitive.

For $\alpha = 0$ this is trivial. For limit $\alpha$, we have $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$, so (a) and (b) are immediate consequences of the induction hypothesis. (In particular, note that any union of transitive sets is transitive.) In order to prove that (a) and (b) for $\alpha + 1$ follow from (a) and (b) for $\alpha$, let us start with (a) for $\alpha + 1$. It clearly suffices to prove that $L_\alpha \subseteq L_{\alpha+1}$. Let $x \in L_\alpha$. Then by (b) for $\alpha$, $x \subseteq L_\alpha$, so by $\Sigma_0$-absoluteness,

$$x = \{y \in L_\alpha \mid \models_{L_\alpha} \text{“}\mathring{y} \in \mathring{x}\text{”}\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

To prove (b) for $\alpha + 1$ now, let $x \in y \in L_{\alpha+1}$. Since $y \in L_{\alpha+1} = \mathrm{Def}(L_\alpha) \subseteq \mathscr{P}(L_\alpha)$, we have $y \subseteq L_\alpha$. But then $x \in L_\alpha$, so by (a) for $\alpha + 1$ just proved, we have $x \in L_{\alpha+1}$, and we are done.

(iii) By induction on $\alpha$. For $\alpha = 0$ we have

$$L_0 = V_0 = \emptyset.$$

At limit stages $\lambda$, we have

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad \text{and} \quad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha,$$

so if $L_\alpha \subseteq V_\alpha$ for all $\alpha < \lambda$, then $L_\lambda \subseteq V_\lambda$. Finally, if $L_\alpha \subseteq V_\alpha$, then

$$L_{\alpha+1} = \mathrm{Def}(L_\alpha) \subseteq \mathscr{P}(L_\alpha) \subseteq \mathscr{P}(V_\alpha) = V_{\alpha+1}.$$

(iv) By (i) it suffices to prove that $\alpha, L_\alpha \in L_{\alpha+1}$ for all $\alpha$. Well, for any $\alpha$,

$$L_\alpha = \{x \in L_\alpha \mid \models_{L_\alpha} \text{“}\mathring{x} = \mathring{x}\text{”}\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

To prove that $\alpha \in L_{\alpha+1}$ we proceed by induction on $\alpha$. Assume that $\gamma \in L_{\gamma+1}$ for all $\gamma < \alpha$. Then by (i), $\gamma \in L_\alpha$ for all $\gamma < \alpha$, i.e. $\alpha \subseteq L_\alpha$. Thus by (ii), $\alpha = L_\alpha \cap \mathrm{On}$. But $\mathrm{On}(v_0)$ is a $\Sigma_0$ formula and is thus absolute for $L_\alpha$. Hence

$$\alpha = \{x \in L_\alpha \mid \models_{L_\alpha} \mathrm{On}(\mathring{x})\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

(Actually we are being a bit sloppy here. As we defined it, $\mathrm{On}(x)$ is a formula of LST, and thus not available for use as above. However, if we take instead the corresponding $\mathscr{L}$-formula, as described in I.9.11, then by I.9.15 we see that for any $x$ and any transitive set $M$ which contains $x$:

$$x \text{ is an ordinal} \leftrightarrow \models_M \mathrm{On}(\mathring{x}).$$

In future we shall not bother too much about fine points of this nature.)

(v) That $L_\alpha \cap \mathrm{On} = \alpha$ was proved during the proof of (iv). In view of (ii), this proves all of the equalities in (v).

(vi) For $\alpha = 0$ we have $L_0 = \emptyset = V_0$. Let $\alpha < \omega$ and assume that $L_\alpha = V_\alpha$. We prove that $L_{\alpha+1} = V_{\alpha+1}$. By (ii) it suffices to prove that $V_{\alpha+1} \subseteq L_{\alpha+1}$. Let $x \in V_{\alpha+1}$. Then $x \subseteq V_\alpha = L_\alpha$, and there are $a_1, \ldots, a_n \in L_\alpha$ such that

$$x = \{a_1, \ldots, a_n\}.$$

(Because $V_\alpha$ is finite for each $\alpha < \omega$.) Hence

$$x = \{z \in L_\alpha \mid \models_{L_\alpha} (\mathring{z} = \mathring{a}_1 \vee \ldots \vee \mathring{z} = \mathring{a}_n)\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1}.$$

Thus by induction, $L_\alpha = V_\alpha$ for all $\alpha < \omega$. It follows at once that

$$L_\omega = \bigcup_{\alpha<\omega} L_\alpha = \bigcup_{\alpha<\omega} V_\alpha = V_\omega.$$

(vii) By (v) we have $|\alpha| \leqslant |L_\alpha|$ for all $\alpha$. By induction on $\alpha \geqslant \omega$ we prove that $|L_\alpha| \leqslant |\alpha|$ for all $\alpha \geqslant \omega$. For $\alpha = \omega$ this holds by (vi), since

$$|L_\omega| = |V_\omega| = \omega.$$

Suppose next that $\lim(\lambda)$ and we know that $|L_\alpha| \leqslant |\alpha|$ for all $\alpha < \lambda$. Then

$$|L_\lambda| = |\bigcup_{\alpha<\lambda} L_\alpha| \leqslant \sum_{\alpha<\lambda} |L_\alpha| \leqslant \sum_{\alpha<\lambda} |\alpha| = |\lambda|.$$

Finally, suppose that $|L_\alpha| \leqslant |\alpha|$. We prove that $|L_{\alpha+1}| \leqslant |\alpha| \,(= |\alpha + 1|)$. Well, since $\mathscr{L}$ is countable, the set of formulas of $\mathscr{L}_{L_\alpha}$ is easily seen to have cardinality $|L_\alpha|$. But this at once implies that

$$|L_{\alpha+1}| = |\mathrm{Def}(L_\alpha)| \leqslant |L_\alpha| \leqslant |\alpha|,$$

and we are done. □

Let $M$ be a transitive proper class, and let $T$ be a theory in LST. We say that $M$ is an *inner model* of $T$ iff $\Phi^M$ for every axiom $\Phi$ of $T$. (The name "inner model" arises from the case where $T$ is the theory ZF, in which case $M$ is a sort of "inner universe" of set theory. But it is convenient to formulate the definition to cover all LST theories $T$.) The following result is fundamental to all work on constructibility theory.

**1.2 Theorem.** *The class* $L$ *is an inner model of* ZF. *More precisely, for every axiom* $\Phi$ *of* ZF,

$$\mathrm{ZF} \vdash \Phi^L.$$

*Proof.* For each axiom $\Phi$ of ZF in turn, we argue in ZF to prove $\Phi^L$.

I. *Extensionality.* We must prove

$$[(\forall x, y)[(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow (x = y)]]^L.$$

Thus, given $x, y \in L$, we must prove

$$[(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow (x = y)]^L.$$

This is the same as

$$(\forall z \in L)(z \in x \leftrightarrow z \in y) \rightarrow (x = y).$$

But since $L$ is transitive, $x, y \subseteq L$, so this is the same as

$$(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow (x = y).$$

And this is true by virtue of the (real) Axiom of Extensionality itself.

II. *Union.* We must prove

$$[\forall x \, \exists y \, \forall z (z \in y \leftrightarrow (\exists u \in x)(z \in u))]^L.$$

Thus, given an $x \in L$ we must find a $y \in L$ such that

$$[\forall z (z \in y \leftrightarrow (\exists u \in x)(z \in u)]^L,$$

i.e. such that

$$(\forall z \in L)(z \in y \leftrightarrow (\exists u \in x)(z \in u)).$$

By the Axiom of Union itself, let

$$y = \bigcup x .$$

Since $x \in L$ there is an ordinal $\alpha$ such that $x \in L_\alpha$. Since $L_\alpha$ is transitive, $y \subseteq L_\alpha$. Moreover,

$$y = \{z \in L_\alpha \mid \models_{L_\alpha} (\exists v_1 \in \mathring{x})(\mathring{z} \in v_1)\},$$

so

$$y \in \mathrm{Def}(L_\alpha) = L_{\alpha+1} \subseteq L .$$

But since $y = \bigcup x$,

$$(\forall z)(z \in y \leftrightarrow (\exists u \in x)(z \in u)),$$

so in particular

$$(\forall z \in L)(z \in y \leftrightarrow (\exists u \in x)(z \in u)),$$

as required.

III. *Infinity*. We must show that

$$[\exists x [\exists y (y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]^L.$$

But by 1.1 (iv), $\omega \in L_{\omega+1} \subseteq L$, so this is immediate.

IV. *Power Set*. We must show that

$$[\forall x \, \exists y \, \forall z (z \in y \leftrightarrow z \subseteq x)]^L .$$

So, given $x \in L$ we must find a $y \in L$ such that

$$(\forall z \in L)(z \in y \leftrightarrow z \subseteq x) .$$

By the Axioms of Power Set and Comprehension, let

$$y = \{z \in \mathscr{P}(x) \mid z \in L\} .$$

We prove that $y \in L$, in which case $y$ is clearly as required.

For each $z \in y$, let $f(z)$ be the least $\alpha$ such that $z \in L_\alpha$. By the Axiom of Collection, let $\alpha$ exceed all $f(z)$ for $z \in y$. Thus $y \subseteq L_\alpha$. But then

$$y = \{z \in L_\alpha \mid \models_{L_\alpha} (\mathring{z} \subseteq \mathring{x})\} \in \mathrm{Def}(L_\alpha) = L_{\alpha+1} .$$

Thus $y \in L$.

V. *Foundation.* We must prove that

$$[\forall x\,[\exists y(y \in x) \rightarrow \exists y(y \in x \wedge (\forall z \in y)(z \notin x))]]^L.$$

Let $x \in L$ be given, $x \neq \emptyset$. We must find a $y \in L$ such that $y \in x$ and

$$[(\forall z \in y)(z \notin x)]^L.$$

By the Axiom of Foundation itself there is a $y \in x$ such that

$$(\forall z \in y)(z \notin x).$$

But $L$ is transitive, so $y \in L$. Clearly, $y$ is as required.

VI. *Comprehension.* Let $\Phi(v_0, \ldots, v_n)$ be a formula of LST. We must prove that

$$[\forall x\,\forall v_1 \ldots \forall v_n \exists y\,\forall z\,[(z \in y) \leftrightarrow (z \in x) \wedge \Phi(z, v_1, \ldots, v_n)]]^L.$$

Let $x, a_1, \ldots, a_n \in L$ be given. We seek a $y \in L$ such that

$$(\forall z \in L)[(z \in y) \leftrightarrow (z \in x) \wedge \Phi^L(z, a_1, \ldots, a_n)].$$

Pick $\alpha$ so that $x, a_1, \ldots, a_n \in L_\alpha$. Applying the Generalised Reflection Principle (I.8.1) to the constructible hierarchy, we can find a $\beta > \alpha$ such that

$$(\forall \vec{z} \in L_\beta)[\Phi^{L_\beta}(\vec{z}) \leftrightarrow \Phi^L(\vec{z})].$$

Let $\varphi(v_0, \ldots, v_n)$ be the $\mathscr{L}$-formula corresponding to $\Phi$, and set

$$y = \{z \in L_\beta \mid \models_{L_\beta} [\varphi(\mathring{z}, \mathring{a}_1, \ldots, \mathring{a}_n) \wedge (\mathring{z} \in \mathring{x})]\}.$$

Then $y \in L_{\beta+1} \subseteq L$. But by I.9.11,

$$y = \{z \in x \mid \Phi^{L_\beta}(z, a_1, \ldots, a_n)\}.$$

So by choice of $\beta$,

$$y = \{z \in x \mid \Phi^L(z, a_1, \ldots, a_n)\}.$$

Since $y \in L$ we are done.

VII. *Collection.* We must show that if $\Phi(v_0, \ldots, v_n)$ is any LST formula, then

$$\begin{aligned}&[\forall v_2 \ldots v_n [\forall x\, \exists y\, \Phi(y, x, v_2, \ldots, v_n)\\ &\qquad \rightarrow \forall u\, \exists v (\forall x \in u)(\exists y \in v)\, \Phi(y, x, v_2, \ldots, v_n)]]^L.\end{aligned}$$

Let $a_2, \ldots, a_n \in L$ be given, and suppose that

$$(\forall x \in L)(\exists y \in L)\, \Phi^L(y, x, a_2, \ldots, a_n).$$

We must show that, if we are given a $u \in L$ then there is a $v \in \mathrm{L}$ such that

$$(\forall x \in u)(\exists y \in v)\, \Phi^L(y, x, a_2, \ldots, a_n).$$

(Since $u, v \subseteq L$ here, by the transitivity of $L$, we do not need to bind $x, y$ by $L$.) Well, for each $x \in u$, let $f(x)$ be the least ordinal $\gamma$ such that

$$(\exists y \in L_\gamma)\, \Phi^L(y, x, a_2, \ldots, a_n).$$

By the Axiom of Collection (in V), let $\alpha$ exceed all $f(x)$ for $x \in u$. Let $v = L_\alpha$. By 1.1 (iv), $v \in L$. Clearly,

$$(\forall x \in u)(\exists y \in v)\, \Phi^L(y, x, a_2, \ldots, a_n),$$

so we are done.

The theorem is proved. $\square$

We shall in fact prove that

$$\mathrm{ZF} \vdash (\mathrm{AC})^L,$$

so $L$ is an inner model of ZFC. This in turn will enable us to prove that AC cannot be disproved in ZF set theory. But first we must establish some further technical results about the constructible hierarchy. This is the business of the next section.

## 2. *The Constructible Hierarchy. The Axiom of Constructibility*

Recall from I.10 that a transitive set $M$ is *amenable* iff:

(i) $(\forall x, y \in M)(\{x, y\} \in M)$;
(ii) $(\forall x \in M)(\bigcup x \in M)$;
(iii) $\omega \in M$;
(iv) $(\forall x, y \in M)(x \times y \in M)$;
(v) if $R \subseteq M$ is $\Sigma_0(M)$, then $(\forall x \in M)(R \cap x \in M)$.

(Intuitively, $M$ is a "model" of the theory BS of I.9.) Our first lemma enables us to apply the results of I.9 and I.10 to the limit levels of the constructible hierarchy.

**2.1 Lemma.** *For each limit ordinal $\alpha > \omega$, $L_\alpha$ is amenable.*

*Proof.* (i) Let $x, y \in L_\alpha$. Since

$$L_\alpha = \bigcup_{\beta < \alpha} L_\beta,$$

there is a $\beta < \alpha$ such that $x, y \in L_\beta$. Then

$$\{x, y\} = \{z \in L_\beta \mid \models_{L_\beta} (\mathring{z} = \mathring{x} \vee \mathring{z} = \mathring{y})\} \in L_{\beta+1} \subseteq L_\alpha.$$

(ii) Let $x \in L_\alpha$. For some $\beta < \alpha$, $x \in L_\beta$. Since $L_\beta$ is transitive, $\bigcup x \subseteq L_\beta$, and we have

$$\bigcup x = \{z \in L_\beta \mid \models_{L_\beta} (\exists u \in \mathring{x})(\mathring{z} \in u)\} \in L_{\beta+1} \subseteq L_\alpha.$$

(iii) By 1.1 (iv), $\omega \in L_\alpha$.

(iv) Let $x, y \in L_\alpha$. For some $\beta < \alpha$, $x, y \in L_\beta$. Since $L_\beta$ is transitive, $x, y \subseteq L_\beta$. Let $a \in x$, $b \in y$. Then $a, b \in L_\beta$, so clearly (see the proof of (i)) $\{a\}, \{a, b\} \in L_{\beta+1}$, and hence $(a, b) = \{\{a\}, \{a, b\}\} \in L_{\beta+2}$. Thus $x \times y \subseteq L_{\beta+2}$ and we have

$$x \times y = \{z \in L_{\beta+2} \mid \models_{L_{\beta+2}} (\exists a \in \mathring{x})(\exists b \in \mathring{y})[\mathring{z} = (a, b)]\} \in L_{\beta+3} \subseteq L_\alpha.$$

(In fact $x \times y \in L_{\beta+2}$. Why?)

(v) Let $R \subseteq L_\alpha$ be $\Sigma_0(L_\alpha)$, $u \in L_\alpha$. We show that $R \cap u \in L_\alpha$. Let $\varphi(v_0, \ldots, v_n)$ be a $\Sigma_0$ formula of $\mathscr{L}$ and let $a_1, \ldots, a_n \in L_\alpha$ be such that

$$(\forall x \in L_\alpha)[x \in R \leftrightarrow \models_{L_\alpha} \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n)].$$

Pick $\beta < \alpha$ such that $u, a_1, \ldots, a_n \in L_\beta$. Since $L_\beta$ is transitive, $u \subseteq L_\beta$, so

$$R \cap u = \{x \mid x \in u \wedge x \in R\} = \{x \in L_\beta \mid x \in u \wedge x \in R\}.$$

Now, being $\Sigma_0$, $\varphi$ is absolute for $L_\beta$, $L_\alpha$ (by I.9.14), so for $x \in L_\beta$,

$$\models_{L_\beta} \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n) \leftrightarrow \models_{L_\alpha} \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n).$$

Hence

$$\begin{aligned} R \cap u &= \{x \in L_\beta \mid x \in u \wedge \models_{L_\alpha} \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n)\} \\ &= \{x \in L_\beta \mid x \in u \wedge \models_{L_\beta} \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n)\} \\ &= \{x \in L_\beta \mid \models_{L_\beta} [\mathring{x} \in \mathring{u} \wedge \varphi(\mathring{x}, \mathring{a}_1, \ldots, \mathring{a}_n)]\} \in L_{\beta+1} \subseteq L_\alpha. \end{aligned}$$

The proof is complete. $\square$

Towards the end of Chapter I, we mentioned on more than one occasion that it would be necessary to carry through two parallel developments concerning logical complexity, one of a metamathematical nature, involving the language LST, the other within set theory, utilising the language $\mathscr{L}$. We are now at the point where we must begin this process.

In I.9, we investigated the logical complexity of the basic syntactical and semantical notions of the language $\mathscr{L}_V$, showing that each concept could be defined by means of a formula of LST which is $\Delta_1^{\text{BS}}$. We shall make *direct* use of these results. However, we shall require analogous results obtained within set theory. More precisely, working within the theory ZF (in fact KP will suffice, as we shall see), we shall need to examine the construction of the constructible hierarchy with regards to its definability properties along the lines of I.10. As a starting point, let us observe that now that we have the language $\mathscr{L}$ available, we can use it to analyse the syntax and semantics of $\mathscr{L}_V$ instead of working in LST. For this, it is convenient to agree to identify each formula of LST with the class it determines. With which convention it should be clear that each of the BS-complexity results of I.9 provides (by means of the replacement of LST by $\mathscr{L}$) a uniform definability result for amenable sets. For example, by repeating the proof of I.9.10 for $\mathscr{L}$ in place of LST, we obtain a proof of the fact that the class Sat ($= \{(u, \varphi) | \operatorname{Sat}(u, \varphi)\}$) is uniformly $\Delta_1^M$ for amenable sets $M$. That is, there is a $\Sigma_1$ formula $\psi(x, y)$ of $\mathscr{L}$ and a $\Pi_1$ formula $\theta(x, y)$ of $\mathscr{L}$ such that for any amenable set $M$, if $u, \varphi \in M$, then

$$\operatorname{Sat}(u, \varphi) \leftrightarrow \models_M \psi(\mathring{u}, \mathring{\varphi}) \leftrightarrow \models_M \theta(\mathring{u}, \mathring{\varphi}).$$

(The formulas $\psi$ and $\theta$ are just the $\mathscr{L}$ analogues of the LST formulas described in I.9.10.)

Let $\operatorname{Seq}(y, x)$ be the LST formula which says that $y$ is the set of all finite sequence from $x$. More precisely (cf. I.9.5), let $\operatorname{Seq}(y, x)$ be the LST formula:

$$\begin{aligned}(\exists f)[&(f \text{ is a function}) \wedge (\operatorname{dom}(f) = \omega) \wedge (f(0) = \emptyset) \wedge (y = \bigcup \operatorname{ran}(f))\\ &\wedge (\forall n \in \omega)(\forall s \in f(n+1))(\exists t \in f(n))(\exists a \in x)(s = t \cup \{(a, n)\})\\ &\wedge (\forall n \in \omega)(\forall s \in f(n))(\forall a \in x)(\exists t \in f(n+1))(t = s \cup \{(a, n)\})].\end{aligned}$$

**2.2 Lemma.**

(i) *The* LST *formula* $\operatorname{Seq}(y, x)$ *is* $\Delta_1^{\mathrm{KP}}$.

(ii) *The class* Seq *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$.

*Proof.* (i) As it stands, $\operatorname{Seq}(y, x)$ is $\Sigma_1$. Or rather, it is $\Sigma_1$ provided we eliminate explicit mention of $\omega$ by means of the prefix (to the entire formula)

$$\begin{aligned}\exists w\,[&\operatorname{On}(w) \wedge (\forall u \in w)(u \text{ is a natural number}) \wedge (\forall u \in w)(\exists v \in w)(u \in v)\\ &\wedge \dots\ \dots\ \dots],\end{aligned}$$

thereafter replacing each mention of $\omega$ by $w$.

Now, in KP, using the Recursion Theorem (I.11.8), for any set $x$ we can construct a function $f$ as in $\operatorname{Seq}(y, x)$, so

$$\mathrm{KP} \vdash \forall x \exists y \operatorname{Seq}(y, x).$$

Clearly, any such $y$ must be unique. Thus,

$$\mathrm{KP} \vdash \operatorname{Seq}(y, x) \leftrightarrow \forall z [\operatorname{Seq}(z, x) \to z = y].$$

This shows that $\operatorname{Seq}(y, x)$ is $\Delta_1^{\mathrm{KP}}$.

(ii) The ideas employed in the proof of this part of the lemma will be used several times in what follows, so we shall first of all consider in general how we can get from a $\Sigma_1^{\mathrm{KP}}$ definability result to a $\Sigma_1^{L_\alpha}$ definability result.

Suppose then that $\Phi(f, x)$ is a $\Sigma_0$ formula of LST, determining the class $A = \{x \mid \exists f \Phi(f, x)\}$. We wish to prove that the class $A$ is $\Sigma_1^{L_\alpha}$ for some limit ordinal $\alpha > \omega$. Consider the $\mathscr{L}$-analogue of $\exists f \Phi(f, x)$, which will be of the form $\exists f \varphi(f, x)$, where $\varphi$ is a $\Sigma_0$ formula of $\mathscr{L}$. We prove that for any $x \in L_\alpha$,

$$x \in A \quad \text{iff } \models_{L_\alpha} \exists f \varphi(f, \mathring{x}).$$

Now by I.9.15, if $x, f \in L_\alpha$, we have

$$\Phi(f, x) \leftrightarrow \models_{L_\alpha} \varphi(\mathring{f}, \mathring{x}).$$

Consequently, for $x \in L_\alpha$,

$$\models_{L_\alpha} \exists f \varphi(f, \mathring{x}) \quad \text{implies } \exists f \Phi(f, x).$$

This leaves us with the proof that

$$\exists f \Phi(f, x) \quad \text{implies } \models_{L_\alpha} \exists f \varphi(f, \mathring{x}).$$

So, in practice what we must prove is that if there is an $f$ such that $\Phi(f, x)$, then there is such an $f$ in $L_\alpha$. (In all the cases we shall encounter, any such $f$ will be unique, so what we shall prove is that if $\Phi(f, x)$, where $x \in L_\alpha$, then $f \in L_\alpha$.) Now let us see how this works in the case of the problem in hand.

Let $\varphi(y, x)$ be the $\mathscr{L}$-analogue of the LST-formula $\mathrm{Seq}(y, x)$. Let $\alpha > \omega$, $\lim(\alpha)$. We prove that for any $x, y \in L_\alpha$,

$$\mathrm{Seq}(y, x) \leftrightarrow \models_{L_\alpha} \varphi(\mathring{y}, \mathring{x}).$$

This will show that the class Seq is uniformly $\Sigma_1^{L_\alpha}$ for limit $\alpha > \omega$. We shall also prove that for any $x \in L_\alpha$ there is a (necessarily unique) $y \in L_\alpha$ such that $\mathrm{Seq}(y, x)$, from which fact it follows as in part (i) that Seq is also $\Pi_1^{L_\alpha}$ (uniformly for limit $\alpha > \omega$).

Let $x \in L_\alpha$. Pick $\gamma < \alpha$ so that $\gamma > \omega$ and $x \in L_\gamma$. If $a \in x$, then we have $(a, n) = \{\{a\}, \{a, n\}\} \in L_{\gamma+1}$ for any $n \in \omega$, so if $s$ is any finite sequence from $x$, then $s \in L_{\gamma+2}$. Thus $\mathrm{Seq}(y, x)$, where

$$y = \{s \in L_{\gamma+2} \mid \models_{L_{\gamma+2}} \text{“}\mathring{s} \text{ is a finite sequence from } \mathring{x}\text{”}\}.$$

But $y \in L_{\gamma+3} \subseteq L_\alpha$. Consider now the function $f$ which figures in the formula $\mathrm{Seq}(y, x)$. If it exists (i.e. if $\mathrm{Seq}(y, x)$), then clearly,

$$f = \{(s, n) \mid s = {}^n x \wedge n \in \omega\}.$$

It is easily seen that for any $n \in \omega$, ${}^n x \in L_{\gamma+3}$, so $({}^n x, n) \in L_{\gamma+5}$, giving $f \in L_{\gamma+6}$.

Thus $f \in L_\alpha$, which implies (see the above discussion):

$$\mathrm{Seq}(y, x) \leftrightarrow \models_{L_\alpha} \varphi(\mathring{y}, \mathring{x}).$$

The proof is complete.[4] □

Let $\mathrm{Pow}(y, x)$ be the LST formula which says that $y$ is the set of all finite subsets of $x$. More precisely, let $\mathrm{Pow}(y, x)$ be as follows:

$$\exists z\, [\mathrm{Seq}(z, x) \wedge y = \{\mathrm{ran}(u) \mid u \in z\}].$$

**2.3 Lemma.**

(i) *The* LST *formula* $\mathrm{Pow}(y, x)$ *is* $\Delta_1^{\mathrm{KP}}$.

(ii) *The class* Pow *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$.

*Proof.* (i) As it stands, $\mathrm{Pow}(y, x)$ is $\Sigma_1$. Moreover,

$$\mathrm{KP} \vdash \forall x\, \exists!\, y\, \mathrm{Pow}(y, x),$$

so as in 2.2 it follows that $\mathrm{Pow}(y, x)$ is in fact $\Delta_1^{\mathrm{KP}}$.

(ii) This follows from part (i) by a straightforward application of the technique discussed above. (The details are left as an exercise for the reader.) □

We shall now write down an LST formula $A(v, u)$ such that

$$A(v, u) \leftrightarrow v = \mathrm{Def}(u).$$

Namely:

$$\begin{aligned}
&(\forall x \in v)(\exists \varphi)\,[\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\}) \wedge (x \subseteq u) \\
&\wedge (\forall z \in u)(z \in x \leftrightarrow \exists \psi\, (\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi)))] \\
&\wedge (\forall \varphi)\,[(\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\})) \\
&\qquad \rightarrow (\exists x \in v)\,[(x \subseteq u) \wedge (\forall z \in u)(z \in x \\
&\qquad\qquad \leftrightarrow \exists \psi\, (\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi)))]].
\end{aligned}$$

Our task now is to modify this formula in order to obtain a $\Sigma_1$ formula equivalent to it. Broadly speaking, the idea is to find a single bound for all of the unbounded quantifiers in $A(v, u)$, much as we did when we formulated the formula $\mathrm{Sat}(u, \varphi)$ prior to I.9.10. What happened there was that we commenced with a formula $S(u, \varphi)$, which embodied the canonical definition of the notion

"$\varphi$ is a sentence of $\mathscr{L}_u$ which is true in $\langle u, \in \rangle$",

and then found a bound for all unbounded quantifiers in $S(u, \varphi)$. Now, the binding set used there is not large enough to handle the quantifiers involved in $A(v, u)$

4 This is not quite accurate, since we did not bother with the quantifier $\exists w$ mentioned in the proof of 2.2. This was because we knew already that $\omega \in L_\alpha$. In practice we shall always restrict our attention only to the "significant" quantifier(s).

(though it will clearly suffice for those quantifiers involved in those parts of $A(v, u)$ concerning Sat). So, as there is clearly no point in rebinding quantifiers which are already bound, let us at once amend $A(v, u)$ by replacing each occurrence of the formula $\mathrm{Sat}(u, \varphi)$ in $A(v, u)$ by $S(u, \varphi)$, denoting the resulting formula by $B(v, u)$. Since $S(u, \varphi)$ is equivalent to $\mathrm{Sat}(u, \varphi)$, $B(v, u)$ will be equivalent to $A(v, u)$. We now seek a bound for all the unbounded quantifiers in $B(v, u)$. (This bound will have to be large enough to rebind all of the quantifiers we have just freed in passing from $A(v, u)$ to $B(v, u)$, of course.) Let $C(w, v, u)$ be the formula obtained from $B(v, u)$ by binding all unbounded quantifiers by $w$. (Thus $C(w, v, u)$ is a $\Sigma_0$ formula.) We must now see what sort of set we can take for the bound $w$.

The unbounded quantifiers involved in $B(v, u)$ fall into three types: those that range over formulas ($\exists \varphi$ and $\forall \varphi$ as in $A(v, u)$), those that range over finite sequences of formulas (such quantifiers occur in Fml, Fr, Sub, and $S(u, \varphi)$), and those ranging over finite sequences of finite sets of variables (these occur in Fr). Hence, all unbounded quantifiers in $B(v, u)$ can (without loss of meaning) be bound by the set

$$\begin{aligned} K(u) = {} & [\text{the set of finite sequences of members of the set } \\ & \quad 9 \cup \{v_i \mid i \in \omega\} \cup \{\mathring{x} \mid x \in u\}] \\ & \cup [\text{the set of finite sequences of finite sequences of members} \\ & \quad \text{of the set } 9 \cup \{v_i \mid i \in \omega\} \cup \{\mathring{x} \mid x \in u\}] \\ & \cup [\text{the set of finite sequences of finite subsets of the} \\ & \quad \text{set } \{v_i \mid i \in \omega\}]. \end{aligned}$$

Let $K(w, u)$ be the LST formula which says "$w = K(u)$", namely:

$$\begin{aligned} (\exists a, b, c, d, e, f)[[(\forall z \in d)\, \mathrm{Vbl}(z) \wedge (\forall i \in \omega)(v_i \in d)] & \\ \wedge\ [(\forall z \in e)\, \mathrm{Const}(z, u) \wedge (\forall z \in u)(\mathring{z} \in e)] & \\ \wedge\ [\mathrm{Seq}(a, 9 \cup d \cup e)] & \\ \wedge\ [\mathrm{Seq}(b, a)] & \\ \wedge\ [\mathrm{Pow}(f, d) \wedge \mathrm{Seq}(c, f)] & \\ \wedge\ [w = a \cup b \cup c]]. & \end{aligned}$$

Provided we remove the explicit mention of $\omega$ as in 2.2, we see that the formula $K(w, u)$ is $\Sigma_1$. If we now let $D(v, u)$ be the formula

$$\exists w [K(w, u) \wedge C(w, v, u)],$$

then clearly,

$$D(v, u) \leftrightarrow v = \mathrm{Def}(u).$$

Moreover, we have:

**2.4 Lemma.**

(i) *The* LST *formula $D(v, u)$ is $\Sigma_1$ and $\Delta_1^{\mathrm{KP}}$.*
(ii) *The class $D$ is uniformly $\Delta_1^{L_\alpha}$ for limit $\alpha > \omega$.*

*Proof.* As in 2.2 and 2.3. (The details are left as an exercise for the reader.) □

Noting that if $\lim(\alpha)$ and $\alpha > \omega$, the set $L_\alpha$ is closed under the function Def (this observation forms part of the proof of 2.4), we often state part (ii) of 2.4 in the following form:

**2.5 Corollary.** *The function* Def *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$. □

We now write down an LST formula, $E(f, \alpha)$ such that

$$E(f, \alpha) \leftrightarrow f = (L_\gamma \mid \gamma \leqslant \alpha).$$

Namely:

$$\begin{aligned}\mathrm{On}(\alpha) &\wedge (f \text{ is a function}) \wedge (\mathrm{dom}(f) = \alpha + 1) \wedge (f(0) = \emptyset) \\ &\wedge (\forall \gamma \in \mathrm{dom}(f))\Big[\Big((\lim(\gamma) \wedge \gamma > 0) \rightarrow \Big(f(\gamma) = \bigcup_{\delta<\gamma} f(\delta)\Big)\Big) \\ &\wedge (\mathrm{succ}(\gamma) \rightarrow D(f(\gamma), f(\gamma - 1)))\Big].\end{aligned}$$

Clearly, $E(f, \alpha)$ says what we want it to, and our next task is to modify this formula to obtain an equivalent $\Sigma_1$ formula, just as we just did for $D(v, u)$. Let $F(w, f, \alpha)$ be the $\Sigma_0$ formula obtained from $E(f, \alpha)$ by replacing the clause $D(f(\gamma), f(\gamma - 1))$ by $C(w, f(\gamma), f(\gamma - 1))$, and rendering the clause

$$f(\gamma) = \bigcup_{\delta<\gamma} f(\delta)$$

in the form

$$(\forall x \in f(\gamma))(\exists \delta \in \gamma)(x \in f(\delta)) \wedge (\forall \delta \in \gamma)(f(\delta) \subseteq f(\gamma)).$$

Comparing the present situation with that which led up to 2.4, we see that all the unbounded quantifiers which figure in $E(f, \alpha)$ (namely as part of the clause $D(f(\gamma), f(\gamma - 1))$) can be bound by the set

$$K(\bigcup \mathrm{ran}(f)).$$

So if we let $G(f, \alpha)$ be the $\Sigma_1$ formula

$$\exists w [K(w, \bigcup \mathrm{ran}(f)) \wedge F(w, f, \alpha)],$$

we have

$$G(f, \alpha) \leftrightarrow f = (L_\gamma \mid \gamma \leqslant \alpha).$$

Moreover:

**2.6 Lemma.**

(i) *The* LST *formula* $G(f, \alpha)$ *is* $\Delta_1^{\mathrm{KP}}$.

(ii) *The class* $G$ *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$.

*Proof.* (i) The proof boils down to proving that

$$\mathrm{KP} \vdash \forall \alpha \, \exists f G(f, \alpha).$$

But this follows from 2.4 (i) together with the KP-Recursion Theorem (I.11.8), which enables us to construct, within KP, the function $(L_\gamma \mid \gamma \leqslant \alpha)$ for any ordinal $\alpha$.

(ii) Here we quickly reduce to proving that for any limit ordinal $\alpha > \omega$, if $\delta < \alpha$ then $(L_\gamma \mid \gamma \leqslant \delta) \in L_\alpha$. In fact it is not hard to see that if $\delta > \omega$, then $(L_\gamma \mid \gamma \leqslant \delta) \in L_{\delta+4}$, so we are done. (We leave all the details to the reader.) □

Let $H(x, \alpha)$ be the LST formula which says that "$x = L_\alpha$", namely:

$$\exists f [G(f, \alpha) \wedge (x = f(\alpha))].$$

The following lemma follows easily from 2.6 using the by now familiar arguments:

**2.7 Lemma.**
(i) *The* LST *formula* $H(x, \alpha)$ *is* $\Delta_1^{\mathrm{KP}}$.
(ii) *The class* $H$ *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$. □

Noting that if $\alpha > \omega$ is a limit ordinal, then $L_\alpha$ is closed under the function $\gamma \mapsto L_\gamma$, we have, by the above:

**2.8 Lemma.** *The function* $\gamma \mapsto L_\gamma$ *is uniformly* $\Delta_1^{L_\alpha}$ *for limit* $\alpha > \omega$. □

The following absoluteness results may now be proved.

**2.9 Lemma.** *Let* $M$ *be an inner model of* KP. *For any* $\alpha \in \mathrm{On}$, $L_\alpha \in M$ *and* $(L_\alpha)^M = L_\alpha$. (*This equality means that if* $[H(x, \alpha)]^M$, *then* $x = L_\alpha$.) *Hence* $(L)^M = L$.

*Proof.* As we observed above, the KP-Recursion Theorem enables us to construct $(L_\gamma \mid \gamma \leqslant \alpha)$ for any ordinal $\alpha$. Hence if $\alpha \in \mathrm{On}$, then we have $(L_\gamma^M \mid \gamma \leqslant \alpha) \in M$, so in particular $L_\alpha^M \in M$. But by 2.7 (i) and I.8.3 (iv), we have the absoluteness result $L_\alpha^M = L_\alpha$. The lemma is proved. □

**2.10 Lemma.** *Let* $M$ *be an admissible set, and let* $\lambda = \sup(M \cap \mathrm{On})$. *For any* $\alpha \in \lambda$, $(L_\alpha)^M = L_\alpha$. *Hence* $(L)^M = L_\lambda$.

*Proof.* Analogous to the proof of 2.9. □

**2.11 Lemma.** *For any* $\alpha$, $(L_\alpha)^L = L_\alpha$. *Hence* $(L)^L = L$.

*Proof.* Directly from 2.9. □

**2.12 Lemma.** *Let* $\alpha > \omega$ *be a limit ordinal. For all* $\gamma < \alpha$, $(L_\gamma)^{L_\alpha} = L_\gamma$. *Hence* $(L)^{L_\alpha} = L_\alpha$.

*Proof.* Much as for 2.10, except that we use the closure properties of $L_\alpha$ rather than admissibility. The details are left to the reader. □

**2.13 Lemma.** *The* LST *formula*

"*$x$ is constructible*"

*is* $\Sigma_1^{\mathrm{KP}}$.

*Proof.*

$$\begin{aligned} x \textit{ is constructible } &\leftrightarrow x \in L \\ &\leftrightarrow \exists \alpha (x \in L_\alpha) \\ &\leftrightarrow \exists \alpha \exists u (u = L_\alpha \wedge x \in u). \end{aligned}$$

The result follows from 2.7 (i) now. □

The *Axiom of Constructibility* is the assertion that all sets are constructible:

$$\forall x (x \in L).$$

This is usually abbreviated as:

$$V = L.$$

For the most part we shall be treating the assertion $V = L$ as a particularly interesting set-theoretical statement, not as a fundamental *axiom* of set theory in the sense of the axioms of ZFC. Thus the use of the word "axiom" in this connection is somewhat different from the more common usage. From our standpoint it would perhaps be more suitable to refer to $V = L$ as the "Hypothesis of Constructibility". However, we shall stick to the accepted usage of the phrase "Axiom of Constructibility".

**2.14 Lemma.** *The* LST *formula* $V = L$ *is* $\Pi_2^{\mathrm{KP}}$ .

*Proof.* $V = L \leftrightarrow \forall x (x \in L)$, which is $\Pi_2^{\mathrm{KP}}$ by virtue of 2.13. □

**2.15 Theorem.** ZF $\vdash (V = L)^L$. *Hence* $L$ *is an inner model of the theory* ZF + $(V = L)$.

*Proof.* By 2.11, $(L)^L = L$. But clearly, $(V)^L = L$. Hence,

$$(V)^L = (L)^L.$$

In other words,

$$(V = L)^L. \quad \square$$

# 3. The Axiom of Choice in $L$

In this section we shall show that

$$\mathrm{ZF} \vdash (\mathrm{AC})^L.$$

We do this in a very strong fashion. We exhibit a formula $\Phi(v_0, v_1)$ of LST such that (suitably expressed)

$$\mathrm{ZF} \vdash [\text{“}\Phi \text{ well-orders the universe”}]^L.$$

In order to describe the formula $\Phi$, it is necessary to look once more at the definition of the constructible hierarchy. Recall that in passing from $L_\alpha$ to $L_{\alpha+1}$, we allow any elements of $L_\alpha$ to figure as parameters in definitions of the new sets appearing in $L_{\alpha+1}$. The following lemma shows that we may be rather more restrictive than this, and provides us with a slightly more convenient characterisation of $L_{\alpha+1}$ in terms of $L_\alpha$.

**3.1 Lemma.** *Let $x \in L_{\alpha+1}$. Then there is a formula $\varphi(v_0, \dots, v_n)$ of $\mathscr{L}$ (so in particular $\varphi$ contains no individual constant symbols) and ordinals $\gamma_1, \dots, \gamma_n < \alpha$ such that*

$$x = \{z \in L_\alpha \mid \models_{L_\alpha} \varphi(\mathring{z}, \mathring{L}_{\gamma_1}, \dots, \mathring{L}_{\gamma_n})\}.$$

*Proof.* By induction on $\alpha$. For $\alpha = 0$ there is nothing to prove, since $\emptyset$ is the only possible set $x$. Let $\alpha > 0$ now, and suppose that the lemma is valid below $\alpha$. If $x \in L_{\alpha+1}$ there is an $\mathscr{L}$-formula $\psi(v_0, \dots, v_n)$ and elements $p_1, \dots, p_n$ of $L_\alpha$ such that

$$x = \{z \in L_\alpha \mid \models_{L_\alpha} \psi(\mathring{z}, \mathring{p}_1, \dots, \mathring{p}_n)\}.$$

Pick $\gamma < \alpha$ so that $p_1, \dots, p_n \in L_{\gamma+1}$. By induction hypothesis, for each $i = 1, \dots, n$ there is an $\mathscr{L}$-formula $\psi_i(v_0, \dots, v_{k(i)})$ and ordinals $\gamma_1^i, \dots, \gamma_{k(i)}^i < \gamma$ such that

$$p_i = \{z \in L_\gamma \mid \models_{L_\gamma} \psi_i(\mathring{z}, \mathring{L}_{\gamma_1^i}, \dots, \mathring{L}_{\gamma_{k(i)}^i})\}.$$

For each $i$, let $\psi_i'(v_0, \dots, v_{k(i)}, v_{k(i)+1})$ be the $\mathscr{L}$-formula obtained from $\psi_i(v_0, \dots, v_{k(i)})$ by binding all unbounded quantifiers by $v_{k(i)+1}$. Then clearly,

$$p_i = \{z \in L_a \mid \models_{L_\alpha} [(\mathring{z} \in \mathring{L}_\gamma) \wedge \psi_i'(\mathring{z}, \mathring{L}_{\gamma_1^i}, \dots, \mathring{L}_{\gamma_{k(i)}^i}, \mathring{L}_\gamma)]\}.$$

Hence,

$$\begin{aligned} x = \{z \in L_\alpha \mid \models_{L_\alpha} & (\exists p_1) \dots (\exists p_n) [\psi(\mathring{z}, p_1, \dots, p_n) \\ & \wedge (\forall v)[(v \in p_1) \leftrightarrow (v \in \mathring{L}_\gamma \wedge \psi_1'(v, \mathring{L}_{\gamma_1^1}, \dots, \mathring{L}_{\gamma_{k(1)}^1}, \mathring{L}_\gamma))] \wedge \dots\dots \\ & \wedge (\forall v)[(v \in p_n) \leftrightarrow (v \in \mathring{L}_\gamma \wedge \psi_n'(v, \mathring{L}_{\gamma_1^n}, \dots, \mathring{L}_{\gamma_{k(n)}^n}, \mathring{L}_\gamma))]]\}. \end{aligned}$$

The lemma is proved. □

We shall now fix a simple, effective well-ordering of the formulas of $\mathscr{L}$. The precise definition is not important. For definiteness, we say that if $\varphi$ and $\psi$ are formulas of $\mathscr{L}$, so in particular, $\varphi$ and $\psi$ are both finite sequences of sets, then $\varphi \dashv \psi$ iff either $\varphi$ is an initial segment of $\psi$ or else $k(\varphi(i)) < k(\psi(i))$, where $i$ is the least integer such that $\varphi(i) \neq \psi(i)$ and where the function $k$ is defined on the set

$$9 \cup \{v_n \mid n \in \omega\}$$

by

$$k(x) = \begin{cases} x, & \text{if } x \in 9 \\ n + 9, & \text{if } x = v_n \, (= (2, n)). \end{cases}$$

We also define $<^*$ to be the lexicographic well-ordering of the finite sequences of ordinals, i.e. if $s$ and $t$ are finite sequences of ordinals, then $s <^* t$ iff

(i) $\operatorname{dom}(s) < \operatorname{dom}(t)$, or else
(ii) $\operatorname{dom}(s) = \operatorname{dom}(t)$ and $s(i) < t(i)$, where $i$ is least such that $s(i) \neq t(i)$.

Using 3.1, we now define a well-ordering of the class $L$. Let $x, y \in L$. We set $x <_L y$ iff either:

(A) The least $\alpha$ such that $x \in L_{\alpha+1}$ is smaller than the least $\beta$ such that $y \in L_{\beta+1}$; or else
(B) there is an $\alpha$ such that $x$ and $y$ both lie in $L_{\alpha+1} - L_\alpha$ and either:
(B1) the $\lhd$-least formula $\varphi(v_0, \ldots, v_n)$ of $\mathscr{L}$ for which there are ordinals $\gamma_1, \ldots, \gamma_n < \alpha$ such that

$$x = \{z \in L_\alpha \mid \models_{L_\alpha} \varphi(\mathring{z}, \mathring{L}_{\gamma_1}, \ldots, \mathring{L}_{\gamma_n})\}$$

$\lhd$-precedes the $\lhd$-least formula $\psi(v_0, \ldots, v_m)$ of $\mathscr{L}$ for which there are ordinals $\delta_1, \ldots, \delta_m < \alpha$ such that

$$y = \{z \in L_\alpha \mid \models_{L_\alpha} \psi(\mathring{z}, \mathring{L}_{\delta_1}, \ldots, \mathring{L}_{\delta_m})\}; \qquad \text{or else}$$

(B2) the formulas $\varphi$ and $\psi$ in (B1) coincide, but the $<^*$-least $n$-sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of ordinals $\gamma_i < \alpha$ which defines $x$ as in (B1) $<^*$-precedes the $<^*$-least $n$-sequence $\langle \delta_1, \ldots, \delta_n \rangle$ of ordinals $\delta_i < \alpha$ which defines $y$.

It is easily seen that $<_L$ is indeed a well-ordering of $L$. Our task now is to investigate the logical complexity of this well-ordering.

The following LST-formula, $N(\alpha, x, \varphi, t)$ says that $\varphi$ is a formula of $\mathscr{L}$, $t$ is a finite sequence of ordinals less than $\alpha$, the free variables of $\varphi$ are $v_0, \ldots, v_n$, where $n = \operatorname{dom}(t)$, and $x = \{z \in L_\alpha \mid \models_{L_\alpha} \varphi(\mathring{z}, \mathring{L}_{t(0)}, \ldots, \mathring{L}_{t(n-1)})\}$:

$$\begin{aligned}
\exists u\, \exists f\, \exists n\, \exists \psi\, [&\operatorname{Fml}(\varphi, \emptyset) \wedge \operatorname{Finseq}(t) \wedge (\operatorname{dom}(t) = n) \wedge (\forall i \in n)(t(i) \in \alpha) \\
&\wedge \operatorname{Fr}(\varphi, u) \wedge (f\colon n+1 \leftrightarrow u) \wedge (\forall i \in n+1)(f(i) = v_i) \\
&\wedge \operatorname{Finseq}(\psi) \wedge (\operatorname{dom}(\psi) = n+1) \wedge (\psi(0) = \varphi) \\
&\wedge (\forall i \in n) \operatorname{Sub}(\psi(i+1), \psi(i), v_{i+1}, \mathring{L}_{t(i)}) \wedge (x \subseteq L_\alpha) \\
&\wedge (\forall z \in L_\alpha)(z \in x \leftrightarrow \exists \theta (\operatorname{Sub}(\theta, \psi(n), v_0, \mathring{z}) \wedge \operatorname{Sat}(L_\alpha, \theta)))].
\end{aligned}$$

The following LST-formula, $M(\alpha, x, \varphi)$, says that $\varphi$ is the $\lhd$-least formula of $\mathscr{L}$ such that $N(\alpha, x, \varphi, t)$ for some $t$:

$$(\exists t) N(\alpha, x, \varphi, t) \wedge (\forall \varphi')[(\exists t') N(\alpha, x, \varphi', t') \rightarrow (\varphi \quad \varphi' \vee \varphi = \varphi')].$$

The next formula of LST, $P(\alpha, x, \varphi, t)$, says that $t$ is the $<^*$-least suitable sequence of ordinals less than $\alpha$ such that $N(\alpha, x, \varphi, t)$:

$$N(\alpha, x, \varphi, t) \wedge (\forall t')[N(\alpha, x, \varphi, t') \rightarrow (t \leqslant^* t')].$$

We are now able to write down a formula of LST which expresses the relation $x <_L y$ outlined above. We shall not bother to replace the relations $\dashv$ and $<^*$ by their LST-definitions, since it should be obvious how this can be done, and is thus not worth causing further complications. Let $X(x, y)$ be the following LST-formula (to express $x <_L y$):

$$(\exists\alpha)[(x \in L_\alpha) \wedge (y \notin L_\alpha)] \vee (\exists\alpha) Q(x, y, \alpha),$$

where $Q(x, y, \alpha)$ is the LST-formula

$$\begin{aligned}&[(x \in L_{\alpha+1}) \wedge (y \in L_{\alpha+1}) \wedge (x \notin L_\alpha) \wedge (y \notin L_\alpha)]\\&\quad \wedge [(\exists\varphi, \psi)[M(\alpha, x, \varphi) \wedge M(\alpha, y, \psi) \wedge (\varphi \dashv \psi)]\\&\quad \vee (\exists\varphi)[M(\alpha, x, \varphi) \wedge M(\alpha, y, \varphi)\\&\qquad\qquad \wedge (\exists s, t)[P(\alpha, x, \varphi, s) \wedge P(\alpha, y, \varphi, t) \wedge (s <^* t)]]].\end{aligned}$$

Now, it is easily seen that any unbounded quantifiers in the formula $Q(x, y, \alpha)$ may be bound by $L_{\max(\omega, \alpha+4)}$. (This includes the quantifiers which are required in order to define $\dashv$ and $<^*$.) So if $R(x, y, \alpha, w)$ is the formula obtained from $Q(x, y, \alpha)$ by binding all quantifiers not already bound by $w$, we see that the relation $x <_L y$ is expressed by the formula

$$(\exists\alpha)[(x \in L_\alpha) \wedge (y \notin L_\alpha)] \vee (\exists\alpha)(\exists w)[w = L_{\max(\omega, \alpha+4)} \wedge R(x, y, \alpha, w)].$$

We denote this formula by $\mathrm{WO}(x, y)$. It is clearly $\Sigma_1$. Moreover,

**3.2 Lemma.**

(i) *The* LST *formula* $\mathrm{WO}(x, y)$ *is* $\Delta_1^{\mathrm{KP}+(V=L)}$.

(ii) $\mathrm{KP} \vdash$ "$\{(x, y) \mid \mathrm{WO}(x, y)\}$ *is a well-ordering of* $L$".

*Proof.* We prove (ii) first. From the way we evolved the formula $\mathrm{WO}(x, y)$ above, it is clear that what we must prove is that (working in KP) if $x, y \in L$, $x \neq y$, then either $\mathrm{WO}(x, y)$ or else $\mathrm{WO}(y, x)$. But if $x, y$ are as stated, then either $x <_L y$ or else $y <_L x$, of course. So we are reduced to proving that if $x <_L y$, then the sets required to exist by virtue of the existential quantifiers involved in the formula $\mathrm{WO}(x, y)$ can all be constructed (from $x, y$) in KP. This is easily seen, and is left as an exercise for the reader.

We now prove (i). We know that the formula $\mathrm{WO}(x, y)$ is $\Sigma_1$. But by (ii) we have

$$\mathrm{KP} \vdash (\forall x, y \in L)[\mathrm{WO}(x, y) \leftrightarrow \neg[(x = y) \vee \mathrm{WO}(y, x)]].$$

Hence $\mathrm{WO}(x, y)$ is also $\Pi_1^{\mathrm{KP}+(V=L)}$. $\square$

Let $\mathrm{wo}(x, y)$ be the analogue of $\mathrm{WO}(x, y)$ in $\mathscr{L}$. Then:

**3.3 Lemma.**

(i) *If* $x, y \in L_\alpha$, *then*

$$\mathrm{WO}(x, y) \leftrightarrow \models_{L_\gamma} \mathrm{wo}(\mathring{x}, \mathring{y}),$$

*where* $\gamma = \max(\omega, \alpha + 5)$.

(ii) *If $\alpha > \omega$ is a limit ordinal, then*

$$\{(x, y) \mid \models_{L_\alpha} \mathrm{wo}(\mathring{x}, \mathring{y})\} \quad \textit{is a well-ordering of } L_\alpha .$$

(iii) *The relation $x <_L y$ is uniformly $\Delta_1^{L_\alpha}$ for limit $\alpha > \omega$.*

*Proof.* (i) This follows from 3.2 by the kind of argument outlined in 2.2. The heart of the proof is to show that the existential quantifiers involved in $\mathrm{wo}(x, y)$ can be bound by $L_\gamma$, where $\gamma$ is as stated. The details are left as an exercise for the reader.

(ii) This follows immediately from (i).

(iii) This also follows from (i). □

We often write $x <_L y$ in place of both $\mathrm{WO}(x, y)$ and $\mathrm{wo}(x, y)$.

The following lemma is clear from the definition of $<_L$, and will often be used without mention.

**3.4 Lemma.**

(i) *If $x <_L y$ and $y \in L_\alpha$, then $x \in L_\alpha$.*
(ii) *If $x \in L_\alpha$ and $y \notin L_\alpha$, then $x <_L y$.*
(iii) *If $x \in y \in L$, then $x <_L y$.* □

For later use we also prove the following result.

**3.5 Lemma.** *Let* pr *be the predecessor function defined on L by*

$$\mathrm{pr}(x) = \{z \mid z <_L x\} .$$

(i) $x \in L \to \mathrm{pr}(x) \in L$.
(ii) *if $\alpha > \omega$ is a limit ordinal, then* $x \in L_\alpha \to \mathrm{pr}(x) \in L_\alpha$.
(iii) pr *is uniformly $\Delta_1^{L_\alpha}$ for limit $\alpha > \omega$.*

*Proof.* (i) follows directly from (ii).

(ii) Let $x \in L_\alpha$. Choose $\beta < \alpha$ so that $x \in L_\beta$. We know that

$$z <_L x \to z \in L_\beta .$$

Moreover by 3.3 (i),

$$(\forall a, b \in L_\beta)[\mathrm{WO}(a, b) \leftrightarrow \models_{L_\gamma} \mathrm{wo}(\mathring{a}, \mathring{b})],$$

where $\gamma = \max(\omega, \beta + 5)$. Hence

$$\mathrm{pr}(x) = \{z \in L_\gamma \mid \models_{L_\gamma} \mathrm{wo}(\mathring{z}, \mathring{x})\} \in L_{\gamma+1} \subseteq L_\alpha .$$

(iii) Let $w(z, x, 1)$ be the $\mathscr{L}$-formula obtained from $\mathrm{wo}(z, x)$ by binding any unbounded quantifiers by 1. By 3.3 (i),

$$\begin{aligned} y = \mathrm{pr}(x) \leftrightarrow \models_{L_\alpha} (\exists \beta)[(\mathring{x} \in L_\beta) \wedge (\mathring{y} \subseteq L_\beta) \\ \wedge (\forall z \in L_\beta)(z \in \mathring{y} \leftrightarrow w(z, \mathring{x}, L_{\max(\omega, \beta+5)}))]. \end{aligned}$$

So pr is uniformly $\Sigma_1^{L_\alpha}$ for limit $\alpha > \omega$. Hence by I.10.4, pr is uniformly $\Delta_1^{L_\alpha}$ for limit $\alpha > \omega$. $\square$

The following result is also fundamental to much of the work on constructibility.

**3.6 Lemma.** *There is a* $\Sigma_1$ *formula* Enum$(\alpha, x)$ *of* LST, *absolute for* $L$, *such that*

$$\mathrm{KP} \vdash \text{“}\textit{If } F = \{(x, \alpha) \mid \mathrm{Enum}(\alpha, x)\}, \textit{ then } F\colon \mathrm{On} \leftrightarrow L\text{”}.$$

*Proof.* Intuitively, Enum$(\alpha, x)$ says that $x$ is the $\alpha$-the member of $L$ under the well-ordering $<_L$. Thus, Enum$(\alpha, x)$ is the formula:

$$\begin{aligned}(\exists f)[(f \text{ is a function}) &\wedge (\mathrm{dom}(f) = \alpha + 1)\\ &\wedge (\forall \xi, \zeta \in \alpha + 1)(\xi < \zeta \to f(\xi) <_L f(\zeta))\\ &\wedge (\exists z)[(z = \mathrm{pr}(x)) \wedge (\forall y \in z)(\exists \beta \in \alpha)(y = f(\beta))] \wedge (f(\alpha) = x)].\end{aligned}$$

It is easily seen that this formula is as stated in the lemma. $\square$

As an illustration of the use of 3.6 we show that $V = L$ can be reduced to an “axiom of constructibility for sets of ordinals”.

**3.7 Lemma.** $\mathrm{KP} \vdash \forall a(a \subseteq \mathrm{On} \to a \in L) \to (V = L)$.

*Proof.* (In KP.) Assume all sets of ordinals are constructible. We prove by $\in$-induction that

$$\forall x(x \in L).$$

Let $x$ be given, and suppose that

$$y \in x \to y \in L.$$

By $\Sigma_1$-Collection, let

$$a = \{\alpha \mid \alpha \in \mathrm{On} \wedge (\exists y \in x)\, \mathrm{Enum}(\alpha, y)\}.$$

By hypothesis, $a \in L$. Hence, using the induction hypothesis,

$$x' = \{y \mid (\exists \alpha \in a)\, \mathrm{Enum}(\alpha, y)^L\} \in L.$$

But by the absoluteness of Enum, $x = x'$, so we have $x \in L$, as required. $\square$

**3.8 Theorem.** $\mathrm{ZF} \vdash (\mathrm{AC})^L$.

*Proof.* An immediate consequence of 3.6. (The function $F$ well-orders $L$.) $\square$

Notice that we made no use of AC in the above proof. This will be important to us in the next section.

# 4. Constructibility and Relative Consistency Results

The construction of inner models such as $L$ provides us with a useful method for obtaining relative consistency results. The idea is as follows. Suppose $\Phi$ is some statement in LST, and that there is a class $M$ such that

$$\mathrm{ZF} \vdash (\mathrm{ZF} + \Phi)^M.$$

Then the consistency of the theory ZF + $\Phi$ follows from the consistency of ZF. Indeed, given a proof of an inconsistency in ZF + $\Phi$ we could, in a highly effective manner, produce from it a proof of an inconsistency in ZF. To see this, let $\Psi_0, \ldots, \Psi_n$ be a proof (in the formal sense) of an inconsistency in ZF + $\Phi$. Thus, for each $i = 0, \ldots, n$, $\Psi_i$ is a formula of LST which is either an axiom of the theory ZF + $\Phi$ or else follows from some of $\Psi_0, \ldots, \Psi_{i-1}$ by an application of a rule of logic, and $\Psi_n$ is a statement such as $(0 = 1)$. Consider the sequence $\Psi_0^M, \ldots, \Psi_n^M$. If $\Psi_i$ is an axiom of ZF + $\Phi$, then $\Psi_i^M$ is a theorem of ZF, by the assumption on $M$. And if $\Psi_i$ follows from some of $\Psi_0, \ldots, \Psi_{i-1}$ by means of a rule of logic, then $\Psi_i^M$ follows from the corresponding members of $\Psi_0^M, \ldots, \Psi_{i-1}^M$ by means of the same rule. Hence $\Psi_n^M$ is a theorem of ZF. But since $\Psi_n$ is an inconsistency, so too is $\Psi_n^M$.

As a particular instance of the above considerations, we have

**4.1 Theorem.** *If* ZF *is a consistent theory, so too is* ZFC.

*Proof.* By 1.2 and 3.8,

$$\mathrm{ZF} \vdash (\mathrm{ZF} + \mathrm{AC})^L. \quad \square$$

Similarly, using 2.15, we obtain

**4.2 Theorem.** *If* ZF *is a consistent theory, so too is* ZFC + $(V = L)$. $\square$

A consequence of this last result is that any statement $\Phi$ which we can prove in the theory ZFC + $(V = L)$ will have automatically been shown to be consistent with ZFC. Thus proofs of results in the theory ZFC + $(V = L)$ have a significance in terms of ZFC set theory, regardless of the light in which $V = L$ is viewed.

We end this short section by giving a characterisation of $L$ in terms of inner models.

**4.3 Theorem** (The Minimal Model Property). *$L$ is the smallest inner model of* ZF.

*Proof.* By 1.1, $L$ is a transitive proper class. By 1.2, $L$ is thus an inner model of ZF. Let $M$ be any other inner model of ZF. By 2.9, $(L)^M = L$. *Thus* $L \subseteq M$. $\square$

In fact the above proof tells us more, namely:

**4.4 Theorem** (The Minimal Model Property for KP). *$L$ is the smallest inner model of* KP. $\square$

# 5. *The Condensation Lemma. The* GCH *in L*

Recall from I.10 the definitions of the notions of *elementary substructure*, $\Sigma_n$-*elementary substructure*, *elementary embedding*, etc., together with the notation we established concerning these notions.

The following lemma is often useful in this connection.

**5.1 Lemma.** *Let* $\mathbf{M} = \langle M, \in, A_1, \ldots, A_n \rangle$, *where* $M$ *is an amenable set, and let* $\mathbf{N} = \langle N, \ldots \rangle$ *be a substructure of* $\mathbf{M}$. *Let* $n > 0$. *The following are equivalent*:

(i) $\mathbf{N} \prec_n \mathbf{M}$;
(ii) *if* $A$ *is a non-empty* $\Sigma_n^{\mathbf{M}}(N)$ *subset of* $M$, *then* $A \cap N \neq \emptyset$.

*Proof.* Before we start, we recall that in the definition of $\Sigma_n$ in the case of the language $\mathscr{L}_V$ we do not allow repeated quantifiers.

(i) → (ii). Let $A$ be a non-empty $\Sigma_n^{\mathbf{M}}(N)$ subset of $M$, and let $\varphi(x, y)$ be a $\Pi_{n-1}$ formula of the $\mathbf{M}$-language, with parameters from $N$, such that

$$A = \{x \in M \mid \models_{\mathbf{M}} \exists y \, \varphi(\mathring{x}, y)\}.$$

Since $A \neq \emptyset$,

$$\models_{\mathbf{M}} \exists x \, \exists y \varphi(x, y).$$

So, as $M$ is amenable,

$$\models_{\mathbf{M}} \exists z \, \varphi((z)_0, (z)_1).$$

So, as $\mathbf{N} \prec_n \mathbf{M}$, by (i),

$$\models_{\mathbf{N}} \exists z \, \varphi((z)_0, (z)_1).$$

So for some $x \in N$,

$$\models_{\mathbf{N}} \exists y \, \varphi(\mathring{x}, y).$$

But $\exists y \varphi(\mathring{x}, y)$ is a $\Sigma_n$ formula of the $\mathbf{M}$-language with parameters from $N$, so by (i) again,

$$\models_{\mathbf{M}} \exists y \, \varphi(\mathring{x}, y).$$

Hence $x \in A$. But $x \in N$ also. Thus, as required,

$$A \cap N \neq \emptyset.$$

(ii) → (i). We prove by induction on the length of formulas that for any sentence $\varphi$ of the $\mathbf{M}$-language with parameters from $N$ which is at most $\Sigma_n$,

$$\models_{\mathbf{N}} \varphi \quad \text{iff} \quad \models_{\mathbf{M}} \varphi.$$

If $\varphi$ is primitive the result is trivial. If $\varphi$ is of the form $\neg \psi$ or else of the form $\psi_1 \wedge \psi_2$, the induction step is immediate. There remains the case where $\varphi$ is of the form $\exists x \psi(x)$. Suppose first that

$$\models_{\mathbf{N}} \varphi .$$

Thus,

$$\models_{\mathbf{N}} \exists x \psi(x) .$$

So for some $x \in N$,

$$\models_{\mathbf{N}} \psi(\mathring{x}) .$$

Now, $\psi(\mathring{x})$ is shorter than $\varphi$ and is at most $\Pi_{n-1}$, so by induction hypothesis,

$$\models_{\mathbf{M}} \psi(\mathring{x}) .$$

Thus

$$\models_{\mathbf{M}} \exists x \psi(x) ,$$

i.e.

$$\models_{\mathbf{M}} \varphi .$$

Conversely, assume now this last fact. Then

$$\models_{\mathbf{M}} \exists x \psi(x) .$$

Let

$$A = \{x \in M \mid \models_{\mathbf{M}} \psi(\mathring{x})\} .$$

Then $A$ is non-empty, and is a $\Sigma_n^{\mathbf{M}}(N)$ subset of $M$. (In fact $A$ is $\Pi_{n-1}^{\mathbf{M}}(N)$.) So by (ii),

$$A \cap N \neq \emptyset .$$

Let $x \in A \cap N$. Then

$$\models_{\mathbf{M}} \psi(\mathring{x}) .$$

But $\psi(\mathring{x})$ has parameters from $N$, is shorter than $\varphi$, and is at most $\Pi_{n-1}$. So by induction hypothesis,

$$\models_{\mathbf{N}} \psi(\mathring{x}) .$$

Thus,

$$\models_{\mathbf{N}} \exists x \psi(x) ,$$

i.e.

$$\models_{\mathbf{N}} \varphi .$$

The proof is complete. $\square$

The following theorem is arguably the most important single result in constructibility theory (as far as applications are concerned).

**5.2 Theorem** (The Condensation Lemma). *Let $\alpha$ be a limit ordinal. If*

$$X \prec_1 L_\alpha,$$

*then there are unique $\pi$ and $\beta$ such that $\beta \leqslant \alpha$ and*:

(i) $\pi\colon \langle X, \in \rangle \cong \langle L_\beta, \in \rangle$;
(ii) *if $Y \subseteq X$ is transitive, then* $\pi \upharpoonright Y = \mathrm{id} \upharpoonright Y$;
(iii) $\pi(x) \leqslant_L x$ *for all* $x \in X$.

*Proof.* Notice first that by an easy induction on $m$ we can prove that $L_m \subseteq X$ for all $m < \omega$. Indeed, if $x \in L_{m+1}$, then $x$ is of the form

$$x = \{a_1, \ldots, a_k\}$$

for some $a_1, \ldots, a_k \in L_m$, and

$$\models_{L_\alpha} \exists x [(\mathring{a}_1 \in x) \wedge \ldots \wedge (\mathring{a}_k \in x) \wedge (\forall z \in x)((z = \mathring{a}_1) \vee \ldots \vee (z = \mathring{a}_k))],$$

so if $L_m \subseteq X$ this sentence is true in $X$, which means that $x \in X$. Since $L_m \subseteq X$ for all $m < \omega$, we have $L_\omega \subseteq X$. Thus in the case $\alpha = \omega$, we have $X = L_\alpha$, and the theorem is trivially valid. So from now on we shall assume that $\alpha > \omega$.

Note first that $X$ is extensional. For suppose that $x, y \in X$, $x \neq y$. Then

$$\models_{L_\alpha} \exists z (z \in \mathring{x} \leftrightarrow z \notin \mathring{y}),$$

so as $X \prec_1 L_\alpha$,

$$\models_X \exists z (z \in \mathring{x} \leftrightarrow z \notin \mathring{y}),$$

which means that for some $z \in X$,

$$z \in x \leftrightarrow z \notin y.$$

Since $X$ is extensional, by the Collapsing Lemma (I.7.1) there is a unique $\pi$ and a unique transitive set $M$ such that

$$\pi\colon X \cong M.$$

We shall show that $M = L_\beta$ for a (unique) ordinal $\beta \leqslant \alpha$.

By 2.7 there is a $\Sigma_0$ formula $\Phi(z, v, \gamma)$ of LST such that

(a) $$\forall \gamma \forall v [v = L_\gamma \leftrightarrow \exists z \Phi(z, v, \gamma)],$$

and moreover, if $\varphi(z, v, \gamma)$ is the $\mathscr{L}$-analogue of $\Phi(z, v, \gamma)$, then (using I.9.15)

(b) $$(\forall \gamma < \alpha)(\forall v)[v = L_\gamma \leftrightarrow v \in L_\alpha \wedge \models_{L_\alpha} \exists z \varphi(z, \mathring{v}, \mathring{\gamma})].$$

Now, $\pi^{-1}: M \prec_1 L_\alpha$, so if $\models_M \mathrm{On}(x)$ then $\pi^{-1}(x) \in \alpha$. Moreover, by (b) we have

(c) $\quad (\forall \gamma < \alpha)[\models_{L_\alpha} \exists v \exists z \varphi(z, v, \mathring{\gamma})]$.

Hence, applying $\pi^{-1}$, we get

(d) $\quad (\forall \gamma \in M)[\models_M \exists v \exists z \varphi(z, v, \mathring{\gamma})]$.

So,

(e) $\quad (\forall \gamma \in M)(\exists v \in M)(\exists z \in M)[\models_M \varphi(\mathring{z}, \mathring{v}, \mathring{\gamma})]$.

Thus as $M$ is transitive, I.9.15 gives

(f) $\quad (\forall \gamma \in M)(\exists v \in M)(\exists z \in M)\, \Phi(z, v, \gamma)$.

Thus by (a)

(g) $\quad (\forall \gamma \in M)(L_\gamma \in M)$.

Now, $M$ is transitive, so

$$M \cap \mathrm{On} = \beta$$

for some ordinal $\beta$. Thus (g) becomes

(h) $\quad (\forall \gamma \in \beta)(L_\gamma \in M)$.

So, as $M$ is transitive, we conclude that

(i) $\quad \bigcup_{\gamma < \beta} L_\gamma \subseteq M$.

Again, since

$$L_\alpha = \bigcup_{\gamma < \alpha} L_\gamma,$$

we have

(j) $\quad (\forall x \in L_\alpha)[\models_{L_\alpha} \exists \gamma \exists v \exists z (\varphi(z, v, \gamma) \wedge (\mathring{x} \in v))]$.

Applying $\pi^{-1}$,

(k) $\quad (\forall x \in M)[\models_M \exists \gamma \exists v \exists z (\varphi(z, v, \gamma) \wedge (\mathring{x} \in v))]$.

Thus,

(l) $\quad (\forall x \in M)(\exists \gamma \in M)(\exists v \in M)(\exists z \in M)[\models_M \varphi(\mathring{z}, \mathring{v}, \mathring{\gamma}) \wedge (\mathring{x} \in \mathring{v})]$.

So by I.9.15,

(m) $\quad (\forall x \in M)(\exists \gamma \in M)(\exists v \in M)(\exists z \in M)[\Phi(z, v, \gamma) \wedge (x \in v)].$

Hence by (a),

(n) $\quad (\forall x \in M)(\exists \gamma \in M)(x \in L_\gamma).$

Thus by definition of $\beta$,

(o) $\quad (\forall x \in M)(\exists \gamma \in \beta)(x \in L_\gamma).$

In other words,

(p) $\quad M \subseteq \bigcup_{\gamma < \beta} L_\gamma.$

Combining (i) and (p) we conclude that

(q) $\quad M = \bigcup_{\gamma < \beta} L_\gamma.$

But $\lim(\alpha)$, so

$$(\forall v \in \alpha)[\models_{L_\alpha} \exists \tau (\mathring{v} < \tau)],$$

which implies that

$$(\forall v \in M)[\models_M \exists \tau (\mathring{v} < \tau)].$$

Thus,

$$(\forall v \in \beta)(\exists \tau \in \beta)(v < \tau).$$

Hence $\lim(\beta)$, and (q) becomes

$$M = L_\beta.$$

That completes the proof of part (i) of the theorem.

Part (ii) follows immediately from I.7.1. We are left with the proof of part (iii).

Suppose that $\pi(x) >_L x$ for some $x \in X$. Let $x_0$ be the $<_L$-least such $x$. Since $x_0 \in X$, $\pi(x_0) \in L_\beta$. But $x_0 <_L \pi(x_0)$. So by 3.4(i), $x_0 \in L_\beta$. Hence $x_0 = \pi(x_1)$ for some $x_1 \in X$. Thus

$$\pi(x_1) = x_0 <_L \pi(x_0).$$

But $<_L$ is uniformly $\Sigma_1^{L_\lambda}$ for limit $\lambda > \omega$ and $\pi^{-1}$: $L_\beta \prec_1 L_\alpha$ (and moreover $\alpha$ and $\beta$ are limit ordinals), so the above inequality yields

$$x_1 <_L x_0.$$

But this means that $x_1 <_L \pi(x_1)$, which contadicts the choice of $x_0$. The proof is complete. $\square$

Using the Condensation Lemma, we shall prove that the GCH is valid in $L$. We require the following lemma, which though stated for limit levels of the constructible hierarchy is really a result about structures with definable well-orders.

**5.3 Lemma.** *Let $\alpha$ be a limit ordinal, and let $X \subseteq L_\alpha$. Let $M$ be the set of all elements of $L_\alpha$ which are definable in $L_\alpha$ from elements of $X$. (i.e. $a \in M$ iff for some formula $\varphi(v_0)$ of $\mathscr{L}_X$, $a$ is the unique element of $L_\alpha$ such that $\models_{L_\alpha} \varphi(\mathring{a})$.)*
*Then*

$$X \subseteq M \prec L_\alpha,$$

*and moreover $M$ is the smallest elementary substructure of $L_\alpha$ which contains all elements of $X$.*

*Proof.* If $\alpha = \omega$ then as in the proof of 5.2 we see at once that $M = L_\alpha$, and that the only elementary submodel of $L_\alpha$ is $L_\alpha$ itself. So we may assume that $\alpha > \omega$ from now on.

If $x \in X$, then $x$ is definable in $L_\alpha$ by means of the formula

$$(v_0 = \mathring{x})$$

so $X \subseteq M$. To show that $M \prec L_\alpha$ we prove that for any formula $\varphi(v_0)$ of $\mathscr{L}_X$,

$$\models_{L_\alpha} \exists x \varphi(x) \quad \text{implies } (\exists x \in M)[\models_{L_\alpha} \varphi(\mathring{x})].$$

(This is *Tarski's Criterion* for being an elementary submodel.) Let $\psi(v_0)$ be the following $\mathscr{L}_X$-formula:

$$\varphi(v_0) \wedge (\forall v_1)(v_1 <_L v_0 \rightarrow \neg \varphi(v_1)).$$

If $\models_{L_\alpha} \exists x \varphi(x)$ then $\models_{L_\alpha} \exists x \psi(x)$. But there is clearly just one $x \in L_\alpha$ such that $\models_{L_\alpha} \psi(\mathring{x})$. Hence the formula $\psi(v_0)$ defines $x$ from elements of $X$ in $L_\alpha$. Thus $x \in M$. Since $\models_{L_\alpha} \varphi(\mathring{x})$, we are done.

Suppose now that $X \subseteq N \prec L_\alpha$. We show that $M \subseteq N$. Let $x \in M$. For some formula $\varphi(v_0)$ of $\mathscr{L}_X$, $x$ is the unique element of $L_\alpha$ such that $\models_{L_\alpha} \varphi(\mathring{x})$. Now, $\models_{L_\alpha} \exists v_0 \varphi(v_0)$, so as $X \subseteq N \prec L_\alpha$, we have $\models_N \exists v_0 \varphi(v_0)$. So for some $y \in N$, $\models_N \varphi(\mathring{y})$. But $N \prec L_\alpha$, so $\models_{L_\alpha} \varphi(\mathring{y})$. Hence $y = x$, and we are done. $\square$

**5.4 Corollary.** *Let $\alpha$ be a limit ordinal. For any $X \subseteq L_\alpha$ there is a unique smallest $M \prec L_\alpha$ such that $X \subseteq M$. For this $M$,*

$$|M| = \max(|X|, \omega).$$

*Proof.* Let $M$ be as in 5.3. Since $\mathscr{L}_X$ has $\max(|X|, \omega)$ many formulas, we clearly have $|M| = \max(|X|, \omega)$. $\square$

Now, one striking difference between the constructible hierarchy and the cumulative hierarchy of sets is the rate of growth. By definition, if $x \in V_\alpha$, then at level $V_{\alpha+1}$, all subsets of $x$ appear. But the same is not true for the constructible hierarchy. For instance, $L_{\omega+2}$ will contain some subsets of $\omega$, but not all of them. More will appear at level $L_{\omega+3}$, still more at level $L_{\omega+4}$, etc. However, as our next lemma shows, there is a bound to this "gradual growth" process.

**5.5 Lemma.** *Assume $V = L$. Let $\kappa$ be a cardinal. If $x$ is a bounded subset of $\kappa$ (or more generally if $x \subseteq L_\alpha$ for some $\alpha < \kappa$), then $x \in L_\kappa$.*

*Proof.* For $\kappa \leqslant \omega$ the result is trivial, since then $L_\kappa = V_\kappa$. So assume $\kappa > \omega$. Pick $\alpha < \kappa$ so that $\alpha \geqslant \omega$ and $x \subseteq L_\alpha$, and let $\lambda$ be a limit ordinal such that $\lambda \geqslant \kappa$ and $x \in L_\lambda$. By 5.4, let $M \prec L_\lambda$ be such that $L_\alpha \cup \{x\} \subseteq M$ and $|M| = |L_\alpha|$. By the Condensation Lemma, let $\pi\colon M \cong L_\gamma$. Since $L_\alpha \cup \{x\}$ is a transitive subset of $M$, $\pi \upharpoonright L_\alpha \cup \{x\} = \mathrm{id} \upharpoonright L_\alpha \cup \{x\}$. In particular, $\pi(x) = x$. Thus $x \in L_\gamma$. Now by 1.1 (vii), $|L_\alpha| = |\alpha|$ and $|L_\gamma| = |\gamma|$. Hence,

$$|\gamma| = |L_\gamma| = |\pi'' M| = |M| = |L_\alpha| = |\alpha| < \kappa .$$

Thus $\gamma < \kappa$. But then $L_\gamma \subseteq L_\kappa$, so $x \in L_\kappa$, and we are done. $\square$

**5.6 Theorem.** *$V = L$ implies* GCH.

*Proof.* By 5.5, $\mathscr{P}(\kappa) \subseteq L_{\kappa^+}$ for all infinite cardinals $\kappa$. So by 1.1 (vii),

$$(\forall \kappa)(2^\kappa \leqslant |L_{\kappa^+}| = \kappa^+).$$

The result follows at once. $\square$

**5.7 Corollary.** $\mathrm{ZF} \vdash (\mathrm{GCH})^L$.

*Proof.* We know that

$$\mathrm{ZF} \vdash [\mathrm{ZFC} + (V = L)]^L .$$

By 5.6,

$$\mathrm{ZFC} + (V = L) \vdash \mathrm{GCH} .$$

The result follows at once. $\square$

**5.8 Corollary.** *If* ZF *is consistent, so too is* ZFC + GCH.

*Proof.* By the discussion in section 4. $\square$

We finish this section by proving two special cases of the Condensation Lemma for later use. First a technical lemma.

**5.9 Lemma.** *Let $\alpha > \omega$ be a limit ordinal, $N \subseteq L_\alpha$. Let $A(x)$ be a non-empty $\Sigma_0^{L_\alpha}(N)$ predicate on $L_\alpha$. Let $x$ be the $<_L$-least element of $L_\alpha$ such that $A(x)$. Then $x$ is $\Sigma_1$-definable from elements of $N$ in $L_\alpha$.*

*Proof.* We can define $x$ in $L_\alpha$ by the predicate

$$A(x) \wedge (\exists u)[u = \mathrm{pr}(x) \wedge (\forall z \in u) \neg A(z)] .$$

By 3.5, this is $\Sigma_1$. $\square$

**5.10 Lemma.** *Assume $V = L$. If $X \prec_1 L_{\omega_1}$, then $X = L_\alpha$ for some $\alpha \leqslant \omega_1$.*

*Proof.* By the condensation lemma, there are $\pi$, $\alpha$ such that $\alpha \leqslant \omega_1$ and $\pi: X \cong L_\alpha$. If $Y \subseteq X$ is transitive, then $\pi \mid Y = \mathrm{id} \restriction Y$. So if we can show that $X$ itself is transitive we shall be done.

Let $x \in X$. Then $x \in L_{\omega_1} = \bigcup_{\gamma < \omega_1} L_\gamma$, so $x \in L_\gamma$ for some $\gamma < \omega_1$. But $|L_\gamma| \leqslant |\gamma| + \omega = \omega$, so as $L_\gamma$ is transitive, $x$ is countable. There is thus a function $f: \omega \xrightarrow{\text{onto}} x$. Let $f$ be in fact the $<_L$-least such function. By 5.5, $f \in L_{\omega_1}$. So by 5.9, $f$ is $\Sigma_1$-definable from $x$ in $L_{\omega_1}$. But $x \in X \prec_1 L_{\omega_1}$. Thus $f \in X$. But clearly $\omega \subseteq X$. Thus $f(n) \in X$ for all $n < \omega$. Thus $x = f'' \omega \subseteq X$. Hence $X$ is transitive, and we are done. □

**5.11 Lemma.** *Assume $V = L$. Let $\kappa > \omega_1$ be a cardinal. If $\omega_1 \in X \prec_1 L_\kappa$, then $X \cap L_{\omega_1} = L_\alpha$ for some $\alpha \leqslant \omega_1$.*

*Proof.* Since $\omega_1 \in X$ and $X \prec_1 L_\kappa$, we have $L_{\omega_1} \in X$ (by 2.8). Clearly,

$$X \cap L_{\omega_1} = \{x \in X \mid \models_X \text{“}\mathring{x} \in \mathring{L}_{\omega_1}\text{”}\}.$$

But $X \prec_1 L_\kappa$. So, using an obvious extension of our established notation, for any $\Sigma_1$ sentence $\varphi$ of $\mathscr{L}_{X \cap L_{\omega_1}}$, we have

$$\models_{L_{\omega_1}} \varphi \quad \text{iff} \quad \models_{L_\kappa} \varphi^{L_{\omega_1}} \quad \text{iff} \quad \models_X \varphi^{L_{\omega_1}} \quad \text{iff} \quad \models_{X \cap L_{\omega_1}} \varphi .$$

Thus $X \cap L_{\omega_1} \prec_1 L_{\omega_1}$, and we are done by 5.10. (Note that in fact the above argument works for any $\varphi$, $\Sigma_1$ or not, so that we actually have $X \cap L_{\omega_1} \prec L_{\omega_1}$.) □

In connection with 5.11, let us just mention that if $\kappa > \omega_1$ is a cardinal and $X \prec_2 L_\kappa$, then we automatically have $\omega_1, L_{\omega_1} \in X$, since $L_{\omega_1}$ is definable in $L_\kappa$ by the $\Sigma_2$-formula (in free variable $u$)

$$(\exists v)(u = L_v) \wedge (\forall x \in u)(\exists f \in u)(f: \omega \xrightarrow{\text{onto}} x) \wedge (\forall f) \neg (f: \omega \xrightarrow{\text{onto}} u).$$

## 6. $\Sigma_n$ Skolem Functions

The notion of a $\Sigma_n$ skolem function for a structure $L_\alpha$ plays an important role in some of the deeper parts of constructibility theory (the so-called *fine structure theory*). In this section we introduce the basic ideas, and in section 7 we give an application, but a detailed study will not be begun until Chapter VI.

Let $(\varphi_i \mid i < \omega)$ enumerate all $\Pi_{n-1}$ formulas of $\mathscr{L}$ with free variables $v_0, v_1$. Fix $\alpha$ a limit ordinal grater than $\omega$. For each $i < \omega$ and each $x \in L_\alpha$, if

$$\models_{L_\alpha} \exists y\, \varphi_i(y, \mathring{x}),$$

let $h_i(x)$ be some element $y$ of $L_\alpha$ such that

$$\models_{L_\alpha} \varphi_i(\mathring{y}, \mathring{x}).$$

This defines an $\omega$-sequence $(h_i \mid i < \omega)$ of partial functions $h_i$ from $L_\alpha$ to $L_\alpha$. By the usual methods of contracting quantifiers and parameters, if $X \subseteq L_\alpha$ is closed under ordered pairs, and if $Y$ is the closure of $X$ under the functions $h_i$, $i < \omega$, then $Y \prec_n L_\alpha$. (The argument is given, in effect, in 6.2 below.) Now, as far as generating $\Sigma_n$ elementary substructures is concerned, the exact definition of the functions $h_i$ is not important. But we shall require something rather more than this, though at this stage the reader will simply have to postpone a proper motivation until later, since the situation we are leading towards is rather complicated. What we require is a particularly "nice", canonical definition of the functions $h_i$. In particular, we want the functions $h_i$ to be definable over $L_\alpha$ in as logically simple as fashion as possible. Now, the most obvious canonical definition of the functions $h_i$ would be to make use of the canonical well-ordering $<_L$ of $L$, setting $h_i(x) \simeq$ the $<_L$-least $y \in L_\alpha$ such that $\models_{L_\alpha} \varphi_i(\mathring{y}, \mathring{x})$. (The symbol $\simeq$ is standard when partial functions are concerned: to write $f(x) \simeq g(x)$ means that $f(x)$ is defined iff $g(x)$ is defined, in which case they are equal.) It is easily seen that each function $h_i$ so defined will be $\Pi_n$. It turns out that we want functions $h_i$ that are $\Sigma_n$ (and in fact rather more than that), so this obvious definition is not adequate for our purposes. To be precise, what we require is the following. Let $h$ be the partial function defined on $\omega \times L_\alpha$ by

$$h(i, x) \simeq h_i(x).$$

Then the function $h$ should be $\Sigma_n(L_\alpha)$.

The construction of a function $h$ as outlined above requires some considerable effort, and will be postponed until much later. For the moment we investigate the general properties such a function will have.

Let $M$ be an amenable set, and let

$$\mathbf{M} = \langle M, \in, A_1, \ldots, A_k \rangle.$$

Let $n \geqslant 1$. A $\Sigma_n$ *skolem function* for $\mathbf{M}$ is a $\Sigma_n(\mathbf{M})$ function $h$ such that $\mathrm{dom}(h) \subseteq \omega \times M$, $\mathrm{ran}(h) \subseteq M$, for which there is a $p \in M$ such that $h$ is $\Sigma_n^{\mathbf{M}}(\{p\})$, and whenever $A$ is a non-empty $\Sigma_n^{\mathbf{M}}(\{p, x\})$ subset of $M$ for some $x \in M$, there is an $i \in \omega$ such that $h(i, x) \in A$. (In this situation we say that $p$ is a *good parameter* for $h$.)

**6.1 Lemma.** *Let $M$ be an amenable set, and let*

$$\mathbf{M} = \langle M, \in, A_1, \ldots, A_k \rangle.$$

*Let $n \geqslant 1$, and let $h$ be a $\Sigma_n$ skolem function for $\mathbf{M}$. Then*:

(i) *if $x \in M$, then*

$$x \in h''(\omega \times \{x\}) \prec_n \mathbf{M};$$

(ii) *if* $q \in M$ *and if* $X \subseteq M$ *is closed under ordered pairs, then*

$$X \cup \{q\} \subseteq h''(\omega \times (X \times \{q\})) \prec_n \mathbf{M}.$$

*Proof.* Let $p$ be a good parameter for $h$. We prove (i) first, Set

$$N = h''(\omega \times \{x\}).$$

Since $\{x\}$ is a $\Sigma_0^{\mathbf{M}}(\{x\})$ subset of $M$,

$$h(i, x) \in \{x\}$$

for some $i < \omega$, so $x \in N$. To show that $N \prec_n \mathbf{M}$ we use 5.1. Let $A$ be a non-empty $\Sigma_n^{\mathbf{M}}(N)$ subset of $M$. We must show that $A \cap N \neq \emptyset$. Pick $y_1, \dots, y_m \in N$ so that $A \in \Sigma_n^{\mathbf{M}}(\{y_1, \dots, y_m\})$. By definition of $N$ there are $j_1, \dots, j_m < \omega$ such that

$$y_1 = h(j_1, x), \dots, y_m = h(j_m, x).$$

Let $\varphi(v_0, \dots, v_m)$ be a $\Sigma_n$ formula of the $\mathbf{M}$-language, having no individual constants, such that for any $a \in M$,

$$a \in A \quad \text{iff } \models_{\mathbf{M}} \varphi(\mathring{a}, \mathring{y}_1, \dots, \mathring{y}_m);$$

and let $\psi(v_0, v_1, v_2, v_3)$ be a $\Sigma_n$ formula of the $\mathbf{M}$-language, also having no individual constants, such that for any $u, v \in M$, $i \in \omega$,

$$v = h(i, u) \quad \text{iff } \models_{\mathbf{M}} \psi(\mathring{v}, \mathring{i}, \mathring{u}, \mathring{p}).$$

Then, for any $a \in M$,

$$a \in A \quad \text{iff } \models_{\mathbf{M}} (\exists y_1, \dots, y_m)[\psi(y_1, \mathring{j}_1, \mathring{x}, \mathring{p}) \wedge \dots \wedge \psi(y_m, \mathring{j}_m, \mathring{x}, \mathring{p}) \wedge \varphi(\mathring{a}, y_1, \dots, y_m)].$$

The predicate $A$ is thus seen to be $\Sigma_n^{\mathbf{M}}(\{x, p\})$. (The parameters $j_1, \dots, j_m$ can be ignored since, being integers they can be replaced by their set-theoretic definitions, i.e. $0 = \emptyset$, $1 = \{\emptyset\}$, etc.) So by the definition of the $\Sigma_n$ skolem function concept there is an $i < \omega$ such that

$$h(i, x) \in A.$$

Thus $A \cap N \neq \emptyset$, and we are done.

We turn to the proof of (ii). Let

$$N = h''(\omega \times (X \times \{q\})).$$

As in part (i) we get $X \cup \{q\} \subseteq N$, and we must show that $N \prec_n \mathbf{M}$. Again, we begin by picking an arbitrary $\Sigma_n^{\mathbf{M}}(N)$ subset $A$ of $M$, and show that if $A \neq \emptyset$ then

$A \cap N \neq \emptyset$. Pick $y_1, \ldots, y_m \in N$ so that $A \in \Sigma_n^{\mathbf{M}}(\{y_1, \ldots, y_m\})$. Pick $j_1, \ldots, j_m < \omega$ and $x_1, \ldots, x_m \in X$ so that

$$y_1 = h(j_1, (x_1, q)), \ldots, y_m = h(j_m, (x_m, q)).$$

Set $x = (x_1, \ldots, x_m)$. Since $X$ is closed under ordered pairs, $x \in X$. As in part (i) it is easily seen that $A$ is $\Sigma_n^{\mathbf{M}}(\{p, (x, q)\})$. It follows that there is an $i < \omega$ such that $h(i, (x, q)) \in A$, giving $A \cap N \neq \emptyset$. $\square$

**6.2 Corollary.** *Let* $\mathbf{M}$, $n$, $h$ *be as in* 6.1. *If* $X \subseteq M$ *and if* $h''(\omega \times X)$ *is closed under ordered pairs, then*

$$X \subseteq h''(\omega \times X) \prec_n \mathbf{M}.$$

*Proof.* Let

$$Y = h''(\omega \times X).$$

By 6.1 it suffices to prove that $h''(\omega \times Y) = Y$. Well, since $X \subseteq Y$ we clearly have

$$Y = h''(\omega \times X) \subseteq h''(\omega \times Y).$$

Conversely, suppose $z \in h''(\omega \times Y)$. Pick $i \in \omega$, $y \in Y$ so that $z = h(i, y)$. For some $j \in \omega$, $x \in X$ we have $y = h(j, x)$. Thus $z = h(i, h(j, x))$, which shows that $z$ is $\Sigma_n$-definable from $p$ and $x$ in $\mathbf{M}$. Thus $\{z\}$ is a $\Sigma_n^M(\{x, p\})$ subset of $M$. So for some $k \in \omega$, $h(k, x) \in \{z\}$. Thus $z \in h''(\omega \times X) = Y$, and we are done. $\square$

If $\alpha > \omega$ is a limit ordinal, then $L_\alpha$ has a $\Sigma_n$ skolem function for every $n \geqslant 1$. For $n > 1$, the proof of this fact is quite tricky, and will not be given until Chapter VI. But for $n = 1$ the proof is both easy and illuminating, so we deal with this case now.

For any limit ordinal $\alpha > \omega$ and any $n \in \omega$, $\models_{L_\alpha}^{\Sigma_n}$ denotes the restriction of $\models_{L_\alpha}$ to the $\Sigma_n$ sentences of $\mathscr{L}_{L_\alpha}$.

**6.3 Lemma.** *Let* $\alpha > \omega$ *be a limit ordinal. Then the relation* $\models_{L_\alpha}^{\Sigma_0}$ *is (uniformly for all such* $\alpha$*)* $\Delta_1^{L_\alpha}$.

*Proof.* For the purposes of this proof, we shall regard the $\Sigma_1$ formula $\mathrm{Sat}(u, \varphi)$ as being expressed in the language $\mathscr{L}$, rather than in LST as defined previously.

If $\varphi$ is a $\Sigma_0$ sentence of $\mathscr{L}_{L_\alpha}$, then by $\Sigma_0$-absoluteness, if $\gamma < \alpha$ is such that $\varphi \in L_\gamma$,

$$\models_{L_\alpha} \varphi \quad \text{iff } \models_{L_\gamma} \varphi.$$

Moreover, absoluteness considerations also tell us that for any $u \in L_\alpha$ and any formula $\psi$ of $\mathscr{L}_u$,

$$\models_u \psi \quad \text{iff } \models_{L_\alpha} \mathrm{Sat}(\mathring{u}, \mathring{\psi}).$$

Hence, for $\varphi$ as above,

$$\models_{L_\alpha} \varphi \quad \text{iff } \models_{L_\alpha} (\exists \gamma)[(\mathring{\varphi} \in L_\gamma) \wedge \mathrm{Sat}(L_\gamma, \mathring{\varphi})],$$

and,

$$\models_{L_\alpha} \varphi \quad \text{iff } \models_{L_\alpha} (\forall \gamma)[(\mathring{\varphi} \in L_\gamma) \to \mathrm{Sat}(L_\gamma, \mathring{\varphi})].$$

By I.9.12, the class $\mathrm{Fml}^{\Sigma_0}$ is $\Delta_1^{L_\alpha}$, and by I.9.10 the class Sat is $\Delta_1^{L_\alpha}$. Also, by 2.8, the relation $(x = L_\gamma)$ is $\Delta_1^{L_\alpha}$. Thus the relation $\models_{L_\alpha}^{\Sigma_0}$ is $\Delta_1^{L_\alpha}$ by virtue of the two definitions above (which are uniform for $\alpha$). $\square$

**6.4 Lemma.** *Let $\alpha > \omega$ be a limit ordinal, and let $n \geqslant 1$. Then the relation $\models_{L_\alpha}^{\Sigma_0}$ is (uniformly in $\alpha$) $\Sigma_n^{L_\alpha}$.*

*Proof.* Let $\varphi$ be a $\Sigma_n$ sentence of $\mathscr{L}_{L_\alpha}$. In case $n$ is odd, the following is clearly equivalent to $\models_{L_\alpha} \varphi$:

$$\begin{aligned}\models_{L_\alpha} [&\exists x_1 \forall x_2 \exists x_3 \ldots \exists x_n \exists \psi \exists u \exists f \exists \theta [\mathrm{Fml}(\psi) \wedge \mathrm{Fr}(\psi, u) \wedge (f\colon n \leftrightarrow u)\\ &\wedge (\mathring{\varphi} = \exists f(0) \forall f(1) \exists f(2) \ldots \exists f(n-1) \psi) \wedge \mathrm{Finseq}(\theta)\\ &\wedge (\mathrm{dom}(\theta) = n+1) \wedge (\theta_0 = \psi) \wedge (\forall i \in n)(\mathrm{Sub}(\theta_{i+1}, \theta_i, f(i), \mathring{x}_{i+1}))\\ &\wedge (\models_{L_\alpha}^{\Sigma_0} \theta_n)]].\end{aligned}$$

In case $n$ is even, then equivalent to $\models_{L_\alpha} \varphi$ we have:

$$\begin{aligned}\models_{L_\alpha} [&\exists x_1 \forall x_2 \exists x_3 \ldots \forall x_n \forall \psi \forall u \forall f \forall \theta [[\mathrm{Fml}(\psi) \wedge \mathrm{Fr}(\psi, u) \wedge (f\colon n \leftrightarrow u)\\ &\wedge (\mathring{\varphi} = \exists f(0) \forall f(1) \exists f(2) \ldots \forall f(n-1) \psi) \wedge \mathrm{Finseq}(\theta)\\ &\wedge (\mathrm{dom}(\theta) = n+1) \wedge (\theta_0 = \psi) \wedge (\forall i \in n)(\mathrm{Sub}(\theta_{i+1}, \theta_i, f(i), \mathring{x}_{i+1})]\\ &\to [\models_{L_\alpha}^{\Sigma_0} \theta_n]]].\end{aligned}$$

In either case, the $\mathscr{L}$-formula which says that

$$\mathring{\varphi} = \exists f(0) \forall f(1) \exists f(2) \ldots - f(n-1) \psi$$

is easily seen to be $\Sigma_0$ (given the values of $f(0), \ldots, f(n-1)$), being simply a long sequence of conjuncts concerning the values of the sequence $\varphi$. So by 6.3, the above expressions give a (uniformly in $\alpha$) $\Sigma_n$ definition of $\models_{L_\alpha}^{\Sigma_n}$. $\square$

Using 6.4, we can now show that for limit $\alpha > \omega$, $L_\alpha$ has a $\Sigma_1$ skolem function.

**6.5 Lemma.** *Let $\alpha > \omega$ be a limit ordinal. Then $L_\alpha$ has a $\Sigma_1$ skolem function. Indeed, there is a $\Sigma_0$ formula $\Theta(v_0, v_1, v_2, v_3)$ of $\mathscr{L}$ such that for any limit ordinal $\alpha > \omega$, $h_\alpha$ is $\Sigma_1$ skolem function for $L_\alpha$, where*

$$y = h_\alpha(i, x) \leftrightarrow \models_{L_\alpha} \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

*Proof.* By an argument similar to the proof of I.9.6 (see also I.9.13), the relation "$\varphi$ is an $\mathscr{L}$-formula of the form $\exists v_2\, \bar{\varphi}(v_0, v_1, v_2)$ where $\bar{\varphi}$ is $\Sigma_0$" is uniformly $\Delta_1^{L_\alpha}$. So, if we define an enumeration $(\varphi_i | i < \omega)$ of all formulas of the form $\varphi_i = \exists v_2\, \bar{\varphi}_i(v_0, v_1, v_2)$ where $\bar{\varphi}_i$ is $\Sigma_0$, in the same way that we well-ordered the formulas of $\mathscr{L}$ in our definition of $<_L$ (see immediately following 3.1), then it is easily seen that this enumeration is uniformly $\Delta_1^{L_\alpha}$.

Define a partial function $r_\alpha$ on $\omega \times L_\alpha$ by:

$$r_\alpha(i, x) \simeq \text{the } <_L\text{-least } w \in L_\alpha \text{ such that}$$
$$\models_{L_\alpha}(\text{“}\mathring{w} \text{ is an ordered pair”}) \wedge \bar{\varphi}_i((\mathring{w})_0, \mathring{x}, (\mathring{w})_1).$$

By (the proof of) 5.9, $r_\alpha$ is $\Sigma_1^{L_\alpha}$. Hence the partial function $h_\alpha$ is $\Sigma_1^{L_\alpha}$, where we define $h_\alpha$ on $\omega \times L_\alpha$ by:

$$h_\alpha(i, x) \simeq (r_\alpha(i, x))_0 .$$

We show that $h_\alpha$ is a $\Sigma_1$ skolem function for $L_\alpha$ with good parameter $\emptyset$ (i.e. effectively with *no* parameter).

Let $A$ be a non-empty $\Sigma_1^{L_\alpha}(\{x\})$ subset of $L_\alpha$. For some $i < \omega$,

$$z \in A \leftrightarrow \models_{L_\alpha} \varphi_i(\mathring{z}, \mathring{x}).$$

Since $A \neq \emptyset$,

$$\models_{L_\alpha} \exists z\, \varphi_i(z, \mathring{x}).$$

Hence,

$$\models_{L_\alpha} \exists w\, [\text{“}w \text{ is an ordered pair”} \wedge \bar{\varphi}_i((w)_0, \mathring{x}, (w)_1)].$$

Thus $r_\alpha(i, x)$ is defined, say $w = r_\alpha(i, x)$. Hence $h_\alpha(i, x)$ is defined and $h_\alpha(i, x) = (w)_0$. Clearly,

$$\models_{L_\alpha} \varphi_i((\mathring{w})_0, \mathring{x}).$$

Hence $h_\alpha(i, x) \in A$, as required.

Notice that the above proof did not depend upon $\alpha$. Hence there is a single $\Sigma_0$ formula $\Theta$ as described in the lemma. □

Notice that in general, the above procedure will not produce a $\Sigma_n$ skolem function if $n > 1$. For if the formulas $\varphi_i$ are $\Sigma_n$, then the formulas $\bar{\varphi}_i$ will be $\Pi_{n-1}$, which means that the function $r_\alpha$ will be $\Pi_n$, as is easily seen by writing out the definition of $r_\alpha$ more fully. (The procedure works in the case $n = 1$ because a bounded universal quantifier prefixing a $\Sigma_0$ formula results in another $\Sigma_0$ formula, whereas if $n > 1$, a bounded universal quantifier prefixing a $\Sigma_{n-1}$ formula gives a $\Pi_n$ formula.)

The function $h_\alpha$ defined in 6.5 is called the *canonical* $\Sigma_1$ skolem function for $L_\alpha$. We illustrate its use in 6.8 below, for which application we require two lemmas.

**6.6 Lemma** (Gödel's Pairing Function). *There is a* $\Delta_1^{\mathrm{KP}}$ *formula* $\Phi(v_0, v_1, v_2)$ *of* LST *such that, if*

$$G = \{(\gamma, (\alpha, \beta)) \mid \Phi(\alpha, \beta, \gamma)\},$$

*then*:

(i) *$G$ is uniformly $\Sigma_1^{L_\alpha}$ for limit $\alpha > \omega$;*
(ii) *$G$*: On $\times$ On $\leftrightarrow$ On;
(iii) *$G(\alpha, \beta) \geqslant \alpha, \beta$ for all $a, \beta$.*

*Proof.* Define a well-ordering $<^*$ of $\mathrm{On} \times \mathrm{On}$ by setting $(\alpha, \beta) <^* (\gamma, \delta)$ iff:

(i) $\max(\alpha, \beta) < \max(\gamma, \delta)$; or
(ii) $\max(\alpha, \beta) = \max(\gamma, \delta)$ and $\alpha < \gamma$; or
(iii) $\max(\alpha, \beta) = \max(\gamma, \delta)$ and $\alpha = \gamma$ and $\beta < \delta$.

Let $G(\alpha, \beta)$ be the order-type of the set of predecessors of $(\alpha, \beta)$ under $<^*$. Thus

$$G\colon (\mathrm{On} \times \mathrm{On}, <^*) \cong (\mathrm{On}, <).$$

It is not hard to see that:

$$\text{(I)} \qquad G(0, \beta) = \sup\nolimits_{\nu < \beta} G(\nu, \nu);$$

$$\text{(II)} \qquad G(\alpha, \beta) = \begin{cases} G(0, \beta) + \alpha, & \text{if } \alpha < \beta, \\ G(0, \alpha) + \alpha + \beta, & \text{if } \alpha \geqslant \beta. \end{cases}$$

By (I) and (II) we can define the unary function $G(0, \beta)$ by means of the recursion

$$\text{(III)} \qquad G(0, \beta) = \sup\nolimits_{\nu < \beta}(G(0, \nu) + \nu + \nu).$$

Thus (c.f. the proof of the KP Recursion Theorem, I.11.8), the function $G(0, \beta)$ can be defined by means of a $\Sigma_1^{\mathrm{KP}}$ formula of LST, and is (by checking that the relevant existential quantifiers can be restricted to $L_\alpha$) uniformly $\Sigma_1^{L_\alpha}$ for limit $\alpha > \omega$. But using (II) we can define the binary function $G(\alpha, \beta)$ from $G(0, \beta)$ in $\Sigma_1$ fashion. Hence $G$ is definable by means of a $\Sigma_1^{\mathrm{KP}}$ formula of LST and is uniformly $\Sigma_1^{L_\alpha}$ for limit $\alpha > \omega$. (In connection with the definability results for $L_\alpha$ in the above, it should perhaps be emphasised that there is no suggestion that $L_\alpha$ should be closed under the function $G$; rather that for each limit $\alpha > \omega$ the class $G \cap (L_\alpha)^3$ is (uniformly) $\Sigma_1^{L_\alpha}$.) Since $G$ is a total function on $\mathrm{On} \times \mathrm{On}$, it is in fact definable by a $\Delta_1^{\mathrm{KP}}$ formula of LST. (But since $G \cap (L_\alpha)^3$ is not total on $\alpha \times \alpha$, it is not the case that $G$ is $\Delta_1^{L_\alpha}$.) $\square$

**6.7 Lemma.** *Let $\alpha > \omega$ be a limit ordinal. Then there is a $\Sigma_1(L_\alpha)$ map of $\alpha$ onto $\alpha \times \alpha$.*

*Proof.* Before we start, we remark that there is no suggestion of any uniformity here, and indeed for many ordinals $\alpha$ we shall make use of parameters in order to define the mapping of $\alpha$ onto $\alpha \times \alpha$.

Set

$$Q = \{\alpha \mid G\colon \alpha \times \alpha \leftrightarrow \alpha\}.$$

It is easily seen that $Q$ is closed and unbounded in On, and that

$$Q = \{\alpha \mid G(0, \alpha) = \alpha\}.$$

We prove the lemma by induction on $\alpha$. Assume that it holds for all $\beta < \alpha$. To prove it for $\alpha$ we consider three cases.

*Case 1.* $\alpha \in Q$.

In this case $G^{-1} \restriction \alpha$ is sufficient.

*Case 2.* $\alpha = \gamma + \omega$ for some limit ordinal $\gamma$.

Define $j\colon \alpha \leftrightarrow \gamma$ by

$$j(\xi) = \begin{cases} 2\xi, & \text{if } \xi < \omega, \\ \xi, & \text{if } \omega \leqslant \xi < \gamma, \\ 2n+1, & \text{if } \xi = \gamma + n. \end{cases}$$

Clearly, $j$ is $\Sigma_1^{L_\alpha}(\{\omega, \gamma\})$.

By induction hypothesis there is a $\Sigma_1(L_\gamma)$ map

$$g\colon \gamma \xrightarrow{\text{onto}} \gamma \times \gamma.$$

Define a map

$$f\colon \alpha \to \alpha \times \alpha$$

by

$$f(\xi) = (j^{-1}((g \circ j(\xi))_0), j^{-1}((g \circ j(\xi))_1)).$$

Clearly, $f$ is $\Sigma_1(L_\alpha)$ and maps $\alpha$ onto $\alpha \times \alpha$. (The function $g$ is an element of $L_\alpha$, and is thus a parameter in the definition of $f$.)

*Case 3.* Otherwise.

Set $(\nu, \tau) = G^{-1}(\alpha)$. Since $\alpha \notin Q$, $\nu, \tau < \alpha$. Let

$$C = \{z \mid z <^* (\nu, \tau)\},$$

where $<^*$ is the well-ordering of $\mathrm{On} \times \mathrm{On}$ defined in the proof of 6.6. Notice that $C \in L_\alpha$. Now, $g = G \restriction C$ maps $C$ one-one onto $\alpha$ (by definition of $G$ from $<^*$). So by 6.6 (i), $g$ is $\Sigma_1^{L_\alpha}(\{C\})$.

Let $\gamma > \omega$ be a limit ordinal such that $\nu, \tau < \gamma < \alpha$. By the induction hypothesis there is a $\Sigma_1(L_\gamma)$ map

$$k\colon \gamma \xrightarrow{\text{onto}} \gamma \times \gamma.$$

Define

$$\bar{k}\colon \gamma \times \gamma \xrightarrow{\text{one-one}} \gamma$$

by setting

$$\bar{k}(\xi, \zeta) = \text{the least } \iota \text{ such that } k(\iota) = (\xi, \zeta).$$

Then $l$ is one-one from $\alpha$ to $\gamma$ where we set $l = \bar{k} \circ g^{-1}$. (Since $\nu, \tau < \gamma$, $C \subseteq \gamma \times \gamma$, so $\operatorname{ran}(g^{-1}) = C \subseteq \gamma \times \gamma = \operatorname{dom}(\bar{k})$.) Now define

$$h\colon \alpha \times \alpha \xrightarrow{\text{one-one}} \gamma \times \gamma$$

by setting $h(\xi, \zeta) = (l(\xi), l(\zeta))$, and define

$$p\colon \alpha \times \alpha \xrightarrow{\text{one-one}} \gamma$$

by $p = \bar{k} \circ h$. Now, $\mathrm{ran}(l) = \bar{k}''(g^{-1}{}''\alpha) = \bar{k}''C$. Hence $\mathrm{ran}(h) = (\bar{k}''C) \times (\bar{k}''C)$. Hence $\mathrm{ran}(p) = \bar{k}''\mathrm{ran}(h) = \bar{k}''[(\bar{k}''C) \times (\bar{k}''C)]$. But $k$ is $L_\gamma$-definable, so $\bar{k} \in L_\alpha$. Since we know that $C \in L_\alpha$, it follows that $D \in L_\alpha$, where $D = \mathrm{ran}(p)$. Thus the function

$$f\colon \alpha \to \alpha \times \alpha$$

is $\Sigma_1(L_\alpha)$, where we set

$$f(\xi) = \begin{cases} p^{-1}(\xi), & \text{if } \xi \in D, \\ (0, 0), & \text{if } \xi \notin D. \end{cases}$$

Since $f$ is clearly onto $\alpha \times \alpha$, we are done. $\square$

We are now able to give our promised use of the $\Sigma_1$ skolem function. It is a "localised" version of 3.6.

**6.8 Lemma.** *Let $\alpha > \omega$ be a limit ordinal. Then there is a $\Sigma_1(L_a)$ map of $\alpha$ onto $L_\alpha$. (The map is not uniformly definable, and may involve parameters in its definition.)*

*Proof.* By 6.7, let $f$ be a $\Sigma_1^{L_\alpha}(\{p\})$ *map of* $\alpha$ *onto* $\alpha \times \alpha$, chosen so that $p$ is the $<_L$-least element of $L_\alpha$ for which such an $f$ exists. Define "inverse functions" $f^0$, $f^1$ to $f$ by the requirement

$$f(v) = (f^0(v), f^1(v)) \qquad (v \in \alpha).$$

For each $n$, define a $\Sigma_1^{L_\alpha}(\{p\})$ function $f_n$ from $\alpha$ onto $\alpha^n$ so that the following conditions are satisfied:

$$f_1 = \mathrm{id} \restriction \alpha;$$
$$f_{n+1}(v) = (f^0(v), f_n \circ f^1(v)).$$

(For each $n$, the precise definition of $f_n$ is obtained by unravelling the above "recursion".)

Let $h = h_\alpha$, the canonical $\Sigma_1$ skolem function for $L_\alpha$, and let $\Theta$ be the canonical $\Sigma_0$ formula of $\mathscr{L}$ which defines $h$ (see 6.5). Set

$$X = h''(\omega \times (\alpha \times \{p\})).$$

*Claim 1.* $X$ is closed under ordered pairs.

To see this, let $x_1, x_2 \in X$. Pick $j_1, j_2 \in \omega$ and $v_1, v_2 \in \alpha$ so that

$$x_1 = h(j_1, (v_1, p)), \qquad x_2 = h(j_2, (v_2, p)).$$

Let $\tau \in \alpha$ be such that

$$(v_1, v_2) = f_2(\tau).$$

Clearly, $\{(x_1, x_2)\}$ is a $\Sigma_1^{L_\alpha}(\{(\tau, p)\})$ predicate on $L_\alpha$. Hence by the properties of $h$,

$$(x_1, x_2) \in X,$$

which proves the claim.

By claim 1 and 6.2,

$$X \prec_1 L_\alpha.$$

By the Condensation Lemma, let

$$\pi\colon X \cong L_\beta.$$

Since $\alpha \subseteq X$, we must have $\beta = \alpha$, so in fact

$$\pi\colon X \cong L_\alpha.$$

*Claim 2.* For all $i \in \omega$, $x \in X$,

$$\pi(h(i, x)) \simeq h(i, \pi(x)).$$

To see this, suppose first that $y = h(i, x)$ is defined. Since $h$ is $\Sigma_1^{L_\alpha}$ and $x \in X \prec_1 L_\alpha$, we have $y \in X$. Since $y = h(i, x)$, we have

$$\models_{L_\alpha} \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

So, as $x, y, i \in X \prec_1 L_\alpha$,

$$\models_X \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

So for some $z \in X$,

$$\models_X \Theta(\mathring{z}, \mathring{y}, \mathring{i}, \mathring{x}).$$

Applying $\pi$ gives

$$\models_{L_\alpha} \Theta(\pi(\mathring{z}), \pi(\mathring{y}), \mathring{i}, \pi(\mathring{x})).$$

Hence,

$$\models_{L_\alpha} \exists z\, \Theta(z, \pi(\mathring{y}), \mathring{i}, \pi(\mathring{x})).$$

This means that

$$\pi(y) = h(i, \pi(x))$$

(and in particular that $h(i, \pi(x))$ is defined).

Now suppose that $h(i, \pi(x))$ is defined. Then $h(i, \pi(x)) \in L_\alpha = \pi'' X$, so for some $y \in X$,

$$h(i, \pi(x)) = \pi(y).$$

By reversing the argument above we obtain

$$\pi^{-1}(h(i, \pi(x))) = h(i, x),$$

and the claim is proved.

Now, $f \subseteq \alpha \times \alpha \times \alpha$, and $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, so

$$\pi'' f = f.$$

Moreover, since $p \in X \prec_1 L_\alpha$ and $\pi\colon X \cong L_\alpha$, $\pi'' f$ is $\Sigma_1^{L_\alpha}(\{\pi(p)\})$. So by choice of $p$ we must have $p \leqslant_L \pi(p)$. But by the properties of $\pi$ (see 5.2), $\pi(p) \leqslant_L p$. Hence $\pi(p) = p$. So by claim 2, if $i \in \omega$ and $v \in \alpha$,

$$\pi(h(i, (v, p))) \simeq h(i, (v, p)).$$

Thus $\pi = \mathrm{id} \restriction X$, which means that $X = L_\alpha$. It follows at once that the function $r$ defined on a subset of $\alpha$ by

$$r(v) \simeq h((f(v))_0, ((f(v))_1, p))$$

is a $\Sigma_1(L_a)$ map such that $r'' \alpha = L_\alpha$. But this does not prove the lemma, since we are looking for a *total* function from $\alpha$ onto $L_\alpha$. However, a simple modification to the function $r$ will suffice. Define $g$ from $\alpha^3$ to $L_\alpha$ by:

$$g(i, v, \tau) = \begin{cases} y, & \text{if } \models_{L_\alpha} \exists w [w = L_\tau \wedge (\exists z \in w)\, \Theta(z, \mathring{y}, \mathring{i}, (\mathring{v}, \mathring{p}))], \\ \emptyset, & \text{if} \cdot \models_{L_\alpha} \exists w [w = L_\tau \wedge (\forall z \in w) \neg\, \Theta(z, \mathring{y}, \mathring{i}, (\mathring{v}, \mathring{p}))]. \end{cases}$$

It is easily seen that $g$ is $\Sigma_1(L_\alpha)$. And clearly,

$$g''(\alpha \times \alpha \times \alpha) = h''(\omega \times (\alpha \times \{p\})) = L_\alpha.$$

Thus $g \circ f_3$ is a required for the lemma. $\square$

We give further applications of the $\Sigma_1$ skolem function in the next section.

## 7. Admissible Ordinals

The notion of an admissible set has already been introduced in I.11. An ordinal $\alpha$ is said to be *admissible* iff there is an admissible set $M$ such that $M \cap \mathrm{On} = \alpha$.

By I.11.2, every uncountable cardinal is an admissible ordinal. The converse is not true, and indeed it is a simple exercise involving the Condensation Lemma to show that if $\kappa$ is an uncountable cardinal, there are $\kappa$ many admissible ordinals below $\kappa$. The starting point for this proof is the following lemma.

**7.1 Lemma.** *An ordinal $\alpha$ is admissible iff $L_\alpha$ is an admissible set.*

*Proof.* If $L_\alpha$ is an admissible set, then $\alpha$ is an admissible ordinal since $L_\alpha \cap \mathrm{On} = \alpha$.

Conversely, suppose that $\alpha$ is an admissible ordinal. Let $M$ be an admissible set such that $M \cap \mathrm{On} = \alpha$. Clearly, $\alpha$ is a limit ordinal greater than $\omega$. Hence by 2.1, $L_\alpha$ is amenable. Thus we must prove that for any $\Sigma_0(L_\alpha)$ relation $R \subseteq L_\alpha \times L_\alpha$, if

$$(\forall x \in L_\alpha)(\exists y \in L_\alpha) R(y, x),$$

then for any $u \in L_\alpha$ there is a $v \in L_\alpha$ such that

$$(\forall x \in u)(\exists y \in v) R(y, x).$$

Let $\varphi(v_0, v_1)$ be a $\Sigma_0$-formula of $\mathscr{L}_{L_\alpha}$ such that for all $x, y \in L_\alpha$,

$$R(y, x) \leftrightarrow \models_{L_\alpha} \varphi(\mathring{y}, \mathring{x}).$$

Thus

$$\models_{L_\alpha} \forall x \exists y \varphi(y, x).$$

Let $u \in L_\alpha$ be given. We seek a $v \in L_\alpha$ such that

$$\models_{L_\alpha} (\forall x \in \mathring{u})(\exists y \in \mathring{v}) \varphi(y, x).$$

Define a function $g$ from $u$ to $\alpha$ by

$$g(x) = \text{the least } \gamma \text{ such that } (\exists y \in L_\gamma)[\models_{L_\alpha} \varphi(\mathring{y}, \mathring{x})].$$

Since $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$, $g$ is well-defined. Now, since $\varphi$ is $\Sigma_0$, for $x, y \in L_\alpha$, we have , by I.9.14,

$$\models_{L_\alpha} \varphi(\mathring{y}, \mathring{x}) \quad \text{iff } \models_M \varphi(\mathring{y}, \mathring{x}).$$

Moreover, by 2.10, $(L_\gamma)^M = L_\gamma$ for all $\gamma < \alpha$. Hence for any $x, \gamma \in M$,

$$\gamma = g(x) \leftrightarrow \models_M [(\mathring{x} \in \mathring{u}) \wedge (\exists w)[(w = L_{\mathring{\gamma}}) \wedge (\exists y \in w) \varphi(y, \mathring{x}) \\ \wedge (\forall v \in w) \neg (\exists y \in v) \varphi(y, \mathring{x})]].$$

Thus $g$ is $\Sigma_1(M)$. So by Localised $\Sigma_1$ Collection (I.11.5) for the admissible set $M$ there is a $v \in M$ such that

$$(\forall x \in u)(\exists \gamma \in v)(\gamma = g(x)).$$

Since $M$ is amenable,

$$\delta = \bigcup (v \cap \mathrm{On}) \in M.$$

Then by definition of $g$,

$$(\forall x \in u)(\exists y \in L_\delta)[\models_{L_\alpha} \varphi(\mathring{y}, \mathring{x})].$$

Since $\delta \in M$ we have $\delta < \alpha$, so $L_\delta \in L_\alpha$ and we are done. $\square$

Using 7.1 we may prove:

**7.2 Lemma.** *Let $\alpha > \omega$ be a limit ordinal. Then $\alpha$ is admissible iff there is no $\Sigma_1(L_\alpha)$ mapping from an ordinal $\delta < \alpha$ cofinally into $\alpha$.*

*Proof.* Suppose first that $\alpha$ is admissible. Let $\delta < \alpha$, and let

$$f\colon \delta \to \alpha$$

be $\Sigma_1(L_\alpha)$. Then

$$(\forall \xi \in \delta)(\exists \zeta \in \alpha)(\zeta = f(\xi)).$$

By $\Sigma_1$ Collection for $L_\alpha$ there is a $\gamma < \alpha$ such that

$$(\forall \xi \in \delta)(\exists \zeta \in \gamma)(\zeta = f(\xi)).$$

Thus $f''\delta \subseteq \gamma < \alpha$, showing that $f$ cannot be cofinal in $\alpha$.

Conversely, suppose there is no $\Sigma_1(L_\alpha)$ function from an ordinal $\delta < \alpha$ cofinally into $\alpha$. Then certainly $\alpha$ cannot be of the form $\gamma + \omega$ for any $\gamma$, so $\alpha$ is a limit of limit ordinals. So by 6.8 there can be no $\Sigma_1(L_\alpha)$ function from any $L_\delta$, $\delta < \alpha$, into $\alpha$ whose range is unbounded in $\alpha$. We show that this implies that $L_\alpha$ is an admissible set. By 2.1, $L_\alpha$ is amenable. So, given a $\Sigma_0(L_\alpha)$ relation $R(y, x)$ on $L_\alpha$ such that

$$(\forall x \in L_\alpha)(\exists y \in L_\alpha)R(y, x),$$

and given a $u \in L_\alpha$, we must find a $v \in L_\alpha$ such that

$$(\forall x \in u)(\exists y \in v)R(y, x).$$

Pick $\delta < \alpha$ so that $u \in L_\delta$, and define a function $f$ from $L_\delta$ *to* $\alpha$ by:

$$f(x) = \begin{cases} \text{the least } \gamma \text{ such that } (\exists y \in L_\gamma)R(y, x), & \text{if } x \in u, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that $f$ is $\Sigma_1(L_\alpha)$. By the above remarks, we know that $f$ cannot be cofinal in $\alpha$, so there is a $\varrho < \alpha$ such that

$$f''L_\delta \subseteq \varrho.$$

By definition of $f$,

$$(\forall x \in u)(\exists y \in L_\varrho)R(y, x),$$

so we are done. $\square$

Our next result strengthens 7.2 considerably. To state the result, it is convenient to introduce the following extension of the concept of amenability, an extension which we shall make frequent use of during our later development.

A structure

$$\mathbf{M} = \langle M, \in, A_1, \ldots, A_k \rangle$$

is said to be *amenable* if $M$ is an amenable set and for each $i = 1, \ldots, k$,

$$u \in M \quad \text{implies} \quad A_i \cap u \in M.$$

(This condition can be regarded as an extension of the "$\Sigma_0$ Comprehension" axiom for amenable sets, that if $R \subseteq M$ is $\Sigma_0(M)$, then $R \cap u \in M$ for all $u \in M$.)

**7.3 Theorem.** *Let $\alpha > \omega$ be a limit ordinal. Then the following are equivalent:*

(i) *$\alpha$ is admissible;*
(ii) *the structure $\langle L_\alpha, A \rangle$ is amenable for any $\Delta_1(L_\alpha)$ set $A \subseteq L_\alpha$;*
(iii) *there is no $\Delta_1(L_\alpha)$ function from an ordinal $\delta < \alpha$ onto $\alpha$.*

*Proof.* (i) $\to$ (ii). This is an immediate consequence of the $\Delta_1$ Comprehension Principle (I.11.1).

(ii) $\to$ (iii). We assume that (ii) holds and (iii) fails and use a diagonalisation argument to obtain a contradiction. By 6.8 and the failure of (iii) there is a $\delta < \alpha$ and a $\Sigma_1(L_\alpha)$ map $f$ from $\delta$ onto $L_\alpha$. Being total, $f$ is in fact $\Delta_1(L_\alpha)$. Hence $D$ is $\Delta_1(L_\alpha)$, where we set

$$D = \{\nu \in \delta \mid \nu \notin f(\nu)\}.$$

By (iii),

$$D = D \cap \delta \in L_\alpha.$$

Hence $D = f(\nu)$ for some $\nu < \delta$. But then

$$\nu \in f(\nu) \leftrightarrow \nu \in D \leftrightarrow \nu \notin f(\nu),$$

a contradiction.

(iii) $\to$ (i). Suppose (iii) holds but (i) fails. By 7.2 and the failure of (i) there is a $\delta < \alpha$ and a $\Sigma_1(L_a)$ map $f$ from $\delta$ cofinally into $\alpha$. Let $f$ be $\Sigma_1^{L_\alpha}(\{p\})$. By (iii), $\alpha$ cannot be of the form $\gamma + \omega$ for any $\gamma$, so we can pick a limit ordinal $\gamma < \alpha$ such that $\delta, p \in L_\gamma$. Set

$$X = h_\alpha''(\omega \times L_\gamma).$$

Since $L_\gamma$ is closed under ordered pairs, 6.1 (ii) tells us that

$$L_\gamma \subseteq X \prec_1 L_\alpha.$$

By the Condensation Lemma, let

$$\pi\colon X \cong L_\beta.$$

Notice that $\pi \restriction L_\gamma = \mathrm{id} \restriction L_\gamma$.

*Claim.* $\pi \upharpoonright X = \mathrm{id} \upharpoonright X$.

To see this, let $\Theta$ be as in 6.5 (namely, the canonical $\Sigma_0$ formula of $\mathscr{L}$ which defines $h_\alpha$ over $L_\alpha$). Let $i \in \omega$, $x \in L_\gamma$, $y \in L_\alpha$ be such that

$$y = h_\alpha(i, x).$$

Then

$$\models_{L_\alpha} \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

Since $y, x \in X \prec_1 L_\alpha$, this gives

$$\models_X \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

Applying $\pi$,

$$\models_{L_\beta} \exists z\, \Theta(z, \pi\mathring{(y)}, \mathring{i}, \mathring{x}).$$

By $U$-absoluteness (I.9.14), it follows that

$$\models_{L_\alpha} \exists z\, \Theta(z, \pi\mathring{(y)}, \mathring{i}, \mathring{x}).$$

In other words,

$$\pi(y) = h_\alpha(i, x) = y.$$

This proves the claim.

By the claim, $X = L_\beta$. Now, $f$ is $\Sigma_1^{L_\alpha}(\{p\})$, and $p \in X \prec_1 L_\alpha$, so $X$ is closed under $f$. But $\delta \subseteq X$ and $f$ is cofinal in $\alpha$. Thus as $X = L_\beta$, which is transitive, we must have $\alpha \subseteq X$. Thus $\beta = \alpha$ and $X = L_\alpha$.

Define a function $g$ from $\omega \times \delta \times L_\gamma$ into $L_\alpha$ by:

$$g(i, v, x) = \begin{cases} y, & \text{if } (\exists z \in L_{f(v)})[\models_{L_\alpha} \Theta(\mathring{z}, \mathring{y}, \mathring{i}, \mathring{x})], \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easily seen that $g$ is $\Sigma_1^{L_\alpha}(\{p\})$. (We leave this to the reader. A similar argument was used towards the end of the proof of 6.8.) Also,

$$g''(\omega \times \delta \times L_\gamma) = h''_\alpha(\omega \times L_\gamma) = X = L_\alpha.$$

(Because $f$ is cofinal in $\alpha$.) But it follows easily from 6.8 that there is a $\Sigma_1(L_\gamma)$ map, $j$, from $\gamma$ onto $\omega \times \delta \times L_\gamma$. Then $g \circ j$ is a $\Sigma_1(L_\alpha)$ map from $\gamma$ onto $L_\alpha$, contradicting (iii). $\square$

It is perhaps woth noting the following fact, used implicitly in the proof of the above lemma.

**7.4 Lemma.** *Let $\alpha, \beta$ be limit ordinals, $\omega < \alpha < \beta$. Then $h_\alpha \subseteq h_\beta$.*

*Proof.* Suppose that

$$y = h_\alpha(i, x).$$

Then with $\Theta$ as in 6.5,

$$\models_{L_\alpha} \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

By $U$-absoluteness,

$$\models_{L_\beta} \exists z\, \Theta(z, \mathring{y}, \mathring{i}, \mathring{x}).$$

Thus

$$y = h_\beta(i, x). \quad \square$$

Clearly, it is the uniformity of the $\Sigma_1$ skolem function which lies behind 7.4. We often use 7.4 without mention.

## Exercises

1. *Primitive Recursive Set Functions* (Section 2)

A function $f: V^n \to V$ is said to be *primitive recursive* (p.r.) iff it is geneated by the following schemas:

(i) $f(x_1, \ldots, x_n) = x_i \quad (1 \leqslant i \leqslant n)$;
(ii) $f(x_1, \ldots, x_n) = \{x_i, x_j\} \quad (1 \leqslant i, j \leqslant n)$;
(iii) $f(x_1, \ldots, x_n) = x_i - x_j \quad (1 \leqslant i, j \leqslant n)$;
(iv) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $h$, $g_1, \ldots, g_k$ are all p.r.;
(v) $f(y, x_1, \ldots, x_n) = \bigcup_{z \in y} g(z, x_1, \ldots, x_n)$, where $g$ is p.r.;
(vi) $f(x_1, \ldots, x_n) = \omega$;
(vii) $f(y, x_1, \ldots, x_n) = g(y, x_1, \ldots, x_n, (f(z, x_1, \ldots, x_n) \mid z \in h(y)))$, where $g$ and $h$ are p.r. and where

$$z \in h(y) \to \operatorname{rank}(z) < \operatorname{rank}(y).$$

(Functions generated by schemas (i) through (v) are said to be *rudimentary*, and play a basic role in our later work on constructibility theory.)

1 A. Show that the following functions are p.r.:

$$f(x_1, \ldots, x_n) = \bigcup x_i \quad (1 \leqslant i \leqslant n);$$
$$f(x_1, \ldots, x_n) = x_i \cup x_j \quad (1 \leqslant i, j \leqslant n);$$
$$f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\};$$
$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n);$$
$$f(x_1, \ldots, x_n) = \emptyset.$$

1 B. Show that if the function $f(y, x_1, \ldots, x_n)$ is p.r., so too is the function $g(y, x_1, \ldots, x_n) = (f(z, x_1, \ldots, x_n) \mid z \in y)$.

A relation $R \subseteq V^n$ is said to be *primitive recursive* (p.r.) iff there is a p.r. function $f: V^n \to V$ such that

$$R = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) \neq \emptyset\}.$$

1 C. Prove the following:

(i) If $f$ and $R$ are p.r., so is

$$g(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n), & \text{if } R(x_1, \ldots, x_n) \\ \emptyset, & \text{if } \neg R(x_1, \ldots, x_n). \end{cases}$$

(ii) $R$ is p.r. iff $\chi_R$ (the characteristic function of $R$) is p.r.

(iii) $R$ is p.r. iff $\neg R$ is p.r.

(iv) Let $f_i: V^n \to V$ be p.r. for $i = 1, \ldots, m$. Let $R_i \subseteq V^n$ be p.r. for $i = 1, \ldots, m$, such that $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{m} R_i = V^n$. Define $f: V^n \to V$ by

$$f(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n) \quad \text{iff } R_i(x_1, \ldots, x_n).$$

Then $f$ is p.r.

(v) If $R(y, x_1, \ldots, x_n)$ is p.r., so too is

$$f(y, x_1, \ldots, x_n) = \{z \in y \mid R(z, x_1, \ldots, x_n)\}.$$

(vi) Let $R(y, x_1, \ldots, x_n)$ be p.r. and such that

$$(\forall x_1 \ldots x_n)(\exists!\, y) R(y, x_1, \ldots, x_n).$$

Define $f$ by

$$f(y, x_1, \ldots, x_n) = \begin{cases} \text{that } z \in y \text{ such that } R(z, x_1, \ldots, x_n), \text{ if such} \\ \text{a } z \text{ exists,} \\ \emptyset, \text{ if no such } z \text{ exists.} \end{cases}$$

Then $f$ is p.r.

(vii) If $R(y, x_1, \ldots, x_n)$ is p.r., so too is $(\exists z \in y) R(z, x_1, \ldots, x_n)$.

(viii) If $R_i \subseteq V^n$ are p.r. for $i = 1, \ldots, m$, so too are $\bigcup_{i=1}^{m} R_i$ and $\bigcap_{i=1}^{m} R_i$.

(ix) The functions $(x)_0$, $(x)_1$, $\mathrm{dom}(x)$, $\mathrm{ran}(x)$ are p.r.

(x) the relations $x = y$ and $x \in y$ are p.r.

1 D. Show that if $f: V^n \to V$ is p.r., then there is a $\Sigma_1$ formula $\Phi$ of LST such that

$$y = f(x_1, \ldots, x_n) \leftrightarrow \Phi(y, x_1, \ldots, x_n).$$

1 E. Show that the ordinal functions $\alpha + 1$, $\alpha + \beta$, $\alpha \cdot \beta$, $\alpha^\beta$ are p.r.

1 F. Let $f(y, x_1, \ldots, x_n)$ be p.r. By recursion, define functions $f^\nu$, $\nu \in \mathrm{On}$, by:

$$\begin{aligned} f^0(y, x_1, \ldots, x_n) &= y; \\ f^{\nu+1}(y, x_1, \ldots, x_n) &= f(f^\nu(y, x_1, \ldots, x_n), x_1, \ldots, x_n); \\ f^\lambda(y, x_1, \ldots, x_n) &= \bigcup_{\nu<\lambda} f^\nu(y, x_1, \ldots, x_n), \qquad \text{if } \lim(\lambda). \end{aligned}$$

Let $g$ be defined by

$$g(\nu, y, x_1, \ldots, x_n) = f^\nu(y, x_1, \ldots, x_n).$$

Show that $g$ is p.r.

1 G. Show that the transitive closure function, TC, is p.r.

1 H. Show that any predicate defined by a $\Sigma_0$ formula of LST is p.r. (Hint: By induction on formulas, using 1 C (iii), (vii), and (viii).)

1 I. Show that the following functions are p.r.:

(i) $f(u) = \{x \mid \mathrm{Const}(x, u)\}$;

(ii) $f(u) = \{x\ \mathrm{Vbl}(x)\}$;

(iii) $f(u) = \{x \mid \mathrm{PFml}(x, u)\}$;

(iv) $f(u) = \{x \mid \mathrm{Fml}(x, u)\}$;

(v) $f(x) = \begin{cases} \text{the set of free variables of } x, \text{ if } \mathrm{Fml}(x), \\ \emptyset, \quad \text{if } \neg\, \mathrm{Fml}(x); \end{cases}$

(vi) $f(x, y, z) = \begin{cases} \text{that } x' \text{ such that } \mathrm{Sub}(x', x, y, z), \\ \qquad \text{if } \mathrm{Fml}(x) \wedge \mathrm{Vbl}(y) \wedge \mathrm{Const}(z), \\ \emptyset, \quad \text{otherwise}; \end{cases}$

(vii) $f(u) = \{x \mid \mathrm{Sat}(u, x)\}$;

(viii) $f(u) = \mathrm{Def}(u)$.

1 J. Show that the function $(L_\nu \mid \nu \in \mathrm{On})$ is p.r.

2. *Relative Constructibility* (Section 2)

Given some set $A$, we define a class $L[A]$ which has many of the nice properties of $L$, but in which the set $A$ is, to some extent, available. The class $L[A]$ is called the *universe of sets constructible relative to* $A$, and is defined by analogy with the definition of $L$.

If $X$ is a set, $\mathrm{Def}^A(X)$ denotes the set of all subsets of $X$ which are definable in the structure $\langle X, \in, A \cap X\rangle$ by means of a formula of $\mathscr{L}_X(\mathring{A})$ having one free variable. (The language $\mathscr{L}_V(\mathring{A})$ was discussed briefly at the end of I.9 and the beginning of I.10.) The *hierarchy of sets constructible relative to A* is defined by the following recursion:

$$L_0[A] = \emptyset; \qquad L_{\alpha+1}[A] = \mathrm{Def}^A(L_\alpha[A]);$$
$$L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A], \qquad \text{if } \lim(\lambda).$$

The class $L[A]$ is then defined thus:

$$L[A] = \bigcup_{\alpha \in \mathrm{On}} L_\alpha[A].$$

2 A. Prove the following analogues of Lemma 1.1:

(i) $\alpha \leqslant \beta$ implies $L_\alpha[A] \subseteq L_\beta[A]$;
(ii) $L_\alpha[A] \subseteq V_\alpha$ for all $\alpha$;
(iii) Each $L_\alpha[A]$ is transitive, and (hence) $L[A]$ is transitive;
(iv) $\alpha < \beta$ implies $\alpha, L_\alpha[A] \in L_\beta[A]$;
(v) $L[A] \cap \alpha = L_\alpha[A] \cap \alpha = L_\alpha[A] \cap \mathrm{On} = \alpha$;
(vi) for $\alpha \leqslant \omega$, $L_\alpha[A] = V_\alpha$;
(vii) for $\alpha \geqslant \omega$, $|L_\alpha[A]| = |\alpha|$.

2 B. Prove that $L[A]$ is an inner model of ZF (in the sense of 1.2).

2 C. A structure of the form $\langle M, \in, A\rangle$ is said to be *amenable* iff $M$ is an amenable set and $A \cap u \in M$ for all $u \in M$. (This notion was introduced in Section 7). Prove that for any limit ordinal $\alpha > \omega$, the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$ is amenable.

Now, the intuition behind the construction of $L[A]$ is that the predicate "$x \in A$" should be available. Consequently, it is common practice to abbrebriate by $L_\alpha[A]$ the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$, just as we used $L_\alpha$ to mean $\langle L_\alpha, \in\rangle$. In particular, to say that $L_\alpha[A]$ is amenable means that the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$ is amenable, as defined above.

2 D. Show that there is a $\Delta_1^{\mathrm{KP}}$ formula $D(v, u, a)$ of LST such that

$$D(v, u, a) \quad \text{iff } v = \mathrm{Def}^a(u).$$

2 E. Show that the function $\mathrm{Def}^A$ is uniformly $\Delta_1^{L_\alpha[A]}$ for all limit $\alpha > \omega$.

2 F. Show that there is a $\Delta_1^{\mathrm{KP}}$ formula $H(x, \alpha, a)$ of LST such that

$$H(x, \alpha, a) \quad \text{iff } x = L_\alpha[a].$$

2 G. Show that the function $v \mapsto L_v[A]$ is uniformly $\Delta_1^{L_\alpha[A]}$ for limit $\alpha > \omega$.

2 H. Show that if $M$ is an admissible set or else an inner model of KP, and if $a \in M$, then for any $\alpha \in M$, $L_\alpha[a] \in M$ and $(L_\alpha[a])^M = L_\alpha[a]$.

2 I. Show that if $\alpha > \omega$ is a limit ordinal, then for any $v < \alpha$,

$$L_\gamma[A] = (L_v[B])^{L_\alpha[A]},$$

where $B = A \cap L_v[A]$. (By 2 C, $B \in L_\alpha[A]$.)

2 J. Prove that if $\alpha > \omega$ is a limit ordinal and $B = A \cap L_\alpha[A]$, then

$$L_\alpha[A] = L_\alpha[B].$$

2 K. Prove that if $B = A \cap L[A]$, then

$$L[A] = L[B] = (L[B])^{L[A]}.$$

Deduce that

$$(V = L[B])^{L[A]}.$$

2 L. Show that there is a $\Sigma_1$ formula $\mathrm{WO}(x, y, a)$ of LST such that

$$\mathrm{KP} \vdash \text{“}\{(x, y) \mid \mathrm{WO}(x, y, a)\} \text{ is a well-ordering of } L[a]\text{”},$$

and such that if $<_{L[A]}$ denotes the well-ordering of $L[A]$ determined by WO, then for any limit ordinal $\alpha > \omega$, $<_{L[A]} \cap (L_\alpha[A])^2$ is $\Sigma_1^{L_\alpha[A]}$.

2 M. Prove $(\mathrm{AC})^{L[A]}$.

2 N. Prove that $L[A]$ is the smallest inner model of ZF which contains the set $A \cap L[A]$. (i.e. $L[A]$ is the smallest inner model $M$ of ZF such that $A \cap M \in M$.)

3. *Use of the Condensation Lemma* (Section 5)

We investigate the question: as $\alpha$ varies over all limit ordinals, how many different sets of $\mathscr{L}$-sentences are theories of some $L_\alpha$?

3 A. Let $\Sigma$ be the set of all sets of $\mathscr{L}$-sentences of the form

$$\{\varphi \mid \vDash_{L_\alpha} \varphi\}$$

for some limit ordinal $\alpha$. Show that

$$|\Sigma| \leqslant |\omega_1^L|.$$

(Is it also the case that

$$|\Sigma|^L \leqslant |\omega_1^L|^L ?).$$

3 B. Let $(\varphi_n \mid n < \omega)$ be the "lexicographic" enumeration of the sentences of $\mathscr{L}$, as described in Section 3. Show that there is no formula $\varphi(v_0)$ of $\mathscr{L}$ such that

$$\vDash_{L_\alpha} \varphi_n \quad \text{iff } \vDash_{L_\alpha} \varphi(\mathring{n}).$$

(Hint: Diagonalisation. Let $(\psi_n \mid n < \omega)$ be the lexicographic enumeration of the formulas of $\mathscr{L}$ with free variable at most $v_0$. Consider the formula

$$\text{“}v_0 \text{ is a natural number”} \wedge \exists k [\text{“}k \text{ is a natural number”} \\ \wedge (\varphi_k = {}^*\psi_{v_0}(v_0)^*) \wedge \neg \varphi(k)],$$

where ${}^*\psi_m(n)^*$ denotes the formula obtained from $\psi_m(v_0)$ by replacing every free occurrence of $v_0$ by the term denoting the integer $n$.)

3 C. Show that $|\Sigma| = |\omega_1^L|$. (Hint: First reduce this to proving that $|\Sigma|^L > \omega$. Then suppose that $|\Sigma|^L = \omega$ and let $T$ be the $<_L$-least subset of $\omega \times \omega$ such that $(T''\{n\} \mid n < \omega)$ enumerates all members of $\Sigma$ via the enumeration $(\varphi_n \mid n < \omega)$ of the $\mathscr{L}$-sentences in 3 B above. Now look at the $\mathscr{L}$-theory of $L_{\omega_1^L}$ and work for a contradiction with 3 B.)

An alternative solution to the original question can be obtained by exhibiting an unbounded set $A \subseteq \omega_1^L$ such that whenever $\alpha, \beta \in A$ and $\alpha \neq \beta$, then the theories of $L_\alpha$ and $L_\beta$ are different. This can be done as follows.

3 D. Set

$$A = \{\alpha \in \omega_1^L \mid \lim(\alpha) \wedge \text{every element of } L_\alpha \text{ is definable (without the use of parameters) in } L_\alpha\}.$$

Show that $A$ is unbounded in $\omega_1^L$. (Hint: To show that $A$ is non-empty, use 5.3 and 5.10. To show that $A$ is unbounded in $\omega_1^L$, suppose otherwise and consider $\lambda = \sup(A)$.)

3 E. Show that if $\alpha, \beta \in A$ and $\alpha \neq \beta$, then $L_\alpha$ and $L_\beta$ have different theories. (Hint: Use 3 B again.)

4. *The Condensation Lemma and the* GCH *in* $L[A]$ (Section 5)

We continue the investigation of $L[A]$ commenced in Exercises 2 above.

4 A. Prove that if $\alpha > \omega$ is a limit ordinal and $X \prec_1 L_\alpha[A]$, there are unique $\pi, \beta$ such that

$$\pi: X \cong L_\beta[B],$$

where $B = \pi''(A \cap X)$.

4 B. Show that if $A \in L_\varrho[A]$, $\alpha > \varrho$, $\alpha > \omega$, $\alpha$ a limit ordinal, and if

$$L_\varrho[A] \subseteq X \prec_1 L_\alpha[A],$$

then there are unique $\pi, \beta$ such that

$$\pi: X \cong L_\beta[A].$$

As we saw in 4 A above, the "condensation lemma" for $L[A]$ does not in general lead to a structure in the $L[A]$ hierarchy. Thus we cannot prove GCH in $L[A]$ as we did for $L$. Indeed, if $\kappa$ were a cardinal such that $2^\kappa = \kappa^{++}$, we could let $A \subseteq \kappa^{++}$ code all subsets of $\kappa$, so $2^\kappa \geqslant \kappa^{++}$ would hold in $L[A]$. However, 4 B enables us to obtain a partial GCH result.

4 C. Prove that if $V = L[A]$, where $A$ is a subset of an infinite cardinal $\kappa$, then $2^\lambda = \lambda^+$ for all cardinals $\lambda \geqslant \kappa$.

A strengthening of 4 C is possible. We require a preliminary result.

4 D. Let $\kappa$ be an uncountable regular cardinal, and let $\mathbf{M} = \langle M, \in, \dots \rangle$ be a structure such that $\kappa \subseteq M$. Let $X \subseteq M$, $|X| < \kappa$. Prove that there is a structure $\mathbf{N} \prec \mathbf{M}$ such that $X \subseteq N$ and $N \cap \kappa \in \kappa$. (Hint: Construct $\mathbf{N}$ as the union of a suitably chosen $\omega$-sequence of submodels of $\mathbf{M}$.)

4 E. Let $V = L[A]$, where $A \subseteq \kappa^+$. Then $2^\kappa = \kappa^+$. (Hint: Use 4 D to prove a special case of 4 A, and note that if $\gamma, \delta < \kappa^+$, then

$$L_\gamma[A \cap \delta] \in L_{\kappa^+}[A].)$$

4 F. Show that if $V = L[A]$, where $A \subseteq \omega_1$, then GCH is valid.

5. $\Sigma_n$ *Skolem Functions* (Section 6)

We show that there is no uniform $\Sigma_2$ skolem function for $L_\alpha$, where $\lim(\alpha)$, $\alpha > \omega$. (It can be shown that each limit $L_\alpha$ does possess a $\Sigma_2$ skolem function, and indeed a $\Sigma_n$ skolem function for any $n$, but the $\Sigma_n$ definitions are not uniform for $n \geqslant 2$. See Chapter VI for details.)

It is convenient to assume $V = L$ throughout. We use $\alpha$ to denote an arbitrary countable limit ordinal.

5 A. Show that the predicate

$$x \leqslant \alpha$$

is uniformly $\Pi_2^{L_{\omega_1+\alpha}}(\{\omega_1\})$.

5 B. Show that the predicate

$$x = \omega_1$$

is uniformly $\Pi_1^{L_{\omega_1+\alpha}}$.

5 C. Show that the predicate

$$x > \alpha$$

is uniformly $\Sigma_2^{L_{\omega_1+\alpha}}$.

5 D. Show that the predicate

$$P_\alpha(x)\colon \lim(x) \wedge (\alpha < x < \omega_1)$$

is uniformly $\Sigma_2^{L_{\omega_1+\alpha}}$.

5 E. Suppose that there were a uniform $\Sigma_2$ skolem function $h_\gamma$ for $L_\gamma$, where $\gamma > \omega$ is a limit ordinal. Let $\varphi$ be a $\Sigma_0$ formula of $\mathscr{L}$ such that for any limit ordinal $\gamma > \omega$,

$$y = h_\gamma(i, x) \quad \text{iff } \models_{L_\gamma} \exists u \forall v \varphi(\mathring{y}, \mathring{i}, \mathring{x}, u, v).$$

Show that for each $\alpha$ there is an integer $i_\alpha$ such that

$$P_\alpha(h_{\omega_1+\alpha}(i_\alpha, \emptyset)),$$

and deduce that for a stationary set $A \subseteq \omega_1$ there is an integer $i$ such that for any $\alpha \in A$,

$$P_\alpha(h_{\omega_1+\alpha}(i, \emptyset)).$$

5 F. Define a sequence $(\alpha_\nu \mid \nu < \omega_1)$ thus:

$$\alpha_0 = 0;$$

$$\alpha_{\nu+1} = h_{\omega_1+\alpha_\nu}(i, \emptyset);$$

$$\alpha_\nu = \sup_{\tau<\nu} \alpha_\tau, \qquad \text{if } \lim(\nu).$$

Show that $(\alpha_\nu \mid \nu < \omega_1)$ is a strictly increasing, continuous sequence of countable limit ordinals.

5 G. Pick a limit ordinal $\nu$ such that $\alpha_\nu \in A$ and for arbitrarily large $\tau < \nu$, $\alpha_\tau \in A$. Let $\gamma = \alpha_{\nu+1}$. By considering $\varphi$, show that there is a $\tau < \nu$ such that $\alpha_\tau \in A$ and $\gamma = \alpha_{\tau+1}$, and deduce that there can be no uniform $\Sigma_2$ skolem function for limit $L_\alpha$.