Chapter XX

Abstract Embedding Relations

by J. A. Makowsky

Abstract model theory deals with the generalization of the concept of a logic. A logic consists of a family of objects called formulas, a family of objects called structures and a binary relation between them, called satisfaction. Various properties of logics, however, can be phrased without direct reference to the formulas, but rather, by considering as the basic concept the class of structures which are the models of some (complete) theory. The previous two chapters have given plenty of evidence for this. In Section XVIII.3 we studied amalgamation properties and in Chapter XIX, the Robinson property, both of which fit this approach. In Chapter XIX we even went a step further: we looked into the possibility of axiomatizing abstract equivalence relations between structures, such as they arise naturally from logics in the form of $\mathscr{L}$-equivalence. There we studied the question under which circumstances such an equivalence relation does indeed come from a logic $\mathscr{L}$.

Algebra, on the other hand, deals with classification of algebraic structures and their extensions. The paradigm of algebraic classification theory, and, for that matter, the paradigm of model-theoretic classification theory, is Steinitz' theory of fields and their algebraic and transcendental extensions. But many of the examples studied in algebra, such as locally finite groups or Banach spaces, are not fit for first-order axiomatizations. Though classes of algebras can be axiomatized, if necessary, with the help of generalized quantifiers, this approach does not necessarily help us to axiomatize the corresponding notion of extensions.

In this chapter we axiomatize the notion of $\mathscr{L}$-extensions, but, contrary to the approach in Chapter XIX, we are not that much interested in the case where it is derived from a logic $\mathscr{L}$. We are rather interested in the question: Under which conditions can certain constructions and proofs from model theory be carried out in a framework which resembles more that of universal algebra or algebra in general?

Very often, axiomatizations grow out of a better understanding of proofs. First, they serve only to structure and clarify the flow of reasoning, but sometimes they gain their own significance and reach maturity. If this happens, new branches of mathematical activity emerge.

Examples from history are the emergence of Hilbert and Banach spaces; universal algebra and model theory of first-order logic, abstract model theory, and here especially, the framework of abstract classes. The abstract classes have their origin in the attempt to better understand certain constructions of models, as they occur in the classification theory of models of first-order theory, and in trying to generalize those constructions so as to fit classes which are not first-order definable. The constructions we have in mind divide sharply into two cases: In the case in which amalgamation fails in an abstract class K, they allow us to construct maximally many non-isomorphic structures of a given cardinality and to show that no universal structures of a given cardinality exist, or, as a combination of both, that there are maximally many structures such that no two of them are mutually embeddable. On the other hand, if some form of amalgamation holds, they allow us to obtain a structure in a higher cardinality. It turns out that the presence or absence of various forms of the amalgamation property acts like a watershed. This is similar to the effect of stability or superstability in first-order classification theory. The transfer of all the technical knowledge of the classification theory of models of first-order theories to models of abstract classes, however, poses challenging difficulties. This chapter presents some of the initial steps towards this aim. The completion of such a program remains the task of future research.

But the axiomatic framework has yet another advantage: It allows us to discern more clearly the set-theoretic and combinatorial structure of the proofs and to separate their combinatorial from their structural contents. Such proofs are usually based on a property P of our abstract class K which is inherently connected to the very definition of K, and a set-theoretic part, whose application does not require more than an axiomatic description of some of the basic aspect of $\mathfrak{K}$ together with the property P. We have encountered such situations in the case of locally finite groups, such as in Giorgetta–Shelah [1983] or in the model theory of ω_1-categorical sentences of extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. It would be interesting to see, if the same applies to recent results in Banach space theory, cf. Bourgain–Rosenthal–Schechtman [1981], for instance, where $\mathfrak{K}$ is the class of all separable Banach spaces with the Radon–Nikodym property.

However, the present chapter is not concerned with such deep results of a very specialized character. Our subject here is the axiomatization of the framework which allows the use of the set-theoretic machinery. What we present are the first steps of a theory still to be developed. The chapter is an exposition of and introduction to three papers by S. Shelah (Shelah [1983b, c, 198?c]), and improvements or elaborations in its exposition due to S. Fuchino, R. Grossberg, and the author. An early version of this chapter consisted of lectures the author and D. Giorgetta have given on the subject in Oberwolfach in January 1980. It contains, for completeness and historical accuracy, also early results of Mal'cev and Jonsson, and some additional material which we include to stress some analogies or give more examples.

In detail the chapter is organized as follows. In Section 1 we present the axiomatic framework and variations thereof. In Section 1.1 we define our program in detail and in Section 1.2 we state and motivate the axioms. The main results of this section, presented in Section 1.3, are various forms of axiomatizability theorems which assert the existence of certain standard logics, in which such classes can be described. One of them, Shelah's presentability theorem, provides us with some cardinal parameters, on which the development of the theory depends. It also gives rather surprising results on the Hanf numbers of abstract classes, depending on those parameters, presented in Section 1.4.

In Section 2 we study the effect of the presence or absence of amalgamation properties in an abstract class $\mathfrak{K}$. In the case when an abstract class has the amalgamation property and the joint embedding property they are called Jonsson classes and were introduced already in 1962 by M. Morley and R. Vaught. It should be mentioned here, that R. Fraïssé was seemingly the first to study amalgamation properties of classes of structures, cf. Fraïssé [1954]. We give a brief survey on what we know about Jonsson classes in Section 2.1 for the sake of completeness and proper perspective. The main advantage of Jonsson classes consists in the existence of universal, homogeneous models, though not necessarily in every cardinality. A substitute of saturated models in many of our constructions, is the limit model, which is introduced in Section 2.2, and some basic properties of limit and superlimit models are proved. Our main interest here, however, is in the absence of amalgamation properties. The thesis, put forward in Shelah's work and in this chapter, states that amalgamation properties should not be part of the axioms, and that, basically, Jonsson's axioms, without amalgamation and joint embedding, provide us with the correct framework for a structure/non-structure theory. The main result, presented in Section 2.3, is Shelah's non-structure theorem for abstract classes and some conjectures for further developments. The non-structure theorem presupposes some weak instance of the GCH, connected to the combinatorial principle weak diamond. In Section 2.4 we present an example which shows that this is necessary. In Section 2.5 we collect the set-theoretic background about the weak diamond, necessary to prove the non-structure theorem. The easier parts of its proof are presented in Section 2.6 and the more complex parts in Section 2.7. The reader interested in the missing proofs will have to get involved with the technical details and conceptual intricacies of Shelah [1984a, b].

In Section 3 we study ω-presentable classes, which, by the presentability theorem, are closely connected to the model theory of $\mathscr{L}_{\omega_1\omega}$. In Section 3.1 we present the present state of art in classification theory for ω-presentable classes and classes defined by a $\mathscr{L}_{\omega_1\omega}$-sentence, and we state some conjectures on how the latter should be true also for ω-presentable classes in general. The main results proved in the sequel are Shelah's reduction theorem and Shelah's abstract ω_1-categoricity theorem. For the proofs of the other theorems the reader will have to consult Shelah [198?c]. In Section 3.2 we present the "soft" aspects of the proof of the abstract ω_1-categoricity theorem, and in Section 3.3 the parts which are more related to the model theory of $\mathscr{L}_{\omega_1\omega}$. In Section 3.4 we prove the reduction theorem. In Section 3.5, finally, we give a narrative account of some aspects of the proof of the existence of superlimits in ω_1.

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1. The Axiomatic Framework

1.1. Prolegomena

In Chapter XVIII we have seen that various $\mathscr{L}$-extension properties play a fruitful role in abstract model theory. In Chapter XIX we have seen how one can replace, under certain circumstances, a logic $\mathscr{L}$ by an abstract equivalence relation or an abstract embedding relation. However, in both cases we still retained the idea of dealing with a logic with various regularity properties concerning the passage from one vocabulary to another. The type of results obtained there also requires such assumptions. If we deal with properties of a fixed class of τ-structures, we are more in the framework of universal algebra. In fact, some of the classical theorems of universal algebra can be viewed as precursors of abstract model theory. Let us elaborate on this a bit.

The first theorem along these lines is Birkhoff's theorem characterizing varieties.

1.1.1 Definitions. (i) A class V of τ-structures closed under isomorphic images, cartesian products, substructures, and homomorphic images is called a *τ-variety*.

(ii) A class of τ-structures K is *automatically definable* if $K = \mathrm{Mod}(\Sigma)$ for some set of atomic τ-formulas Σ.

1.1.2 Theorem (Birkhoff). *The τ-varieties are exactly the atomically definable classes of τ-structures.*

1.1.3 Definitions. (i) A class V of τ-structures closed under isomorphic images, cartesian products, and substructures is called a *quasi-variety*.

(ii) An *infinitary Horn formula* is a formula of the form $\bigwedge_{i \in I} \varphi_i \to \psi$, where I is any set and φ_i, ψ are atomic formulas.

(iii) A class of τ-structures K is *Horn definable*, if $K = \mathrm{Mod}(\Sigma)$ for a (possibly proper) class of infinitary Horn formulas.
(iv) An *infinitary clause* is a formula of the form $\bigvee_{i \in I} \varphi_i$, where I is any set and the φ_i's are atomic or negated atomic (i.e., *basic*) formulas.
(v) A class of τ-structures K is *clause definable* if $K = \mathrm{Mod}(\Sigma)$ for a (possibly proper) class of infinitary clauses over τ.
(vi) A class of τ-structures is *basic compact*, if for every set Σ of basic formulas over some vocabulary τ_1, $\tau \subset \tau_1$, such that every finite subset $\Sigma_0 \subset \Sigma$, Σ_0 has a model $\mathfrak{A}$ with $\mathfrak{A} \upharpoonright \tau \in K$, the Σ has too.

1.1.4 Theorem. (i) (Cudnovskii). *A class K of τ-structures closed under isomorphisms and substructures iff K is clause definable.*
(ii) (Cudnovskii [1968]). *The quasi-varieties are exactly the Horn definable classes.*
(iii) (McKinsey [1943]). *If additionally K is basic compact then the class defining K is a set of finitary clauses or finitary Horn formulas.*
(iv) (E. Fisher [1977]). *The assumption that in* (i) *or* (ii) *K is always definable by a set (Horn) clauses is equivalent to Vopenka's principle.*

For a definition of Vopenka's principle see Section XVIII.1.3. Similar theorems hold for classes closed under unions of chains and other closure properties.

Quasi-varieties are particularly interesting because they allow the construction of *free objects* (*initial objects*) and Mal'cev [1954] has given the following characterization of quasi-varieties.

1.1.5 Definition (Free Structures). (i) Let K be a class of structures for a vocabulary τ, $\mathfrak{A} \in K$ and $X \subset A$ such that $\mathfrak{A}$ is generated (as a substructure) by X. We say that $\mathfrak{A}$ is *free* in K, if for every $\mathfrak{B} \in K$ and any relation preserving mapping $f: X \to B$ there is a homomorphism $g: \mathfrak{A} \to \mathfrak{B}$ extending f.
(ii) Let a class K of τ-structures be called *free*, if for every variety V of τ'-structures such that $K \cap V \neq \emptyset$, $K \cap V$ has a $\tau \cup \tau'$-structure which is free in $K \cap V$.

1.1.6 Theorem (Mal'cev [1954]). *A class K is free iff it is a quasi-variety.*

For a discussion of Mal'cev's theorem cf. also Mahr–Makowsky [1983].

1.1.7 Stating the Problem. The aim of this chapter is to give an introduction in to a sequence of papers by S. Shelah entitled "Classification theory for non-elementary classes Ia, Ib, and II." (Shelah [1983b, c, 198?c]). The idea here is very simple. Instead of having a logic $\mathscr{L}$ we are given a class K of τ-structures satisfying certain properties. We would like to ask questions concerning the existence of various models in such a class K. In the following we list the paradigms of our questions together with a typical instance of a theorem answering such a question in some special case.

1.1.8 Categoricity. Under what conditions is K categorical in some cardinal?

The paradigm of such questions concerns categoricity in ω for first-order model theory. There the characterization theorem due independently to Engeler, Ryll-Nardzewski, and Svenonious, connects categoricity of a theory with its Lindenbaum algebras being atomic. In the case of $\mathscr{L}_{\omega_1\omega}$ Scott's theorem states that every complete sentence is categorical in ω. For other cardinalities characterization of categoricity is more connected to transfer properties, such as Morley's theorem, stating that a countable first-order theory is categorical in one uncountable cardinal iff it is categorical in every uncountable cardinal. Attempts to generalize this to $\mathscr{L}_{\omega_1\omega}$ have only partially succeeded, cf. Keisler [1971]. Much of Section 3 is devoted to related questions.

1.1.9 The Spectrum. More generally, denote by $I(K, \kappa)$ the number of isomorphism types of models in K of cardinality κ. If $K = \text{Mod}(T)$ for some first-order theory, we write $I(T, \kappa)$ instead of $I(K, \kappa)$. What can we say about $I(K, \kappa)$?

In the case of countable first-order theory, twenty years of research have led to the following theorem of Shelah, proving therewith a conjecture due to Morley.

1.1.10 Theorem (Shelah). *Let T be a countable first-order theory. Then $I(T, \kappa)$ is not-decreasing on uncountable cardinals and, in fact, either*:

(i) $I(T, \kappa) = 2^\kappa$; or
(ii) $I(T, \omega_\alpha) < \beth_{\omega_1}(\text{card}(\alpha))$.

The proof of this theorem was complete with Shelah [1982f], based on Shelah [1978a].

Much of Section 2 is devoted to prove similar theorems for abstract classes.

1.1.11 Rigid Models. A model is *rigid*, if it has no non-trivial automorphisms. Let $R(K, \kappa)$ be the number of isomorphism types of rigid structures in K of cardinality κ. Interest in rigid models arose, after it was shown by Ehrenfeucht and Mostowski, that every first-order theory has models with many automorphisms. Generalizations of this to abstract model theory are discussed in Section XVIII.4.5. The following theorem shows, unfortunately, that very little can be said about the function $R(T, \kappa)$ in the case of first-order logic.

1.1.12 Theorem (Shelah [1976b]). *Assume $\lambda^\omega \leq \lambda^+$ for every λ. For every Σ^1_2-class $\mathbf{C}$ of cardinals there is a sentence $\varphi \in \mathscr{L}_{\omega\omega}$ such that $\mathbf{C} = \{\kappa \in \text{Card}: R(\varphi, \kappa) \neq 0\}$.*

This refutes a conjecture of Ehrenfeucht, which tried to describe $R(T, \kappa)$. It seems that one should ask for rigid models which are also $\text{card}(T)^+$-saturated. In Shelah [1983d] there are partial results indicating that at least the existence of rigid models in some class K can be settled in an abstract framework. In this chapter we shall not deal with rigid models, but we would like to draw attention to this promising direction of research. A sample theorem is the following result due to Shelah, refuting a conjecture (unpublished) of H. Salzmann, suggesting that every rigid real closed field is archimedian:

1.1.13 Theorem (Shelah). *Assume GCH. There are arbitrarily large rigid ω_1-saturated real closed fields.*

1.1.14 Problem. Characterize the abstract classes (defined below) which have arbitrarily large rigid models.

1.1.15 Homogeneous and Saturated Models. Similar problems can be stated for homogeneous models. In Section 2 we shall study this question. Theorem 2.1.11 gives some information about the spectrum of homogeneous models $H(K, \kappa)$. In first-order model theory saturated models are suitable described as universal and homogeneous. Already in the early days of classification theory, axiomatic frameworks have been studied. Jonsson [1956, 1960] and Fraïssé [1954] proposed axioms for the existence of universal and homogeneous structures in a class K and Morley–Vaught [1962] used this framework to construct saturated structures. We shall return to a detailed discussion of these axioms in Section 1.2 and for Jonsson's work in Section 2.1. What we want to note here, is that the construction of the saturated model heavily depends on the *amalgamation* property of K. We shall see that there are good reasons for this. The question arises if there is a suitable substitute for saturated models? One of the key notions introduced in this chapter is the *limit model.* The similarity consists less in the definition, than in its use in various proofs. Section 2.2 gives the definitions and its presence is felt through the rest of the chapter.

1.2. The Axioms

Here K is a class of τ-structures and $<_K$ is a two-place relation between members $\mathfrak{A}$, $\mathfrak{B}$ of K. If the context is clear we omit the K in $<_K$ and assume that all structures $\mathfrak{A}, \mathfrak{B} \in K$.

The axioms presented below are modeled after various examples of model theory. It is good to have some of these at disposal when reading the axioms, so we present them before stating the axioms.

1.2.1 Examples. (i) Let T be a complete first-order theory over some vocabulary τ and put $K_T = \mathrm{Mod}(T)$ and $<$ be first-order elementary extension.

(ii) Let K_{wo} be the class of well-orderings and $\mathfrak{A} <_{\mathrm{wo}} \mathfrak{B}$ hold if $\mathfrak{B}$ is an end extension of $\mathfrak{A}$, i.e., every $b \in B - A$ is bigger than every $a \in A$.

(iii) Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\omega_1})$ be the logic with the quantifier "there exist uncountably many." Let a weak τ-model $(\mathfrak{A}, q)$ consist of a τ-structure together with a family q of subsets of A. Let the formulas of $\mathscr{L}_{\mathrm{weak}}$ be as for $\mathscr{L}$ but define $\mathfrak{A} \models_{\mathrm{weak}} Qx\varphi(x)$ if $\{a \in A: \mathfrak{A} \models_{\mathrm{weak}} \varphi(a)\} \in q$. Let K be the class of all weak τ-models for some fixed vocabulary τ. We define $<^{**}$ as in Keisler [1970, 1971a] by $\mathfrak{A} <^{**} \mathfrak{B}$ iff $\mathfrak{A} <_{\mathscr{L}_{\mathrm{weak}}} \mathfrak{B}$ and for every $\bar{a} \in A^m$ and for every formula $\varphi = \varphi(x, \bar{y}) \in \mathscr{L}(\tau)$ we have that if $\mathfrak{A} \models \neg Qx\varphi(x, \bar{a})$ then $\{b \in A: \mathfrak{A} \models \varphi(b, \bar{a})\} = \{b \in B: \mathfrak{B} \models \varphi(b, \bar{a})\}$.

We shall return to this example in more detail in Section 4.

(iv) The category of *universal locally finite groups, with* K_{ULF} *the class* of those groups and $<_{\mathrm{ULF}}$ the ordinary subgroup relation, cf. Kegel–Wehrfritz [1973]. The model theory of uncountable universal locally finite groups was studied in Macintyre–Shelah [1976] and Grossberg–Shelah [1983].

(v) (Elementary Classes with Omitting Types). Let τ be a fixed vocabulary, T be a first-order theory over τ, i.e., $T \subset \mathscr{L}_{\omega\omega}(\tau)$, and Γ be a set of types over τ. Let $K = \{\mathfrak{A} \in \mathrm{Str}(\tau)\colon \mathfrak{A} \models T$ *and* $\mathfrak{A}$ *omits every* $p \in \Gamma\}$ and $\mathfrak{A} <_K \mathfrak{B}$ if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. It is easy to see (cf. Keisler [1970]) that example (iii) is a special case of this.

We shall return to this example in Section 1.3.

Having these examples in mind, we now state the axioms. They come in several groups of various degree of strength. First some (almost) trivial axioms concerning transitivity of our embedding relation:

Axiom 1 (Substructure Axiom). If $\mathfrak{A} < \mathfrak{B}$ then $\mathfrak{A} \subset \mathfrak{B}$, i.e., $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

1.2.2 Definitions. (i) If $\mathfrak{A} \subset \mathfrak{B}$ are τ-structures, and f is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, say that *f is an K-embedding*, if $f(\mathfrak{A}) <_K \mathfrak{B}$.

(ii) If $\mathfrak{A}$, $\mathfrak{B}$ are τ-structures and f_{AB} is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we denote by $[\mathfrak{A}; \mathfrak{B}, f_{AB}]$ the two-sorted structure consisting of the two structures $\mathfrak{A}$, $\mathfrak{B}$ expanded by a function symbol F interpreted by the embedding f_{AB} and a new unary predicate symbol U, both not in τ, such that $\mathfrak{B} \restriction U \cong f_{AB}(\mathfrak{A})$. If $\mathfrak{A} \subset \mathfrak{B}$ and f_{AB} is the identity on $\mathfrak{A}$ we just write $[\mathfrak{A}; \mathfrak{B}]$. Note the difference between our notation $[\mathfrak{A}; \mathfrak{B}]$ and $[\mathfrak{A}, \mathfrak{B}]$ for the disjoint pair construction in Chapter XVIII.

Axiom 2 (Isomorphism Axiom). (i) If $\mathfrak{A} \in K$ and $\mathfrak{A}_1 \cong \mathfrak{A}$ then $\mathfrak{A}_1 \in K$.

(ii) If $\mathfrak{A} < \mathfrak{B}$ and $[\mathfrak{A}; \mathfrak{B}] \cong [\mathfrak{A}_1; \mathfrak{B}_1]$ then $\mathfrak{A}_1 < \mathfrak{B}_1$.

Axiom 3 (Transitivity Axiom). (i) If $\mathfrak{A}_1 < \mathfrak{A}_2 < \mathfrak{A}_3$ then $\mathfrak{A}_1 < \mathfrak{A}_3$.

(ii) If $\mathfrak{A}_1 \subset \mathfrak{A}_2 < \mathfrak{A}_3$ and $\mathfrak{A}_1 < \mathfrak{A}_3$ then $\mathfrak{A}_1 < \mathfrak{A}_2$.

Clearly examples (i)–(v) satisfy these axioms.

1.2.3 Definition. Let $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ be a family of structures in $\mathfrak{K}$.

(i) $\mathfrak{A}_\alpha$ is *K-increasing* if $\alpha > \beta < \gamma$ implies that $\mathfrak{A}_\alpha < \mathfrak{A}_\beta$.

(ii) $\mathfrak{A}_\alpha$ is *continuous* if for every limit ordinal $\delta < \gamma$ we have $\mathfrak{A}_\delta = \bigcup_{\alpha<} \mathfrak{A}_\alpha$.

(iii) $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a *K-chain* if it is both K-increasing and continuous.

Axiom 4 (Chain Axiom). (i) If $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a K-chain then $\mathfrak{A}_0 < \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$.

(ii) If $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a K-chain, $\mathfrak{N} \in K$ and for each $\alpha < \gamma$ $\mathfrak{A}_\alpha < \mathfrak{N}$ then $\bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha < \mathfrak{N}$.

Again all our examples from above satisfy this axiom.

We denote by K_λ ($K_{<\lambda}$, $K_{\leq\lambda}$) the class of structures of K of cardinality *exactly* (*less than, less than or equal to*) λ.

Our next axiom is an analogue of the Löwenheim–Skolem–Tarski theorem for first-order logic and introduces a cardinal parameter, which we shall call the *Löwenheim number* $l(K)$ of $\langle K, < \rangle$.

Axiom 5 (Existence of Löwenheim Number). There is first a cardinal $l(K) \geq \text{card}(\tau(K))$ such that:

(i) $K_{l(K)} \neq \varnothing$; and

(ii) whenever $\mathfrak{A} \in K$ and X is a subset of the universe A of $\mathfrak{A}$ then there is a $\mathfrak{B} \in K$ such that $X \subset B$, $\text{card}(\mathfrak{B}) \leq l(K) + \text{card}(X)$ and $\mathfrak{B} < \mathfrak{A}$.

1.2.4 Examples. (i) In the example of well-orderings with end-extensions (Example 1.2.1(ii)) has no Löwenheim number. To see this, take any well ordering $\mathfrak{A}$ of cofinality ω of cardinality κ. If X is a countable cofinal set then for every $\mathfrak{B} <_{\text{wo}} \mathfrak{A}$ with $X \subset B$ we have $\mathfrak{B} = \mathfrak{A}$.

(ii) The Löwenheim numbers of Examples 1.2.1(i), (iii), (iv), and (v) are ω.

(iii) If we modify Example 1.2.1(iii) such that the interpretation of the quantifier $Qx\varphi(x)$ ensures that the set defined by φ is uncountable, then the Lowenheim number is ω_1.

(iv) In Gurevic [1982] Löwenheim properties of general categories are studied. The situation described there consists of a logic $\mathscr{L}$ and an abstract class K together with a family $\mathbf{H}$ of homomorphisms. Supposing that K has Löwenheim number λ and $\mathscr{L}$ has Löwenheim number μ, we are interested in the existence of a cardinal $g(\lambda, \mu)$ such that for every $\mathfrak{A} \in K$ there is $\mathfrak{B} < \mathfrak{A}$ with $\text{card}(B) < g(\lambda, \mu)$ such that for every $H \in \mathbf{H}$ we have $H(\mathfrak{B}) < H(\mathfrak{A})$ is also an $\mathscr{L}$-embedding.

1.2.5 Remark. We could state Axiom 5 only for $X \subset A$ with $\text{card}(X) \leq l(K)$ and use Axiom 4 to prove Axiom 5 from this weaker assumption.

1.2.6 Definitions. (i) A class K together with a relation $<_K$ satisfying the Axioms 1–4 is called a *abstract class*.

(ii) A class K together with a relation $<_K$ satisfying the Axioms 1–5 is called a *abstract class of Löwenheim number* $l(K)$.

(iii) Let K_i be abstract classes over vocabularies τ_i, $i \in I$. We define the intersection $K = \bigcap_{i \in I} K_i$ to be the class of $\bigcup_{i \in I} \tau_i$-structures such that for $\mathfrak{A}, \mathfrak{B} \in K$, $\mathfrak{A} < \mathfrak{B}$ iff $\mathfrak{A} \upharpoonright \tau_i < \mathfrak{B} \upharpoonright \tau_i$ holds in K_i, $i = 1, 2$.

1.2.7 Proposition. (i) *The intersection of any family of abstract classes is again an abstract class.*

(ii) *If K_i, $i \in I$ is a family of abstract classes of Löwenheim number κ_i then the intersection $\bigcap_{i \in I} K_i$ is an abstract class of Löwenheim number $\sum_{i \in I} \kappa_i$.*

1.2.8 Remark. Unions of abstract classes need not be abstract classes. It is easy to construct examples violating Axiom 3(i) and also Axiom 3(ii). But then Axiom 4(i) and 4(ii) become meaningless. For disjoint unions only Axiom 3(ii) may be violated, but unions of disjoint abstract classes are admittedly uninteresting.

1.3. Presentability of Abstract Classes

Our next theorem establishes a connection between abstract classes of a given Löwenheim number and some infinitary logics and will give as a more precise cardinal parameter than the Löwenheim number.

1.3.1 Definitions. Let τ be a fixed vocabulary, T be a first-order theory over τ, i.e., $T \subset \mathscr{L}_{\omega\omega}(\tau)$, and Γ be a set of types over τ.

(i) A class $K = \text{MOT}(T, \Gamma)$ if $K = \{\mathfrak{A} \in \text{Str}(\tau)$: $\mathfrak{A} \models T$ and $\mathfrak{A}$ *omits every* $p \in \Gamma\}$.
MOT(T, Γ) stands for *Models of T Omitting the Types* from Γ.

We say that K is an *elementary class omitting some types* and write $K \in \text{ECOT}$ if there are T, Γ such that $K = \text{MOT}(T, \Gamma)$.

(ii) If $\tau_0 \subset \tau$ and K is a class of τ_0-structures we write $K = \text{MOT}_{\tau_0}(T, \Gamma)$ if $K = \{\mathfrak{A} \in \text{Str}(\tau_0)$: $\mathfrak{A}$ *has an expansion* $\mathfrak{A}' \in \text{MOT}(T, \Gamma)\}$.

We say that K is a *projective class omitting some types* and write $K \in \text{PCOT}$ if there are T, Γ, τ_0 such that $K = \text{MOT}_{\tau_0}(T, \Gamma)$.

(iii) We say that $K \in \text{ECOT}(\lambda, \mu)$ or $K \in \text{PCOT}(\lambda, \mu)$ if for T, Γ as above we have that $\text{card}(T) \leq \lambda$, $\text{card}(\Gamma) \leq \mu$.
(iv) If $\langle K, <_K \rangle$ is an abstract class, we say that K is (λ, μ)*-presentable* if $K \in \text{PCOT}(\lambda, \mu)$ and $K_< = \{[\mathfrak{A}, \mathfrak{B}]: \mathfrak{A} <_K \mathfrak{B}\} \in \text{PCOT}(\lambda, \mu)$.

If $\lambda = \mu$ we omit μ and just speak of λ*-presentable classes.*

1.3.2 Examples. From the examples in 1.2.1 in the previous section, (i) and (iii) are ω-presentable and (ii) is not presentable for any cardinals λ, μ. This follows from the non-characterizability of the class of well-orderings in $\mathscr{L}_{\infty\omega}$ (cf. Theorem 3.3.1) and the theorem below. However, they are axiomatizable in $\mathscr{L}_{\omega_1\omega_1}$.

Clearly (λ, μ)-presentable classes are projective classes in the logic $\mathscr{L}_{\nu\omega}$ with $\nu = (\sup(\lambda, \mu)^+)$, but from the infinitary operations we only use once universal quantification over infinitary formulas. Example (v) is just an instance of an PCOT-class.

Clearly, a λ-presentable class has Löwenheim number λ.

1.3.3 Theorem (Shelah's Presentability Theorem). *Let* $\langle K, < \rangle$ *be an abstract class over a vocabulary* τ, $\text{card}(\tau) = \lambda$, *and with Löwenheim number* $\mu \geq \lambda$. *Then* $\langle K, > \rangle$ is $(\mu, 2^\mu)$*-presentable.*

Proof. The proof uses two lemmas.

1.3.4 Lemma (Direct Limit Lemma). *Let I be a directed set (i.e., partially ordered by $\leq$, such that any two elements have a common upper bound). Let $\langle K, <\rangle$ be an abstract class and $\mathfrak{M}_i$ $(i \in I)$ be a family of structures in K with $i, j \in I$, $i \leq j$ implies that $\mathfrak{M}_i < \mathfrak{M}_j$. Then*

(i) *for every $i \in I$ the structure $\mathfrak{M}_i < \bigcup_{j \in I} \mathfrak{M}_j$ and*
(ii) *if $\mathfrak{N} \in K$ and for every $j \in I$, $\mathfrak{M}_j < \mathfrak{N}$ then $\bigcup_{j \in I} \mathfrak{M}_j < \mathfrak{N}$.*

Proof. We prove (i) and (ii) simultaneously by induction on card(I). If I is finite there is nothing to prove, since I has a maximal element.

Suppose card(I) $= \mu$ and we have proved the lemma for card(I) $< \mu$. We can find a family I_α $(\alpha < \mu)$ such that:

(a) card(I_α) $<$ card(I);
(b) $\alpha < \beta < \mu$ implies that $I_\alpha \subset I_\beta \subset I$;
(c) $\bigcup_{\alpha < \mu} I_\alpha = I$;
(d) for every limit ordinal $\delta > \mu$ $\bigcup_{\alpha < \delta} I_\alpha = I_\delta$; and
(e) for each $\alpha < \mu$ I_α is directed and non-empty.

Let $\mathfrak{M}^\alpha = \bigcup_{j \in I_\alpha} \mathfrak{M}_j$. So by indication hypothesis from (i), $j \in I_a$ implies $\mathfrak{M}_j < \mathfrak{M}^\alpha$ and by induction hypothesis from (ii) $\mathfrak{M}^\alpha < \mathfrak{N}$. If $\alpha > \beta$ then $j \in I_\alpha$ implies $\mathfrak{M}_j < \mathfrak{M}^\beta$. Hence, by the induction hypothesis from (ii) $\mathfrak{M}^\alpha = \bigcup_{j \in I_\alpha} \mathfrak{M}_j < \mathfrak{M}^\beta$. So by the chain axiom $\mathfrak{M}^\alpha < \bigcup_{\beta < \mu} \mathfrak{M}^\beta = \bigcup_{j \in I} \mathfrak{M}_j$, and as $j \in I_\alpha$ implies $\mathfrak{M}_j < \mathfrak{M}^\alpha$, we can conclude by the transitivity axiom that $\mathfrak{M}_j < \bigcup_{i \in I} \mathfrak{M}_D$. To conclude that $\bigcup_{i \in I} \mathfrak{M}_i = \bigcup_{\alpha < \mu} \mathfrak{M}^\alpha < \mathfrak{N}$ we use the second part of the chain axiom. ☐

1.3.5 Lemma (Skolemization Lemma). *Let $\langle K, >\rangle$ be an abstract class over a vocabulary τ with Löwenheim number $l(K)$ and let $\tau_1 = \tau \cup \{F_i^n : i < l(K), n \in \omega\}$ a new vocabulary where all the F_i^n are n-place function symbols not in τ. If $\mathfrak{M}$ is a τ-structure and $\mathfrak{M}^*$ is an expansions of $\mathfrak{M}$ to an τ_1-structure and $\bar{a} \in M^n$ we denote by $\mathfrak{M}^*_{\bar{a}}$ the minimal substructure of $\mathfrak{M}^*$ containing $\bar{a}$ and put $\mathfrak{M}_{\bar{a}} = \mathfrak{M}^*_{\bar{a}} \restriction \tau$. Then every $\mathfrak{M} \in K$ has an expansion $\mathfrak{M}^*$ such that for every $n \in \omega$ and $\bar{a} \in M^n$:*

(i) $\mathfrak{M}_{\bar{a}} < \mathfrak{M}$;
(ii) card($\mathfrak{M}_{\bar{a}}$) $\leq l(\mathfrak{K})$;
(iii) *if $\bar{b}$ is a subsequence of $\bar{a}$ then $\mathfrak{M}_{\bar{b}} < \mathfrak{M}_{\bar{a}}$; and*
(iv) *for every τ_1-substructure $\mathfrak{N}^*$ of $\mathfrak{M}^*$ we have that $\mathfrak{N}^* \restriction \tau < \mathfrak{M}$.*

Proof. We define by induction on $n \in \omega$ for every $\bar{a} \in M^n$ the values of $f_i(\bar{a})$, the interpretation of $F_i^n(\bar{a})$, where $i < l(K)$. By our assumption on the Löwenheim number of K there is for every subsequence $\bar{b}$ of $\bar{a}$ an $\mathfrak{M}_{\bar{b}}$ of cardinality less or equal than $l(K)$ such that $\mathfrak{M}_{\bar{b}} < \mathfrak{M}$. So we can find $\mathfrak{M}_{\bar{a}}$ of cardinality less or equal than $l(K)$ such that:

(a) $\mathfrak{M}_{\bar{a}} < \mathfrak{M}$;
(b) for every subsequence $\bar{b}$ of $\bar{a}$, $\mathfrak{M}_{\bar{b}} < \mathfrak{M}_{\bar{a}} < \mathfrak{M}$; and
(c) the choice of $\mathfrak{M}_{\bar{a}}$ does not depend on the order of $\bar{a}$.

To secure (b) we need Axiom 3(ii).

Now let $\{c_i: i < j \leq l(K)\}$ be an enumeration of the universe of $\mathfrak{M}_{\bar{a}}$ and put $f_i^n(\bar{a}) = c_i$ for $i < j$ and $f_i^n(\bar{a}) = c_0$ for $j < i < l(K)$.

Clearly, (i)–(iii) hold for $\mathfrak{M}^*$. To verify (iv) we use Lemma 1.3.4. ▯

1.3.6 Proof of Theorem 1.3.3. Let $\mathfrak{M}^*$ be as in Lemma 1.3.5 and let Γ_n be the set of complete n-types $p = p(x_0, \ldots, x_{n-1})$ in $\mathscr{L}_{\omega\omega}(\tau_1)$ such that:

(a) if $\bar{a} \in M^n$ realizes p in $\mathfrak{M}^*$ and $\bar{b}$ is a subsequence of $\bar{a}$ then $\mathfrak{M}_{\bar{b}} <_K \mathfrak{M}_{\bar{a}}$.

Clearly, (a) can be expressed by a first-order type over τ_1.

Now let Γ the set of complete n-types in $\mathscr{L}_{\omega\omega}(\tau_1)$ which are not in $\bigcup_{m \in \omega} \Gamma_m$ and put $K' = \mathrm{MOT}(\varnothing, \Gamma)$.

Claim 1. *If* $\mathfrak{A} \in K'$ *then* $\mathfrak{A} \upharpoonright \tau \in K$.

If $\mathfrak{A}$ is finitely generated, this is true since the only types realized in $\mathfrak{A}$ take care of this. Otherwise we write $\mathfrak{A}$ as the union of its finitely generated substructures and apply Lemma 1.3.4.

Claim 2. *If* $\mathfrak{A} \in K$ *then it has an expansion* $\mathfrak{A}^* \in K'$.

This clearly follows from Lemma 1.3.5.

This proves that $K \in \mathrm{PCOT}$. To prove that $\{[\mathfrak{A}; \mathfrak{B}]: \mathfrak{A} <_K \mathfrak{B}\}$ is also in PCOT we repeat the same proof for pairs of structures. ▯

Shelah's presentability theorem uses additional function symbols, even in the case where $<_K$ is just the substructure relation. On the other hand it guarantees axiomatizability in $\mathscr{L}_{\kappa\omega}$ for some κ depending on the Löwenheim number of K. One should compare this with the following easy generalization of the classical Chang–Łos–Suszko theorem:

1.3.7 Proposition*. *Let* κ *be a strongly inaccessible cardinal,* τ *a vocabulary with* $\mathrm{card}(\tau) < \kappa$ *and* K *an abstract class of* τ*-structures with Löwenheim number* $l(K) < \kappa$ *and* $<_K$ *the ordinary substructure relation. Then there is a prenex* $\forall\exists$*-sentence* $\varphi \in \mathscr{L}_{\kappa\kappa}(\tau)$ *such that* $K = \mathrm{Mod}(\varphi)$.

1.4. Hanf Numbers

Hanf numbers were defined in Chapter II for arbitrary logics. In Section IX.3.2 Hanf numbers for infinitary logics are studied. We want to apply these results together with the presentability theorem and characterizability theorem to abstract classes. We first define Hanf numbers for abstract classes and recall some material from Chapter IX.

1.4.1 Definitions (Hanf Numbers). (i) Let K be any class of structures closed under isomorphisms. We define the *Hanf number* $h(K)$ to be

$$h(K) = \bigcup \{\mathrm{card}(\mathfrak{A})^+ : \mathfrak{A} \in K\}.$$

If $h(K) > \kappa$ for every cardinal κ we write $h(K) = \infty$.

(ii) If **C** is a family of classes of structures closed under isomorphisms, we define the *Hanf number* $h(\mathbf{C})$ to be

$$h(\mathbf{C}) = \bigcup \{h(K): K \in \mathbf{C} \text{ and } h(K) < \infty\}.$$

$h(\mathbf{C})$ is the smallest cardinal κ such that if some some $K \in \mathbf{C}$ has a model of cardinality κ then it has arbitrary large models.

The concept of a Hanf number is only interesting for families of classes K, such as Jonsson classes, abstract classes with Löwenheim number $l(K) = \lambda$, ECOT(λ, μ), PCOT(λ, μ), etc.

1.4.2 Examples. (i) If all models $\mathfrak{A} \in K$ are of cardinality strictly less than κ then $h(K) \leq \kappa$.

(ii) If K is $\mathrm{PC}_{\mathscr{L}_{\omega_1\omega}}$ then $h(K) \leq \beth_{\omega_1}$, by Theorem VIII.6.4.4.

(iii) If K is PCOT(λ, μ) and $\lambda \leq \mu$ then $h(K) < \beth_{(2^\mu)^+}$, by corollary IX.3.2.14.

1.4.3 Theorem*. *Let K be an abstract class over a vocabulary τ with* card$(\tau) = \lambda$ *and with Löwenheim number $l(K) = \mu$. Put $\kappa_0 = 2^{\lambda+\mu}$ and $\kappa = \beth_{(2^{\kappa_0})^+}$. Then $h(K) < \kappa$.*

Proof. Use the presentability theorem (1.3.3) and Example 1.4.1(iv) above. □

2. *Amalgamation*

2.1. Jonsson Classes and Universal and Homogeneous Models

We did not require in our definition of abstract classes any form of amalgamation. In fact, the point of our approach is, that amalgamation is *not* needed to get a nice structure/non-structure theory. It turns out that the presence or absence of amalgamation is like a watershed: The resulting model theories differ considerably. In this section we look at the case where amalgamation is true for any triple of models, a case which had been studied in the literature already in Jonsson [1956]. In Morley–Vaught [1962] they are called Jonsson classes. Jonsson classes are special cases of our abstract classes in the sense that the axioms of abstract classes are part of the axioms of Jonsson classes which we shall discuss now. Note that our terminology will differ slightly from the terminology scattered in the literature.

Let K be an abstract class. We shall introduce some more axioms:

Axiom 6 (Amalgamation). If $\mathfrak{A}_i \in K$, $i = 0, 1, 2$ and $\mathfrak{A}_0 < \mathfrak{A}_j$, $j = 1, 2$ then there is $\mathfrak{A} \in K$ such that $\mathfrak{A}_j < \mathfrak{A}$, $j = 1, 2$ and such that the diagram of the embeddings commutes.

Axiom 7 (Joint Embedding). If $\mathfrak{A}_j \in K$, $j = 1, 2$ then there is $\mathfrak{A} \in K$ such that $\mathfrak{A}_j < \mathrm{A}$.

Axiom 8 (Unboundedness). K contains structures of arbitrarily unbounded cardinality.

2.1.1 Definitions. (i) K is a *weak Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axiom 6.

(ii) K is a *Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axioms 6 and 7.

(iii) K is an *unbounded Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axioms 6, 7, and 8.

2.1.2 Proposition*. (i) *Every weak Jonsson class K is a disjoint union of* (*possibly a proper class*) *of Jonsson classes.*

(ii) *If K is a weak Jonsson class and* $l(K) = \lambda$ *then K is a disjoint union of at most* 2^λ *many Jonsson classes.*

Proof. To see (i), we define an equivalence relation $\mathfrak{A} \equiv \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in K$ by: $\mathfrak{A} \equiv \mathfrak{B}$ if there is $\mathfrak{C} \in K$ such that $\mathfrak{A} < \mathfrak{C}$ and $\mathfrak{B} < \mathfrak{C}$. By the amalgamation axiom this is indeed an equivalence relation and every such equivalence class is a Jonsson class.

(ii) is obvious. ▯

2.1.3 Examples. (i) If Σ is a complete set of first-order sentences with an infinite model, then $\mathrm{Mod}(\Sigma)$ with the elementary embedding $<$ is a unbounded Jonsson class.

(ii) Jonsson classes are not necessarily unbounded: Let $K(\alpha)$ be the class of well-orderings embeddable into $\langle \alpha, < \rangle$ with end-extensions. As noted already in Example 1.2.1(ii) this gives rise to an abstract class and amalgamation and joint embedding hold trivially.

Unbounded Jonsson classes are the right framework for the construction of universal, homogeneous, and saturated structures. A fair exposition of this approach may be found in Bell–Slomson [1969, Chapter 10].

However we note that Jonsson classes are rather rare. In fact we have:

2.1.4 Proposition. *Let* $\mathscr{L}$ *be a logic with occurrence number below the first uncountable measurable cardinal such that for every complete set of sentences* $\Sigma \subset \mathscr{L}(\tau)$ *with an infinite model,* $\mathrm{Mod}(\Sigma)$ *together with* $\mathscr{L}$*-extensions is a weak Jonsson class. Then* $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.

Proof. From the abstract amalgamation theorem (Theorem XVIII.3.4.2) we get that $\mathscr{L}$ is compact. Now we apply Theorem 3.1.9 also from Chapter XVIII. ▯

2.1.5 Definitions. Let K be a fixed abstract class with Löwenheim number $l(K) = \lambda$ and $\kappa \geq \lambda$.

(i) A structure $\mathfrak{M} \in K$ is (K, κ)-*universal*, if whenever $\mathfrak{A} \in K$ is of cardinality strictly less than κ then there is a K-embedding of $\mathfrak{A}$ into $\mathfrak{M}$.
(ii) A structure $\mathfrak{M} \in K$ is K-*universal*, if it is $\text{card}(\mathfrak{M})^+$-universal.
(iii) A structure $\mathfrak{M} \in K$ is (K, κ)-*homogeneous*, if whenever $\mathfrak{A} <_K \mathfrak{B} <_K \mathfrak{M}$, $\text{card}(\mathfrak{B}) < \kappa$ and $f: \mathfrak{A} \to \mathfrak{M}$ is a K-embedding, then there is a K-embedding $f': \mathfrak{B} \to \mathfrak{M}$ such that $f' \upharpoonright \mathfrak{A} = f$.
(iv) A structure $\mathfrak{M} \in K$ is K-*homogeneous*, if its $\text{card}(\mathfrak{M})$-homogeneous.

The following theorem is at the origin of Jonsson classes. It was first proved in Jonsson [1960] for countable vocabularies. The general treatment occurs first in Morley–Vaught [1962]. A fair treatment is in Bell–Slomson [1969] and Comfort–Negrepontis [1974].

2.1.6 Theorem (Jonsson). *Let K be a unbounded Jonsson class with Löwenheim number $l(K) = \lambda$. Let further $\kappa \geq \lambda$ be a regular beth number. Then there is $\mathfrak{M} \in K$ which is K-homogeneous and K-universal and $\mathfrak{M}$ is unique up to isomorphism.*

If the Jonsson class K is not unbounded, we can still get universal and homogeneous structures, even if we relax the amalgamation axiom a bit.

2.1.7 Definitions. Let K be an abstract class.

(i) Let $\mathfrak{A} \in K$. We say that $\mathfrak{A}$ *is an* (λ, μ)-*amalgamation base for* K, if for every $\mathfrak{B}_1, \mathfrak{B}_2 \in K$ with $\text{card}(\mathfrak{B}_1) = \lambda$, $\text{card}(\mathfrak{B}_2) = \mu$, $\mathfrak{A} <_K \mathfrak{B}_i$ $(i = 1, 2)$ there is $\mathfrak{M} \in K$ and K-embeddings $f_i: \mathfrak{B}_i \to \mathfrak{M}$ such that $f_1 \upharpoonright \mathfrak{A} = f_2 \upharpoonright \mathfrak{A}$. We call $\mathfrak{M}$ also an *amalgamating structure for* $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$.
(ii) We say that K *has the* (κ, λ, μ)-*amalgamation property*, if every $\mathfrak{A} \in K$ with $\text{card}(\mathfrak{A}) = \kappa$ is a (λ, μ)-amalgamation base.
(iii) If $\kappa = \lambda$ we just speak of the (λ, μ)-amalgamation property. If $\kappa = \lambda = \mu$ we just say that K_λ has the amalgamation property.
(iv) We write $(<\lambda, \mu)$-amalgamation property, if K has the (λ', μ)-amalgamation property for every $\lambda' < \lambda$ and similarily for the other parameters.

The precise theorem on the existence of homogeneous and universal models, using basically the same proof, is the following:

2.1.8 Theorem (Shelah). *Let K be an abstract class with Löwenheim number $l(K)$, $\kappa \leq \lambda$ and $\lambda = \lambda^{<\kappa}$.*

(i) *If K has the $(<\kappa, \lambda)$-amalgamation property, then for every $\mathfrak{A} \in K$ of cardinality λ there is κ-homogeneous model $\mathfrak{M}$ of cardinality λ such that $\mathfrak{A} <_K \mathfrak{M}$.*
(ii) *If in* (i) $\kappa = \lambda$ *and additionally, K has the joint embedding property* (*i.e., satisfies Axiom* 7), *then there is a universal, homogeneous model $\mathfrak{M}$ of cardinality λ.*
(iii) *If in* (i) *additionally $l(K) < \lambda$ and $\kappa = \lambda$ then the universal and homogeneous model of cardinality λ is unique up to isomorphism.*

2.1.9 Remarks. (i) If K is an unbounded Jonsson class then the universal and homogeneous model of cardinality λ has a proper K-extension. In fact, if λ is regular and $\lambda > l(K)$, then it is a (λ, λ)-limit, as defined in the next section.

(ii) If K is not unbounded, then the universal and homogeneous model can be rigid and have no proper K-extensions. Take, for example, the class of well-orderings of order type less or equal to some fixed cardinal κ together with end-extensions. Then $\langle \kappa, \in \rangle$ has all the above properties.

(iii) If we drop Axiom 4(ii) in our definition of abstract classes we still can prove an analogue to Theorem 2.1.8(ii), losing universality only. More precisely, there is a homogeneous, $(<\lambda)$-universal model in K which is smooth, i.e., the union of a continuous K-chain of models of cardinality strictly smaller than λ. Axiom 4(ii) is used to get the universality from $(<\lambda)$-universality and smoothness. An example of a class K, where this situation applies, is given in Section XVIII.3.4.

(iv) In the literature before 1980 Axiom 4(ii) is usually not required for the definition of a Jonsson class. Presentations of the original theory of Jonsson classes may be found in Bell–Slomson [1969] and Comfort–Negrepontis [1974]. The latter also contains detailed historical remarks.

Given an abstract class K we might also be interested in the number of homogeneous models K has in a given cardinality:

2.1.10 Definition. Let K be an abstract class. We denote by $H(K, \lambda)$ the number of isomorphism classes of K-homogeneous models of cardinality λ.

2.1.11 Theorem (Shelah). *Let K be an abstract class (over a vocabulary τ) with Löwenheim number $l(K) = \lambda$ and $\kappa > \lambda$. Then $H(K, \kappa) \leq 2^{\lambda + \mathrm{card}(\tau)}$.*

Outline of Proof. We observe that two K-homogeneous structures $\mathfrak{A}$, $\mathfrak{B}$ of cardinality $\kappa > \lambda$ are isomorphic iff they have the same substructures of cardinality λ. □

It remains an open problem to characterize $H(K, \kappa)$ further.

We conclude this subsection with a theorem on the existence of universal models in big cardinals.

2.1.12 Theorem (Grossberg–Shelah [1983]). *Let κ be a compact cardinal and $\lambda > \kappa$ with λ strong limit and of cofinality ω. Let K be an abstract class with Löwenheim number $l(K) < \kappa$ which satisfies the joint embedding property. Then there is a universal model in K_λ.*

There are non-trivial applications of the above theorem in the case of locally finite groups.

Proof. In Grossberg–Shelah [1983] this is proved for K the class of all models of some $\mathscr{L}_{\kappa\kappa}$-sentence φ which satisfies the joint embedding property for 2^κ many models simultaneously. In the case of an abstract class the latter can be replaced by the simple JEP, using Axiom 4 (unions of chains). It is easy to see how the proof in

Grossberg–Shelah [1983] can be adapted to abstract classes: We use the Lowenheim number and the presentability theorem to get that K is a projective class in $\mathscr{L}_{\kappa\kappa}$. Next we observe that the proof in Grossberg–Shelah [1983] also works for projective classes. □

2.2. Limit Models

One of the more powerful tools in classical model theory is the use of saturated or special models. Their construction can be carried out in the context of Jonsson classes as described in the previous section. However, we have also seen there that Jonsson classes are very rare outside of first-order model theory. So we need a substitute for saturated models whose existence does not depend on the amalgamation axiom.

2.2.1 Definition ((λ, κ)-Limit Models). Let K be an abstract class with $<_K$.

(i) A model $\mathfrak{N} \in K$ is a *weak λ-limit* if the following properties (a), (b), and (c) are satisfied.

(a) card($\mathfrak{N}$) $= \lambda$.
(b) $\mathfrak{N}$ has a proper extension $\mathfrak{M}$ with $\mathfrak{N} <_K \mathfrak{M}$.
(c) For every $\mathfrak{M} \in K$ such that card($\mathfrak{M}$) $= \lambda$ and $\mathfrak{N} <_K \mathfrak{M}$ there is a $\mathfrak{N}' \in K$ such that $\mathfrak{N} \cong \mathfrak{N}'$ and $\mathfrak{M} <_K \mathfrak{N}'$.

(ii) A structure $\mathfrak{N} \in K$ is a *(λ, κ)-limit model in K*, if it is a weak λ-limit and additionally the following property (d) holds.

(d) If $\{\mathfrak{N}_i: i < \kappa \leq \lambda\}$ is a K-chain and for each $i < \kappa$, $\mathfrak{N}_i \cong \mathfrak{N}$, then $\bigcup_{i<\kappa} \mathfrak{N}_i \cong \mathfrak{N}$.

(iii) A model $\mathfrak{N} \in K$ is a *λ-superlimit* if it is a (λ, κ)-limit for every $\kappa \leq \lambda$.

Superlimits are closely related to saturated models:

2.2.2 Proposition. (i) *If $\mathfrak{M}$ is saturated or special and of cardinality λ then $\mathfrak{M}$ is a (λ, cf(λ))-limit in $K = \{\mathfrak{A} \in \mathrm{Str}(\tau): \mathfrak{M} \equiv_{\mathscr{L}_{\omega\omega}} \mathfrak{A}\}$ with elementary embeddings.*

(ii) *If K is an abstract class and $\mathfrak{N} \in K$ is K-universal and K-homogeneous of cardinality λ, then $\mathfrak{N}$ is weak λ-limit iff $\mathfrak{N}$ is not K-maximal.*

Proof. (i) We have to verify (a), (b), (c), and (d). (a) is true by hypothesis. (b) follows from the compactness of first-order logic and (c) follows from the fact that saturated models are universal. For (d) we have to show that if for every $i <$ cf(λ) $\mathfrak{M}_i$ is saturated then $\bigcup_{i<\mathrm{cf}(\lambda)} \mathfrak{M}_i$ is saturated, too. For λ regular this is easy (Chang–Keisler [1973, Exercise 5.1.1]). For λ singular, see Shelah [1978a]. From this, together with the uniqueness of saturated models, we conclude that $\bigcup_{i<\mathrm{cf}(\lambda)} \mathfrak{M}_i \cong \mathfrak{M}$. The proof for special models is similar and left to the reader.

(ii) is trivial. □

The following two simple propositions will be used in the later sections.

2.2.3 Proposition. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda^+$ which has a weak λ-limit model $\mathfrak{N} \in K$. Then there is a model $\mathfrak{M} \in K$ with* $\operatorname{card}(\mathfrak{M}) = \lambda^+$.

Proof. By (b) there is $\mathfrak{M}' \in K$ such that $\mathfrak{N} <_K \mathfrak{M}'$. If $\operatorname{card}(\mathfrak{M}') > \lambda^+$ we get $\mathfrak{M}$ from the Löwenheim number. If $\operatorname{card}(\mathfrak{M}') = \lambda$ we apply (c) to get $\mathfrak{N}' \cong \mathfrak{N}$ with $\mathfrak{N} <_K \mathfrak{N}'$ and use this to construct a K-chain of length λ^+. Now we apply the chain axiom. ▯

2.2.4 Proposition. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ which has, up to isomorphism, exactly one model $\mathfrak{N} \in K$ of cardinality λ. Then $\mathfrak{N}$ is λ-superlimit iff K has a model $\mathfrak{M}$ of cardinality strictly bigger than λ.*

Proof. If $\mathfrak{N}$ is weak λ-limit we can apply Proposition 2.2.3. So assume that $\mathfrak{M} \in K$ is of cardinality strictly bigger than λ. Using the Löwenheim number we can get $\mathfrak{M}_0 <_K \mathfrak{M}_1 <_K \mathfrak{M}$ with both $\mathfrak{M}_0, \mathfrak{M}_1$ of cardinality λ and isomorphic to $\mathfrak{N}$. This proves (b) of the definition of the superlimit (Definition 2.2.1). Properties (c) and (d) are trivial under the hypothesis of categoricity in λ ▯

We conclude this section with a few observations on the uniqueness of superlimits, whose proofs are trivial.

2.2.5 Proposition. *Let K be an abstract class with a λ-superlimit $\mathfrak{M}$.*

(i) *If $\mathfrak{N}$ is also a λ-superlimit then $\mathfrak{N} \cong \mathfrak{M}$ iff either $\mathfrak{N} < \mathfrak{M}$ or $\mathfrak{M} < \mathfrak{N}$ (modulo some K-embedding).*
(ii) *If K has the joint embedding property, then the superlimit is unique, up to isomorphism.*
(iii) *If $\mathfrak{M}$ is universal, then it is unique.*

2.2.6 Example. Here is an example of an abstract class K_P which has exactly $\alpha + 1$ λ-superlimits of cardinality ω_α. Let K consist of structures with one unary predicate R, whose interpretation is infinite. We put $\langle A, R_A \rangle < \langle B, R_B \rangle$ iff $A \subset B$ and $R_A = R_B$. Clearly $\langle A, R_A \rangle$ is λ-superlimit iff $A - R_A$ has cardinality λ.

We shall often deal with a situation where an abstract class K with Löwenheim number $l(K) < \lambda$ has a λ-superlimit $\mathfrak{M}$ which is universal, homogeneous and is an amalgamation basis for K_λ. Clearly then, only by universality and homogeneity, K has the $(<\lambda, \lambda)$-amalgamation property.

2.2.7 Problem. Does K_λ in this case also have the (λ, λ)-amalgamation property?

In Section 2.3 we state a conjecture, whose proof would follow from a positive answer to this problem.

2.3. Counting Models in the Absence of an Amalgamation Bases

In this section we assume K is an abstract class, which is not a Jonsson class and therefore does not have the amalgamation property, but still does have a λ-superlimit $\mathfrak{M} \in K_\lambda$. Our main theorem of this section is:

2.3.1 Theorem (Shelah's Non-structure Theorem for Abstract Classes). *Assume $2^\lambda < 2^{\lambda^+}$. Let K be an abstract class such that:*

(i) *there is a λ-superlimit $\mathfrak{M} \in K_\lambda$;*
(ii) *$\mathfrak{M}$ is not an amalgamation basis for K_{λ^+}.*

Then $I(K, \lambda^+) = 2\lambda^+$ and there is no universal model in K_{λ^+}.

At this point it is appropriate to state some conjectures. The first one deals with the existence of universal and homogeneous superlimits.

2.3.2 Conjecture (Shelah). Let K be an abstract class with Löwenheim number $l(K) < \lambda$ such that $I(K, \lambda^+) < 2^{\lambda^+}$.

(i) If K additionally satisfies the joint embedding property (Axiom 7), then there is K-universal and K-homogeneous λ-superlimit $\mathfrak{M} \in K$.
(ii) If K has arbitrarily large models, then there is a K-universal and K-homogeneous λ-superlimit $\mathfrak{M} \in K$.

(It may be enough to assume that there is a model of cardinality bigger than 2^{λ^+}.)

An instance of this conjecture is Theorem 3.1.8, with $\lambda = \omega_1$ and K ω-presentable. A proof of this conjecture would give us, with the help of the previous theorem, also a proof of the following conjecture:

2.3.3 Conjecture. Assume GCH. Let K be an abstract class with Löwenheim number $l(K) = \omega$ which has arbitrary large models and such that for every $\lambda > \omega$, $I(K, \lambda) < 2^\lambda$. Then K has the amalgamation property and therefore is a weak Jonsson class.

2.3.4 Problem. Could we replace $l(K) = \omega$ by arbitrary λ in the above conjecture?

Finally we state a conjecture which presents an improvement on Theorem 2.3.1.

2.3.5 Conjecture. Assume $2^\lambda < 2^{\lambda^+}$. Let K be an abstract class with Löwenheim number $l(K) < \lambda$ such that:

(i) there is a universal and homogeneous λ-superlimit $\mathfrak{M} \in K_\lambda$;
(ii) $I(K, \lambda^+) < 2^{\lambda^+}$.

Then K_λ has the amalgamation property.

Clearly, from Theorem 2.3.1, $\mathfrak{M}$ is an amalgamation basis for K_λ, and, by universality and homogeneity, K has the $(<\lambda, \lambda)$-amalgamation property.

We conclude this section with another conjecture, generalizing Morley's categoricity theorem for first-order logic to λ-presentable classes.

2.3.6 Conjecture (Shelah). Let K be an abstract λ-presentable class and let h_λ be the Hanf number for λ-presentable classes. If $I(K, \kappa) = 1$ for some $\kappa > h_\lambda$ then $I(K, \kappa) = 1$ for every $\kappa > h_\lambda$.

Added in Proof. Recently R. Grossberg and S. Shelah announced the following Theorem:

2.3.7 Theorem. *Let K be an unbounded abstract λ-presentable class. If there is a $\mu > \lambda$ such that for every $n \in \omega$ $I(K, \mu^{+n}) = 1$ then for every $\kappa > \lambda$ $I(K, \kappa) = 1$.*

2.4. Martin's Axiom Disproves the Non-structure Theorem

Before we discuss the proof of Theorem 2.3.1 we want to comment on its set-theoretic hypothesis $2^\lambda = \lambda^+$. For this we have to recall Martin's Axiom MA from set theory.

2.4.1 Definitions (Partial Orders). (i) A *partial order* is a pair $\langle P, \leq \rangle$ such that P is not empty and $\leq$ is a transitive and reflexive relation on P.

(ii) Given $p, q \in P$ we say that *p and q are compatible* if there is $r \in P$ such that $r \leq p$ and $r \leq q$ and *p and q are incompatible* if they are not compatible. A *antichain in P* is a set $A \subset P$ such that for every $p, q \in P$ either $p = q$ or p and q are incompatible.

(iii) A partial order $\langle P, \leq \rangle$ satisfies the *countable chain condition* (c.c.c) if every antichain in P is countable.

(iv) A set $D \subset P$ is *dense*, if for every $p \in P$ there is a $q \in D$ such that $q \leq p$.

(v) A set $G \subset P$ is *filter in P*, if any two elements in G are compatible and whenever $p \in G$ and $q \geq p$ then $q \in G$.

2.4.2 Martin's Axiom. (i) MA(κ) is the statement: If $\langle P, \leq \rangle$ is a partial order satisfying c.c.c and $\{D_i: i < \kappa\}$ is a family of dense subsets of P then there is a filter G in P such that for every $i < \kappa$, $D_i \cap G \neq \emptyset$.

(ii) MA is the statement: For every $\kappa < 2^\omega$ MA(κ).

For more references the reader may consult Kunen [1980] or Shelah [1982c].

2.4.3 Proposition (Shelah). *Assume* ZFC + MA + $\neg$CH (*and therefore* $2^\omega = 2^{\omega_1}$). *Then there is an ω-presentable abstract class $K_0 \in$* ECOT *such that*:

(i) $I(K, \kappa) = 1$ *for every* $\kappa < 2^\omega$;

(ii) $I(K, \kappa) = \emptyset$ *for every* $\kappa > 2^\omega$; but

(iii) K_ω *does not have the amalgamation property.*

Proof. We first define the class K_0. The vocabulary τ_0 consist of a binary predicate **C** and a unary predicate **P**. A τ_0-structure $\mathfrak{A} = \langle A, E, P\rangle$ is in K_0 if:

(1) P is countable.
(2) If xEy then $x \in P$ but $y \notin P$.

For every $y \notin P$ we define $S_y = \{x \in A: xEy\}$. Clearly $S_y \subset P$.

(3) (Extensionality of E). If $x \neq y$, $x, y \notin P$ then $S_x \neq S_y$.
(4) For every $x \notin P$ and for every finite set $C \subset P$ there is a $y \notin P$ such that the symmetric difference $S_x \Delta\ S_y = C$.

We define an equivalence relation on $A - P$ by $x \equiv y$ iff $S_x\Delta\ S_y$ is finite. Clearly every equivalence class is countable. Let the number of such equivalence classes be the dimension $\dim(\mathfrak{A})$ of $\mathfrak{A}$. Now we require that:

(5) If $x_1, x_2, \ldots, x_n$ are mutually inequivalent and not in P, then every finite boolean combination of the sets $S_{x_1}, S_{x_2}, \ldots, S_{x_n}$ is infinite.

This concludes the definition of K_0.

Next we define the substructure relation $<_0$ for $\mathfrak{A} = \langle A, E_A, P_A\rangle$, $\mathfrak{B} = \langle B, E_B, P_B\rangle$, both in K_0 by $\mathfrak{A} <_0 \mathfrak{B}$ if $\mathfrak{A} \subset \mathfrak{B}$ and $P_A = P_B$.

We have to verify that this defines an abstract class with (i)–(iii). We leave the verification of the axioms to the reader. To verify (i) and (ii) we prove five claims:

Claim 1. *There are no models of cardinality greater than* 2^ω.

By (1) P is countable and by (4) every element is either in P or in some S_x for $x \in P$. So the claim follows from (3). This proves Proposition 2.4.3(ii).

Claim 2. K_0 *is categorical in* ω.

This one can prove using (5) and a Cantor-style back-and-forth argument.

Claim 3. $\mathrm{MA}(\kappa)$ *implies that* K_0 *is categorical in* $\kappa < 2^\omega$.

Clearly $\mathfrak{A}, \mathfrak{B} \in K_0$ have the same cardinality iff they have the same dimension. So let $\mathfrak{A}, \mathfrak{B} \in K_0$ be of the same dimension $\kappa < 2^\kappa$. So let $E_i^A, E_i^B, i < \kappa$ be an enumeration of the equivalence classes in $\mathfrak{A}, \mathfrak{B}$, respectively. Let F be the family of all finite partial isomorphisms $f: \mathfrak{A} \to \mathfrak{B}$ such that additionally to the isomorphism conditions we have:

(a) for every $x \in \mathrm{dom}(f)$, $x \in E_i^A$ iff $f(x) \in E_i^B$; and
(b) if $x, y \in \mathrm{dom}(f)$, $x, y \notin P$, and $x \equiv y$ then the finite set $S_x\Delta\ S_y \subset \mathrm{dom}(f)$.

Clearly F is a partial order by the natural extension relation of partial isomorphisms: $f \leq g$ iff f extends g. To show that F satisfies c.c.c., we show:

Claim 4. *If* $\{f_i: i < \kappa\} \subset F$ *for some* κ *such that* $\omega < \kappa < 2^\omega$ *then there is* $I \subset \kappa$, $\mathrm{card}(I) = \kappa$, *such that* $\{f_i: i \in I\}$ *are all compatible.*

This follows from the fact that all the sets P_A, E_i^A are countable.

Now we define $D_a = \{f \in F: a \in \mathrm{dom}(f)\}$ and $D_b = \{f \in F: b \in \mathrm{rg}(f)\}$. Clearly, all the D_a, D_b are dense in F. So let G be a filter in F which intersects all the D_a, D_b with $a \in P_A$ and $b \in P_B$. Such a filter exists by MA(κ). Next, we define $g = \bigcup_{f \in G} f$.

Claim 5. *g: $\mathfrak{A} \to \mathfrak{B}$ is an isomorphism.*

g is one–one and onto by our choice of the D_a, D_b, and g is an isomorphism, since every finite restriction of g has an extension in F.

So Claims 3–5 prove Proposition 2.4.3(i).

We still have to prove Proposition 2.4.3(iii). For this let $\mathfrak{A} = \langle A, E_A, P_A \rangle \in K_0$ be countable. Let $S^1 \subsetneq S^2 \subsetneq P_A$ be two generic subsets different from all the S_x, $x \in A - P_A$. We now form $\mathfrak{A}_i$ ($i = 1, 2$) by adding the necessary new points to $A - P_A$ to ensure that S^i is of the form S_x for some $x \in A_i - P_A$ and to make (2)–(4) true. No points are added in P_A. Clearly $\mathfrak{A}_i$ can be constructed to be countable and in K_0, and $\mathfrak{A} <_0 \mathfrak{A}_i$. Now assume $\mathfrak{B}$ is an amalgamating structure. Then there are $z_i \in B - P_A$ ($i = 1, 2$) such that $S_{z_i} = S^i$ and $S_{z_1} \cap S_{z_2} = \emptyset$, contradicting (5). Therefore $\mathfrak{B} \notin K_0$. □

2.5. *Preliminaries for the Weak Diamond*

In this section we collect the set-theoretic preliminaries needed in Section 2.6. They are concerned about the relation between various instances of the GCH and combinatorial principles related to $\diamond$. First we present a variation of Ulam's theorem (cf. Lemma XVIII.4.3.9). Recall that an *ideal J on a set I* is the dual of a filter F on the set I, and that an ideal is *normal*, if the dual filter is normal. A subset $S \subset I$ is called *J-positive*, if $S \notin J$. Since the filter D_κ of closed and unbounded sets on κ is normal, the stationary sets on κ are D_κ-positive.

2.5.1 Ulam's Theorem. *Let J be a normal ideal on κ^+.*

(i) (Ulam). *Let κ be an infinite cardinal. If $S \subset \kappa^+$, $S \notin J$, S may be decomposed into κ^+ disjoint J-positive subsets.*

(ii) *There is a family* $\mathbf{S}$ *of 2^{κ^+} many J-positive subsets of κ^+ such that for any $S_1, S_2 \in \mathbf{S}$ the symmetric difference $S_1 \Delta S_2$ is J-positive as well.*

Proof. (i) is standard, e.g., Theorem 3.2 in Chapter B.3 of the *Handbook of Mathematical Logic* [Barwise 1977], where it is stated for stationary rather than J-positive sets. But the same proof works for this generalized version.

To prove (ii) let $\{S_\alpha: \alpha < \kappa^+\}$ be the disjoint family of J-positive sets from (i). Let $X \subset \kappa^+$, $X \neq \emptyset$. Define $T_X = \bigcup_{\alpha \in X} S_{2\alpha} \cup \bigcup_{\alpha \notin X} S_{2\alpha+1}$. Clearly each T_X is J-positive and $X \neq Y$ implies that $T_X \Delta T_Y$ is J-positive. □

2.5.2 Jensen's $\diamond$. Jensen's $\diamond$ for $\omega_1(\diamond_{\omega_1})$ can be formulated as: There exists a family of functions $\{g_\alpha: \alpha \to \alpha: \alpha < \omega_1\}$ such that for every $f: \omega_1 \to \omega_1$ we have that $\{\alpha < \omega_1: f \restriction \alpha = g_\alpha\}$ is stationary.

2.5.3 The Principles Φ and Θ of Devlin and Shelah. Let F be a function which maps (0, 1)-sequences of length $\alpha < \lambda$ into $\{0, 1\} = 2$, and let $S \subset \lambda$.

(i) The principle $\Phi_\lambda^2(S)$ says that for every such function F there is a function $g: \lambda \to 2$ such that for every other function $f: \lambda \to 2$ the set

$$\{\alpha \in S: F(f \upharpoonright \alpha) = g(\alpha)\}$$

is stationary on λ.

(ii) The principle Φ is just $\Phi_{\omega_1}^2(\omega_1)$.

(iii) The principle $\Phi_\lambda^\kappa(S)$ is obtained from Φ_λ^2 by replacing every occurrence of 2 by κ, both in the range and domain of F as well as in the range of g and the domain of f.

(iv) If $S = \lambda$ we omit it.

(v) The principle Θ says that if $\{f_\eta: \eta \in {}^{\omega_1}2\}$ is a family of functions with each $f_\eta: \omega_1 \to 2^\omega$, then there is $\eta \in {}^{\omega_1}2$ such that the set

$$\{\delta \in \omega_1: (\exists \rho \in {}^{\omega_1}2)[f_n \upharpoonright \delta = f_\eta \upharpoonright \delta \text{ and } \rho \upharpoonright \delta = \eta \upharpoonright \delta \text{ and } \rho(\delta) \neq \eta(\delta)]\}$$

is stationary.

2.5.4 Theorem. (i) (Jensen). *$\diamondsuit_{\omega_1}$ implies $2^\omega = \omega_1$.*

(ii) (Devlin–Shelah). *$\diamondsuit_{\omega_1}$ implies the principle Φ.*

(iii) (Devlin–Shelah). *$2^\omega < 2^{\omega_1}$ implies Θ.*

A proof of (i) may be found in textbooks like Kunen [1980]. (ii) and (iii) are proved in Devlin–Shelah [1978]. The important fact about the principle Φ is the following theorem:

2.5.5 Theorem (Devlin–Shelah [1978]). *The principle Φ is equivalent to $2^\omega < 2^{\omega_1}$.*

2.5.6 Definition (Small Sets). (i) A subset $S \subset \lambda$ is (λ, κ)-*small*, if $\Phi_\lambda^\kappa(S)$ fails.

(ii) Let us denote by $\mathrm{S}(\lambda, \kappa)$ the set of all (λ, κ)-small subsets of λ.

2.5.7 Remarks. (i) Clearly, (λ, κ)-small sets are stationary in λ.

(ii) The principle Φ is equivalent to $\omega_1 \notin \mathrm{S}(\omega_1, 2)$.

2.5.8 Proposition (Shelah). (i) $\mathrm{S}(\lambda, \kappa)$ *forms a normal ideal on λ.*

(ii) *Φ_λ^κ holds iff $\mathrm{S}(\lambda, \kappa)$ forms a non-trivial normal ideal on λ.*

Proof. (i) is a special case of Lemma 14.1.9 in Shelah [1982, Book] and (ii) follows trivially from the definitions and (i). □

2.5.9 Strong Negations of Φ. First we write out the negation of $\Phi_{\omega_1}^2$: there is a function F which maps (0, 1)-sequences of length $\alpha < \lambda$ into $\{0, 1\} = 2$, such that for every function $g: \lambda \to 2$ there is a function $f: \lambda \to 2$ such that the set

$$\{\alpha < \lambda: F(f \upharpoonright \alpha) = g(\alpha)\}$$

is closed and unbounded in λ. We want to generalize and parametrize this further.

Let λ be a regular cardinal and $\bar{\mu} = \langle \bar{\mu}(i): i < \lambda \rangle, \bar{\chi} = \langle \bar{\chi}(i): i < \lambda \rangle$ be sequences of cardinals. We want to generalize the above negation of Φ for a function F with domain

$$\mathrm{dom}(F) = D(\bar{\mu}) = \bigcup_{\alpha<\lambda} \prod_{i<\alpha} \bar{\mu}(i).$$

Now we denote by $\mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ the statement:

There is a function F such that:

(a) for every $\alpha < \lambda$, if $\eta \in \prod_{i<\alpha} \bar{\mu}(i)$, then $F(\eta) < \bar{\chi}(\alpha)$; and
(b) for every $h \in \prod_{\alpha<\lambda} \bar{\chi}(\alpha)$ there exists $\eta \in \prod_{\alpha<\lambda} \bar{\mu}(\alpha)$ such that

$$\{\alpha < \lambda: F(\eta \restriction \alpha) = h(\alpha)\}$$

is closed and unbounded in λ.

Such a function F is said to *exemplify* $\mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$.

If $\bar{\mu}, \bar{\chi}$ are singletons we use the obvious notation. If $\bar{\mu} = \mu(0), \mu(1)$ we use the obvious abuse of notation. In 14.1.5 of Shelah [1982c] it is proved that we can always assume that $\bar{\mu}$ is a sequence of length two. Clearly $\mathrm{Unif}(\lambda, 2, 2)$ is just the negation of Φ^2_λ, and $\mathrm{Unif}(\lambda, \kappa, \kappa)$ is just the negation of Φ^κ_λ.

The version of the weak diamond needed in Section 2.7, and its connection to the continuum hypothesis, is captured in the following proposition:

2.5.10 Proposition. *Assume* $2^\kappa < 2^{\kappa^+}$.

(i) $\Phi^2_{\kappa^+}$ holds.
(ii) $\mathrm{Unif}(\kappa^+, \mu, 2, 2)$ *fails for every* μ *with* $\mu^\omega < 2^{\kappa^+}$.

This proposition follows from the following two results from Shelah [1982c]:

2.5.11 Theorem (Shelah). *Assume that* λ *is regular and*

(i) $2^{<\lambda} < 2^\lambda$;
(ii) $\mu^\omega < 2^\lambda$.

Then $\mathrm{Unif}(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ *fails.*

Proof. Shelah [1982c, Theorem 14.1.10]. □

2.5.12 Proposition (Shelah). $\mathrm{Unif}(\lambda^+, \mu, 2, 2)$ *implies* $\mathrm{Unif}(\lambda^+, \mu, 2^\lambda, 2^\lambda)$.

Proof. Shelah [1982c, Lemma 14.1.7(1)] for the case λ replaced by λ^+. □

Proof of Proposition 2.5.10. We prove (ii) since (i) follows from (ii) by putting $\mu = 1$. We apply Theorem 2.5.11 with $\lambda = \kappa^+$ and therefore $2^{<\lambda} = 2^\kappa$. So we get that $\mathrm{Unif}(\kappa^+, \mu, 2^\kappa, 2^\kappa)$ fails, for every $\mu^\omega < 2^{\kappa^+}$. So by Proposition 2.5.12 $\mathrm{Unif}(\kappa^+, \mu, 2, 2)$ fails, for every $\mu^\omega < 2^{\kappa^+}$. □

2.6. Proof of the Non-structure Theorem: The Countable Case

The purpose of this section is to prove completely some special cases of the non-structure theorem (2.3.1) and to understand some of the intricacies of its complete proof. The complete proof appears in Section 2.7. For expository (and historical) reasons we shall work our way from the moderately simple case to the more difficult.

2.6.1 Theorem. *Assume $2^\omega < 2^{\omega_1}$. Let K be an abstract class with Löwenheim number $l(K) = \omega$ such that:*

(i) *$I(K, \omega) = 3$;*
(ii) *$I(K, \omega_1) \neq \varnothing$; and*
(iii) *K_ω does not have the amalgamation property.*

Then there is no universal model in K_{ω_1} and therefore $I(K, \omega_1) > 1$.

Outline of Proof. The proof consists of several stages: A construction of a system of countable models, a construction of uncountable models, and a verification that no model of cardinality ω_1 is universal. The same pattern will be followed in subsequent proofs, so we try to give this first proof a modular structure.

2.6.2 Construction of Countable Models. Clearly, by (i) and (ii) and Proposition 2.2.4 the unique countable model $\mathfrak{N} \in K$ is an ω-superlimit. Let $\mathfrak{M}^* \in K$ be of cardinality ω_1, so without loss of generality we can assume that its domain $M^* = \omega_1$. By (iii) there are countable $\mathfrak{M} <_K \mathfrak{M}_i$ $(i = 0, 1)$ which exemplify the failure of the amalgamation property and $\mathfrak{M} \cong \mathfrak{M}_i \cong \mathfrak{N}$. Since the Löwenheim number $l(K) = \omega$ we can assume that $\mathfrak{M} <_K \mathfrak{M}^*$. We shall show that $\mathfrak{M}^*$ is not universal.

For this we define by induction on $\alpha < \omega_1$ countable models $\mathfrak{M}_\eta$ where $\eta \in {}^{\alpha}2$, i.e., η ranges over sequences of 0's and 1's of length $l(\eta) \leq \alpha$. If η, ν are two such sequences, we write $\eta \subset \nu$ if η is an initial segment of ν, and if $\beta < \alpha$ we denote by $\eta \upharpoonright \beta$ the restriction of η to β.

Now we require that:

(1) $\mathfrak{M}_\eta$ is countable and the universe $M_\eta = \omega(1 + l(\eta))$.
(2) If $\eta \subset \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.
(3) If $\delta \in \omega_1$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha < \delta} \mathfrak{M}_{\eta \upharpoonright \alpha}.$$

To construct all the $\mathfrak{M}_\eta$'s we put for $\alpha = 0$, $\mathfrak{M}_\phi \cong \mathfrak{N}$, and for the $\alpha = \delta$ limit we take the limits, as K is an abstract class. For $\alpha = \beta + 1$ we have to work a bit. For each η of length $\leq \beta$ we choose an isomorphism f_η from $\mathfrak{M}$ onto $\mathfrak{M}_\eta$. Here we use (i). Now we define functions f_η^i and models $\mathfrak{M}_{\eta^\frown\langle i\rangle}$ such that f_η^i extends f_η and is an isomorphism from $\mathfrak{M}_i$ onto $\mathfrak{M}_{\eta^\frown\langle i\rangle}$. In other words, at every stage we copy our original counterexample to the amalgamation property.

2.6.3 Construction of Models in ω_1. Now we construct models of cardinality ω_1. For every η of length ω_1 we put $\mathfrak{M}_\eta = \bigcup_{\alpha<\omega_1} \mathfrak{M}_{\eta \restriction \alpha}$.

2.6.4 $\mathfrak{M}^*$ is not Universal. Assume, for contradiction, that $\mathfrak{M}^*$ were universal. Then for each η of length ω_1 there is an embedding g_η from $\mathfrak{M}_\eta$ into $\mathfrak{M}^*$ such that $g_\eta(\mathfrak{M}_\eta) <_K \mathfrak{M}^*$.

Now we use the principle Θ and get two sequences η, ν of length ω_1 and $\alpha < \omega_1$ with $\alpha = \omega\alpha$ such that:

(4) $\eta \restriction \alpha = \nu \restriction \alpha$, $\eta(\alpha) = 0$, $\nu(\alpha) = 1$; and
(5) $g_\eta \restriction \mathfrak{M}_{\eta \restriction \alpha} = g_\nu \restriction \mathfrak{M}_{\nu \restriction \alpha}$.

But this shows that $\mathfrak{M}_0$, $\mathfrak{M}_1$ can be amalgamated over $\mathfrak{M}$ with amalgamating structure $\mathfrak{M}^*$ by setting

$$h_0 = (g_\eta \restriction \mathfrak{M}_{\eta \restriction (\alpha+1)}) f^0_{\eta \restriction \alpha} \colon \mathfrak{M}_0 \to \mathfrak{M}^*,$$

and

$$h_1 = (g_\nu \restriction \mathfrak{M}_{\nu \restriction (\alpha+1)}) f^1_{\eta \restriction \alpha} \colon \mathfrak{M}_1 \to \mathfrak{M}^*,$$

a contradiction to our choice of $\mathfrak{M}$, $\mathfrak{M}_0$, $\mathfrak{M}_1$. $\square$

This proof describes the basic structure of all the further proofs. We have, till now, avoided two problems: How to get maximally many models in $\lambda^+ = \omega_1$, rather than just no universal models, and how to replace ω by general cardinals λ. Historically, Shelah solved these two problems one after the other, and the proof of the general theorem evolved while various versions of Shelah [1983b, c, 198?c] were written. For instance, the following theorem can be proven with just slightly more combinatorial effort:

2.6.5 Theorem. *Assume $2^\omega < 2^{\omega_1}$ and let K be an abstract class with Löwenheim number $l(K) = \omega$ such that:*

(i) $I(K, \omega) = 1$;
(ii) $I(K, \omega_1) \neq \emptyset$; *and*
(iii) K_ω *does not have the amalgamation property.*

Then $I(K, \omega_1) > 2^\omega$.

The best possible results for $\lambda = \omega$ was first proved using the additional hypothesis that K be ω-presentable. However, in the following section we present the general case with a complete proof, taken from Shelah [1983b, c].

2.7. *Proof of the Non-structure Theorem: The General Case*

In the general case we have to analyze closer, how the λ-superlimit fails to be an amalgamation basis. We distinguish two cases:

2.7.1 Definitions (Failures of Amalgamation). Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K.

(i) We say that $\mathfrak{M}$, $\mathfrak{M}_0$, $\mathfrak{M}_1$, all isomorphic to $\mathfrak{N}$, form a *maximal counterexample*, if $\mathfrak{M} <_K \mathfrak{M}_i$ cannot be amalgamated in K, but for every $\mathfrak{A}_{i,k} \in K_\lambda$ $(i, k = 0, 1)$ such that $\mathfrak{M}_i <_K \mathfrak{A}_{i,k}$ there is an amalgamating structure $\mathfrak{B}_i \in K$ for $\mathfrak{M} <_K \mathfrak{A}_{i,k}$ $(k = 0, 1)$.

(ii) We say that $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{M}_0$, $\mathfrak{M}_1$, form an *extendible counterexample*, if they are all isomorphic to $\mathfrak{N}$, $\mathfrak{N} <_K \mathfrak{M}$ and for every $\mathfrak{B} \in K_\lambda$ with $\mathfrak{N} <_K \mathfrak{B}$ there are K-embeddings $f_i^B \colon \mathfrak{B} \to \mathfrak{M}_i$ such that $\mathfrak{N} <_{K, f_i^B} \mathfrak{M}_i$ has no amalgamating structure.

2.7.2 Lemma. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K. Then either:*
Case 1. *There is a maximal counterexample; or*
Case 2. *There is an extendible counterexample.*

Proof. Let $\overline{\mathfrak{M}}, \overline{\mathfrak{M}}_i$ $(i = 0, 1)$ be a counterexample which is not maximal. So without loss of generality for every $\mathfrak{K}$-extension $\mathfrak{B} \in \mathfrak{K}_\lambda$ of $\overline{\mathfrak{M}}_0$ there are $\mathfrak{K}$-extensions $\mathfrak{B}_i \in \mathfrak{K}_\lambda$ $(i = 0, 1)$ of $\mathfrak{B}$ such that $\overline{\mathfrak{M}}$, $\mathfrak{B}_i$ have no amalgamating structure. Put $\mathfrak{N} = \overline{\mathfrak{M}}$ and $\mathfrak{M} = \overline{\mathfrak{M}}_0$. Clearly, using property (c) of the definition of superlimits (Definition 2.2.1), for every $\mathfrak{K}$-extension $\mathfrak{B} \in \mathfrak{K}_\lambda$ of $\overline{\mathfrak{M}}_0$ there are $\mathfrak{M}_i$ $(i = 0, 1)$ isomorphic to $\mathfrak{N}$ and embeddings f_i^B such that $\mathfrak{N} <_{K, f_i^B} \mathfrak{M}_I$ has no amalgamating structure. So $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{M}_i$ is our extendible counterexample. □

This lemma is the key to the proof of the non-structure theorem for abstract classes. For the sake of readability we state it once more, in a sharpened form:

2.7.3 Theorem (Shelah's Non-structure Theorem for Abstract Classes). *Assume $2^\lambda < 2^{\lambda^+}$. Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K. Then $I(K, \lambda^+) = 2^{\lambda^+}$. (In fact there are 2^{λ^+} many structures in K_{λ^+} such that for no two of them is there a K-embedding from one into the other.)*

Proof. The proof uses Lemma 2.7.2 and therefore treats the two cases separately. In each case the proof proceeds along the pattern of the proof of Theorem 2.6.1: Construction of a system of models is of cardinality λ, each of them isomorphic to the superlimit; construction of models is of cardinality λ^+ and the verification that there are many non-isomorphic models.

2.7.4 Case 1: Construction of Models of Cardinality λ. We define by induction on $\alpha < \lambda^+$ models $\mathfrak{M}_\eta$ indexed by (0, 1)-sequences $\eta \in {}^\alpha 2$ such that:

(1) $\mathfrak{M}_\eta$ is isomorphic to the λ-superlimit $\mathfrak{M}$ the universe of $\mathfrak{M}_\eta$ is the set $M_\eta = \lambda(1 + l(\eta))$.
(2) If $\eta < \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.
(3) If $\delta \in \lambda^+$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha<\delta} \mathfrak{M}_{\eta \upharpoonright \alpha}.$$

The construction is the same as in subsection 2.6.2, using the maximal counterexample and the properties of the λ-superlimit.

2.7.5 Case I: Construction of Models of Cardinality λ^+. For every (0, 1)-sequence $\eta \in^{\lambda^+} 2$ we put $\mathfrak{M}_\eta = \bigcup_{\alpha<\lambda^+} \mathfrak{M}_{\eta \upharpoonright \alpha}$.

2.7.6 Case I: Counting the Models of Cardinality λ^+. Let $\delta < \lambda^+$ be such that $\lambda\delta = \delta$, η, $\nu \in {}^\delta 2$ $\mathfrak{M}_\eta$, $\mathfrak{M}_\nu$ models of cardinality λ as constructed above, and $h: \mathfrak{M}_\eta \to \mathfrak{M}_\nu$, a $\mathfrak{K}$ embedding. We now define a function $F(\eta, \nu, h)$ such that $F(\eta, \nu, h) = 1$ iff $\mathfrak{M}_\eta <_K \mathfrak{M}_{\eta\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{K,h} \mathfrak{M}_{\eta\langle 0\rangle}$ have an amalgamating structure, and $F(\eta, \nu, h) = 0$, otherwise. Note that, by our assumptions on δ the universe of $\mathfrak{M}_\eta$ is the set $M_\eta = \delta$. Use now $2^\lambda < 2^{\lambda^+}$ and Proposition 2.5.10 to conclude that λ^+ is not $(\lambda^+, 2)$-small. Then apply Ulam's theorem (2.5.1) and Proposition 2.5.8(ii) to partition λ^+ into a family $\{S_\alpha : \alpha < \lambda^+\}$ of disjoint non-$(\lambda^+, 2)$-small subsets. Now apply $\Phi^2_{\lambda^+}$ to find a family $\{\rho_\alpha \in^{\lambda^+} 2 : \alpha < \lambda^+\}$ such that for each $\alpha < \lambda^+$, η, $\nu \in \lambda^+$ and $h: \lambda^+ \to \lambda^+$ the set $\{\delta < \lambda^+ : F(\eta \upharpoonright \delta, \nu \upharpoonright \delta, h \upharpoonright \delta) = \rho_\alpha(\delta)\}$ is stationary in λ^+.

For each $I \subset \lambda^+$ we define a (0, 1)-sequence $\eta_I \in^{\lambda^+} 2$ such that $\eta_I(i) = \rho_\alpha(i)$ if $i \in \bigcup_{\alpha \in I} S_\alpha$ and $\eta_I(i) = 0$, otherwise. This is well defined since the S_α's form a partition of λ^+.

Our next goal is:

2.7.7 Lemma. *Given $I, J \subset \lambda^+, I - J \neq \varnothing$, then there is no K-embedding*

$$h: \mathfrak{M}_{\eta_I} \to \mathfrak{M}_{\eta_J}.$$

Proof of Lemma. Assume, for contradiction, that $h: \mathfrak{M}_\eta \to \mathfrak{M}_\eta$ is a K-embedding, but there is $\gamma \in I - J$. Clearly the set

$$C = \{\delta < \lambda^+ : h \upharpoonright \delta \text{ is a function into } \delta \text{ and } \lambda\delta = \delta\}$$

is closed and unbounded. Look at S_γ. We use C and F, ρ defined above, to define

$$S'_\gamma = \{\delta \in S_\gamma : F(\eta_I \upharpoonright \delta, \eta_J \upharpoonright \delta, h \upharpoonright \delta) = \rho_\gamma(\delta)\} \cap C.$$

By the choice of ρ above we conclude that $S'_\gamma \neq \varnothing$. Now we choose $\delta \in S'_\gamma$ and put $\eta = \eta_I \upharpoonright \delta$ and $\nu = \eta_J \upharpoonright \delta$. From the definition of η and the fact that $\{S_\alpha : \alpha < \lambda^+\}$ forms a partition, it is clear that $\eta_J(\delta) = 0$.

We now proceed to show that both possibilities, $\eta_I(\delta) = 0$ and $\eta_I(\delta) = 1$, lead to a contradiction.

Case 1: $\eta_I(\delta) = 0$.

Then $\rho_\gamma(\delta) = 0$ and, since $\delta \in S'_\gamma$, we have $F(\eta, \nu, h \upharpoonright \delta) = 0$. But by the choice of h and δ we know that $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure, contradicting the definition of the function F.

Case 2: $\eta_I(\delta) = 1$.

Then both $\rho_\gamma(\delta) = 1$ and $F(\eta, v, h \upharpoonright \delta) = 1$, and, by the definition of F, $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure. On the other hand we have

Fact 1: $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 1\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure inside $\mathfrak{M}_\eta$.

But $h \upharpoonright \delta: \mathfrak{M}_\eta \to \mathfrak{M}_\nu$ is a K-embedding, by the choice of δ.

We now construct two models $\mathfrak{N}_1$, $\mathfrak{N}_2$ of cardinality λ such that:

(i) $\mathfrak{M}_{\nu^\frown\langle 0\rangle} < \mathfrak{N}_1$ and $\mathfrak{M}_{\eta^\frown\langle 0\rangle}$ is embeddable into $\mathfrak{N}_1$ by some h_0 extending $h \upharpoonright \delta$.

(ii) $\mathfrak{N}_2 \stackrel{\text{def}}{=} \mathfrak{M}_{\eta_J\upharpoonright\gamma}$ for some γ with $\delta + 1 < \gamma < \lambda^+$ and $\mathfrak{M}_{\eta^\frown\langle 1\rangle}$ is embeddable into $\mathfrak{N}_2$ by some mapping h_1 extending $h \upharpoonright \delta$.

To get (i) is trivial. To get (ii) we use Fact 1.

Fact 2. $\mathfrak{M}_{\nu^\frown\langle 0\rangle} < \mathfrak{N}_1$, and $\mathfrak{M}_\nu < \mathfrak{N}_1$ and $\mathfrak{M}_\nu < \mathfrak{N}_2$ have an amalgamating structure.

This follows from (i) and (ii) and the fact that our construction is based on a maximal counterexample.

But we have

Fact 3. $\mathfrak{M}_{\eta^\frown\langle 0\rangle} <_{h_0} \mathfrak{N}_1$, $\mathfrak{M}_{\eta^\frown\langle 1\rangle} <_{h_1} \mathfrak{N}_1$ and $h_0 \upharpoonright \delta = h_1 \upharpoonright \delta = h \upharpoonright \delta$.

Furthermore, since our construction is based on counterexamples to amalgamation, we have

Fact 4. $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 1\rangle}$ have no amalgamating structure.

But Facts 2 and 3 contradict Fact 4, which concludes the proof of the lemma.

2.7.8 Case 2: Construction of Models of Cardinality λ. We define by induction on $\alpha < \lambda^+$ models $\mathfrak{M}_\eta$ indexed by (0, 1)-sequences $\eta \in {}^\alpha 2$ such that:

(1) $\mathfrak{M}_\eta$ is isomorphic to the λ-superlimit $\mathfrak{M}$ and for the empty sequence $\langle\ \rangle$ we put $\mathfrak{M}_{\langle\ \rangle} = \mathfrak{M}$.

(2) If $\eta \subset \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.

(3) If $\delta \in \omega_1$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha<\delta} \mathfrak{M}_{\eta\upharpoonright\alpha}.$$

(4) For each η the structures $\mathfrak{N}$, $\mathfrak{M}_{\eta^\frown\langle 0\rangle}$, $\mathfrak{M}_{\eta^\frown\langle 1\rangle}$ have no amalgamating structure.

The definition of Case 2 is ready tailored for the construction of the $\mathfrak{M}_\eta$'s. The construction of models of cardinality λ^+ is the same as in Case 1.

2.7.9 Case 2: Counting the Models of Cardinality λ^+. If $\eta, \nu \in {}^{\lambda^+}2$, $\eta \neq \nu$, there is no K-embedding $f: \mathfrak{M}_\eta \to \mathfrak{M}$ such that $f \restriction \mathfrak{N} = \mathrm{id}$. For, otherwise, let α be minimal such that $\eta(\alpha) \neq \nu(\alpha)$ and put $\delta = \eta \restriction \alpha$. Then f would allow us to find an amalgamating structure for $\mathfrak{N}$, $\mathfrak{M}_{\delta\langle 0\rangle}$, $\mathfrak{M}_{\delta\langle 1\rangle}$.

So there are 2^{λ^+} many models of cardinality λ^+ which are not isomorphic over $\mathfrak{N}$. Since $\mathfrak{N}$ has cardinality λ there are at most $(\lambda^+)^\lambda = 2^\lambda$ many ways of interpreting $\mathfrak{N}$ in $\mathfrak{M}_\eta$. Since we assumed that $2^\lambda < 2^{\lambda^+}$, we conclude that $I(K, \lambda^+) = 2^{\lambda^+}$. This completes the proof of Theorem 2.7.3, and therefore the nonstructure theorem. The statement in the parentheses now follows with a subtle counting argument which we leave to the reader. □

2.7.10 Remark. If we just want to show that there are at least 2^λ many non-isomorphic models in K_{λ^+} we can use Proposition 2.5.10(ii) instead of 2.5.10(i) and simplify the proof a bit. We change the definition of the function F to be a function of four variables where the new variable ranges over the indexes of a list of $\mu < \mu^+ = 2^\lambda$ many non-isomorphic models of K. Instead of using first Ulam's theorem to partition λ^+ we can now apply Unif$(\lambda^+, \mu, 2, 2)$ directly.

3. ω-Presentable Classes

3.1. Classification Theory for ω-Presentable Classes

In this section we shall study some examples which illustrate that some of the classification theory of first-order model theory can be carried over to abstract classes, provided they are ω-presentable. For $\mathscr{L}_{\omega_1\omega}$ this was initiated by G. Cudnovskii, J. Keisler, and S. Shelah, cf. Keisler [1971] and was carried out to considerable extent in Shelah [1984a, b, c]. It seems that, with enough effort and ingenuity, many results should be provable, in some form or another, also for λ-presentable classes. This is still in the making, but we think that this direction of future research is among the most challenging tasks of "higher model theory."

The first two theorems along these lines are direct descendants of two theorems in Shelah [1975c]. The proofs, which we are going to sketch, appear, in this streamlined form, here for the first time in print.

3.1.1 Theorem (Shelah's Reduction Theorem). *Let K with $<_K$ be an abstract ω-presentable class over a vocabulary τ such that:*

$$I(K, \omega_1) < 2^{\omega_1}.$$

Then there is a ω-presentable abstract class K with $<_{K'}$ over a vocabulary τ', $\tau \subset \tau'$ such that

(i) *if* $\mathfrak{A} \in K'$ *then* $\mathfrak{A} \upharpoonright \tau \in K$;
(ii) *if* $\mathfrak{A}, \mathfrak{B} \in K'$ *and* $\mathfrak{A} <_{K'} \mathfrak{B}$ *then* $\mathfrak{A} \upharpoonright \tau <_K \mathfrak{B} \upharpoonright \tau$;
(iii) *if* $\mathfrak{A}, \mathfrak{B} \in K'$ *and* $\mathfrak{A} <_{K'} \mathfrak{B}$ *then* $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$; *and still*
(iv) $I(K, \omega_1) \neq \varnothing$ *iff* $(K', \omega_1) \neq \varnothing$.

In particular, $I(K', \omega) = 1$ *by* (iii).

Recall that $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$ here means that for every finite set of constant symbols A_0 the expansion $\langle \mathfrak{A}, A_0 \rangle \equiv \langle \mathfrak{B}, A_0 \rangle$ in the logic $\mathscr{L}_{\infty\omega}$.

3.1.2 Remarks. (i) The reduction theorem allows us to construct Scott sentences of uncountable structures. We shall return to this in Section 3.4.
(ii) In Shelah [1975c] the reduction theorem is proved by constructing what is called there "nice" sentences.
(iii) In the reduction theorem above, we can replace the assumption

$$I(K, \omega_1) < 2^{\omega_1}$$

by the assumption that K has arbitrary large models and Löwenheim number ω, and get the same result.

3.1.3 Theorem (Shelah's Abstract ω_1-Categoricity Theorem, 1977). *Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) = 1$.

Then $I(K, \omega_2) \neq \varnothing$.

3.1.4 Corollary (Shelah). *Let φ be a sentence of the logic $\mathscr{L}_{\omega_1\omega}(Q_1)$ which has exactly one model of cardinality ω_1. Then φ has a model of cardinality ω_2.*

3.1.5 Historical Remark. Corollary 3.1.4 shows that there are no theories in $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$ which have exactly one uncountable model. This had been asked by J. T. Baldwin (Friedman [1975c]) and actually was the origin of Theorem 3.1.3. In Shelah [1975c, Corollary 3.1.4] was proved with the additional set-theoretic hypothesis $\diamondsuit$, and in Shelah [1983b, c] under the hypothesis $2^{\omega} < 2^{\omega_1}$. Without any set-theoretic hypothesis Corollary 3.1.4 was proved by S. Shelah in 1976 (my personal notes).

3.1.6 Theorem (Shelah 1977). *Assume that $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $1 \leq I(K, \omega_1) < 2^{\omega_1}$.

Then $I(K, \omega_2) \neq \varnothing$.

Assumption (i) in the two theorems above is not essential. Though it does not follow from (ii), we can always replace K satisfying (ii) by K which also satisfies (i) using the reduction theorem.

The main tool in the proof of Theorem 3.1.6 is the use of a ω_1-superlimit. The concept of superlimit models was introduced with generalizations in mind. The following theorem guarantees its existence. If our only purpose was to prove Theorem 3.1.6 we could also avoid the construction of superlimits. S. Fuchino [1983] has presented such a direct proof.

3.1.7 Theorem (Existence of Superlimits). *Assume that* $2^\omega < 2^{\omega_1} < 2^{\omega_2}$. *Let* K *with* $<_K$ *be an abstract* ω*-presentable class such that*:

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then there is a ω_1*-superlimit model* $\mathfrak{M}$ *in* K_{ω_1} *which is homogeneous and universal.*

Clearly, Theorem 3.1.6 follows from Theorem 3.1.7 together with Proposition 2.2.3. Actually we shall only need that there is a weak limit in ω_1. We shall give a narrative account of the proof of Theorem 3.1.7 in Section 3.5. The existence of a weak limit in ω_1 will be proved as Claim 3.5.2.

3.1.8 Corollary. *Assume* K *is as in the theorem above. Then the* ω_1*-superlimit model* $\mathfrak{M}$ *is an amalgamation basis for* K_{ω_1}.

Proof. Use the non-structure theorem (2.3.1) together with Theorem 3.1.7. □

This corollary is somehow not satisfactory. What we really would like to obtain is the following conjecture:

3.1.9 Conjecture (ω_1-Amalgamation Conjecture). Assume that $2^\omega < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:

(i) $I(K, \omega) = 1$; and
(ii) $I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then K_{ω_1} has the amalgamation property.

Note that this conjecture follows from Conjecture 2.3.2.

In the remainder of this section we shall prove Theorems 3.1.1 and 3.1.3 completely, and sketch the proof of Theorem 3.1.7, from which Theorem 3.1.6 follows.

We conclude this section with the statement of the main theorem of the classification theory for $\mathscr{L}_{\omega_1\omega}$ (Shelah [1984a, b]) and a conjecture on how this should generalize for ω-presentable classes.

3.1.10 Theorem (Shelah's Classification Theorem for $\mathscr{L}_{\omega_1\omega}$). *Assume $2^{\omega_n} = \omega_{n+1}$ for every $n < \omega$. Let $K = \text{Mod}(\psi)$ for some sentence $\psi \in \mathscr{L}_{\omega_1\omega}$. If K has an uncountable model then at least one of the following is true. Either*:

(i) *for some $n > 0$ $I(K, \omega_1) = 2^{\omega_n}$; or*
(ii) *K has models in every infinity cardinality, and if it is categorical in some $\lambda > \omega_1$ then it is categorical in every $\mu \geq \omega_1$.*

3.1.11 Remark. Theorem 3.1.10 is not true, when we replace K by some $\text{PC}_{\mathscr{L}_{\omega_1\omega}}$-class. To see this consider the class K of structures (with equality only) of cardinality at most ω_1. Clearly K is categorical in every infinite power and has no models bigger than ω_1. Using the fact that the natural numbers are characterizable in $\mathscr{L}_{\omega_1\omega}$, one easily sees that the class of ω_1-like orderings in $\text{PC}_{\mathscr{L}_{\omega_1\omega}}$. Therefore also $K \in \text{PC}_{\mathscr{L}_{\omega_1\omega}}$. For a discussion of categoricity in $\mathscr{L}_{\omega_1\omega}$ see Keisler [1971, p. 91ff.].

For generalizations of Theorem 3.1.10 we shall finally discuss several conjectures:

3.1.12 Conjecture (Shelah). If an abstract ω-presentable class K has one uncountable model then it has at least 2^{ω_1} many non-isomorphic uncountable models.

3.1.13 Comments. Possibly one has to use some set-theoretic hypothesis such as in the classification theorem for $\mathscr{L}_{\omega_1\omega}$, or $2^{\omega_n} < 2^{\omega_{n+1}}$ for every $n \in \omega$ to prove this conjecture. Theorem 3.1.7 was proved in Shelah [198?c] as a basis for a proof of Conjecture 3.1.12. A special case of this conjecture consists in showing, for example, that if K has exactly one model in ω_2 then it has a model in ω_3. As we shall see in the next section, however, there is one application of the non-characterizability of well-orderings (Theorem 3.2.1), which cannot be adapted in an obvious way: We cannot prove that the superlimit $\mathfrak{M}$, whose existence is stated in Theorem 3.1.7, can be embedded into itself such that it forms a dense pair as defined in the next section (Definition 3.2.4). Only a deeper analysis of the types realized in models in K reveals that such dense pairs do not exist. What one really does is more in the spirit of stability theory, than in the original spirit of abstract model theory. But it seems that this is where the future lies: To use the concepts and methods of stability theory in the framework of abstract classes. The following remarks show, however, that this is more complicated than one might be ready to believe at first glance.

Next we look at the logic $\mathscr{L}_{\omega\omega}(\text{pos})$, which was introduced in Chapter II, and its infinitary extensions $\mathscr{L}_{\omega_1\omega}(\text{pos})$. These logics were studied in Makowsky–Shelah [1981] and Makowsky [1978a]. $\mathscr{L}_{\omega\omega}(\text{pos})$ is a countably compact extension of $\mathscr{L}_{\omega\omega}(Q_1)$ which is properly contained in $\mathscr{L}_{\omega\omega}(\text{aa})$. The reader may also want to consult Chapter IV.

3.1.14 Conjecture (Classification Theorem for $\mathscr{L}_{\omega_1\omega}(\text{pos})$, Makowsky–Shelah)). Assume $2^{\omega_n} = \omega_{n+1}$ for every $n < \omega$. Let $K = \text{Mod}(\psi)$ for some sentence

$$\psi \in \mathscr{L}_{\omega_1\omega}(\text{pos}).$$

If K has an uncountable model then at least one of the following is true. Either:

(i) for some $n > 0$, $I(K, \omega_n) = 2^{\omega_n}$; or
(ii) K has models in every infinite cardinality, and if it is categorical in some $\lambda > \omega_1$ then it is categorical in every $\mu \geq \omega_1$.

3.1.15 Remarks. (i) The straightforward notions of stability theory (Shelah [1978a]) do not adapt readily to our situation. In fact, it is consistent with ZFC + $2^\omega = \omega_2$ that there is an ω-presentable abstract class which is categorical in ω_1 but is unstable. Also all its models are of cardinality at most 2^ω. Take the $\mathscr{L}_{\omega\omega}$(pos) sentence which says that $<$ is a dense linear order with no first or last element, that each interval is uncountable, but that there is a dense countable subset. Categoricity in ω_1 follows from Baumgartner [1973], the bound on the cardinality of the models and instability are obvious.

(ii) Conjecture 3.1.14 becomes false for $\mathscr{L}_{\omega\omega}$(aa): There is a sentence

$$\psi \in \mathscr{L}_{\omega\omega}(\mathrm{aa})$$

such that ψ has, up to isomorphism, exactly one model and this model is of cardinality ω_1. To see this, let ψ be the sentence which says that $<$ is a dense linear order with no first or last element, each initial segment is countable, but the model is not, and aa$s$ $\exists x \, \forall y(s(y) \leftrightarrow y < x)$. The only model of ψ is, up to isomorphism, the structure $\langle \eta \times \omega_1, < \rangle$. (See also Remark IV.4.1.2(v).)

(iii) The analogue of Theorem 3.1.6 for $\mathscr{L}_{\omega_1\omega}$(pos) has been proved in Makowsky–Shelah [198?a]. At the time of completion of this chapter, this paper was still in the process of being checked.

3.2. Extensions With and Without First Elements

Let K with $<_K$ be an abstract ω-presentable class such that: (i) $I(K, \omega) = 1$, and (ii) $I(K, \omega_1) = 1$. We want to show that $I(K, \omega_2) \neq \emptyset$. For this purpose we show first:

3.2.1 Lemma. *Under the above hypotheses the following are equivalent*:

(i) $I(K, \omega_2) \neq \emptyset$.
(ii) *There are* $\mathfrak{A}, \mathfrak{B} \in K_{\omega_1}$ *such that* $\mathfrak{A} < \mathfrak{B}$ *and* $\mathfrak{A} \neq \mathfrak{B}$, *i.e.*, $\mathfrak{A} \in K_{\omega_1}$ *is not maximal.*

Proof. (i) → (ii) We just apply Axiom 5.

(ii) → (i) Since here $\mathfrak{A} \cong \mathfrak{B}$ we can construct a K-chain of length ω_2 which gives us the required model. □

3.2.2 Definition. A structure $\mathfrak{A}$ in an abstract class K (K_λ) is *K-maximal* (*K_λ-maximal*), if there is no $\mathfrak{B} \in K$ ($\mathfrak{B} \in K_\lambda$) such that $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$.

In this section we write $\mathfrak{A} < \mathfrak{B}$ only for proper extensions, and we shall use $\mathfrak{A} \leq \mathfrak{B}$ if we allow also the identity.

What we really prove to get Theorem 3.1.3 is the following:

3.2.3 Theorem. *Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$;
(ii) $I(K, \omega_1) \neq 0$; *and*
(iii) *every $\mathfrak{A} \in K_{\omega_1}$ is K_{ω_1}-maximal.*

Then $I(K, \omega_1) = 2^{\omega_1}$.

Clearly, in the above situation, the structures in K_ω are not maximal, since there is an uncountable model.

3.2.4 Definitions. Let K with $<$ be an abstract class, λ a cardinal, and $\mathfrak{A} < \mathfrak{B}$ with $b \in B - A$ and $\mathfrak{A}, \mathfrak{B} \in K_\lambda$.

(i) We say that b is *a first element for* $\mathfrak{A} < \mathfrak{B}$ if for every $\mathfrak{A}_1, \mathfrak{B}_1$ such that $\mathfrak{A} < \mathfrak{A}_1 < \mathfrak{B}_1, \mathfrak{B} < \mathfrak{B}_1$ we have that $b \in A_1$. (We assume here for simplicity that the embeddings are the identity. The reader can easily formulate the definition for the more general case.)

Note that, if there is no first element for $\mathfrak{A} < \mathfrak{B}$, we can think of this as an amalgamation property: For every $b \in B - A$ there is a structure $\mathfrak{A}_1 \in K$ and an amalgamating structure $\mathfrak{B}_1$ such that $b \notin A_1$. If there is no first element for $\mathfrak{A} < \mathfrak{B}$, this can happen in a strong form:

(ii) We say that $\mathfrak{A} < \mathfrak{B}$ is *a dense pair* if for every $b \in B - A$ there is a structure $\mathfrak{A}_1$ in K_λ such that $\mathfrak{A} < \mathfrak{A}_1 < \mathfrak{B}$ with $b \in B - A_1$.

The above definitions are our key tools in the proof of Theorem 3.2.3 and therefore of the abstract categoricity theorem.

3.2.5 Example. To illustrate the proof idea let us recall a simple theorem about the number of non-isomorphic dense linear orderings of cardinality ω_1. We take here K_{end} to be the class of all dense linear orderings without extremal elements, and define for $\mathfrak{A}, \mathfrak{B} \in K_{\text{end}}$ the substructure relation $\mathfrak{A} <_{\text{end}} \mathfrak{B}$ as the end-extensions. Clearly $K_{\text{end}, \omega}$ has, up to isomorphism, only one element.

3.2.6 Proposition. *There are 2^{ω_1} many non-isomorphic linear dense ω_1 like orderings.*

Proof. Let $I \subset \omega_1$. We define $\mathfrak{A}_I = \bigcup_{\alpha \in \omega_1} \mathfrak{A}_\alpha$ where each $\mathfrak{A}_\alpha$ is isomorphic to a copy of the rationals $\mathfrak{Q} = \langle Q, < \rangle$. Let $\mathfrak{Q}_{\text{first}} = \langle [b, 1), < \rangle$ be a copy of the rationals with a first element b and put $\mathfrak{Q}_1 = \mathfrak{Q} + \mathfrak{Q}_{\text{first}}$ and $\mathfrak{Q}_2 = \mathfrak{Q} + \mathfrak{Q}$. Clearly $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_2$ is a dense pair and b is a first element for $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_1$. Now we put $\mathfrak{A}_0 = \mathfrak{Q}$ and $\mathfrak{A}_\delta = \bigcup_{\alpha < \delta} \mathfrak{A}_\alpha$ for δ a limit ordinal. To get $\mathfrak{A}_{\alpha+1}$ we make $\mathfrak{A}_\alpha <_{\text{end}} \mathfrak{A}_{\alpha+1}$ isomorphic to $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_1$ if $\alpha \in I$ and isomorphic to $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_2$ if $\alpha \notin I$.

Let $I, J \subset \omega_1$ and F be the c.u.b. filter on ω_1. We claim that $\mathfrak{A}_I \simeq \mathfrak{A}_J$ implies that $I = J$ (mod F). By Ulam's theorem (cf. Theorem 2.5.1 or Lemma XVIII.4.3.9) there are 2^{ω_1} many non-equivalent stationary subsets of ω_1, hence the result. ☐

The next lemmas will allow us to copy this proof for our abstract classes.

3.2.7 Lemma. *Let K with < be an abstract class with Löwenheim number λ and*

(i) $I(K, \lambda) = 1$;
(ii) $I(K, \lambda^+) \neq \emptyset$;
(iii) *every* $\mathfrak{A} \in K_{\lambda^+}$ *is K-maximal.*

Then there are $\mathfrak{A}, \mathfrak{B} \in K_\lambda$ *and* $b \in B - A$ *such that b is a first element for* $\mathfrak{A} < \mathfrak{B}$.

In other words, if no pair $\mathfrak{A} < \mathfrak{B}$ *of structures from* K_λ *has a first element, then there is a non-maximal* $\mathfrak{A}_1 \in K_{\lambda^+}$.

Proof. Assume for contradiction that $\mathfrak{A}_0 < \mathfrak{B}_0$ are given in K_λ with no $b \in B_0 - A_0$ a first element. Fix $b_0 \in B_0 - A_0$. So there are $\mathfrak{A}_1 < \mathfrak{B}_1$ with $\mathfrak{A}_0 < \mathfrak{A}_1$ and $\mathfrak{B}_0 < \mathfrak{B}_1$ and $b_0 \in B_1 - A_1$.

From this situation we construct K-chains $\mathfrak{A}_\alpha, \mathfrak{B}_\alpha$ ($\alpha < \lambda^+$) with $b_0 \in B_\alpha - A_\alpha$, using that $I(K, \lambda) = 1$. Now we put $\mathfrak{A} = \bigcup_\alpha \mathfrak{A}_\alpha$ and $\mathfrak{B} = \bigcup_\alpha \mathfrak{B}_\alpha$ and find that $\mathfrak{A}, \mathfrak{B} \in K_{\lambda^+}$, $\mathfrak{A} < \mathfrak{B}$ and $b_0 \in B - A$. ☐

3.2.8 Lemma. *Let K with* $<_K$ *be an abstract* ω*-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) \neq \emptyset$.

Then there is a dense pair $\mathfrak{A} < \mathfrak{B}$ in K_ω.

The proof of Lemma 3.2.8 consists in an application of the Morley–Lopez-Escobar theorem on the non-expressibility of well-orderings in $\mathscr{L}_{\infty\omega}$ which was first used in Shelah [1975]. We shall return to this in Section 3.3.

Proof of Theorem 3.2.3. We are now in a position to copy the proof of Proposition 3.2.6. We put now $\mathfrak{Q}$ to be the only countable model of K, $\mathfrak{Q}_1$ a countable extension of $\mathfrak{Q}$ with $b \in Q_1$ a first element (Lemma 3.2.6), and $\mathfrak{Q}_2$ a countable extension of $\mathfrak{Q}$ such that $\mathfrak{Q} <_K \mathfrak{Q}_2$ is a dense pair (Lemma 3.2.8). The rest of the argument remains unchanged. ☐

3.3. *Some Model Theory for* $L_{\omega_1\omega}$

In Section 1.3 Shelah's presentability theorem tells us that every ω presentable class K is actually a PC-class in $\mathscr{L}_{\omega_1\omega}$. Some of the model theory of $\mathscr{L}_{\omega_1\omega}$ has been developed in Chapter VIII, but for the reader's sake we make this section as self-contained as possible. Our aim here is to prove Lemma 3.2.8 and Shelah's reduction theorem (3.1.1). Both theorems use heavily the non-characterizability of the class of well-orderings as a PC-class in $\mathscr{L}_{\omega_1\omega}$, which we state here precisely (cf. Section VIII.1.3, Section II.5.2 and Proposition IX.3.2.16)

3.3.1 Theorem (Non-characterizability of Well-Orderings). *Let* $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$ *and let* $U, < \in \tau$ *be a unary and a binary relation symbol of* τ. *Suppose that for each* $\alpha \in \omega_1$, φ *has a model* $\mathfrak{A} = \langle A, U, <, \ldots \rangle$ *such that* $<$ *linearly orders* U *and* $\langle \alpha, < \rangle \subset \langle U, < \rangle$. *Then*:

(i) φ *has a countable model* $\mathfrak{B} = \langle B, V, <, \ldots \rangle$ *such that* $<$ *linearly orders* V *and* $\langle V, < \rangle$ *contains a copy of the rationals* $\langle Q, < \rangle$;
(ii) φ has an uncountable model $\mathfrak{B} = \langle B, V, <, \ldots \rangle$ *such that* $<$ *linearly orders* V *and* $\langle V, < \rangle$ *contains a copy of the rationals* $\langle Q, < \rangle$.

(i) is due to Morley [1965] and Lopez-Escobar [1966]. A proof may be found in Keisler [1971a]. (ii) can be proved by combining (i) with the construction and characterization of the existence of suitable end-extensions, as described in Keisler [1971a]. But it was Shelah who first observed that this theorem can be used in many situations as a substitute for compactness. This is the main theme of this section. We shall use Theorem 3.3.1(ii) to construct, in certain situations, Scott sentences of uncountable models, and also, if such Scott sentences exist, to construct dense pairs of countable models. Let us recall some definitions:

3.3.2 Definition (Scott Sentences). (i) Let $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$. We say that φ is a *Scott sentence*, if all models of φ are $\mathscr{L}_{\infty\omega}$-equivalent.
(ii) Let $\varphi \in \mathscr{L}_{\omega_1\omega}(\tau')$ and $\tau \subset \tau'$. We say that φ is a *weak Scott sentence* (*for* τ), if all τ-reducts of models of φ are $\mathscr{L}_{\infty\omega}$-equivalent.
(iii) If $\mathfrak{A}$ is a τ-structure then we say that $\mathfrak{A}$ *has a Scott sentence*, if there is a Scott sentence $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$ with $\mathfrak{A} \models \varphi$. Similarly for weak Scott sentences.
(iv) If $\mathfrak{A}$ is a τ-structure which has a (weak) Scott sentence φ, we denote by $\sigma(\mathfrak{A})$ a formula logically equivalent to φ.

In Theorem VIII.4.1.6 these definitions are justified.

3.3.3 Lemma. *If a* τ*-structure* $\mathfrak{A}$ *has a weak Scott sentence* σ_w *over a vocabulary* τ' *then it has also a Scott sentence* σ.

Proof. Let $\mathfrak{B}'$ be a countable model of σ_w and $\mathfrak{B} = \mathfrak{B}' \upharpoonright \tau$. Put $\sigma = \sigma(\mathfrak{B})$. By the completeness theorem for $\mathscr{L}_{\omega_1\omega}$ $\sigma_\omega \models \sigma$, so $\mathfrak{A} \models \sigma$. $\square$

3.3.4 Definition. (i) (Fragments of $\mathscr{L}_{\omega_1\omega}$). A *countable fragment* $\mathscr{L}$ *of* $\mathscr{L}_{\omega_1\omega}$ is a countable subset of $\mathscr{L}_{\omega_1\omega}$ closed under taking subformulas, name changing, applying the finitary connectives and quantification.
(ii) ($\mathscr{L}$-embeddings). Let $\mathscr{L}$ be a fragment of $\mathscr{L}_{\omega_1\omega}$ and $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We say that $\mathfrak{A}$ *is an* $\mathscr{L}[\tau]$*-substructure of* $\mathfrak{B}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every finite subset $A_0 \subset A$ the expansions by constants for elements of A_0, $\langle \mathfrak{A}, A_0 \rangle$ and $\langle \mathfrak{B}, A_0 \rangle$, are $\mathscr{L}$-equivalent.
(iii) (Karp Substructures). Let $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We say that $\mathfrak{A}$ *is a Karp-substructure of* $\mathfrak{B}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every finite subset $A_0 \subset A$ the expansions by constants for elements of A_0, $\langle \mathfrak{A}, A_0 \rangle$ and $\langle \mathfrak{B}, A_0 \rangle$, are $\mathscr{L}_{\infty\omega}$-equivalent.

(iv) (ω-Presentable Substructure Relation). Let $\propto$ be a binary relation between τ-structures such that $\mathfrak{A} \propto \mathfrak{B}$ implies that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. We say that $\propto$ *is an ω-presentable substructure relation, if*:

(a) for every $\mathfrak{A}$ we have $\mathfrak{A} \propto \mathfrak{A}$;
(b) $\propto$ satisfies the transitivity axiom;
(c) $\propto$ satisfies the chain axiom; and
(d) the class of τ_{sr}-structures $[\mathfrak{A}; \mathfrak{B}]$ such that $\mathfrak{A} \propto \mathfrak{B}$ is PCOT(ω, ω).

Obviously we define τ_{sr} such that the universe of $\mathfrak{A}$ is the interpretation of a distinguished unary predicate of τ_{sr}. Note that (d) ensures that we have Lowenheim number ω.

3.3.5 Lemma. (i) *Let $\mathscr{L}$ be a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Then the notion of a $\mathscr{L}$-substructure gives rise to an ω-presentable substructure relation.*

(ii) *The notion of a* Karp-*substructure is also an ω-presentable substructure relation.*

Proof. Both statements are easy coding exercises. For (i) we use the truth adequacy of $\mathscr{L}_{\omega_1\omega}$ for countable fragments. Details are discussed in Section XVII.1. For (ii) we use the characterization of $\mathscr{L}_{\infty\omega}$-equivalence in terms of partial isomorphisms, as described in Section II.4 and Chapter VIII. The questions which interest us now, are whether an uncountable structure $\mathfrak{A}$ has a (weak) Scott sentence, and under what conditions a Scott sentence has uncountable models? The following is a variation on a special case of Theorem XVIII.7.3.1, which is due to Gregory [1973].

3.3.6 Theorem. *Let φ be a weak Scott sentence. Then the following are equivalent*:

(i) *φ has an uncountable model*;
(ii) *for every countable fragment $\mathscr{L}$ containing φ there are countable models $\mathfrak{B}$, $\mathfrak{C}$ of φ such that $\mathfrak{B} \precneqq_{\mathscr{L}} \mathfrak{C}$ and $\mathfrak{B} \cong \mathfrak{C}$*;
(iii) *for every ω-presentable substructure relation $\propto$ there are countable models $\mathfrak{B}$, $\mathfrak{C}$ of φ such that $\mathfrak{B} \propto \mathfrak{C}$, $\mathfrak{B}$ is a proper substructure of $\mathfrak{C}$ and $\mathfrak{B} \cong \mathfrak{C}$.*

Proof. (ii) implies (i) trivially (in contrast to the proof of Theorem XVIII.7.3.1), since $\mathfrak{A} \cong \mathfrak{B}$ allows us to construct a chain of length ω_1 whose limit is the desired model.

(iii) implies (ii) by the lemma above.

So assume (i). To prove (iii) we just use the reflexivity of $\propto$ and the Lowenheim–Skolem theorem for $\mathscr{L}_{\omega_1\omega}$ together with the properties of the weak Scott sentence. ☐

For weak Scott sentences with uncountable models we can already construct dense pairs for any countable fragment $\mathscr{L}$ of $\mathscr{L}_{\omega_1\omega}$.

3.3.7 Theorem. *Let φ be a weak Scott sentence with an uncountable model $\mathfrak{D}$ and $\propto$ an ω-presentable substructure relation. Then there are two countable τ-structures $\mathfrak{B}$, $\mathfrak{C}$ such that $\mathfrak{B} \propto \mathfrak{C}$ is a dense pair for $\propto$.*

Proof. We can write $\mathfrak{D}$ as the union of a $\propto$-chain of length ω_1 $\{\mathfrak{D}_\alpha : \alpha \in \omega_1\}$ with $\mathfrak{D}_\alpha \propto \mathfrak{D}_\beta$ for every $\alpha < \beta < \omega_1$. We can code this situation in a model $\mathfrak{M}$ and describe it by a formula $\vartheta \in \mathscr{L}_{\omega_1\omega}[\tau']$ over some vocabulary τ' extending τ, which satisfies the hypothesis of Theorem 3.3.1. Here we use the ω-presentability of $\propto$. The universes of the models $\mathfrak{D}_\alpha$ are coded by a binary predicate symbol and constants $R(-, c_\alpha)$. The second argument of R ranges over some linearly ordered set $\langle U, < \rangle$, the index set.

Now we apply Theorem 3.3.1(ii) and get a model $\mathfrak{N}$ such that a copy of the rationals $\langle Q, < \rangle$ can be embedded into the index set. Let $\{d_n : n \in \omega\}$ be a decreasing sequence in $\mathfrak{N}$ and d be a lower bound for it. Put now $\mathfrak{C}$ to be the model defined in $\mathfrak{N}$ by $R(-, d_0)$ and $\bigcup_{a<d_n} R(-, a) = \bigcap_{n\in\omega} R(-, d_n) = \mathfrak{B}$. This is not empty, since the structure defined by $R(-, d)$ is contained in it. Clearly, ϑ can be chosen such that $\mathfrak{B} \propto \mathfrak{C}$ is a dense pair. ☐

3.3.8 Proof of lemma 3.2.8. Our first step in the proof is the construction of a Scott sentence. So let K be an ω-presentable class with $K_{\omega_1} \neq \varnothing$ and $I(K, \omega_1) < 2^{\omega_1}$. Then there is a $\mathfrak{B} \in K_{\omega_1}$ which has a weak Scott sentence σ. To see this, we apply Shelah's reduction theorem (3.1.1) to K. So let K' be as in Theorem 3.1.1 and let $\mathfrak{B} \in K'_{\omega_1}$. Since K' is ω-presentable, there is a countable $\mathfrak{A} \in K'$ with $\mathfrak{A} <_{K'} \mathfrak{B}$ and therefore $\mathfrak{A} <_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$. Let $\sigma = \sigma(\mathfrak{A})$. Clearly, $\mathfrak{B} \models \sigma$. Now the lemma follows from Theorem 3.3.7 ☐

3.4. *Constructing Scott Sentences for Uncountable Models*

Our second application of Theorem 3.3.1(ii) is the proof of the reduction theorem (3.1.1). First we need a lemma on the minimal number of types realized in models in K_{ω_1}. Let us recall the definition of types.

3.4.1 Definition. (i) ($\mathscr{L}[\tau]$-types). Let $\mathfrak{M}$ be a τ-structure, $A \subset M$ a subset of the universe of $\mathfrak{M}$, $\bar{a} \in M^m$ and let $\varphi(\bar{x})$ range over $\mathscr{L}[\tau]$-formulas with all the free variables among $\bar{x} = (x_0, x_1, \ldots)$. For $b \in A$ let **b** be a constant symbol whose interpretation in $\mathfrak{M}$ is b. We define

$$\mathrm{tp}(\bar{a}, A, \mathscr{L}, \mathfrak{M}) = \{\varphi(\bar{x}, \mathbf{b}) : \varphi \in \mathscr{L}(\tau), \mathfrak{M} \models \varphi(\bar{x}, \bar{y})[\bar{a}, \bar{b}], \bigwedge \bar{b} \in A^n\}$$

be the *m-type of $\bar{a}$ in $\mathfrak{M}$ over A*.

(ii) If $t = \mathrm{tp}(\bar{a}, A, \mathscr{L}, \mathfrak{M})$ is a countable type we define by ϑ_t the conjunction of all the formulas of t. Note that ϑ_t is not necessarily a formula of $\mathscr{L}$.

3.4.2 Lemma. *Let $\tau \subset \tau'$, $\psi \in K_{\omega_1\omega}[\tau']$ and $\mathscr{L}$ a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Put* $K = \mathrm{Mod}(\psi) \upharpoonright \tau$. *Then*:

(i) (Keisler [1970, Theorem 5.10]). *If in some uncountable model $\mathfrak{M}$ of ψ uncountably many $\mathscr{L}[\tau]$-types are realized, then* $I(K, \omega_1) = 2^{\omega_1}$.

(ii) (Shelah). *Here we assume* $2^{\omega_1} > 2^{\omega}$. *If in some uncountable model* $\mathfrak{M}$ *of* ψ *there is a countable subset* $A \subset M$ *such that in* $\mathfrak{M}$ *uncountably many* $\mathscr{L}[\tau]$-*types are realized over* A, *then* $I(K, \omega_1) = 2^{\omega_1}$.

Proof. To prove (ii) from (i) we observe that there are at most $(\omega_1)^{\omega} = 2^{\omega}$ many ways of interpreting countably many constants in a model of cardinality ω_1. More details may be found in Shelah [1978a, Chapter 8, Lemma 1.3]. □

The next theorem extends this to $\mathscr{L}_{\omega_1\omega}$ proper.

3.4.3 Theorem (Shelah). *Let* $\tau \subset \tau'$ *be two vocabularies,* $\psi \in \mathscr{L}_{\omega_1\omega}[\tau']$ *a formula, and* $\mathfrak{M}$ *be a* τ'-*structure of cardinality* ω_1 *such that* $\mathfrak{M} \models \psi$.

(i) *If for every countable fragment* $\mathscr{L}$ *only countably many* $\mathscr{L}[\tau]$-*types are realized in* $\mathfrak{M}$, *then* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized.*

(ii) *If for every countable fragment* $\mathscr{L}$ *and for every countable subset* $A \subset M$ *only countably many* $\mathscr{L}[\tau]$-*types are realized in* $\mathfrak{M}$ *over* A, *then* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which over every countable* $A \subset N$ *only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized over* A.

(iii) *If* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which over every countable* $A \subset N$ *only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized over* A, *then* $\mathfrak{N} \upharpoonright \tau$ *has a Scott sentence* $\sigma = \sigma(\mathfrak{N} \upharpoonright \tau)$.

Proof. (i) For every $\alpha < \omega_1$ we define a countable fragment $\mathscr{L}_\alpha$ of $\mathscr{L}_{\omega_1\omega}$. $\mathscr{L}_0 = \mathscr{L}_{\omega\omega}$ and $\mathscr{L}_\delta = \bigcup_{\beta<\delta} \mathscr{L}_\beta$ for δ a limit ordinal. $\mathscr{L}_{\alpha+1}$ is the minimal fragment of $\mathscr{L}_{\omega_1\omega}$ containing $\mathscr{L}_\alpha$ and for every $\bar{a} \in M^m$ the formula $\vartheta_{t(\bar{a})}$ where $t(\bar{a}) = \text{tp}(\bar{a}, \varnothing, \mathscr{L}_\alpha, \mathfrak{M})$. Clearly, for every $\alpha < \omega_1$ the fragment $\mathscr{L}_\alpha$ is indeed countable. Let τ'' be $\tau' \cup \{\mathbf{C}_n, \mathbf{F}_n : n \in \omega\}$. We now expand $\mathfrak{M}$ to a τ''-structure $\mathfrak{M}''$ in the following way:

First we assume without loss of generality that $M = \omega_1$. Now

$$\mathfrak{M}'' = \langle \mathfrak{M}, <, E_0, \ldots, E_n, \ldots, F_0, \ldots, F_n, \ldots \rangle_{n \in \omega},$$

where

(a) $<$ is the natural ordering on ω_1.

(b) E_n is a $(2n+1)$-ary relation and $(\alpha, \bar{a}, \bar{b}) \in E_n$ iff $\bar{a}, \bar{b} \in M_n$ and

$$\text{tp}(\bar{a}, \varnothing, \mathscr{L}_\alpha, \mathfrak{M}) = \text{tp}(\bar{b}, \varnothing, \mathscr{L}_\alpha, \mathfrak{M}).$$

(c) F_n is an $(n+1)$-ary function with the finite ordinals as its range and $F_n(\alpha, \bar{a}) = F_n(\alpha, \bar{b})$ iff $(\alpha, \bar{a}, \bar{b}) \in E_n$. Such an F_n can be chosen because the number of $\mathscr{L}_\alpha$-types realized in $\mathfrak{M}$ is countable by our hypothesis.

We note the following facts:

Fact 1. Every E_n defines a family of equivalence relations $E_{\alpha,n}$ on n-tuples of $\mathfrak{M}$ indexed by the first argument.

Fact 2. If $\alpha < \beta$ then $E_{\beta,n}$ refines $E_{\alpha,n}$.

Fact 3. Each $E_{\alpha,n}$ has at most countably many equivalence classes.

Fact 4. $<$ is an ordering with a first element, which we call 0, and $(0, \bar{a}, \bar{b}) \in E_n$ iff the $\mathscr{L}_0$-types of $\bar{a}$ and $\bar{b}$ are equal.

Fact 5. If $(\alpha + 1, \bar{a}, \bar{b}) \in E_n$ then for every $c \in M$ there is a $d \in M$ such that

$$(\alpha, \bar{a}, c, \bar{b}, d) \in E_{n+1}.$$

Clearly, Facts 1–5 can be expressed by a sentence $\chi \in \mathscr{L}_{\omega_1\omega}[\tau'']$. To express Fact 3 we need the functions F_n.

Now we apply Theorem 3.3.1(ii) to the sentence $\psi \wedge \chi$. We get a model $\mathfrak{N}'' \models \psi \wedge \chi$ of cardinality ω_1 where $<$ contains a copy of the rationals. Put $\mathfrak{N} = \mathfrak{N}'' \upharpoonright \tau$. Let $\{d_n: n \in \omega\}$ be an infinite decreasing sequence of elements in $\mathfrak{N}''$. We use it to define equivalence relations E_n^+ on n-tuples of $\mathfrak{N}''$ by putting

$$(\bar{a}, \bar{b}) \in E_n^+ \text{ iff } \mathfrak{N}'' \models \mathbf{C}_n(d_m, \bar{a}, \bar{b})$$

for some $m \in \omega$. It is easy to check, that for this equivalence relation we have

Fact 6. If $(\bar{a}, \bar{b}) \in E_n^+$ then for every $c \in N$ there is a $d \in N$ such that $(\bar{a}, c, \bar{b}, d) \in E_{n+1}^+$; and

Fact 7. Each $E_{\alpha,n}$ has at most countably many equivalence classes.

We just use the fact that $\mathfrak{N}'' \models \psi \wedge \chi$ and the definition of E_n^+.

Using Fact 6 we can show by induction on φ:

Fact 8. For every $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$, if $(\bar{a}, \bar{b}) \in E_n^+$ then $\mathfrak{N} \models \varphi(\bar{a})$ iff $\mathfrak{N}' \models \varphi(\bar{b})$.

This together with Fact 7 shows that in $\mathfrak{N}$ only countably many $\mathscr{L}_{\omega_1\omega}[\tau]$-types are satisfied. This ends the proof of (i).

To prove (ii) we repeat the same proof but change the definition of the fragments such as to include the constants required.

To prove (iii) we remark that $\psi \wedge \chi$ is a weak Scott sentence. To obtain a Scott sentence we apply Lemma 3.3.3. □

3.4.4 Corollary (Shelah). *Let K be a* PC-*class in* $\mathscr{L}_{\omega_1\omega}$ *with at least one, but less than* 2^{ω_1}, *many models of cardinality* ω_1. *Then there is an uncountable model* $\mathfrak{A} \in K$ *which has a Scott sentence.*

This corollary was proved by different methods (admissible sets) in Makkai [1977] under the stronger hypothesis that there are less than 2^ω many models of cardinality ω_1.

We are now in a position to prove Theorem 3.1.1.

3.4.5 Proof of the Reduction Theorem. Assume that $2^\omega < 2^{\omega_1}$. Let K with $<_K$ be an abstract ω-presentable class over a vocabulary τ such that $I(K, \omega_1) < 2^{\omega_1}$. Let $\psi \in \mathscr{L}_{\omega_1\omega}[\tau']$ be the sentence defining K. By our assumption on K we can apply

Lemma 3.4.1 and find an uncountable model $\mathfrak{M} \models \psi$ such that the hypothesis of Theorem 3.4.2(ii) is satisfied. So we can use Theorem 3.4.2(iii) to find a model $\mathfrak{N} \models \psi$ of cardinality ω_1 such that $\mathfrak{N} \upharpoonright \tau$ has a Scott sentence σ.

We have to show that there is a ω-presentable abstract class K' with $<_{K'}$ over a vocabulary τ', $\tau \subset \tau'$ such that:

(i) if $\mathfrak{A} \in K'$ then $\mathfrak{A} \upharpoonright \tau \in K$;
(ii) if $\mathfrak{A}, \mathfrak{B} \in K'$ and $\mathfrak{A} <_{K'} \mathfrak{B}$ then $\mathfrak{A} \upharpoonright \tau <_K \mathfrak{B} \upharpoonright \tau$;
(iii) if $\mathfrak{A}, \mathfrak{B} \in K'$ and $\mathfrak{A} <_{K'} \mathfrak{B}$ then $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$; and still
(iv) $I(K, \omega_1) \neq \varnothing$ iff $I(K', \omega_1) \neq \varnothing$.

So we put K' to be $\mathrm{Mod}(\psi \wedge \sigma)$. Clearly, (i) is satisfied. To define $<_{K'}$ we define it as an ω-presentable substructure relation such that $\mathfrak{A} <_{K'} \mathfrak{B}$ iff $\mathfrak{A} <_K \mathfrak{B}$ and $\mathfrak{A}$ is a *Karp*-substructure of $\mathfrak{B}$, applying Theorem 3.3.5(iii). Clearly, this ensures that (ii), (iii), and (iv) are now satisfied. ☐

3.5. How to Construct Super Limits

The purpose of this section is to give a brief survey on the difficulties in the proof of Theorem 3.1.7. Let us state it once more:

3.5.1 Theorem (Existence of Superlimits). *Assume that $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $1 \leq I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then there is a ω_1-superlimit model $\mathfrak{M}$ in K_{ω_1} which is homogeneous and universal.

3.5.2 Amalgamation and Joint Embedding Property in ω. First we observe that the unique countable model $\mathfrak{M}_\omega$ of K is a ω-superlimit, by Proposition 2.2.4, since K has uncountable models and is ω-categorical. Therefore, using Theorem 2.3.1, $\mathfrak{M}_\omega$ is an amalgamation basis for K_ω. Again by ω-categoricity, K_ω has the joint embedding property.

Now we are in a position to apply Theorem 2.1.8. We need the above hypothesis to ensure that $\omega_1 = \lambda = \lambda^{<\lambda}$. So there is a universal and homogeneous model $\mathfrak{M}$ in K_{ω_1}.

We would like next to prove the following:

3.5.3 Claim. *$\mathfrak{M}$ is a weak-limit.*

Note that Claim 3.5.3 is enough to prove Theorem 3.1.6 as pointed out immediately after Theorem 3.1.7.

Proof. We have to verify the conditions (a–d) of Definition 2.2.1. Clearly, the cardinality of $\mathfrak{M}$ is ω_1, so (a) is satisfied.

To verify (b), i.e., to show that $\mathfrak{M}$ is not maximal, we use a modification of Lemma 3.2.7, respectively, Theorem 3.2.3, stating that if $\mathfrak{M} \in K_{\omega_1}$ is universal and maximal, then $I(K, \omega_1) = 2^{\omega_1}$.

We recall property (c): Given $\mathfrak{N} \in K_{\omega_1}$ with $\mathfrak{M} <_K \mathfrak{N}$ there is $\mathfrak{M} \cong \mathfrak{M}'$ such that $\mathfrak{N} <_K \mathfrak{M}'$. To construct $\mathfrak{M}'$ we write $\mathfrak{N}$ as a union of an increasing K-chain of isomorphic copies of the ω-superlimit and reconstruct a universal and homogeneous model in K_{ω_1} along this chain. Then we use the uniqueness of the universal and homogeneous model (Theorem 2.1.8(iii)). $\square$

Next we want to establish the following claim:

3.5.4 Claim. *$\mathfrak{M}$ is a (ω_1, ω_1)-limit.*

We only have to show that unions of K-chains of ω_1 many isomorphic copies of $\mathfrak{M}$ are again isomorphic to $\mathfrak{M}$. To see this, we show that such an union is again homogeneous. For this, we use the homogeneity of $\mathfrak{M}$ and the following lemma:

3.5.5 Lemma. *If $\mathfrak{M}_0, \mathfrak{M}_1 \in K_{\omega_1}$ are both homogeneous and $\mathfrak{A} \in K_\omega$ with $\mathfrak{A} <_K \mathfrak{M}_0$, then every K-embedding of $\mathfrak{A}$ into $\mathfrak{M}_1$ can be extended to an isomorphism from $\mathfrak{M}_0$ onto $\mathfrak{M}_1$.*

Proof. Besides homogeneity, we use that K is ω-categorical and that K also satisfies closure under directed systems. $\square$

To end the proof of Claim 3.5.4 we apply the lemma cofinally often along the chain and use that every countable substructure already appears in an element of this chain. $\square$

3.5.6 Types and Forcing. The main difficulty in the proof of Theorem 3.5.1 is to prove that it gives a (ω_1, ω)-limit. For this we need a better description of the homogeneous model $\mathfrak{M}$ in K_{ω_1}. We would like to build $\mathfrak{M}$ as a union of countable models $\mathfrak{A}_\alpha, \alpha < \omega_1$, such that in every $\mathfrak{A}_{\alpha+1}$ all the types over $\mathfrak{A}_\alpha$, satisfied in $\mathfrak{M}$, are already satisfied. This leads us to a natural, but rather complicated, definition of forcing, a corresponding definition of "types" and a machinery to apply techniques connected with non-forking, symmetry, and finite bases of types, stationarization, etc., as in Shelah [1978a].

3.5.7 The Big Two-Dimensional Picture. All this machinery is needed to cope with the following situation. Let $\mathfrak{M}_i$, $i \in \omega$ be a countable K-chain of isomorphic copies of $\mathfrak{M}$ and let each $\mathfrak{M}_i = \bigcup_\alpha \mathfrak{N}_{\alpha, i}$ be the union of countable models. To show that $\bigcup_i \mathfrak{M}_i \cong \mathfrak{M}$ we have to verify that various finite configurations of countable models in this system allow amalgamation within this system. This is needed to replace this two-dimensional presentation of $\bigcup_i \mathfrak{M}_i$ by an ω_1 long K-chain of countable models from the two-dimensional presentation, which will enable us to show that $\mathfrak{M} \cong \bigcup_i \mathfrak{M}_i$.

3.5.8 The Underlying Philosophy. The underlying philosophy in all of this is, that instead of types, as in first-order model theory, we have to deal with certain generalizations of amalgamation properties of countable structures. Proving the existence of uncountable structures with certain properties is then reduced to proving more and more complicated countable amalgamation properties.

A proof of Theorem 3.1.6 which does not use Theorem 3.5.1 can be found in Fuchino [1983]. There also the condition $2^{\omega_1} < 2^{\omega_2}$ is not needed.

3.5.9 A Gourmet End (Joint Work with Irit M. Manskleid). In the tradition of some of the books of this *Perspectives of Mathematical Logic*, I would like to conclude this last chapter with a gourmet treat. The following recipe is connected with my work with Saharon Shelah in two ways: In 1974, when we started to work together on abstract model theory, I also visited Florence, Italy. There I dined at Sabatini's, a restaurant renowned for its combination of Italian and French cooking. Italian cooking puts the emphasis on the main ingredients of a dish by letting them have their optimal gustatory and olfactory effects; French cooking is famous for refining the ingredients of a dish by the addition of ornamental, but dominant, components, especially sauces. The most exciting dish I tasted at Sabatini's was "vermicelli colla salsa di tartufi" (homemade, very thin spaghetti with a truffle cream sauce). Truffles are ugly, potato-shaped mushrooms, but inside they hide, like many of Shelah's proofs, a delicate core. In a multitude of attempts I tried to find appropriate truffles and to reconstruct the dish. Here is the result.

3.5.10 The Truffles (Fungus; Tuber, Hebrew: Kmehin). Truffles are famous, rare, and expensive, especially the French and Italian kind. They grow on calcarious ground in symbiosis with oaks, beeches, or some desert bushes. Less fancy truffles grow in North Africa, the Carmel mountain, and the Negev desert. They also grow, though rarely, in California and Oregon. These truffles are much cheaper but they are good enough, if pickled for one month in dry white wine with bay leaves. We need about 200 gr of them, after washing and peeling. If they come from sandy areas, such as the Carmel or the Negev, this is equivalent to more than half a kilo bought on the market.

3.5.11 The Vermicelli. Prepare a dough (standard pasta dough, possibly with half the flour whole grain). Let it rest. Using a pasta machine, roll as thin as possible and cut into the thinnest possible slices. Separate them by hand and let them dry for an hour. This is like counting to ω_1, naming every ordinal. You will get acquainted with every slice personally. Boil in water with salt and olive oil added. You need two tablespoons of olive oil per liter of water and half a liter of water per 100 gr of pasta.

3.5.12 The Sauce (The quantities are for half a kilo pasta). A quarter liter of sweet (fat) cream is heated with 100 gr of butter till the butter is melted. Add the truffles, chopped very thin. Simmer for about ten minutes. Add 150 gr of ground dry cheese (parmesan). Stir well till the cheese is melted like in cheese fondue. Add salt and fresh ground pepper.

3.5.13 Serving. When the pasta is ready (*al dente*, not too soft), pour it into a sieve, but do not rinse in cold water. Return the pasta into a heatable dish and add the hot sauce. Stir well and reheat if necessary. Eat and enjoy. Serves four to six.

3.5.14 Postscript. This recipe may look complicated. But here is another analogy to many of the proofs in this chapter: Once you are through, you understand that it was worth it, and moreover, that it was the appropriate way to do it.