Chapter XVI
Borel Structures and Measure and Category Logics

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Two very significant ways in which the theory of models has been extended beyond first-order logic are the enrichment of the syntax to include additional quantifiers and the restriction of the class of structures to be considered. These two means will be brought together in this chapter. The focus here will be on the model theory of structures whose domain and some subset of whose definable relations and functions can be built from the subsets of $\mathbb{R}^n$ that are most frequently encountered in analysis and topology: the Borel sets (see Section 1.1 for precise definitions). Such first-order structures are studied in Sections 1.2 and 1.3. The most widely-used notions of size for Borel sets are category, measure, and uncountability. The model theory of "Borel structures," when the syntax is expanded to allow quantifiers capable of expressing one or more of these concepts will be explored in the final two sections of the chapter.

Friedman initiated the study of the structures and logics which are the subject of this chapter in the series of abstracts (Friedman [1978], [1979a] and [1979b]). Most of the major results presented here are due to him. Friedman has expressed the hope that by restricting the available class of structures for a theory to those which are in some sense Borel, the negative results obtained by using arbitrary uncountable or non-separable structures can be largely eliminated. That is to say, the abundance of positive results found in many areas of mathematical practice for countable, separable, or even well-behaved uncountable and non-separable structures may also be discovered for the classes of structures to be discussed here.

At present it is not at all clear that the study of these structures and logics can quite realize the aims sketched above. Nevertheless, the techniques and notions that have already been developed seem powerful, and the wealth of interesting problems that arise in this area surely warrants our further attention.

1. Borel Model Theory

1.1. Measure and Category Logic, and Borel Structures

In this chapter, all theories will be built from a countable vocabulary τ, even though we will usually suppress explicit reference to τ. First-order logic and several finitary extensions of it obtained by adjoining various combinations of the new quantifiers Q, Q_c, and Q_m will be considered in this chapter. The intended interpretation of Q is "there exist uncountably many," and that of Q_c, "there exist non-meager (= not first-category) many," while that of Q_m is "there exist non-measure 0 many." Thus, for example, the logic $\mathscr{L}(Q, Q_c)$ whose formulas consist of those finitary formulas that are constructed from the symbols of first-order logic and the additional first-order quantifiers Q and Q_c will be studied.

The domain of any structure mentioned in this chapter will be a subset of $\mathbb{R}$, the set of real numbers. The extra clauses in the definition of satisfaction dealing with the new quantifiers are then given naturally. The definition is as usual for Q (see Chapter II). For Q_c and Q_m, if $\mathscr{M}$ is a τ-structure, $\operatorname{dom}(\mathscr{M}) = M \subseteq \mathbb{R}$, and $\bar{a} \in {}^nM$, then

$$\mathscr{M} \models Q_c x\, \psi(x, \bar{a}) \quad \text{iff} \quad \{x \in M : \mathscr{M} \models \psi(x, \bar{a})\} \text{ is non-meager,}$$

and

$$\mathscr{M} \models Q_m x\, \psi(x, \bar{a}) \quad \text{iff} \quad \{x \in M : \mathscr{M} \models \psi(x, \bar{a})\} \text{ is not of measure 0.}$$

Structures whose domain and/or relations and functions are arbitrary subsets of $\mathbb{R}$ or $\mathbb{R}^m$ will not be considered here. Rather, this chapter will focus attention on various specializations that are obtained in different ways when the subsets to be considered are required to be Borel.

1.1.1 Definition. A *Borel structure* will be a structure whose domain is a non-empty Borel subset of $\mathbb{R}$ and whose relations and functions are all Borel. A structure for one of the logics $\mathscr{L}$ just described is said to be *totally Borel* if all relations definable by $\mathscr{L}$-formulas with parameters are Borel. Moreover, if φ is a formula in one of these logics then an $\mathscr{L}$-structure $\mathscr{M}$ is *Borel for* φ if it is Borel and every relation definable over $\mathscr{M}$ from a subformula of φ is also Borel. If T is an $\mathscr{L}$-theory, then the $\mathscr{L}$-structure $\mathscr{M}$ is *Borel for* T if it is Borel for every $\varphi \in T$.

We now illustrate the expressive capabilities of some of the logics to be considered. The reader is encouraged to produce further examples.

Example 1. A definable instance of the property that a countable union of meager sets is meager can be expressed by the $\mathscr{L}(Q, Q_c)$ formula

$$\forall y \neg Q_c x\, \varphi(x, y) \wedge \neg Q y\, \exists x\, \varphi(x, y) \to \neg Q_c x\, \exists y\, \varphi(x, y).$$

Obviously, this formula is true in any $\mathscr{L}(Q, Q_c)$-structure.

Example 2. The following $\mathscr{L}(Q_m)$-formula expresses a definable form of the Fubini theorem:

$$Q_m x\, Q_m y\, \varphi(x, y) \leftrightarrow Q_m y\, Q_m x\, \varphi(x, y).$$

This formula certainly is valid for all totally Borel $\mathscr{L}(Q_m)$-*structures.*

Example 3. If "Q_m" is replaced everywhere in the above formula by "Q_c," then the resulting formula asserts a definable form of the Kuratowski–Ulam theorem (see Oxtoby [1971, Theorem 15.1]), which is true in all totally Borel $\mathscr{L}(Q_c)$-structures.

1.2. The Borel Completeness Theorem and Some Classical Applications

The primary result to be examined in this section is

1.2.1 Borel Completeness Theorem (Friedman [1978]). *A first-order theory T has an uncountable totally Borel model iff it has an infinite model.* ▯

Remarks. Before we prove this theorem, some comments are in order. First, since every countable subset of $\mathbb{N}$ is Borel, the ordinary completeness theorem implies that every consistent first-order theory has a totally Borel model. Thus, the real content of Theorem 1.2.1 lies in the construction of an *uncountable* totally Borel structure for T. Second, this result can be considerably sharpened. It is well known that every uncountable Borel subset of $\mathbb{R}$ is Borel isomorphic to $\mathbb{R}$ itself. Consequently, if T has an infinite model, then T has a totally Borel model whose domain is $\mathbb{R}$.

Proof of Theorem 1.2.1. All that requires proof is the assertion that if T has an infinite model, then T has an uncountable totally Borel model. So we assume that T has an infinite model. The uncountable totally Borel model that we will construct will be the Skolem hull of a sequence of indiscernibles.

First, extend the theory T to a theory T^* in an expanded language so that T^* has built-in Skolem functions. As usual, T^* has a model with an infinite sequence of indiscernibles $(I, <)$ having the same order type as the rational numbers. This sequence of indiscernibles may then be stretched to obtain a sequence of indiscernibles $(J, <)$ which also has the same order type as the irrational numbers. We will construct the totally Borel model of T from $\mathscr{H}\langle J\rangle$, the Skolem hull of J.

Any element of $H\langle J\rangle$ can be generated as $t(i_1, \ldots, i_n)$, where $t(v_1, \ldots, v_n)$ is a term having exactly the free variables $v_1, \ldots, v_n$ and $i_1 < i_2 < \cdots < i_n$ are distinct elements of J. We will restrict our attention to such representations for the remainder of the proof. The following two statements may be verified with only a little effort:

(1) For any term $t(v_1, \ldots, v_n)$, there exists a smallest $S \subseteq \{1, \ldots, n\}$, so that for any $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ from J, $t(i_1, \ldots, i_n) = t(j_1, \ldots, j_n)$ iff $i_k = j_k$, for every $k \in S$.
(2) Suppose $t(v_1, \ldots, v_n)$ and $t'(u_1, \ldots, u_m)$ are two terms with associated $S_t \subseteq \{1, \ldots, n\}$ and $S_{t'} \subseteq \{1, \ldots, m\}$. If for some $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_m$ from J, it is true that $t(i_1, \ldots, i_n) = t'(j_1, \ldots, j_m)$, then $\{k\colon i_k = j_l$ for some $l\} = S_t$ and $\{l\colon j_l = i_k$, for some $k\} = S_{t'}$.

Also, we fix an enumeration $\langle t_k : k < \omega \rangle$ of terms, each with its corresponding S_k given by (1) above.

As in descriptive set theory, we will identify the irrational numbers with ${}^{\omega}\omega$. Furthermore, for any k, $({}^{\omega}\omega)^k$ is homeomorphic to ${}^{\omega}\omega$. Since the irrational numbers are homeomorphic to their intersection with any open interval of $\mathbb{R}$, it follows that for any k, $({}^{\omega}\omega)^k$ is homeomorphic to the intersection of the irrationals and any open interval contained in $\mathbb{R}$. With these facts in hand, we can now undertake the construction of the totally Borel model of T.

Our main task is to map $H\langle J \rangle$ properly to a Borel subset of $\mathbb{R}$. To accomplish this, we first map $(J, <)$ ordermorphically onto ${}^{\omega}\omega \cap (-1, 0)$. Then, by induction on k, this mapping will be extended to include all elements generated by t_k. Thus, assume that we have extended the mapping to include those elements generated by t_l, $l < k$. We show how to extend it to include those elements generated by t_k.

The construction for t_k splits into two cases according to whether or not there is some $l < k$ and sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ from T so that $t_l(i_1, \ldots, i_m) = t_k(j_1, \ldots, j_n)$. Suppose first that there are such an l and sequences $\vec{i}$ and $\vec{j}$. Let l_0 be the least such l. From (2), $|S_{l_0}| = |S_k| = p$. Then, from (1) it follows that the value of t_k, for any increasing n-tuple from J and that of t_{l_0} for any increasing m-tuple from J, depend only on the p coordinates in S_k and S_{l_0}, respectively. To simplify notation, we assume that both S_k and S_{l_0} are $\{1, \ldots, p\}$. It then follows that

$$t_k(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n) = t_{l_0}(i_1, \ldots, i_p, j_{p+1}, \ldots, j_m),$$

for any $i_1 < \cdots < i_n$ and $i_p < j_{p+1} < \cdots < j_m$ from J. We may then naturally identify the elements generated from t_k with that subset of $\mathbb{R}$ to which the terms generated from t_{l_0} have already been mapped via the correspondence

$$t_k(i_1, \ldots, i_n) \mapsto t_{l_0}(i_1, \ldots, i_p, j_{p+1}, \ldots, j_m)$$

Notice that if $S_k = \varnothing$, then t_k is constant for all increasing n-tuples from J. In this case, t_k (and hence t_{l_0}) generates only one element of $H\langle J \rangle$.

Let us now carry out the construction in the second case. Hence, assume that there is no $l < k$ and sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ from J such that $t_l(i_1, \ldots, i_m) = t_k(j_1, \ldots, j_n)$. Suppose also that $|S_k| = p$. In this case we will map the elements generated by t_k into the intersection of the irrationals and $(k, k + 1)$. Since the value of t_k depends only upon the p coordinates of S_k, we may map the elements generated by t_k canonically to

$$\mathcal{O} = \{(x_1, \ldots, x_p) : x_l \in {}^{\omega}\omega \quad \text{for each} \quad l \leq p, \\ \text{and} \quad x_1 < x_2 < \cdots < x_p\}.$$

Then, since $({}^{\omega}\omega)^p$ is homeomorphic to $({}^{\omega}\omega) \cap (k, k + 1)$, the open set $\mathcal{O}$ of "elements" can be identified with a relatively open subset of $({}^{\omega}\omega) \cap (k, k + 1)$. Again, if $S_k = \varnothing$, then t_k generates one element of $H\langle J \rangle$, which can be mapped to the real number $(2k + 1)/2$. Regardless, the elements of $H\langle J\rangle$ have been identified with a subset of $\mathbb{R}$ that is at worst π_2^0.

To complete the proof of the theorem, we must show that every definable relation of $\mathscr{H}\langle J\rangle$— the domain of $\mathscr{H}\langle J\rangle$ being identified with the Borel subset of $\mathbb{R}$ constructed above—also is Borel. To this end, observe that in $\mathscr{H}\langle J\rangle$ every formula is equivalent to a quantifier free one. Then it is routine, but tedious, to check that the model just constructed is totally Borel. □

Some additional information may be extracted from the proof of the theorem. First, it is apparent that all of the definable Borel relations of the model constructed fall within the first ω levels of the Borel hierarchy. Second, we could begin the construction with a Borel subset of arbitrary complexity, and thereby build a Borel model of complexity as high as we please. We must pay the price, unfortunately, for such simultaneous control and flexibility: it is not always easy to build models with desired properties from a Skolem hull of indiscernibles. This point is emphasized by the proof of the first application of the Completeness theorem given as Theorem 1.2.2. A more versatile technique for building models will be introduced in Section 3.

The first use of the completeness theorem establishes the existence of recursively saturated totally Borel structures. That there are limits to the saturation of Borel structures beyond the *a priori* ones will be shown by Theorem 1.3.3. The proof of the next theorem is essentially an adaptation of that given by Barwise [1975] in proving that any structure has a recursively saturated elementary extension of the same cardinality.

1.2.2 Theorem. *Every countably infinite $\mathscr{L}$-structure $\mathscr{M} = \langle M, R_1, \ldots, R_n\rangle$ has an uncountable totally Borel elementary extension that is recursively saturated.*

Proof. We will use the notation and definitions from Barwise [1975]. Hence, let $\mathscr{M}$ be as in the hypothesis of the theorem. To $\mathbb{HYP}_{\mathscr{M}}$, considered as the one-sorted structure $\mathfrak{A} = \langle M \cup A, M, A, R_1, \ldots, R_n, \in\rangle$, add a one-to-one function $F\colon M \cup A \to M$. By the compactness theorem, we obtain a countable $\langle \mathfrak{B}, G\rangle \succ \langle \mathfrak{A}, F\rangle$ having non-standard natural numbers. Applying the Borel completeness theorem to $\langle \mathfrak{B}, G\rangle$, there exists an uncountable totally Borel

$$\langle \mathfrak{C}, H\rangle = \langle N \cup C, N, C, R_1, \ldots, R_n, E, H\rangle$$

The well-founded part of $\mathfrak{C}$, $\mathfrak{C}'$, is admissible and $N \in \mathfrak{C}'$. Consequently, $o(\mathfrak{C}') = \omega$, and $\mathscr{N}$ is recursively saturated. Furthermore, $\mathscr{N}$ clearly is totally Borel, and H insures that $\mathscr{N}$ is uncountable. □

The second application of Theorem 1.2.1 that we will present deals with the existence of Borel two-cardinal models.

1.2.3 Theorem. *Let T be a stable first-order theory in a vocabulary have a distinguished unary predicate P. If T has a model $\mathscr{M}$ so that $|M| > |P(\mathscr{M})|$, where $P(\mathscr{M}) = \{a \in M\colon \mathscr{M} \models P(a)\}$, then T has an uncountable totally Borel model $\mathscr{N}$ with $|P(\mathscr{N})| = \aleph_0$.* □

Before proving this result, we observe that it is best possible, as an infinite Borel subset of $\mathbb{R}$ has power $\aleph_0$ or $2^{\aleph_0}$. The proof uses the following proposition.

1.2.4 Lemma. *Any first-order theory T with distinguished unary predicate P that has a model $\mathscr{M}$ of power $\beth_\omega$ in which $|P(\mathscr{M})| = \aleph_0$ has an uncountable totally Borel model $\mathscr{N}$ with $|P(\mathscr{N})| = \aleph_0$.*

Proof. Expand T to a theory T^* with built-in Skolem functions in an expanded vocabulary τ^*. Then make $\mathscr{M}$ into a model of T^*, and let $\{t_n(v_1, \ldots, v_{m_n}): n < \omega\}$ be an enumeration of the terms in τ^*. By the Erdös–Rado theorem, the theory consisting of sentences of the kinds given in (1) through (4) below in the vocabulary $\tau^* \cup \{c_i: i < \omega\}$ can be seen to be consistent:

(1) all sentences of T^*;

(2) $c_i \neq c_j$, for $i \neq j$;

(3) $\varphi(c_{i_1}, \ldots, c_{i_p}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_p})$, where φ is a τ^*-formula and $i_1 < \cdots < i_p$, $j_1 < \cdots < j_p$;

(4) $P(t_n(c_{i_1}, \ldots, c_{i_{m_n}})) \to t_n(c_{i_1}, \ldots, c_{i_{m_n}}) = t_n(c_{j_1}, \ldots, c_{j_{m_n}})$, where $n < \omega$ and $i_1 < \cdots < i_{m_n}, j_1 < \cdots < j_{m_n}$.

It is clear that the interpretation of $\{c_i: i < \omega\}$ will be a sequence of indiscernibles in a model of the set of sentences above. Furthermore, the Skolem hull of a stretched sequence of such indiscernibles having the order type of the irrationals will be a model of T of power $2^{\aleph_0}$ in which the interpretation of P has power $\aleph_0$. This structure can be turned into a totally Borel model of T by repeating the relevant part of the proof of Theorem 1.2.1. ☐

Proof of Theorem 1.2.3. From Lachlan [1973] the stability of T implies that T has a model $\mathscr{M}$ of power $\beth_\omega$ in which the interpretation of P has power $\aleph_0$. The result then follows from Lemma 1.2.4. ☐

1.3. Two Theorems on Borel Structures

The results in the preceding sections do not seem to support the reasons given in the introduction for studying Borel structures. The two theorems of this section by contrast reveal in striking ways the effect of restricting one's model theory to Borel structures.

1.3.1 Theorem (Shelah). *A Borel linear order is either separable or has uncountably many pairwise disjoint open intervals.*

A linear order which is not separable and which has no uncountable collection of pairwise disjoint open intervals is known as a *Suslin line* (See Kunen [1980] for a detailed discussion). The statement asserting the non-existence of Suslin lines is known as Suslin's hypothesis, and it is of interest since it is tantamount to asking whether or not ($\mathbb{R}$, $<$) is, up to isomorphism, the only unbounded, dense, complete linear order that does not have an uncountable collection of pairwise disjoint open intervals. It is now well known that Suslin's hypothesis is independent of Zermelo–Fraenkel set theory. In contrast, the theorem shows that the Borel version of Suslin's hypothesis is true.

Proof of Theorem 1.3.1. (For unexplained notions from set theory, refer to Jech [1978] or Kunen [1980]). Work in a countable transitive model of set theory. To obtain a contradiction, we will suppose that the conclusion fails. Assume, then, that there is a Borel linear order $\langle B, <\rangle$ which is a Suslin line. In the standard way, a *Suslin tree* (that is, a tree of height ω_1 having neither an uncountable anti-chain nor an uncoutable branch) can be constructed from $(B, <)$. The nodes of the tree can be taken to be open intervals of $(B, <)$ that are ordered by reverse inclusion so that incomparable intervals are disjoint. In the generic extension obtained by forcing with the partial order taken from the tree, the linear order $\langle B^*, <^*\rangle$ given by the Borel code for $\langle B, <\rangle$ has uncountably many pairwise disjoint open intervals. This last assertion may be written as a sentence of $\mathscr{L}_{\omega_1\omega}(Q)$, where Q means "there exist uncountably many." Since the sentence is consistent (in the generic extension), the completeness theorem for $\mathscr{L}_{\omega_1\omega}(Q)$ in Keisler [1970] implies that the sentence is satisfiable in the ground model by a structure $\langle C, <c\rangle$. Then, since $\langle C < c\rangle$ looks sufficiently like $\langle B, <\rangle$ (it will actually be a submodel!), it can be seen that $\langle B, <\rangle$ must have uncountably many pairwise disjoint open intervals contrary to the hypothesis. $\square$

Theorem 1.3.1 has been strengthened in Friedman [1979a]:

1.3.2 Theorem. *A Borel linear order is either separable or contains a perfect totally isolated set* (*that is, a perfect set A such that, for any $a \in A$, there is an open I containing a with $I \cap A = \varnothing$*). $\square$

The final result of this section demonstrates that the restriction to Borel structures does not permit the full saturation of models that we can obtain in classical model theory. Indeed, if we adopt the view that uncountable chains in a linear order reflect pathology and prefer to work with separable orders instead, then Theorem 1.3.3 points out the naturalness of the class of Borel structures.

1.3.3 Theorem (Harrington and Shelah [1982]). *A Borel linear order cannot have an uncountable increasing or decreasing chain.* $\square$

The proof of Theorem 1.3.3 uses forcing and may be found in Harrington and Shelah [1982]. Since the proof is indirect and, consequently, does not say much about Borel linear orders, we can ask:

1.3.4 Problem. Is there a structure theorem for Borel linear orders that accounts for Theorem 1.3.3?

2. *Axiomatizability and Consequences for Category and Measure Logics*

2.1. *Axiomatizability of $\mathscr{L}(Q, Q_c)$*

The first subsection is devoted to the proof of axiomatizability, while the second contains a survey of consequences. We remark that an explicit set of axioms for $\mathscr{L}(Q, Q_c)$ will be given in Section 3. The main result of the present discussion is due to Friedman.

2.1.1 Theorem (Friedman [1978]). *The set of sentences of $\mathscr{L}(Q, Q_c)$ that are valid in all totally Borel $\mathscr{L}(Q, Q_c)$-structures is recursively enumerable.*

Proof. We will work both *with* and *in* second-order arithmetic Z_2 (see Apt–Marek [1974] for the axioms of Z_2 as well as other facts about second-order arithmetic). It is known that the theory of Borel sets and functions and category and measure for the same can be carried out in Z_2. In particular, any model of Z_2 will have its own version of the syntax and semantics for $\mathscr{L}(Q, Q_c)$. Let φ be an $\mathscr{L}(Q, Q_c)$-sentence. The theorem will be proved if we show

(1) There is a totally Borel model of φ iff there is a model $\mathscr{M}$ of Z_2 that also satisfies "there is a totally Borel model of φ."

The left-to-right implication in (1) is trivial, and it remains to prove the other direction.

Consequently, we assume that $\mathscr{M} \models Z_2$ + that "there exists a totally Borel model of φ." To ease the exposition, we further assume that $\mathscr{M}$ is a countable, ω-standard model. The argument to be given can be modified to deal with the possibility that $\mathscr{M}$ is a non-standard model.

In what follows, we will make liberal use of forcing over $\mathscr{M}$. We will obtain $\mathscr{M}$-generic extensions of $\mathscr{M}$ by adding $\mathscr{M}$-generic subsets of ω, or "reals" to $\mathscr{M}$. The reader can check that this may be done just as for set theory and that all the usual facts about forcing—for example, the Generic Model Theorem—hold in this context.

First, we will construct a certain extension, $\mathscr{M}[T]$, of $\mathscr{M}$. To do this, we observe that by standard techniques, we can build a perfect

$$T \subseteq 2^{<\omega} = \bigcup_{n \in \omega} {}^n 2,$$

considered as a tree ordered by inclusion, such that any path through T is a *Cohen generic real* over $\mathscr{M}$. And, moreover, any finite number of such reals are mutually generic. Let $[T]$ be the set of all infinite paths through T. We then let

$$\mathscr{M}[T] = \bigcup_{\substack{G \subseteq [T] \\ G \text{ finite}}} \mathscr{M}[G],$$

where each $\mathscr{M}[G]$ is just the generic extension obtained by iterating k times the construction used to add a single Cohen real to a model of Z_2. It can be shown that

$\mathscr{M}[T]$ is the model constructed from $\mathscr{M}$ and $[T]$ in the sense appropriate for arithmetic (for a detailed discussion, see Halpern–Levy [1971]). It may be shown (the reader should consult Friedman [1975b]) that $\mathscr{M}[T]$ will be a model of Z_2. For what follows, it is crucial to observe that each element of $\mathscr{M}[T]$ is really constructed from finitely many elements of $[T]$.

For some $a \in M$, we have that $\mathscr{M} \models$ "a is a totally Borel model of φ." We will have to show that this fact is absolute. Without loss of generality, a Borel structure may be thought of as having as its domain a subset of ${}^{\omega}2$. Since there are only countably many formulas in $\mathscr{L}(Q, Q_c)$, a totally Borel $\mathscr{L}(Q, Q_c)$-structure $\mathscr{N}$ can be considered as a Borel subset $B \subseteq \bigcup_{n\in\omega} \omega \times ({}^{\omega}2)^n$, where $\mathscr{N} \models \theta(a_1, \ldots, a_n)$ iff $(\ulcorner \theta(v_1, \ldots, v_n) \urcorner, a_1, \ldots, a_n) \in B (\ulcorner . \urcorner$ represents some fixed coding of the formulas of $\mathscr{L}(Q, Q_c)$ on ω). In other words, that $\mathscr{N}$ is totally Borel implies that its satisfaction relation is totally Borel. Let $b_a \in M$ be the version in $\mathscr{M}$ of the B that corresponds to a. It is well known that Borel subsets admit codings by single reals c so that the property "c codes a Borel set" is a π^1_1-statement (see Jech [1978]). If we let c_a be the code for b_a, then

(2) $\mathscr{M} \models$ "c_a codes a Borel set"
$\wedge$ "the set that c_a codes is the complete diagram of an $\mathscr{L}(Q, Q_c)$-structure"
$\wedge$ "φ is true in this structure."

Hence, we must analyze the statement in (2) to see that it is absolute. The first conjunct in (2), is π^1_1, as we have already observed, and the last is elementary. If we show that the second conjunct is no worse than π^1_2, then by the Schoenfield Absoluteness Lemma (see Jech [1978]) it follows that the statement in (2) will be absolute, as we wish to establish. To prove that the second conjunct is as claimed, we simply write down the conjunction of the clauses in the definition of $\mathscr{L}(Q, Q_c)$-satisfaction in a π^1_2 manner. We only indicate how this is done for Q_c clause, as the other clauses are easy. (For the Q clause, we use the fact that a countable set of reals may be coded by a single real and that any uncountable Borel set contains a perfect subset). We must show that the statement

$$S \equiv \text{"}\{e : (\ulcorner \psi(x, v_1, \ldots, v_n) \urcorner, e, a_1, \ldots, a_n) \in b_a\} \text{ is not meager"}$$

has both a π^1_2 and a Σ^1_2 form so that

$$(\ulcorner Q_c x \psi(x, v_1, \ldots, v_n) \urcorner, a_1, \ldots, a_n) \in b_a \leftrightarrow S$$

will be π^1_2.

For the π^1_2 version, observe that

$$\begin{aligned} S \leftrightarrow \neg \exists c[&\text{"}c \text{ codes a Borel set"} \\ &\wedge \text{"the set coded by } c\text{"} = \textstyle\bigcup_{n\in\omega} \text{"set coded by } c_n\text{"} \\ &\quad \wedge \forall n(\text{"the set coded by } c_n \text{ is closed"} \\ &\wedge \text{"the complement of the set coded by } c_n \text{ is dense and open"}) \\ &\wedge \forall e((\ulcorner \psi(x_1, v_1, \ldots, v_n) \urcorner, e, a_1, \ldots, a_n) \in b_a \to \text{the set coded by } c\text{"})]. \end{aligned}$$

The expression under the scope of the outermost quantification can be seen to be π_1^1 (For instance, the property of being open and dense involves only quantification over ω if one uses codings of intervals with rational endpoints; see Jech [1978] for further details). Hence, the entire expression is π_2^1, as desired. To produce a Σ_2^1 rendition of S, we invoke the fact that Borel subsets have the property of Baire. Now,

$$\begin{aligned} S \leftrightarrow \exists c\, \exists e[&\text{“the set coded by } c \text{ is a non-empty open set”} \\ &\wedge \text{“the set coded by } e \text{ is of first category”} \\ &\wedge \forall h(\text{“} h \in (\text{set coded by } c) - (\text{set coded by } e)\text{”} \\ &\rightarrow (\ulcorner \psi(x, v_1, \ldots, v_n) \urcorner, h, a_1, \ldots, a_n) \in b_a)]. \end{aligned}$$

It is easy to check that the right hand side of the equivalence is Σ_2^1.

Consequently, $\mathscr{M}[T] \models$ "c_a is a Borel code for a totally Borel model of φ" by absoluteness. The remainder of the argument is given to the extraction of a real totally Borel model of φ from that which $\mathscr{M}[T]$ understands to be coded by c_a. To this end, we first will define a mapping $f: {}^{\omega}2 \cap \mathscr{M}[T] \xrightarrow{1-1} \omega_2$ so that the image of any $\mathscr{M}[T]$-Borel set is Borel, and moreover,

(3) an $\mathscr{M}[T]$-Borel subset of $\mathscr{M}[T]$ is $\mathscr{M}[T]$ meager iff its image under f is meager.

Let $S = \{\sigma_n: 2^{<\omega} \rightarrow 2^{<\omega}: n < \omega\}$ be a list, which includes the identity, of order-preserving permutations, σ, of $2^{<\omega}$ so that:

(4) for some k, if $lh(s) = k$ then $\sigma(s^\frown t) = \sigma(s)^\frown t$ (i.e. each such σ is determined by its restriction to some ${}^k2$);
(5) for the least k as in (4), there is no $s \in {}^k2$ and $m < k$ such that $\sigma(s) = \sigma(s \restriction m)^\frown s(m)^\frown \cdots ^\frown s(k-1)$;
(6) for each k, there is exactly one $\sigma \in S$ satisfying (4) and (5); and
(7) For any k and any distinct $s, t \in {}^k2$, there is $\sigma \in S$ with $\sigma(s) = t$.

The permutations all will be in $\mathscr{M}$. Let

$$[T_n] = \{g: \omega \rightarrow 2: (\exists h \in [T])(\forall m) g \restriction m = \sigma_n(h \restriction m)\}.$$

Clearly, each $[T_n]$ is a perfect subtree of ${}^{\omega}2$. More importantly, if $\sigma_n \neq \sigma_m$, then $[T_n] \cap [T_m] = \varnothing$. Indeed, suppose (with a permissible abuse of notation) that $\sigma_n g = \sigma_m h$. Since all the elements of $[T]$ are mutually generic, we have that $g = h$, and so $\sigma_n g = \sigma_m g$. But then the requirements imposed on S imply that $\sigma_m = \sigma_n$.

The mapping f satisfying (3) will be defined separately for $\bigcup_{n<\omega} [T_n]$ and for

$$R = ({}^{\omega}2 \cap \mathscr{M}[T]) \setminus \bigcup_{n<\omega} [T_n].$$

Before defining f, though, we must make one more observation. Each $r \in \mathscr{M}[T] \cap {}^{\omega}2$, as noted earlier, is generated by some unique smallest finite $G = \{g_1, \ldots, g_n\} \subseteq [T]$, where $g_1 < \cdots < g_n$ under the lexicographic order on ${}^{\omega}2$. That is, r is given by the G-interpretation, $t(g_1, \ldots, g_n)$ of a name t consisting of order pairs of elements of ω and n-tuples of forcing conditions. It can be seen that the mapping

$$f_t(g_1, \ldots, g_n) = t(g_1, \ldots, g_n)$$

defined on the Borel domain

$$\{(g_1, \ldots, g_n): g_1 < \cdots < g_n\} \subseteq ({}^{\omega}2)^n$$

has a Borel image in ${}^{\omega}2$. Then, since for each n, there are only countably many names ($\mathscr{M}$ is countable), we conclude that $\mathscr{M}[T] \cap {}^{\omega}2$ is Borel.

We may now define f. Let $Q \subseteq {}^{\omega}2$ be a nowhere dense perfect set and, for each $n < \omega$ let $P_n = \{g \in {}^{\omega}2 : g \upharpoonright (n+2) = \langle 0 \ldots 01 \rangle\}$. For each n, $P_n \backslash Q$, and with the exception of one point, $[T_n]$, can be written as the disjoint union of basic open sets. Hence, $f \upharpoonright [T_n]: [T_n] \xrightarrow{1-1} P_n \backslash Q$ can be defined so that the image of every relatively open subset of $[T_n]$ is not meager in ${}^{\omega}2$. Finally, since ${}^{\omega}2 \cap \mathscr{M}[T]$ is Borel, we can define $f \upharpoonright R: R \xrightarrow{1-1} Q$ so as to insure that f sends Borel sets to Borel sets, as required.

We still must prove that f satisfies (3). First, suppose that B is an $\mathscr{M}[T]$-meager, $\mathscr{M}[T]$-Borel set. A Cohen generic real is characterized (see Jech [1978, Section 42]) by not belonging to any meager Borel set coded in the ground model over which the real is generic. It follows then that B cannot contain any reals that are generic with respect to the finitely many reals of $[T]$ that generate the Borel codes for B and the countable union of closed, nowhere dense sets that compel B to be meager. Consequently, for every n, $B \cap [T_n]$ is finite, whence $B \cap \bigcup_{n<\omega} [T_n]$ is countable, and so $f(B)$ must be meager. On the other hand, suppose that B is an $\mathscr{M}[T]$-non-meager, $\mathscr{M}[T]$-Borel set. Then, for some $s \in 2^{<\omega}$, relative to

$$[s] = \{f \in {}^{\omega}2 : f \upharpoonright n = s, \text{ for } n = lh(s)\},$$

the set $B \cap [s]$ is $\mathscr{M}[T]$-comeager. For some $\sigma_n \in S$, a basic open subset of $[T_n]$ lies in $[s]$. As before, by Cohen genericity, only finitely many elements of $[T_n] \cap [s]$ may lie outside B. Thus a basic open subset of $[T_n]$ must be contained in $B \cap [s]$; and, finally, $f(B)$ itself must be non-meager.

Consequently, we see that the $\mathscr{M}[T]$-totally Borel structure that c_a codes in $\mathscr{M}[T]$ becomes a real totally Borel structure under f. By (3), Q_c is preserved under this transformation. Also, Q is preserved because $\mathscr{M}[T] \cap {}^{\omega}2$ is uncountable, and internally $\mathscr{M}[T]$ will reflect the property that any uncountable Borel set contains a perfect subset which must be isomorphic to its understanding of ${}^{\omega}2$. Therefore, the real totally Borel structure will satisfy φ, completing the proof of the theorem. $\square$

2.2. *Consequences of Theorem 2.1.1 and Its Proof*

We first remark that a random real is characterized by not being an element of any measure 0 Borel set with code in the ground model over which it is generic. Thus, if the proof of Theorem 2.1.1 is carried out using random reals instead of Cohen generic reals, the result is:

2.2.1 Theorem (Friedman [1978]). *The set of sentences of $\mathscr{L}(Q, Q_m)$ that are valid in all totally Borel $\mathscr{L}(Q, Q_m)$-structures is recursively enumerable.* ☐

The definition of f in the proof of Theorem 2.1.1 could be modified in another way. That is, Q could be chosen to be of measure 0 as well as meager, and each relatively open subset of $[T_n]$ could be mapped to a set of positive measure. With these changes, we are able to prove the left-to-right implication in:

2.2.2 Theorem (Friedman [1978]). *For any sentence φ of $\mathscr{L}(Q, Q_c)$ let φ^* be the $\mathscr{L}(Q, Q_m)$-sentence obtained from φ by replacing each "Q_c" by "Q_m". Then, φ has a totally Borel $\mathscr{L}(Q, Q_c)$-model iff φ^* has a totally Borel $\mathscr{L}(Q, Q_m)$-model.* ☐

The reverse implication can be shown by appropriately modifying the proof of Theorem 2.2.1. Theorem 2.2.2 might be seen as a transfer theorem between $\mathscr{L}(Q, Q_c)$ and $\mathscr{L}(Q, Q_m)$. Of further interest would be a duality theorem for $\mathscr{L}(Q, Q_c, Q_m)$ that parallels Erdös–Sierpinski duality on the real line (see Oxtoby [1971], Theorem 19.5). More explicitly, for an $\mathscr{L}(Q, Q_c, Q_m)$-sentence φ, let φ^* be the $\mathscr{L}(Q, Q_c, Q_m)$-sentence obtained by interchanging "Q_c" and "Q_m". Such a duality principle would state that φ has a totally Borel model iff φ^* does.

2.2.3 Problem. Is there such a duality principle? Also, are the validities of $\mathscr{L}(Q, Q_c, Q_m)$ recursively enumerable?

The proof of Theorem 2.1.1 can be adapted in yet another way to yield results for sets of sentences $\Phi = \{\varphi_i : i < \omega\}$ of $\mathscr{L}(Q, Q_c)$. It can be shown that Φ has a totally Borel model iff there is $\mathscr{M} \models Z_2$ and an $\mathscr{M}$-non-standard formula φ^* having each $\varphi_i \in \Phi$ as a conjunct so that $\mathscr{M} \models$ "φ^* has a totally Borel model"—the left-to-right implication follows from the ordinary compactness theorem. Therefore, we have:

2.2.4 Theorem (Friedman [1978]). *The logic $\mathscr{L}(Q, Q_c)$ is countably compact.* ☐

Again, by suitably interchanging the roles of category and measure, the same result follows for $\mathscr{L}(Q, Q_m)$.

2.2.5 Theorem (Friedman [1978]). *The logic $\mathscr{L}(Q, Q_m)$ is countably compact.* ☐

A central thesis of this chapter is that Borel structures permit a more manageable model theory. We offer some further evidence of this with the observation given below. Let Q^* be interpreted by "there exist $2^{\aleph_0}$ many," then we have

2.2.6 Theorem. *Given an $\mathscr{L}(Q)$ sentence φ, let φ^* be the $\mathscr{L}(Q^*)$ sentence obtained by replacing each "Q" by "Q^*". Then φ is valid on all totally Borel structures iff φ^* is.* ☐

The proof rests simply on the fact that any Borel set has power either $\aleph_0$ or $2^{\aleph_0}$. Observe in support of our claim, that for arbitrary structures, the analogue of Theorem 2.2.6 depends on set theory: It is true if the continuum hypothesis holds, but false if, for example, $2^{\aleph_0} = \aleph_{\omega_2}$. This raises the following question:

2.2.7 Problem. How badly behaved can the set of valid sentences of $\mathscr{L}(Q^*)$ be?

We remark that further refinements of the results above (they can be proved using similar techniques) are announced in Friedman [1979a]. In particular, all the results above are true, if suitably modified, for countable admissible languages.

3. *Completeness Theorems*

Although the results of Section 2 suffice to establish that $\mathscr{L}(Q, Q_c)$ and $\mathscr{L}(Q, Q_m)$ have recursively enumerable validities, explicit sets of axioms are not exhibited and the proofs of the theorems do not contribute very much towards building a model theory for these logics. Here we present simple complete sets of axioms for these logics and various sublogics. Furthermore, in proving these theorems a genuine model building tool which we might call a "continuous" Henkin construction is developed.

3.1. *The Completeness Theorem for $\mathscr{L}(Q_m)$*

The axioms for $\mathscr{L}(Q_m)$ are as follows:

(A) All the usual axiom schemas for first-order logic (as in Chang–Keisler [1973], for example).

(M0) $\neg(Q_m x)(x = y)$.

(M1) $(Q_m x)\psi(x, \ldots) \leftrightarrow (Q_m x)\psi(y, \ldots)$, where $\psi(x, \ldots)$ is an $\mathscr{L}(Q_m)$-formula in which y does not occur and $\psi(y, \ldots)$ is the result of replacing each free occurrence of x by y.

(M2) $(Q_m x)(\varphi \vee \psi) \rightarrow (Q_m x)\varphi \vee (Q_m x)\psi$.

(M3) $[(Q_m x)\varphi \wedge (\forall x)(\varphi \rightarrow \psi)] \rightarrow (Q_m x)\psi$.

(M4) $(Q_m x)(Q_m y)\varphi \rightarrow (Q_m y)(Q_m x)\varphi$.

Notice that axiom (M4) represents a definable form of Fubini's theorem. The rules of inference for $\mathscr{L}(Q_m)$ are the same as for first-order logic: *modus ponens* and generalization. Let the system just described be denoted by K_m.

3.1.1 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q_m)$ has a totally Borel model iff T is consistent in K_m.* $\square$

Before proving the theorem, we require one further notion. An *equivalence Borel structure* is a Borel structure equipped with a Borel equivalence relation defined on its domain so that in addition:

(a) equality is interpreted by E;
(b) each E-equivalence class is both meager and null (that is, it is of measure 0);
(c) the quantifier Q counts the number of E-equivalence classes; and
(d) the relations and functions of the structure are preserved by E.

This last clause, (d), simply means that if $\mathscr{M} \models E(a_i, b_i)$ for $i = 0, \ldots, n-1$, R is an n-place relation symbol and F an n-place function symbol, then

$$\mathscr{M} \models R(a_0, \ldots, a_{n-1}) \leftrightarrow R(b_0, \ldots, b_{n-1})$$

and

$$\mathscr{M} \models E(F(a_0, \ldots, a_{n-1}), F(b_0, \ldots, b_{n-1})).$$

In other words, E is a congruence relation.

Proof of Theorem 3.1.1. To simplify the presentation, we assume that no sentence in T contains a universal or existential quantifier. The argument sketched here can be modified to yield the theorem in full generality; and, along the way, we indicate the changes that must be made. Clearly, only the direction from right-to-left requires proof. Moreover, we assume that "$(Q_m x)(x = x)$" $\in T$. For if $T \cup \{(Q_m x)(x = x)\}$ were not consistent in K_m, then $T \vdash \neg(Q_m x)(x = x)$. In this case, any countable model of the $\mathscr{L}$-theory T' obtained from T by replacing the outermost subformulas of members of T of the form $Q_m x\varphi$ by $(\exists x)(x \neq x)$ will suffice. It can be verified that T' could be derived from T within K_m in this case.

The proof will be carried out in two steps: An equivalence Borel model for T will be constructed first, and from this a totally Borel model for T will be built.

Let us fix an enumeration $\{\varphi_i : i < \omega\}$ of the formulas in $\mathscr{L}(Q_m)$ without first-order quantifiers. We add a new set of variables $V = \{x_s : s \in \omega^{<\omega} \wedge (\forall n > 0)s(n) \in \{0, 1\}\}$ to the vocabulary of $\mathscr{L}(Q_m)$. Next, by induction on n, we define sets $V_n \subseteq V$ and F_n contained in the set of $\mathscr{L}(Q_m)(V)$-formulas in which only variables in V_n are free and only $\mathscr{L}(Q_m)$-variables are bound.

Let $V_0 = \{x_{\langle 0 \rangle}\}$ and $F_0 = \varnothing$. Given V_n and F_n, we define V_{n+1} and F_{n+1} so that the following conditions are met:

(1) $x_s \neq x_t \in F_{n+1}$ for all $x_s, x_t \in V_n$, $s \neq t$;
(2) if $\varphi(x_{s_1}, \ldots, x_{s_k}) \in F_n$, then for all $i_1, \ldots, i_k \in \{0, 1\}$, $\varphi(x_{s_1 \wedge i_1}, \ldots, x_{s_k \wedge i_k}) \in F_{n+1}$, where "$\wedge$" represents concatenation;
(3) for all $i \leq n$, if the free variables of φ_i are $v_1, \ldots, v_k$, $k \leq n$, then for all distinct $x_{s_1}, \ldots, x_{s_k} \in V_n$, either $\varphi_i(x_{s_1}, \ldots, x_{s_k}) \in F_{n+1}$ or $\neg\varphi_i(x_{s_1}, \ldots, x_{s_k}) \in F_{n+1}$;
(4) if $(Q_m x)\psi \in F_n$ and all free variables of ψ are in V_n, then $\psi(x_{\langle p \rangle}) \in F_{n+1}$, where p is the least $q \in \omega$ with $x_{\langle q \rangle} \notin V_n$;

(5) $V_{n+1} = V_n \cup V'_n$, where $V'_n \subseteq V$ consists of all members of $V \setminus V_n$ introduced in (1)–(4);
(6) if $x_{s_1}, \ldots, x_{s_r}$ lists V_{n+1} in lexicographic order, then the sentence $(Qx_{s_1}) \ldots (Qx_{s_r})(\bigwedge_{\varphi \in F_{n+1}} \varphi)$ is consistent with T in K_m.

By extensive syntactic manipulation and making heavy use of the "Fubini" axiom given in (M4), it can be seen that V_{n+1} and F_{n+1} can be defined. If first-order quantifiers were present, then Skolem functions would have to be introduced before defining V_n and F_n, and (6) would have to be replaced by a modified consistency criterion. Let $F = \bigcup_{n<\omega} F_n$ and notice that $V = \bigcup_{n<\omega} V_n$.

We define an equivalence Borel structure by first defining a structure $\mathscr{M}$ and then mapping $\mathscr{M}$ suitably to a Borel subset of $\mathbb{R}$. The universe of $\mathscr{M}$ will consist of the union of

$$BT = \{f : \omega \to \omega : (\forall n > 0) f(n) \in \{0, 1\}\},$$

called the set of basic terms, and the set of all proper formal terms,

$$FT = \{t(f_1, \ldots, f_n) : t(v_1, \ldots, v_n) \text{ is an } \mathscr{L}\text{-term and } f_1, \ldots, f_n \in B\}.$$

For a relation symbol R, we define

$$\begin{aligned} \mathscr{M} \models R(t_1(f_{11}, \ldots, f_{1k_1}), \ldots, t_l(f_{l_1}, \ldots, f_{lk_l})) \\ \text{iff} \quad & \text{for some finite initial segments, } s_{11}, \ldots, s_{lk_l} \\ & \text{of } f_{11}, \ldots, f_{lk_l}, \text{ respectively,} \\ & R(t_1(x_{_{11}}, \ldots, x_{s_{1k_1}}), \ldots, t_l(x_{s_{l1}}, \ldots, x_{s_{lk_l}})) \in F. \end{aligned}$$

The equality relation is defined in exactly the same way.

Let $B \subseteq \mathbb{R}$ be a Borel set consisting of countably many disjoint perfect subsets, so that $\mathbb{R} \setminus B$ has measure 0, $\mathbb{R} \setminus B$ has power $2^{\aleph_0}$, and any basic open subset of any of the perfect sets has positive measure. The map $g : \mathscr{M} \xrightarrow{1-1} \mathbb{R}$ is then defined so that BT is mapped canonically to B, and FT is mapped in any Borel way into $\mathbb{R} \setminus B$.

Once this has been done, it can be shown by an easy induction on complexity that for any $\mathscr{L}(Q_m)$-formula $\varphi(v_1, \ldots, v_l)$ without ordinary quantifiers, we have

$$\begin{aligned} (7) \quad \mathscr{M} \models \varphi(t_1(f_{11}, \ldots, f_{1k_1}), \ldots, t_l(f_{l1}, \ldots, f_{lk_l})) \\ \text{iff} \quad & \text{for some finite initial segments } s_{11}, \ldots, s_{lk_l} \\ & \text{of } f_{11}, \ldots, f_{lk_l}, \text{ respectively,} \\ & \varphi(t_1(x_{s_{11}}, \ldots, x_{s_{1k_1}}), \ldots, t_l(x_{s_{l1}}, \ldots, x_{s_{lk_l}})) \in F, \end{aligned}$$

where the universe of $\mathscr{M}$ is now identified with its image under g. In particular, $\mathscr{M}$ is an equivalence Borel model of T.

If remains only to convert $\mathscr{M}$ into a totally Borel model. To accomplish this, it will suffice to produce $h: \mathscr{M} \to \mathscr{M}$, where again $\mathscr{M}$ is identified with its image in $\mathbb{R}$ under g, so that h takes Borel sets to Borel sets, and also if

$$\mathscr{M} \models t(f_1, \ldots, f_m) = t'(f'_1, \ldots, f'_n), \quad \text{then} \\ h(t(f_1, \ldots, f_m)) = h(t'(f'_1, \ldots, f'_n)).$$

Although we omit the details, that such an h can actually be defined follows from (7) above. To see that $h(\mathscr{M})$ still satisfies T, observe that (1) implies that no basic terms will be identified under h. Consequently, the property of having measure greater than 0 is preserved. $\square$

3.2. Further Completeness Results

We indicate, without proof, several more completeness theorems that can be established using techniques similar to the one employed in Section 3.1. Let axioms (C0)–(C4) be the results of replacing "Q_m" everywhere in (M0)–(M4) by "Q_c", and let K_c represent the resulting proof system.

3.2.1 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q_c)$ has a totally Borel model iff T is consistent in K_c.* $\square$

The axiom systems for logics involving Q will entail expanding the vocabularies to contain a unary predicate, $N(\cdot)$, and a binary function symbol $F(\cdot,\cdot)$. To make the axioms more comprehensible, we remark that the intended interpretation for $N(\cdot)$ is, of course, $\mathbb{N}$. Also, as x varies, $F(x, \cdot)$ is intended to represent one-to-one maps from the universe of the structure to all perfect subsets of the structure. The axioms thus are as follows.

(Q0) The usual axioms for $\mathscr{L}(Q)$ in the expanded vocabulary (see Chapter IV).
(Q1) $(\forall x)(\forall y)(\forall z)[F(x, y) = F(x, z) \to y = z]$.
(Q2) $\neg(Qx)N(x)$.
(Q3) $(Qy)\varphi \to (\exists x)(\forall y)(\forall z)(z = F(x, y) \to \varphi(z))$.
(Q4) $\neg(Qy)\varphi \to (\exists x)(\forall y)[\varphi \to (\exists z)(N(z) \wedge F(x, z) = y)]$.
(Q5) $(Qx)(x = x) \to (\exists x)(Qy)(\forall z)(F(x, z) \neq y)$.
(Q6) $Qx(x = x) \to [(\forall y_1) \ldots (\forall y_n)\, \exists z(N(z) \wedge \varphi(y_1, \ldots, y_n, z)) \to (\exists z)(\exists x_1) \ldots (\exists x_n)(\forall y_1) \ldots (\forall y_n)(N(z) \wedge \varphi(F(x_1, y_1), \ldots, F(x_n, y_n), z))]$, where $x_1, \ldots, x_n$ are not free in φ.

Axiom (Q5) asserts, in effect, that any perfect set contains two disjoint perfect subsets (by composition of functions, using F), and (Q6) expresses the statement that a partition of the product of perfect sets, $\prod_{i=1}^n P_i$, into countably many pieces admits a homogeneous subset of the form $\prod_{i=1}^n P'_i$, where $P'_i \subseteq P_i$ and P'_i is perfect, for each $i = 1, \ldots, n$. We let K_U represent the proof system based on axioms (A), (U0)–(U6) with the rules of inference as above.

3.2.2 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q)$ (in the original vocabulary) has a totally Borel model iff T is consistent in K_u.* □

We will not prove this theorem, but a sketch of why K_u is sound deserves mention. Indeed, suppose that φ is a sentence in the original vocabulary that is provable in K_u. We must show that φ is true in every totally Borel $\mathscr{L}(Q)$-structure $\mathscr{M}$. To do this, we might attempt to expand $\mathscr{M}$ to obtain a structure for the new vocabulary in which the axioms of K_u are satisfied. However, for this to be possible, it might be necessary that complicated uncountable projective sets contain perfect subsets. Such a strong property cannot be guaranteed (see Jech [1978], for example, in Section 41). To circumvent this difficulty, we move up to a generic extension in which at least every uncountable projective set contains a perfect subset. Assuming that there exists an inaccessible cardinal, it is well known that this can be done (see Jech [1978], Section 42); and, by means of a more delicate construction, it can be done without the additional assumption. Let a be the Borel code for $\mathscr{M}$. By absoluteness, the interpretation of a, $\mathscr{M}^*$, in the generic extension remains a totally Borel model of $\mathscr{L}(Q)$ (recall the proof of Theorem 2.1.1). Within the generic extension, $\mathscr{M}^*$ can be expanded to a model of the axioms of K_u, and so φ is true in $\mathscr{M}^*$. Again by absoluteness—this time, however, only for the elementary statement asserting that "$\ulcorner\varphi\urcorner \in a$"—it follows that $\mathscr{M} \models \varphi$, as required.

Finally, we consider the logics $\mathscr{L}(Q, Q_m)$ and $\mathscr{L}(Q, Q_c)$. Consider

$$(\mathrm{MU})(Q_m x)(\exists y)(\varphi(x, y) \wedge N(y)) \to (\exists y)(N(y) \wedge (Q_m x)\varphi(x, y)),$$

which has the obvious interpretation that the union of countably many sets of measure 0 has measure 0. Let (CU) be the result of everywhere replacing "Q_m" by "Q_c" in (MU). Let $K_{m,u}$ be the system based on axioms (A), (M0)–(M4), (Q0)–(Q6), and (MU), and let $K_{c,u}$ be the corresponding system for $\mathscr{L}(Q, Q_c)$.

3.2.3 Theorem (Friedman [1979a]). *A set of sentences T in the original vocabulary for $\mathscr{L}(Q, Q_m)$ respectively $\mathscr{L}(Q, Q_c)$) has a totally Borel model iff T is consistent in $K_{m,u}$ (respectively $K_{c,u}$).* □

We close with some pertinent remarks. Several refinements of the results above are announced in Friedman [1979a]. Of particular interest are results described there whose hypotheses are stronger than ZFC. As perhaps the soundness argument for Theorem 3.2.2 has foreshadowed, such theorems will concern Borel rather than *totally Borel* structures.

Finally, we state some problems. Pursuing a question raised in Section 2.2, we can consider

3.2.4 Problem. If the set of valid sentences of $\mathscr{L}(Q, Q_c, Q_m)$ is recursively enumerable, find a simple set of axioms for it.

3.2.5 Problem. Develop the model theory for these logics.

Dependency Chart for Chapter 16

This chapter may be read independently of others in the book.

—— indicates essential dependence.
---- indicates non-essential, but useful background.