Chapter II

Extended Logics: The General Framework

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The contents of this chapter are intended to serve as preparation for the more specific or more advanced topics of the chapters that follow. We will pay equal attention to general notions and concrete systems. The first part of the material is concerned with basic notions and examples. In Section 1 we define general logical systems. Section 2 contains a description of numerous concrete examples together with an elaboration of their essential properties—as far as this can be given without greater effort. Section 3 is concerned with elementary and projective classes as a tool to compare the expressive power of logical systems. Applications include the systematic use of PC-reducibility for compactness proofs. In Section 4 numerous preceding examples are systematized by the notion of the Lindström quantifier, and an analogue of the Ehrenfeucht–Fraissé characterization of elementary equivalence for logics with monotone quantifiers is proved. The second part of the chapter is concerned with a more systematic representation of central model-theoretical notions, divided into three groups around compactness (Section 5), Löwenheim–Skolem phenomena (Section 6) and interpolation (Section 7).

We assume that the reader is acquainted with basic notions and facts of first-order model theory. In general we will consider only one-sorted structures; however, since in some cases many-sortedness leads to a methodological enrichment even for one-sorted model theory (see, for instance, Examples 7.1.2), we give the definitions for the many-sorted case (provided the many-sorted formulation is not too tedious and is of practical value). If not stated otherwise, examples, results and proofs refer to the one-sorted version. In most cases it is not hard to give the many-sorted extensions. For example, this can be done by reduction to the one-sorted version using additional predicates ("Unification of Domains", see Feferman [1968a, p. 13]). However, there are exceptions and the warning following Definition 2.1.1 should be consulted.

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1. General Logics

What is a logic? The answer to this question is a pragmatic one: we collect some basic features common to well-known logical systems and use them as defining properties of a logic. In order to cover all important systems, we would have to be rather general. On the other hand we wish to provide convenient definitions to work with. In order to escape this dilemma we do not fix a single definition, but leave it to the working logician to choose a suitable notion according to the needs of specific situations. Having thus created the general framework, we then list some further properties of logics that serve as a means for describing numerous important examples of stronger logics in Section 2.

1.1. The Framework

For the purposes of exposition, we shall restrict ourselves to notions of logics based on conventional algebraic structures. For natural generalizations to other structures such as topological ones, see Chapters III and XV. We begin by listing our notational conventions and by recalling standard concepts from model theory.

Many-sorted *vocabularies* τ, σ, ... are non-empty sets that consist of *sort symbols* $s, \ldots$, finitary *relation symbols* $P, R, \ldots$, finitary *function symbols* $f, g, \ldots$ and *constants* $c, d, \ldots$. Each constant and each function symbol of a vocabulary τ is equipped with a sort symbol of τ as are the argument places of relation and function symbols of τ.

Let R be a binary relation symbol whose argument places are equipped with sort symbols s_2, s_1, respectively, f be a unary function symbol equipped with s_2, whose argument place is equipped with s_1, and c be a constant equipped with s_1. Then

$$(*) \qquad \tau = \{s_1, s_2, s_3, R, f, c\}$$

is a vocabulary. The τ-*terms* are built up and equipped with a sort symbol in the obvious way. For instance, $f(c)$ is a τ-term. It is assigned the sort symbol s_2, the symbol with which f is equipped. $f(f(c))$ is not a τ-term because $f(c)$ is not equipped with s_1. In first-order logic the atomic τ-sentences are of shape Rt_0t_1 where t_0, t_1 are τ-terms equipped with s_2, s_1, respectively, or of shape $t_0 = t_1$ either for arbitrary τ-terms t_0, t_1 or—a variant that we shall adopt—only for τ-terms t_0, t_1 which are equipped with the same sort symbol.

We use self-explanatory denotations of vocabularies such as

$$\tau = \{s, \ldots, R, \ldots, f, \ldots, c, \ldots\}.$$

In the one-sorted case we drop the sort symbol, writing for instance

$$\tau = \{R, \ldots, f, \ldots, c, \ldots\}.$$

A many-sorted structure $\mathfrak{A}$ of vocabulary τ (called a "τ-*structure*") possesses non-empty domains $A_s, \ldots$, corresponding to the sort symbols $s, \ldots$ of τ, and interprets the other symbols in τ as usual. The elements of A_s are called the *elements of sort s* of $\mathfrak{A}$.

For instance, with τ as in $(*)$ above, a τ-structure $\mathfrak{A}$ consists of domains A_{s_1}, A_{s_2}, A_{s_3}, of a subset $R^{\mathfrak{A}}$ of $A_{s_2} \times A_{s_1}$, a function $f^{\mathfrak{A}}: A_{s_1} \to A_{s_2}$ and an element $c^{\mathfrak{A}} \in A_{s_1}$.

We denote structures in obvious ways such as

$$\mathfrak{A} = (A_s, \ldots, R^{\mathfrak{A}}, \ldots, f^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

in the many-sorted case, and

$$\mathfrak{A} = (A, R^{\mathfrak{A}}, \ldots, f^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

in the one-sorted case. The class of τ-structures will be denoted by Str[τ], and for any structure $\mathfrak{A}$ we let $\tau_{\mathfrak{A}}$ be the vocabulary of $\mathfrak{A}$.

If $\sigma \subseteq \tau$ and $\mathfrak{A} \in \mathrm{Str}[\tau]$, then we define $\mathfrak{A} \upharpoonright \sigma$, the σ-*reduct* of $\mathfrak{A}$, to be the σ-structure that arises from $\mathfrak{A}$ by "forgetting" A_s for $s \notin \sigma$ and $R^{\mathfrak{A}}, \ldots$ for $R, \ldots \notin \sigma$. If τ is as in $(*)$ above, then for instance

$$(A_{s_1}, A_{s_2}, A_{s_3}, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}) \upharpoonright \{s_1, s_2, R\} = (A_{s_1}, A_{s_2}, R^{\mathfrak{A}}).$$

Let τ be one-sorted, $\mathfrak{A} \in \mathrm{Str}[\tau]$, and $C \subseteq A$. C is τ-*closed in* $\mathfrak{A}$ if $C \neq \emptyset$, if moreover $c^{\mathfrak{A}} \in C$ for $c \in \tau$, and C is closed under $f^{\mathfrak{A}}$ for $f \in \tau$. If C is not empty, $[C]^{\mathfrak{A}}$ denotes the substructure of $\mathfrak{A}$ generated by C, sometimes also written $\mathfrak{A}|C$ if C is τ-closed in $\mathfrak{A}$. If $P \in \tau$ is unary, $\sigma \subseteq \tau$, and $P^{\mathfrak{A}}\sigma$-closed in $\mathfrak{A} \upharpoonright \sigma$, we can form the structure $(\mathfrak{A} \upharpoonright \sigma)|P^{\mathfrak{A}}$. This gives what is called a *relativized reduct* of $\mathfrak{A}$.

A map $\rho: \tau \to \sigma$ is called a *renaming* (*from* τ *onto* σ) if it is a bijection from τ onto σ that maps sort symbols onto sort symbols, relation symbols onto relation symbols of the same arity, function symbols onto function symbols of the same arity, and constants onto constants such that the sort symbols the latter ones are equipped with correspond via ρ. For instance, if $R \in \tau$ is as in $(*)$ above, then the argument places of $\rho(R)$ are equipped with $\rho(s_1)$, $\rho(s_2)$, respectively. Given a renaming $\rho: \tau \to \sigma$ and a τ-structure $\mathfrak{A}$, we can "rename" $\mathfrak{A}$ by ρ, thus obtaining the σ-structure $\mathfrak{B} = \mathfrak{A}^{\rho}$ with $B_{\rho(s)} = A_s$ for $s \in \tau$ and $\rho(\S)^{\mathfrak{B}} = \S^{\mathfrak{A}}$ for the other symbols § from τ.

With this preparation, we can now come to the central notion of this chapter.

1.1.1 Definition. A *logic* is a pair $(\mathscr{L}, \models_{\mathscr{L}})$, where $\mathscr{L}$ is a mapping defined on vocabularies τ such that $\mathscr{L}[\tau]$ is a class (the class of $\mathscr{L}$-*sentences* of vocabulary τ)

and $\models_{\mathscr{L}}$ (the *$\mathscr{L}$-satisfaction relation*) is a relation between structures and $\mathscr{L}$-sentences. Moreover, the following properties (i)–(v) hold:

(i) *If* $\tau \subseteq \sigma$, *then* $\mathscr{L}[\tau] \subseteq \mathscr{L}[\sigma]$;
(ii) *If* $\mathfrak{A} \models_{\mathscr{L}} \varphi$, *then* $\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}]$;
(iii) *Isomorphism Property*. If $\mathfrak{A} \models_{\mathscr{L}} \varphi$ and $\mathfrak{B} \cong \mathfrak{A}$, then $\mathfrak{B} \models_{\mathscr{L}} \varphi$.
(iv) *Reduct Property*. If $\varphi \in \mathscr{L}[\tau]$ and $\tau \subseteq \tau_{\mathfrak{A}}$, then

$$\mathfrak{A} \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \mathfrak{A} \upharpoonright \tau \models_{\mathscr{L}} \varphi.$$

(v) *Renaming Property*. Let $\rho: \tau \to \sigma$ be a renaming. Then for each $\varphi \in \mathscr{L}[\tau]$ there is a sentence, say φ^{ρ}, from $\mathscr{L}[\sigma]$ such that for all τ-structures $\mathfrak{A}$,

$$\mathfrak{A} \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \mathfrak{A}^{\rho} \models_{\mathscr{L}} \varphi^{\rho}.$$

Remark. The renaming property expresses the following simple fact: Given an $\mathscr{L}$-sentence φ of vocabulary $\tau = \{s, \ldots, R, \ldots\}$, then the symbols (and the sorts) in φ can be renamed in any reasonable way ρ, and the resulting $\{\rho(s), \ldots, \rho(R), \ldots\}$-sentence φ^{ρ} has, for any τ-structure $\mathfrak{A}$, the same meaning in the "renamed" $\{\rho(s), \ldots, \rho(R), \ldots\}$-structure $\mathfrak{B} = (B_{\rho(s)}, \ldots, \rho(R)^{\mathfrak{B}}, \ldots)$ as φ has in $\mathfrak{A}$.

The reader will have noticed here that we did not incorporate conditions concerning rules of inference or other "logical" properties in our definition. Hence it would seem more appropriate to use a term such as *model-theoretic language* (see Feferman [1974b]) instead of the term *logic*. However, the latter is shorter and has become customary. (See also Chapter I for a discussion concerning the choice of this terminology.)

In order to avoid overburdening the notation, we often denote logics simply by $\mathscr{L}, \mathscr{L}^*, \ldots$ and write "$\models$" instead of "$\models_{\mathscr{L}}$". Basic model-theoretic notions are introduced as usual. For instance, if $\varphi \in \mathscr{L}[\tau]$, we write $\mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi)$ (or simply $\mathrm{Mod}(\varphi)$, if τ and $\mathscr{L}$ are given) for $\{\mathfrak{A} \in \mathrm{Str}[\tau] \mid \mathfrak{A} \models_{\mathscr{L}} \varphi\}$; and for $\Phi \cup \{\varphi\} \subseteq \mathscr{L}[\tau]$ the $\mathscr{L}$-consequence relation is defined by

$$\Phi \models_{\mathscr{L}} \varphi \quad \text{iff for all } \mathfrak{A} \in \mathrm{Str}[\tau], \quad \mathfrak{A} \models_{\mathscr{L}} \Phi \text{ implies } \mathfrak{A} \models_{\mathscr{L}} \varphi.$$

Two τ-structures $\mathfrak{A}, \mathfrak{B}$ are *$\mathscr{L}$-equivalent*, $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, iff for all $\varphi \in \mathscr{L}[\tau]$, $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\mathfrak{B} \models_{\mathscr{L}} \varphi$. We write $\mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$ for $\{\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}] \mid \mathfrak{A} \models_{\mathscr{L}} \varphi\}$; it is called the *$\mathscr{L}$-theory* of $\mathfrak{A}$.

1.1.2 A First Variant. For some purposes it is especially convenient to have *variables* and *formulas* available in a logic. This can be accomplished by a natural generalization of Definition 1.1.1: For each sort symbol s we specify a class of variables $x^s, \ldots$ for objects of sort s and replace $\mathscr{L}$ by two functions $\mathrm{Sent}_{\mathscr{L}}$ and $\mathrm{Form}_{\mathscr{L}}$, where, for all τ, we let $\mathrm{Form}_{\mathscr{L}}[\tau]$ be the class of *$\mathscr{L}$-formulas* of vocabulary τ and $\mathrm{Sent}_{\mathscr{L}}[\tau]$ be the class of *$\mathscr{L}$-sentences* of vocabulary τ. Exact definitions, even including that for the free occurrence of variables, can be obtained along the lines of Definition 1.1.1 in a canonical way. For the general theory we will usually assume that logics are given without variables, although in most concrete examples we will follow the definition just sketched. Since free variables behave like constants, there are only minor differences between the two variants.

The traditional *first-order logic*, $\mathscr{L}_{\omega\omega}$, can be regarded as a logic in the forgoing sense. Moreover, it is also a logic in the sense of Definition 1.1.1, if, for any $\boldsymbol{\tau}$, we define $\mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$ to be the set of first-order sentences of vocabulary $\boldsymbol{\tau}$. Similarly, *second-order logic*, $\mathscr{L}^2$, *weak second-order logic*, $\mathscr{L}^{w2}$, *infinitary logics* such as $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}_{\infty\omega}$ and logics with *cardinality quantifiers* such as $\mathscr{L}_{\omega\omega}(Q_\alpha)$ (where Q_α is interpreted as *there are* $\aleph_\alpha$ *many*) are logics in both sense, with or without free variables.

1.1.3 A Second Variant. The so-called *ω-logic* arises from first-order logic by fixing a vocabulary $\{s, <\}$ and allowing only structures $\mathfrak{A}$ such that $\{s, <\} \subseteq \boldsymbol{\tau}_{\mathfrak{A}}$ and $\mathfrak{A} \upharpoonright \{s, <\}$ is isomorphic to the standard structure $(\omega, <)$ of the ordering of the natural numbers. Of course ω-logic does not fit into the present framework because the renaming property fails. In order to cover it by a notion of logic, we are led to a generalization of Definition 1.1.1: In addition we demand that a logic $\mathscr{L}$ have a further component, namely a map $\mathrm{Str}_{\mathscr{L}}$ defined on vocabularies where, for all $\boldsymbol{\tau}$, we let $\mathrm{Str}_{\mathscr{L}}[\boldsymbol{\tau}]$ be a class of $\boldsymbol{\tau}$-structures, the *$\boldsymbol{\tau}$-structures admitted for $\mathscr{L}$*. Then we modify the basic properties of Definition 1.1.1 in the obvious way (see Section 2.6).

1.2. Basic Closure Properties

Practically all investigations of logics need stronger assumptions than those of the last section. The following closure properties are met by most of the well-known systems and provide much technical facilitation.

1.2.1 Definition. The purpose of the *basic closure properties* is to guarantee that we have at least the expressive power of first-order logic. We have:

(i) *Atom Property*. For all $\boldsymbol{\tau}$ and all atomic $\varphi \in \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$ there is a sentence $\psi \in \mathscr{L}[\boldsymbol{\tau}]$ such that

$$\mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}_{\omega\omega}}(\varphi);$$

(ii) *Negation Property*. For all $\boldsymbol{\tau}$ and all $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ there is a sentence $\psi \in \mathscr{L}[\boldsymbol{\tau}]$ such that

$$\mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\psi) = \mathrm{Str}[\boldsymbol{\tau}] \backslash \mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\varphi);$$

(iii) *Conjunction Property*. For all $\boldsymbol{\tau}$ and all $\varphi_0, \varphi_1 \in \mathscr{L}[\boldsymbol{\tau}]$ there is a sentence $\psi \in \mathscr{L}[\boldsymbol{\tau}]$ such that

$$\mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\varphi_0) \cap \mathrm{Mod}^{\boldsymbol{\tau}}_{\mathscr{L}}(\varphi_1);$$

(iv) *Particularization Property.* If c is of sort s, $c \in \tau$, then for any $\varphi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau \backslash \{c\}]$ such that for all $(\tau\backslash\{c\})$-structures $\mathfrak{A}$, $\mathfrak{A} \models \psi$ iff $(\mathfrak{A}, a) \models \varphi$ for some $a \in A_s$.

If $\mathscr{L}$ has the *boolean property*, that is, (ii) and (iii) together, then we use $\neg\varphi$, $\varphi_0 \wedge \varphi_1$ to denote the required sentence ψ. If $\mathscr{L}$ has the particularization property, we write $\exists c\varphi$ for a corresponding ψ.

For technical convenience we formulate the following properties only in the one-sorted case.

1.2.2 Definition. All the basic examples of logics above (but not ω-logic) allow relativizations in the sense of:

Relativization Property. If $c \notin \tau \cup \sigma$, $\chi \in \mathscr{L}[\sigma \cup \{c\}]$ and $\varphi \in \mathscr{L}[\tau]$, then there is a sentence $\psi \in \mathscr{L}[\tau \cup \sigma]$ such that for all $(\tau \cup \sigma)$-structures $\mathfrak{B}$, if the set $\chi^{\mathfrak{B}} = \{b \in B \mid (\mathfrak{B}, b) \models \chi\}$ is τ-closed in $\mathfrak{B}$, then

$$\mathfrak{B} \models \psi \quad \text{iff} \quad (\mathfrak{B} \restriction \tau) | \chi^{\mathfrak{B}} \models \varphi.$$

Intuitively, ψ is the (more exactly, it is *a*) relativization of φ to $\{c \mid \chi(c)\}$, often written as $\varphi^{\{c|\chi(c)\}}$ or simply φ^P, if $\chi = Pc$.

If constants are present, relativizations can cause difficulties. For instance, if a vocabulary τ contains constants, it is impossible to represent two τ-structures with distinct domains as relativized reducts of a third structure. For the usual logics this difficulty can be overcome, because one can eliminate constants via descriptions by unary relations. We formulate this in a general context, giving an even stronger version that is needed on various occasions: the substitution property. In the simplest case this property guarantees that for any σ, τ the following holds[2]: If $R \notin \tau$ is n-ary and $\psi(\mathbf{c}) \in \mathscr{L}[\sigma \cup \{c_0, \ldots, c_{n-1}\}]$, then for every $\varphi \in \mathscr{L}[\tau \cup \{R\}]$ there is $\varphi[R/\lambda\mathbf{c}\psi(\mathbf{c})] \in \mathscr{L}[\tau \cup \sigma]$ with the meaning

$$\exists R(\forall \mathbf{c}(R\mathbf{c} \leftrightarrow \psi(\mathbf{c})) \wedge \varphi).$$

Similarly for n-ary $f \notin \tau$ and constants $c \notin \tau$, where for instance $\varphi[f/\lambda\mathbf{c}c\psi(\mathbf{c}, c)]$ has the meaning

$$\exists f(\forall \mathbf{c}c(f(\mathbf{c}) = c \leftrightarrow \psi(\mathbf{c}, c)) \wedge \varphi)$$

and $\varphi[c/\lambda d\psi(d)]$ has the meaning

$$\exists c(\forall d(c = d \leftrightarrow \psi(d)) \wedge \varphi).$$

1.2.3 Definition. In generality $\mathscr{L}$ has the *substitution property* iff for any τ, τ' with $\tau \subseteq \tau'$, if $\varphi \in \mathscr{L}[\tau']$ and, for all $R, \ldots, f, \ldots, c, \ldots \in \tau' \backslash \tau$, there are given predicates

[2] We use $\mathbf{a}, \ldots, \mathbf{c}, \ldots, \mathbf{x}, \ldots$ to stand for finite sequences of elements, constants, variables, respectively, of appropriate length.

$\psi_R(\mathbf{c}_R), \ldots$ then there exists an $\mathscr{L}$-sentence that arises from φ by simultaneously replacing $R, \ldots$ by $\lambda \mathbf{c}_R \psi(\mathbf{c}_R), \ldots$, respectively.

It is easy to give a more precise formulation of this definition and to see that any logic $\mathscr{L}$ with the atom property and the substitution property *allows elimination of function symbols* in the following sense: If $\boldsymbol{\sigma}$ arises from $\boldsymbol{\tau}$ by replacing any $f \in \boldsymbol{\tau}$, where f is n_f-ary, and any $c \in \boldsymbol{\tau}$ by new relation symbols R_f and R_c of arity $(n_f + 1)$ and 1, respectively, then for each $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ there exists $\psi \in \mathscr{L}[\boldsymbol{\sigma}]$ such that the $\boldsymbol{\sigma}$-models of ψ arise from the $\boldsymbol{\tau}$-models of φ by replacing the functions and constants by their graphs. A similar consideration shows that the substitution property yields the renaming property, at least in the one-sorted case. The many-sorted version needs a diligent treatment of sort symbols.

Logics satisfying the properties given in Definitions 1.2.1 to 1.2.3 are well-suited for general investigations, and we call them *regular logics*. A regular logic contains for each first-order sentence φ a sentence ψ of the same vocabulary and with the same models. When working with such a logic, it is convenient (and will be done tacitly) to assume that φ itself can be taken as such a ψ.

Further basic properties of logics can be of value in specific situations. One can, for example, demand that for any $\mathscr{L}$-sentence φ there is a smallest $\boldsymbol{\tau} = \boldsymbol{\tau}_\varphi$ such that $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ ("*occurrence property*"). Concerning questions of effectiveness it is reasonable to assume that $\boldsymbol{\tau}_\varphi$ exists and is finite. In order to have precise definitions of such notions for the examples that follow, we complete this section with a translation of crucial properties known from first-order logic into our general framework. More detailed definitions will follow in Sections 5 through 7.

1.2.4 Definition. Let $\mathscr{L}$ be a logic. Then

(i) For an infinite cardinal κ, $\mathscr{L}$ is *κ-compact* iff for all $\boldsymbol{\tau}$ and all $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$ of power $\leq \kappa$, if each finite subset of Φ has a model, then Φ has a model.

(ii) $\mathscr{L}$ is *compact* iff $\mathscr{L}$ is κ-compact for all infinite κ.

(iii) $\mathscr{L}$ is *effective* iff for all $\boldsymbol{\tau} \subseteq \mathrm{HF}$ (the set of hereditarily finite sets),

$$\mathscr{L}[\boldsymbol{\tau}] = \bigcup_{\substack{\boldsymbol{\tau}_0 \subseteq \boldsymbol{\tau} \\ \boldsymbol{\tau}_0 \text{ finite}}} \mathscr{L}[\boldsymbol{\tau}_0],$$

and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$, $\mathscr{L}[\boldsymbol{\tau}_0]$ is a recursive subset of HF. (Of course, it is the usual encoding of first-order formulas by hereditarily finite sets that leads to this definition.)

(iv) $\mathscr{L}$ is *effectively regular* iff $\mathscr{L}$ is regular and effective and all regularity properties are effective. For instance, effectiveness of the negation property means that for each $\boldsymbol{\tau}_0 \in \mathrm{HF}$ there is a recursive function $\dot{\neg}: \mathscr{L}[\boldsymbol{\tau}_0] \to \mathscr{L}[\boldsymbol{\tau}_0]$ such that for any $\varphi \in \mathscr{L}[\boldsymbol{\tau}_0]$, $\dot{\neg}(\varphi)$ is a negation of φ.

(v) $\mathscr{L}$ is *recursively enumerable for validity* iff $\mathscr{L}$ is effective and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$ the set $\{\varphi \in \mathscr{L}[\boldsymbol{\tau}_0] \mid \models \varphi\}$ is recursively enumerable.

(vi) $\mathscr{L}$ is *recursively enumerable for consequence* iff $\mathscr{L}$ is effective and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$ and all recursively enumerable subsets Φ of $\mathscr{L}[\boldsymbol{\tau}_0]$ the set $\{\varphi \in \mathscr{L}[\boldsymbol{\tau}_0] \mid \Phi \models \varphi\}$ is recursively enumerable.

(vii) $\mathscr{L}$ has the *Löwenheim–Skolem property* (*down to* κ) iff each satisfiable $\mathscr{L}$-sentence has a model of power $\leq \aleph_0 (\leq \kappa)$. (Here, the power of a τ-structure $\mathfrak{A}$ is defined as $|A|$ in the one-sorted case and as $\sum_{s \in \tau} |A_s|$ in the many-sorted case.)

(viii) $\mathscr{L}$ has the *Craig* or *interpolation property* iff for all τ_0, τ_1: if $\varphi_i \in \mathscr{L}[\tau_i]$ $(i = 0, 1)$ and $\varphi_0 \models \varphi_1$, then there is an *interpolant*, that is, a sentence $\psi \in \mathscr{L}[\tau_0 \cap \tau_1]$ such that $\varphi_0 \models \psi$ and $\psi \models \varphi_1$ (provided—in the many-sorted case—that $\tau_0 \cap \tau_1$ contains at least one sort symbol).

(ix) $\mathscr{L}$ has the *Beth property* (that is, $\mathscr{L}$ satisfies Beth's definability theorem) iff for all τ, all symbols § from τ different from sort symbols and all $\varphi \in \mathscr{L}[\tau]$, if § is implicitly defined by φ, then § is explicitly definable relative to φ.

The notions of implicit and explicit definability are given, say for unary R according to the following definition.

1.2.5 Definition. (i) R is *implicitly defined by* φ, if every $(\tau \backslash \{R\})$-structure has at most one expansion to a τ-structure satisfying φ.

(ii) R is *explicitly definable relative to* φ, if for a new constant c of the same sort s as the argument place of R, there is a sentence $\psi(c)$ in $\mathscr{L}[(\tau \backslash \{R\}) \cup \{c\}]$ such that for all τ-structures $\mathfrak{A}$ with $\mathfrak{A} \models \varphi$ one has

$$R^{\mathfrak{A}} = \{a \in A_s \mid (\mathfrak{A}, a) \models \psi(c)\}.$$

Intuitively this last means that

$$\varphi \models \forall c(Rc \leftrightarrow \psi(c)).$$

Inspection shows that the usual proof in $\mathscr{L}_{\omega\omega}$ of Beth's theorem via the interpolation theorem needs only the regularity properties of $\mathscr{L}_{\omega\omega}$ given by (i)–(v) in Definition 1.1.1 together with the basic closure properties given in Definition 1.2.1. Hence, any regular logic $\mathscr{L}$ with the interpolation property has the Beth property. This simple fact may be considered as the first theorem of abstract model theory that we have met. And, of course, there is also a first problem: Under what conditions can one conclude that the definability property yields the interpolation property? For an answer, the reader is referred to Section XVIII.4.

Historical Remarks. The impetus to treat general logical systems goes back to Mostowski [1957]. Definitions similar to the ones above were given first by Lindström [1969] and H. Friedman [1970a]. Barwise [1974a] develops a more systematic approach in a categorical framework. A fairly general definition covering, for instance, topological logics is given in Mundici [1984b]. A thorough discussion of properties of logics—from basic ones to more specific ones—can be found in Feferman [1975].

2. *Examples of Principal Logics*

The study of general logics should provide us with means to investigate concrete logics. On the other hand the study of concrete systems can indicate paths that should be followed in the abstract theory. Led by this insight, we now briefly describe numerous systems beyond first-order logic and sketch their most important features. According to our agreement we restrict ourselves to the one-sorted case. An exception is the higher-order case in Section 2.1. More details can be found in Chapter VI.

2.1. *Logics of Higher Order*

Among the possible higher-order logics, we will restrict ourselves to those of just the next level.

2.1.1 Definition. *Second-order logic*, $\mathscr{L}^2$, is built up as usual, allowing for each sort s quantification over n-ary relations on the domain of sort s.

Obviously, $\mathscr{L}^2$ is regular. Its expressive power, however, contrasts with the fact that practically all useful model-theoretic properties of first-order logic fail. Moreover, because of our weakness in governing the notion of subset, we quickly run into set-theoretical dependencies as well. For instance, via a suitable formulation of the continuum hypothesis (CH), one can obtain an $\mathscr{L}^2$-sentence that is valid iff CH holds. Nevertheless the situation is not quite hopeless since many of the logics developed up to now can be considered as parts of $\mathscr{L}^2$. Hence investigations of stronger logics can be seen as aimed at providing a model-theoretic treatment for more and more of $\mathscr{L}^2$. In particular, Chapters XII and XIII will demonstrate that it is even possible to venture into the "real realm" of second-order logic.

Warning. We are usually correct in taking it for granted that properties of a logic do not change if we pass from the many-sorted case to the one-sorted case or vice versa; however, the interpolation property does fail for $\mathscr{L}^2$ in the two-sorted case, even though it is trivially true in the one-sorted case. The proof uses a far-reaching method that goes back to Craig [1965]. A version of it is given in Section 7.3, and a systematic treatment can be found in Section XVII.1.2.

2.1.2 Definition. *Weak second-order logic*, $\mathscr{L}^{w2}$, in contrast to $\mathscr{L}^2$, has the relation variables ranging only over *finite* relations.

It would appear that $\mathscr{L}^{w2}$ deprives the notion of subset of its teeth. In $\mathscr{L}^{w2}$, however, one can easily express the notion of finiteness, because the finiteness of the domain of sort s is guaranteed by the sentence $\exists X^s \, \forall x^s \, X^s x^s$. In this way, one can characterize, for example, the standard model of arithmetic, torsion groups, etc. Hence, $\mathscr{L}^{w2}$ is neither $\aleph_0$-compact nor recursively enumerable for validity. On the other hand, it is easy to prove the Löwenheim–Skolem property. As arithmetical $\mathscr{L}^{w2}$-truth is implicitly definable, it can be seen by the method mentioned in the warning above (see Section 7.3) that the Beth property and hence the interpolation property fail.

2.2. *Examples of Logics with Cardinality Quantifiers*

If $\mathscr{L}_{\omega\omega}$ is enlarged by a unary quantifier Q that is monotone in the sense defined for Theorem 4.2.3, then, according to Theorem III.4.1, the resulting logic $\mathscr{L}_{\omega\omega}(Q)$ is regular just in case Q is some Q_α. (For any ordinal α, $Q_\alpha x\varphi(x)$ means that there are at least $\aleph_\alpha$ many x such that $\varphi(x)$.) We shall deal here with the logic $\mathscr{L}_{\omega\omega}(Q_1)$ and some of its relatives. For considerably more information and historical notes see Chapter IV, and for $\mathscr{L}_{\omega\omega}(Q_\alpha)$ with $\alpha > 1$ see Chapter V.

Example 1. The logic $\mathscr{L}_{\omega\omega}(Q_1)$ has some useful properties: It is $\aleph_0$-compact (Fuhrken [1964]) and recursively enumerable for consequence (Vaught [1964]). Keisler [1970] gives a completeness proof using an elegant system of rules that arises from a complete first-order calculus by addition of the following four axiom schemata:

(i) "*2 is countable*": $\neg Q_1 x(x = y \vee x = z)$;
(ii) "*Q_1 is monotone*": $\forall x(\varphi \rightarrow \psi) \rightarrow (Q_1 x\varphi \rightarrow Q_1 x\psi)$;
(iii) "*Countable unions of countable sets are countable*": $Q_1 x\, \exists y\varphi \rightarrow \exists x\, Q_1 y\varphi \vee Q_1 y\, \exists x\varphi$;
(iv) *Renaming of bound variables*: $Q_1 x\varphi(x, \mathbf{x}) \leftrightarrow Q_1 y\varphi(y, \mathbf{x})$ for any y not free in $\varphi(x, \mathbf{x})$.

For details see Section IV.3. Alternative proofs will be given in Sections 3.1 and 3.2. As we shall see there, the expressive power of $\mathscr{L}_{\omega\omega}(Q_1)$ beyond first-order logic comes down to the characterization of $\aleph_1$-*like orderings*, i.e. structures $\mathfrak{A} = (A, <^{\mathfrak{A}})$ that are models of the axioms for linear orderings plus the sentence

$$Q_1 x\, x = x \wedge \forall y \neg Q_1 x\, x < y.$$

For the strength of $\mathscr{L}_{\omega\omega}(Q_1)$ in mathematical contexts, see Chapter VII.

The set $\{\neg Q_1 x\, x = x\} \cup \{\neg c_\alpha = c_\beta \mid 0 \leq \alpha < \beta < \aleph_1\}$ shows us that $\mathscr{L}_{\omega\omega}(Q_1)$ is not $\aleph_1$-compact. Of course, the Löwenheim–Skolem property fails, but the Löwenheim–Skolem property down to $\aleph_1$ (even for sets of sentences of power $\leq\aleph_1$) can be proved similarly to the downward Löwenheim–Skolem–Tarski theorem in $\mathscr{L}_{\omega\omega}$. Also, $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies an omitting types theorem (cf. Section IV.3.3). But the hope of having found a useful logic was weakened by several points. For instance, by the up-to-now unsuccessful search for satisfactory preservation theorems, and by the failure of the interpolation property (Keisler 1971) and the Beth property (H. Friedman [1973]).

Keisler's counterexample to interpolation in $\mathscr{L}_{\omega\omega}(Q_1)$ can be described as follows. Let $\varphi_0(E, R)$ express that E is an equivalence relation with only uncountable equivalence classes and that R is a countable set of representatives. Furthermore, let $\varphi_1(E, S)$ express a similar statement with S being an *un*countable set of representatives. Then the entailment

$$(*) \qquad \varphi_0(E, R) \models \neg\varphi_1(E, S)$$

holds, but there is no $\mathscr{L}_{\omega\omega}(Q_1)$-interpolant (cf. 4.2.7).

Example 2. What might be called the "*Ramseyfication*" of the quantifier Q_1 leads to the regular logics $\mathscr{L}_{\omega\omega}(Q_1^n)$, for $n \geq 1$, and $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ of Magidor–Malitz [1977a]. Q_1^n is an n-ary quantifier, the meaning of which is defined by the following satisfaction condition:

$$\mathfrak{A} \models Q_1^n \mathbf{x}\varphi(\mathbf{x}) \quad \text{iff} \quad \text{there is an uncountable subset } M \text{ of } A \text{ where } \mathfrak{A} \models \varphi[\mathbf{b}] \text{ for all } \mathbf{b} \in M^n.$$

Sometimes one uses the variation with "for all $\mathbf{b} \in M^n$" replaced by "for all *distinct* $b_0, \ldots, b_{n-1} \in M$"; however, the quantifiers resulting in either version can easily be defined from each other.

Assuming $V = L$ (or even $\diamondsuit_{\aleph_1}$), Magidor and Malitz showed that $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ is $\aleph_0$-compact. A proof is given in Section IV.5.2. On the other hand, according to a result of Shelah, it is consistent to assume that $\mathscr{L}_{\omega\omega}(Q_1^2)$ is not $\aleph_0$-compact. The dependence on set-theoretical principles beyond usual set theory (such as $\diamondsuit_{\aleph_1}$) becomes intelligible if one takes into consideration that Suslin trees, for instance, are characterizable in $\mathscr{L}_{\omega\omega}(Q_1^2)$ (see Example IV.5.1.4).

In $\mathscr{L}_{\omega\omega}(Q_1^2)$, the entailment $(*)$ of Example 1 has the interpolant

$$\varphi(E) \wedge \neg Q_1^2 xy(x = y \vee \neg Exy)$$

where $\varphi(E)$ states that E is an equivalence relation with only uncountable equivalence classes. Nevertheless, for no $n \geq 1$ does $\mathscr{L}_{\omega\omega}(Q_1^n)$ have the Beth property (see Badger [1980]). For a counterexample to interpolation see 7.1.3(b). Because $\mathscr{L}_{\omega\omega}(Q_1^2)$ overcomes Keisler's counterexample, it is strictly stronger than $\mathscr{L}_{\omega\omega}(Q_1)$; moreover, as was shown by Garavaglia and Shelah, the expressive power of $\mathscr{L}_{\omega\omega}(Q_1^{n+1})$ is greater than that of $\mathscr{L}_{\omega\omega}(Q_1^n)$, for all $n \geq 1$. Details and further results of this kind can be found in Rapp [1983], [1984].

Example 3. "*Positive*" *logic*, $\mathscr{L}_{\omega\omega}(\text{pos})$, and "*negative*" *logic*, $\mathscr{L}_{\omega\omega}(\text{neg})$. As has been pointed out, mainly by Feferman, it would be interesting to have a regular $\aleph_0$-compact extension of $\mathscr{L}_{\omega\omega}(Q_1)$ that is recursively enumerable for consequence and has the interpolation property. Such a logic would combine the usefulness of $\aleph_0$-compactness *and* interpolation with the expressive power of $\mathscr{L}_{\omega\omega}(Q_1)$. The search has been unsuccessful so far. (Reasons can be found, for instance, in Proposition XVII.2.4.6.) However, the attempts to date have led to various systems possessing all desired properties up to interpolation.

In order to find a candidate besides $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ we observe that $Q_1 x\varphi(x)$ means the same as

$$(*) \qquad \exists \text{ uncountable } X\, \forall x(\neg Xx \vee \varphi(x))$$

or as

$$(**) \qquad \neg\exists \text{ countable } X\, \forall x(Xx \vee \neg\varphi(x)).$$

Thus we are led to logics that arise from $\mathscr{L}_{\omega\omega}$ by allowing quantifications over either uncountable or over countable subsets. In both cases, however, $\aleph_0$-compactness fails, since we can characterize in these logics $(\omega_1, <)$ and $(\omega, <)$, respectively. For instance, a linear ordering is isomorphic to $(\omega_1, <)$ iff it is ω_1-like and each uncountable subset has a least element.

Let us say that a set variable X occurs *negatively* (*positively*) in a formula φ, if there is an occurrence of X in φ that lies in the scope of an odd (even) number of negation signs provided $\neg$, $\wedge$, $\vee$ are the only propositional connectives in φ. Obviously, X occurs only negatively in the matrix of $(*)$ and only positively in the matrix of $(**)$. Hence, in our second, and more modest attempt, we define the logics $\mathscr{L}_{\omega\omega}(\text{neg})$ and $\mathscr{L}_{\omega\omega}(\text{pos})$ that arise from $\mathscr{L}_{\omega\omega}$ by allowing existential quantifications such as $\exists X\varphi$, with the variable X ranging over uncountable (countable) subsets, only in case X occurs at most negatively (positively) in φ.

$\mathscr{L}_{\omega\omega}(\text{neg})$ extends $\mathscr{L}_{\omega\omega}(Q^n \mid n \geq 1)$, but, according to a result of Stavi, $(\omega_1, <)$ is still characterizable (see Theorem IV.5.1.2). On the other hand, $\mathscr{L}_{\omega\omega}(\text{pos})$ turns out to be $\aleph_0$-compact and recursively enumerable for consequence. It is strictly stronger than $\mathscr{L}_{\omega\omega}(Q_1)$, because the entailment $(*)$ in Example 1 has the $\mathscr{L}_{\omega\omega}(\text{pos})$-interpolant

$$\varphi(E) \wedge \exists X\, \forall y\, \exists x(Xx \wedge Exy).$$

An easy induction shows the validity of:

$(***)$ *If $\varphi(X, \ldots)$ is an $\mathscr{L}_{\omega\omega}(\text{pos})$-formula and $\mathfrak{A} \models \varphi[M, \ldots]$ holds for some countable $M \subseteq A$, then for any countable M' such that $M \subseteq M' \subseteq A$, we have $\mathfrak{A} \models \varphi[M', \ldots]$.*

Intuitively, this means that $\mathscr{L}_{\omega\omega}(\text{pos})$ allows existential quantifications over large countable sets. The next example provides a natural generalization of this feature.

Example 4. *Stationary logic* is denoted by $\mathscr{L}_{\omega\omega}(\text{aa})$. Here we restrict ourselves to a short description that will be sufficient to give a compactness proof for models of power $\aleph_1$ in Section 3.2. A comprehensive treatment is given in Section IV.4.

We first need some set-theoretical terminology. For any set A, a subset S of the set $P_{\omega_1}(A)$ of countable subsets of A is *unbounded* (in $P_{\omega_1}(A)$), if for any $s \in P_{\omega_1}(A)$ there is some $s' \in S$ such that $s \subseteq s'$. The set S is *closed* (in $P_{\omega_1}(A)$), if the union of any countable $\subseteq$-chain in S belongs to S. The set S is said to be *cub*, if it is both closed and unbounded. The *cub filter*, $D(A)$, over A (and it really is a filter!) consists of those subsets of $P_{\omega_1}(A)$ which contain a cub set. If $A = \omega_1$, then those subsets of ω_1 which are closed and unbounded in the usual sense of ordinal number theory form a basis of $D(A)$. Intuitively, $D(A)$ may be considered as the set of those subsets of $P_{\omega_1}(A)$ which consist of "almost all" elements of $P_{\omega_1}(A)$.

The logic $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ arises from $\mathscr{L}_{\omega\omega}$ by adding new variables $X, Y, \ldots$ for countable subsets. These lead to new atomic formulas Xt (for first-order terms t). Besides the usual first-order operations, quantifications over set variables are allowed *only* by means of a new unary quantifier (aa). The meaning of (aa) is specified by the satisfaction condition:

$$\mathfrak{A} \models (\mathrm{aa})X\varphi(X) \quad \text{iff} \quad \{s \in P_{\omega_1}(A) \mid \mathfrak{A} \models \varphi[s]\} \in D(A).$$

In other words the condition means that $\mathfrak{A} \models \varphi[s]$ holds for "almost all" countable subsets s of A.

The name "stationary" suggests several features: For instance, the dual quantifier $\neg(\mathrm{aa})\neg$ to (aa) means intuitively "for stationary many" (where a stationary set is one intersecting every cub set). As the results in Chapter IV will illustrate, stationary logic is a nice resting point in the ladder of extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. According to (***) of Example 3 above, any $\mathscr{L}_{\omega\omega}(\mathrm{pos})$-formula $\exists X\varphi$ has the same meaning as $(\mathrm{aa})X\varphi$. Therefore $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ can be considered as an extension of $\mathscr{L}_{\omega\omega}(\mathrm{pos})$. It is even a strict extension (see Remark IV.4.1.2(v)).

2.3. *Cardinality Quantifiers with Complex Scopes*

There are some interesting quantifiers which are applied to pairs of formulas. The *Rescher quantifier*, Q^{R}, from Rescher [1962], is defined by the satisfaction condition:

$$\begin{aligned}\mathfrak{A} \models Q^{\mathrm{R}}xy[\varphi(x), \psi(y)] \quad &\text{iff}\\ &|\{a \in A \mid \mathfrak{A} \models \varphi[a]\}| < |\{b \in A \mid \mathfrak{A} \models \psi[b]\}|.\end{aligned}$$

The *equicardinality* or *Härtig quantifier*, I, from Härtig [1965], is defined similarly but with " = " instead of " < ". Q^{R} and I lead to the regular logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$ and $\mathscr{L}_{\omega\omega}(I)$.

Clearly, the quantifier I can be expressed by Q^{R}. On the other hand it can be seen that there is no $\mathscr{L}_{\omega\omega}(I)$-sentence of vocabulary $\{U\}$ that has the same models as $Q^{\mathrm{R}}xy[Ux, \neg Uy]$. (See also Hauschild [1981].)

Since $(\omega, <)$ can be characterized in $\mathscr{L}_{\omega\omega}(I)$ by adjoining to the usual axioms of linear orderings without last element the sentence

$$\forall xy(x = y \leftrightarrow Iuv[u < x, v < y]),$$

we see that neither $\mathscr{L}_{\omega\omega}(I)$ nor $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$ is $\aleph_0$-compact. Even more, if φ is the $\mathscr{L}_{\omega\omega}(I)$-sentence in the vocabulary $\{<, U\}$ formed from the axioms of a linear ordering by adjoining the sentence

$$\forall xy(Ux \wedge Uy \wedge Iuv[u < x, v < y] \rightarrow x = y)$$

then the $\{<\}$-reducts of the models of φ relativized to the predicate U form the class of all linear orderings that are isomorphic to the natural ordering on a set of cardinals, and this is nothing more than the class of *all* well-orderings. In the terminology to come (see Definition 3.1.1) the class of all well-orderings in RPC in $\mathscr{L}_{\omega\omega}(I)$ and hence in $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$.

2.4. Logics with Cofinality Quantifiers

Is there a regular logic strictly stronger than first-order logic that is fully compact? In Shelah [1975d] one finds a variety of examples. We mention the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$, where $Q^{\mathrm{cf}\,\omega}$ is a binary quantifier the meaning of which is given by

$$\mathfrak{A} \models Q^{\mathrm{cf}\omega}xy\ \varphi(x, y) \quad \text{iff} \quad \{(a, b) \in A \times A \mid \mathfrak{A} \models \varphi[a, b]\} \text{ is a linear ordering of its field with cofinality } \omega.$$

In Section 3.2 we sketch a proof that $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\omega})$ is fully compact and recursively enumerable for consequence. For the failure of the interpolation property see Counterexample 7.1.3(c), and for larger cofinalities, see Chapter V.

2.5. Logics with Quantifiers of Partially Ordered Prefixes

A usual first-order prefix is of "linear character" in the sense that each existential variable depends on *all* preceding universal ones. This becomes obvious by the introduction of *Skolem functions*. For instance, a formula such as

$$\forall u\ \exists v\ \forall wx\ \exists y\ \varphi(u, v, w, x, y)$$

is equivalent to

$$\exists fg\ \forall uwx\ \varphi(u, f(u), w, x, g(u, w, x)),$$

where f is a unary and g a ternary function variable. One of the simplest examples of a prefix that is not of this kind leads to the 4-ary *Henkin-quantifier* Q^{H} (Henkin [1961]). Its meaning is given by:

$$\mathfrak{A} \models Q^{\mathrm{H}}x_0 y_0 x_1 y_1\ \varphi(x_0, y_0, x_1, y_1) \quad \text{iff} \quad \text{there are functions } f_0, f_1 \colon A \rightarrow A \text{ such that for all } a_0, a_1 \in A \text{ we have } \mathfrak{A} \models \varphi[a_0, f_0(a_0), a_1, f_1(a_1)].$$

Usually $Q^{\mathrm{H}}x_0 y_0 x_1 y_1\, \varphi(x_0, y_0, x_1, y_1)$ is written more intuitively as

$$\begin{matrix} \forall x_0\ \exists y_0 \\ \forall x_1\ \exists y_1 \end{matrix} \varphi(x_0, y_0, x_1, y_1),$$

in order to display the functional dependence of the variables.

The *Henkin logic* $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$ is regular. But, if φ is the sentence

$$\exists z \begin{matrix} \forall x_0\ \exists y_0 \\ \forall x_1\ \exists y_1 \end{matrix} (z \neq y_0 \wedge (y_0 = x_1 \rightarrow y_1 = x_0))$$

and $A \neq \emptyset$, then we have $A \models \varphi$ iff there are $a \in A$ and $f_0, f_1\colon A \rightarrow A$ such that $a \notin \mathrm{rg}(f_0)$ and $f_1(f_0(b)) = b$ for all $b \in A$. This simply means that A is infinite. Hence, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$ is not $\aleph_0$-compact. Moreover, the adjunction to $\mathscr{L}_{\omega\omega}$ of quantifiers like Q^{H} that stem from partially ordered prefixes leads to the full expressive power of second-order logic. Details can be found in Section VI.1. For the mathematical relevance of these quantifiers see Barwise [1976].

2.6. *Logics with Standard Part*

An immediate way to obtain a logic in which, say, $(\omega, <)$ is characterizable, is to incorporate $(\omega, <)$ into the semantics of first-order logic as done in ω-logic. The following definition provides a generalization.

Let $\mathscr{L}$ be a logic, τ_0 a vocabulary, U a unary relation symbol not in τ_0, and $\mathfrak{K}$ a class of τ_0-structures closed under isomorphism. We define a logic $\mathscr{L}(\mathfrak{K})$ in the sense of the generalization under 1.1.3 as follows:

$$\mathscr{L}(\mathfrak{K})[\tau] = \begin{cases} \mathscr{L}[\tau], & \text{if } \tau_0 \cup \{U\} \subseteq \tau; \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Str}_{\mathscr{L}(\mathfrak{K})}[\tau] = \begin{cases} \{\mathfrak{A} \in \mathrm{Str}[\tau] \mid U^{\mathfrak{A}}\ \tau_0\text{-closed in } \mathfrak{A} \text{ and} \\ \qquad (\mathfrak{A} \restriction \tau_0) \mid U^{\mathfrak{A}} \in \mathfrak{K}\}, & \text{if } \tau_0 \cup \{U\} \subseteq \tau; \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$\mathfrak{A} \models_{\mathscr{L}(\mathfrak{K})} \varphi \quad \text{iff} \quad \mathfrak{A} \in \mathrm{Str}_{\mathscr{L}(\mathfrak{K})}[\tau_{\mathfrak{A}}],\ \varphi \in \mathscr{L}(\mathfrak{K})[\tau_{\mathfrak{A}}], \text{ and } \mathfrak{A} \models_{\mathscr{L}} \varphi.$$

In the many-sorted case one can proceed similarly (and even dispense with the analogues of U by introducing new sorts, see also Remark 3.1.2).

If $\mathfrak{K} = \{\mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A}\}$, we write $\mathscr{L}(\mathfrak{A})$ instead of $\mathscr{L}(\mathfrak{K})$.

An interesting example in addition to ω-logic, $\mathscr{L}_{\omega\omega}(\omega, <)$, is $\mathscr{L}_{\omega\omega}(\mathfrak{K})$, where $\mathfrak{K}$ is the class of $\aleph_1$-like orderings. In both cases one can dispense with U, as the task of U can be taken over by the field of $<$.

The following fact plays a key role in the compactness proof for $\mathscr{L}_{\omega\omega}(Q_1)$ as given in Section 3.1.

2.6.1 Theorem. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is $\aleph_0$-compact.*

Proof. Let $\tau_{\mathfrak{A}} = \{<\}$ for $\mathfrak{A} \in \mathfrak{K}$ and τ a fixed *countable* vocabulary, $< \in \tau$. The τ-*regularity scheme* $\Sigma = \Sigma(\tau)$ consists of the $\mathscr{L}_{\omega\omega}[\tau]$-sentences of the form

$$\forall \mathbf{x}\, \forall x\, \exists y\, \forall u < x(\exists v \in \mathit{field}(<)\varphi(u, v, \mathbf{x}) \to \exists v < y\, \varphi(u, v, \mathbf{x})).$$

It is sufficient to prove that for all $\Phi \subseteq \mathscr{L}_{\omega\omega}[\tau]$,

$$(*) \qquad \Phi \text{ has an } \mathscr{L}_{\omega\omega}(\mathfrak{K})\text{-model} \quad \text{iff} \quad \Psi \text{ has an } \mathscr{L}_{\omega\omega}\text{-model},$$

where $\Psi = \Phi \cup \Sigma \cup \{<$ *is a linear ordering of its field without last element*$\}$.

The implication from left to right is clear, because $\aleph_1$ is regular.

For the other direction assume that Ψ has a τ-model $\mathfrak{A}$, where $\mathfrak{A}$ can be chosen countable. We show that there exists a countable τ-structure $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $<^{\mathfrak{B}}$ is a proper end extension of $<^{\mathfrak{A}}$. Then we can repeat this process $\aleph_1$-times, taking unions at limit stages, and arrive at an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model of Φ.

Let $\Delta(\mathfrak{A})$ be the elementary diagram of $\mathfrak{A}$ formulated with new constants $\underline{a}$ for $a \in A$, c a new constant, and let $\Xi = \Delta(\mathfrak{A}) \cup \{\underline{a} < c \mid a \in \text{field}(<^{\mathfrak{A}})\}$. We have to show that Ξ has a model which, for all $a_0 \in \text{field}(<^{\mathfrak{A}})$, omits the type $\{x \neq \underline{a} \mid a <^{\mathfrak{A}} a_0\} \cup \{x < \underline{a_0}\}$. In order to prove this, let $a_0 \in \text{field}(<^{\mathfrak{A}})$ be given and a formula $\chi(x, y)$ of vocabulary $\tau \cup \{\underline{a} \mid a \in A\}$ be such that

$$(1) \qquad \Xi \cup \{\exists x\, \chi(x, c)\} \text{ has a model.}$$

We have to show that

$$(2) \qquad \Xi \cup \left\{\exists x\left(\chi(x, c) \wedge \left(\bigvee_{a <^{\mathfrak{A}} a_0} x = \underline{a} \vee \underline{a_0} \leq x \vee x \notin \mathit{field}(<)\right)\right)\right\}$$

has a model.

Let us write $\exists$ arb. lg. $w\psi(w, \ldots)$ for $\forall u \in \mathit{field}(<)\, \exists w > u\psi(w, \ldots)$. By an easy compactness argument we see that (1) is equivalent to:

$$(1') \qquad (\mathfrak{A}, (a)_{a \in A}) \models \exists \text{ arb. lg. } w\, \exists x\, \chi(x, w),$$

and that it is sufficient to prove instead of (2):

$$(2') \qquad \begin{aligned} &(\mathfrak{A}, (a)_{a \in A}) \models \bigvee_{a <^{\mathfrak{A}} a_0} \exists \text{ arb.lg. } w\chi(\underline{a}, w), \quad \text{or} \\ &(\mathfrak{A}, (a)_{a \in A}) \models \exists \text{ arb.lg. } w\, \exists x(\chi(x, w) \wedge (\underline{a_0} \leq x \vee x \notin \mathit{field}(<))). \end{aligned}$$

For a proof of (2′) assume the first disjunct to be false. Then for each $a <^{\mathfrak{A}} a_0$ there is some $b \in \text{field}(<^{\mathfrak{A}})$ such that for all $d \in \text{field}(<^{\mathfrak{A}})$, if $\chi(\underline{a}, \underline{d})$ holds in $(\mathfrak{A}, (a)_{a \in A})$, then $d <^{\mathfrak{A}} b$. As $\mathfrak{A}$ satisfies the τ-regularity scheme, there is a uniform bound b_0 of this kind for all $a <^{\mathfrak{A}} a_0$. Hence, because of (1′), the second disjunct must be true. □

The proof yields more. From (∗) we obtain for $\Phi \cup \{\varphi\} \subseteq \mathscr{L}_{\omega\omega}(\mathfrak{K})[\tau]$:

$$\Phi \models_{\mathscr{L}_{\omega\omega}(\mathfrak{K})} \varphi \quad \text{iff} \quad \Psi \models_{\mathscr{L}_{\omega\omega}} \varphi.$$

If Φ is recursively enumerable, then so is Ψ. Thus we have:

2.6.2 Corollary. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is recursively enumerable for consequence.* □

2.7. Infinitary Logics

We shall not go into details here. Infinitary logics of type $\mathscr{L}_{\kappa\lambda}$ and admissible fragments will be treated in Chapters VIII and IX. Infinitary quantifiers such as the game quantifier G are described in Chapter X. For $\mathscr{L}_{\infty\omega}$ and arguments for its naturalness, see Section III.3 and, in particular, Section XVII.2.2. Occasionally we shall also consider logics such as $\mathscr{L}_{\kappa\lambda}(Q_1)$.

In $\mathscr{L}_{\omega\omega}$, the set $\{\neg, \wedge, \vee\}$ forms a complete system of propositional connectives. Of course, in $\mathscr{L}_{\omega_1\omega}$, where we use only $\neg$ and the generalizations of $\wedge, \vee$, we are far away from propositional completeness. Hence the question arises whether there are other reasonable (infinitary) propositional connectives for $\mathscr{L}_{\omega_1\omega}$. The answer is, in some sense, positive; details can be found in the references given in Section III.3.8.

3. Comparing Logics

In the preceding section we intuitively compared logics with respect to their expressive power. The aim of this section is to give precise definitions for the comparison of logics by use of elementary and projective classes and to present some concrete examples that will illustrate the methodological importance of the latter notions.

3.1. Elementary and Projective Classes

We begin with a basic definition.

3.1.1 Definition. Let $\mathscr{L}$ be a logic and $\mathfrak{K}$ a class of τ-structures.

We say that $\mathfrak{K}$ is an *elementary class* in $\mathscr{L}$ (or that $\mathfrak{K}$ is EC in $\mathscr{L}$, or that $\mathfrak{K} \in \text{EC}_{\mathscr{L}}$) iff there is $\varphi \in \mathscr{L}[\tau]$ such that $\mathfrak{K} = \text{Mod}^{\tau}_{\mathscr{L}}(\varphi)$.

We say that $\mathfrak{K}$ is a *projective class* in $\mathscr{L}$ (or that $\mathfrak{K}$ is PC in $\mathscr{L}$, or that $\mathfrak{K} \in \mathrm{PC}_{\mathscr{L}}$) iff there is $\tau' \supseteq \tau$, having, in the many sorted case, the same sort symbols as τ, and a class $\mathfrak{K}'$ of τ'-structures, $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that $\mathfrak{K} = \{\mathfrak{A} \restriction \tau \mid \mathfrak{A} \in \mathfrak{K}'\}$, the class of τ-reducts of $\mathfrak{K}'$.

On the other hand $\mathfrak{K}$ is a *relativized projective class* in $\mathscr{L}$ (or $\mathfrak{K}$ is RPC in $\mathscr{L}$, or $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$) iff (in the one-sorted case) there is $\tau' \supseteq \tau$, a unary relation symbol $U \in \tau' \backslash \tau$, and a class $\mathfrak{K}'$ of vocabulary τ', $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that

$$\mathfrak{K} = \{(\mathfrak{A} \restriction \tau) | U^{\mathfrak{A}} \mid \mathfrak{A} \in \mathfrak{K}' \text{ and } U^{\mathfrak{A}} \text{ is } \tau\text{-closed in } \mathfrak{A}\};$$

or (in the many-sorted case) there is $\tau' \supseteq \tau$ and a class $\mathfrak{K}'$ of τ'-structures, $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that $\mathfrak{K} = \{\mathfrak{A} \restriction \tau \mid \mathfrak{A} \in \mathfrak{K}'\}$.

Using an intuitive notation, we can say for instance that $\mathfrak{K}$ is RPC in $\mathscr{L}$ in the many-sorted version, if there is some $\tau' \supseteq \tau$ and $\psi \in \mathscr{L}[\tau']$ such that $\mathfrak{K} = \mathrm{Mod}^{\tau}_{\mathscr{L}}(\exists_{\tau' \backslash \tau} \psi)$.

3.1.2 Remarks. For all usual logics $\mathscr{L}$ and classes $\mathfrak{K}$ of one-sorted structures, we have $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$ in the one-sorted version iff $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$ in the many-sorted version. The same is true for all regular logics, if we restrict ourselves to finite vocabularies. (The direction from right to left can be shown by unification of domains, and that from left to right by the dual procedure.) Obviously, we have "PC $\subseteq$ RPC" for any logic $\mathscr{L}$ containing sentences such as $\forall x U x$; the inclusion is strict for $\mathscr{L}_{\omega\omega}$, but not for $\mathscr{L}_{\omega_1\omega}$. For details concerning these and other well-known logics, see Oikkonen [1979c].

In general it is not true that every class PC in $\mathscr{L}$ is EC in $\mathscr{L}$, even if $\mathscr{L} = \mathscr{L}_{\omega\omega}$. A counterexample for $\mathscr{L}_{\omega\omega}$ is given by the class of infinite sets. The question whether any class $\mathfrak{K}$ of τ-structures such that $\mathfrak{K}$ *and* $\overline{\mathfrak{K}} = \mathrm{Str}[\tau] \backslash \mathfrak{K}$ are (R)PC in $\mathscr{L}$, is EC in $\mathscr{L}$, will lead to an interesting interpolation property, the so-called Δ-*interpolation* (see Section 7.2). The following simple equivalence shows that interpolation is a generalization of Δ-interpolation.

3.1.3 Proposition. *For any logic $\mathscr{L}$ having the negation property, the following are equivalent:*

(i) *$\mathscr{L}$ has the interpolation property.*
(ii) *For all τ, any two disjoint classes $\mathfrak{K}_0$, $\mathfrak{K}_1$ of τ-structures that are* PC *in $\mathscr{L}$ (one-sorted case) or* RPC *in $\mathscr{L}$ (many-sorted case), can be separated by an elementary class; that is, there is a class $\mathfrak{K} \in \mathrm{EC}_{\mathscr{L}}$ such that $\mathfrak{K}_0 \subseteq \mathfrak{K}$ and $\mathfrak{K}_1 \subseteq \overline{\mathfrak{K}}$.* □

What does it mean to say that a logic $\mathscr{L}^*$ is as strong as $\mathscr{L}$? The model-theoretical point of view offers several ways that lead to a precise definition, starting for example from the following concepts:

(*) *For any $\mathscr{L}$-sentence φ there is an $\mathscr{L}^*$-sentence φ^* having the same meaning as φ.*

(**) *Structures that can be distinguished in $\mathscr{L}$, can also be distinguished in $\mathscr{L}^*$.*

From (*) we obtain the usual definition if we identify the meaning of a sentence with its class of models:

3.1.4 Definition. Let $\mathscr{L}$, $\mathscr{L}^*$ be logics. We say that $\mathscr{L}^*$ is *as strong as* $\mathscr{L}$, in symbols $\mathscr{L} \leq \mathscr{L}^*$, iff every class EC in $\mathscr{L}$ is EC in $\mathscr{L}^*$. Similarly, $\mathscr{L}$ and $\mathscr{L}^*$ are *equally strong* or *equivalent*, in symbols $\mathscr{L} \equiv \mathscr{L}^*$, iff both $\mathscr{L} \leq \mathscr{L}^*$ and $\mathscr{L}^* \leq \mathscr{L}$. Finally, we say that $\mathscr{L}^*$ is *stronger than* $\mathscr{L}$, in symbols $\mathscr{L} < \mathscr{L}^*$, iff $\mathscr{L} \leq \mathscr{L}^*$ and not $\mathscr{L} \equiv \mathscr{L}^*$.

Obviously, $\leq$ is a partial ordering on logics.

Concept (**) can be made precise by the notion of $\mathscr{L}$-equivalence of structures:

3.1.5 Definition. $\mathscr{L} \leq_{\equiv} \mathscr{L}^*$ iff for all τ and all $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}[\tau]$, if $\mathfrak{A} \equiv_{\mathscr{L}^*} \mathfrak{B}$, then $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$.

When we compare the two notions, we immediately see that $\mathscr{L} \leq \mathscr{L}^*$ implies $\mathscr{L} \leq_{\equiv} \mathscr{L}^*$. The other direction can be false; for instance $\mathscr{L}_{\infty G} \leq_{\equiv} \mathscr{L}_{\infty\omega}$, as $\mathscr{L}_{\infty G}$ has the Karp property, but $\mathscr{L}_{\infty\omega} < \mathscr{L}_{\infty G}$ (see the remark following Theorem 4.3.2 and Section X.3.1). Whereas we shall refer to $\leq_{\equiv}$ only occasionally, the relation $\leq$ and its generalizations (see Definition 3.1.6 below) will actually turn out to be of great methodological importance.

From the examples in Section 2 and the results there stated, we obtain that

$$\mathscr{L}_{\omega\omega} < \mathscr{L}^{w2} < \mathscr{L}^2;$$

$$\mathscr{L}_{\omega\omega} < \mathscr{L}_{\omega\omega}(Q_1) < \mathscr{L}_{\omega\omega}(Q_1^2) < \cdots;$$

$$\mathscr{L}_{\omega\omega}(Q_1) < \mathscr{L}_{\omega\omega}(\mathrm{pos}) < \mathscr{L}_{\omega\omega}(\mathrm{aa});$$

$$\mathscr{L}_{\omega\omega} < \mathscr{L}_{\omega\omega}(I) < \mathscr{L}_{\omega\omega}(Q^{\mathrm{R}}).$$

Moreover, one can easily prove that $\mathscr{L}^{w2} < \mathscr{L}_{\omega_1\omega}$, but $\mathscr{L}^2 \nleq \mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\omega_1\omega} \nleq \mathscr{L}^2$. For the class $\mathfrak{K}$ of $\aleph_1$-like orderings we have $\mathscr{L}_{\omega\omega}(\mathfrak{K}) \leq \mathscr{L}_{\omega\omega}(Q_1)$. However, the other direction is false as can be seen from the sentence $Q_1 x\, x = x$. In order to remedy this situation to some extent, we introduce some new relations between logics, taking (relativized) projective classes instead of elementary ones in Definition 3.1.4.

3.1.6 Definition. For logics $\mathscr{L}$ and $\mathscr{L}^*$, $\mathscr{L} \leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}^*$ iff every class that is (R)PC in $\mathscr{L}$, is (R)PC in $\mathscr{L}^*$. Analogously $<_{(\mathrm{R})\mathrm{PC}}$ and $\equiv_{(\mathrm{R})\mathrm{PC}}$ can be defined.

Now we can state:

3.1.7 Proposition. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(Q_1) \leq_{\mathrm{RPC}} \mathscr{L}_{\omega\omega}(\mathfrak{K})$, provided that for $\mathscr{L}_{\omega\omega}(Q_1)$ we do not allow the symbol $<$ that is used for the orderings in $\mathfrak{K}$.*

Proof. Let $\varphi \in \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ be given such that φ contains a subformula $Q_1 x\, \psi(x, \mathbf{y})$ with Q_1 not in ψ. We take an appropriate new function symbol f and then, writing $f_{\mathbf{y}}(x)$ for $f(x, \mathbf{y})$, we replace $Q_1 x \psi(x, \mathbf{y})$ in φ by a formula $\chi = \chi(\mathbf{y})$ expressing

$\{f_{\mathbf{y}}(x) \mid \psi(x, \mathbf{y})\}$ *is an unbounded subset of field*$(<)$.

Also we add to the resulting sentence, as a conjunct, the sentence $\forall \mathbf{y}(\chi \vee \vartheta)$, where ϑ means that

$\lambda x f_{\mathbf{y}}(x)$ *is injective on* $\{x \mid \psi(x, \mathbf{y})\}$ *and* $\{f_{\mathbf{y}}(x) \mid \psi(x, \mathbf{y})\}$ *is a bounded subset of field*$(<)$.

Repeating this process until all occurrences of Q_1 are eliminated, we arrive at some $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-sentence $\tilde{\varphi}$ in some vocabulary $\tilde{\tau} \supseteq \tau$ such that

$$\mathrm{Mod}^{\tau}_{\mathscr{L}_{\omega\omega}(Q_1)}(\varphi) = \{\mathfrak{A} \upharpoonright \tau \mid \mathfrak{A} \in \mathrm{Mod}^{\tilde{\tau}}_{\mathscr{L}_{\omega\omega}(\mathfrak{K})}(\tilde{\varphi})\}. \quad \square$$

3.1.8 Corollary. *$\mathscr{L}_{\omega\omega}(Q_1)$ is $\aleph_0$-compact.*

Proof. Let $\Phi \subseteq \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ be countable such that every finite subset of Φ has an $\mathscr{L}_{\omega\omega}(Q_1)$-model. We may suppose $< \notin \tau$. Then every finite subset of $\tilde{\Phi}$ has an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model, where $\tilde{\Phi} = \{\tilde{\varphi} \mid \varphi \in \Phi\}$ and all the function symbols used in the construction of the sentences $\tilde{\varphi}$ are chosen to be different. By $\aleph_0$-compactness of $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ (see Theorem 2.6.1) $\tilde{\Phi}$ has an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model, and hence Φ has an $\mathscr{L}_{\omega\omega}(Q_1)$-model. $\square$

When we analyze the preceding argument, we see that it is essentially based on the ordering $\mathscr{L}_{\omega\omega}(Q_1) \leq_{\mathrm{RPC}} \mathscr{L}_{\omega\omega}(\mathfrak{K})$. Generalizing, we obtain the first part of:

3.1.9 Proposition. *Assume $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ and κ to be infinite. Then*:

(i) *If $\mathscr{L}^*$ is κ-compact, then so is $\mathscr{L}$. Hence, if $\mathscr{L}^*$ is compact, then so is $\mathscr{L}$.*
(ii) *If $\mathscr{L}^*$ has the Löwenheim–Skolem property down to κ, then so does $\mathscr{L}$.*

Proof. To prove part (ii) for instance in the many-sorted case, assume that $\mathscr{L}^*$ has the Löwenheim–Skolem property down to κ and that φ is a satisfiable sentence from $\mathscr{L}[\tau]$. As $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$, there is some $\tau^* \supseteq \tau$ and a sentence $\varphi^* \in \mathscr{L}^*[\tau^*]$ such that $\emptyset \neq \mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi) = \mathrm{Mod}^{\tau^*}_{\mathscr{L}^*}(\varphi^*) \upharpoonright \tau$. By assumption, φ^*, having a model, has a model $\mathfrak{A}^*$ of power $\leq \kappa$. Hence $\mathfrak{A}^* \upharpoonright \tau$ is a model of φ of power $\leq \kappa$. $\square$

Proposition 3.1.9(i) is used in numerous compactness proofs. Similar to Corollary 3.1.8, the (κ-) compactness of the logic $\mathscr{L}$ in question is reduced to the (κ-) compactness of some other logic $\mathscr{L}^*$ by showing $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ and proving (κ-) compactness for $\mathscr{L}^*$. Often $\mathscr{L}^*$ is first-order logic with some additional restrictions (for instance $\aleph_1$-like orderings). Some further examples will be presented in Section 3.2.

The general scheme underlying all these proofs can be formulated as follows: In order to show that a logic $\mathscr{L}$ has some property P, one

(a) *first finds a logic $\mathscr{L}^*$ such that $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$;*
(b) *then proves that $\mathscr{L}^*$ has property P;*
(c) *and finally verifies that P descends from $\mathscr{L}^*$ to $\mathscr{L}$.*

If P means (κ-) compactness or the Löwenheim–Skolem property down to κ, step (c) above becomes superfluous because of Proposition 3.1.9. In later sections we will see that numerous other properties are inherited downward along $\leq_{\mathrm{RPC}}$, thus enlarging the applicability of the reduction method considerably.

In many cases, if $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$, completeness properties also descend from $\mathscr{L}^*$ to $\mathscr{L}$. Rather than give a general theorem, we will confine ourselves to examples. In order to present the first one, we again let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. In the terminology of the proofs of Proposition 3.1.7 and Corollary 3.1.8 we have for any $\Phi \subseteq \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ and any $\varphi \in \mathscr{L}_{\omega\omega}(Q_1)[\tau]$, that if $< \notin \tau$, then

$$\Phi \models_{\mathscr{L}_{\omega\omega}(Q_1)} \varphi \quad \text{iff} \quad \tilde{\Phi} \models_{\mathscr{L}_{\omega\omega}(\mathfrak{K})} \tilde{\varphi}.$$

As the transition from an $\mathscr{L}_{\omega\omega}(Q_1)$-sentence ψ to $\tilde{\psi}$ is effective, we obtain the following result from Corollary 2.6.2:

3.1.10 Theorem. *$\mathscr{L}_{\omega\omega}(Q_1)$ is recursively enumerable for consequence.* $\square$

3.2. A Reduction Method

Many applications of the reduction scheme given in Section 3.1 can be systematized in a way first made explicit in Hutchinson [1976b]. The method applies to logics $\mathscr{L}$ that admit a nice set-theoretical description, and the corresponding logics $\mathscr{L}^*$ are based on specific models of set theory. Without exhausting its full power, we illustrate the method by some examples. (See also Section XVII.2.3.) First, we treat $\mathscr{L}_{\omega\omega}(Q_1)$. Then we sketch a similar procedure for $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ and for $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$. Besides Corollary 3.1.8 and Theorem 3.1.10 ($\aleph_0$-compactness and recursive enumerability for consequence) we show that $\mathscr{L}_{\omega\omega}(Q_1)$ has the Löwenheim–Skolem property down to $\aleph_1$. The reader is urged to compare the following proofs with those given in Section 3.1.

We set $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$. Our first considerations aim at a suitable logic $\mathscr{L}^*$ based on models of set theory which is $\geq_{\mathrm{RPC}} \mathscr{L}$, $\aleph_0$-compact and has the Löwenheim–Skolem property down to $\aleph_1$. For our purposes it will be sufficient to have an intuitive description of $\mathscr{L}^*$. A precise definition is left to the reader.

Let τ be a countable vocabulary, which is kept fixed for the argument to follow, and let $\boldsymbol{\sigma} = \{\varepsilon, c_0\} \cup \{c^{\S} \mid \S \in \tau\}$, where ε is a new binary relation symbol for the $\in$-relation between sets, and c_0 and the $c^{\S}$ are new constants. Next we define a set

$\Gamma = \Gamma(\boldsymbol{\tau})$ of $\mathscr{L}_{\omega\omega}[\boldsymbol{\sigma}]$-sentences that provides us with a set-theoretical description of $\boldsymbol{\tau}$-structures. In fact, we set

$$\Gamma = \{\psi_0\} \cup \{\psi^{\S} \mid \S \in \boldsymbol{\tau}\},$$

where

$$\psi_0 \text{ is } c_0 \neq \emptyset, \text{ i.e. } \psi_0 = \exists x\, x \varepsilon c_0, \text{ and}$$

$$\psi^{\S} = \begin{cases} c^{\S}\ \varepsilon\ c_0, & \text{if } \S \text{ is a constant;} \\ c^{\S} \subseteq c_0^n, & \text{if } \S \text{ is an } n\text{-ary relation symbol;} \\ c^{\S}: c^n \to c_0, & \text{if } \S \text{ is an } n\text{-ary function symbol.} \end{cases}$$

If $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$, let φ^* be a natural set-theoretic translation of φ. For example, if φ is

$$\exists z\, Q_1 x(Rzx \wedge \neg f(x) = d)$$

put φ^* equal to

$$\exists z \varepsilon c_0 \,|\{x \varepsilon c_0 \,|(z, x) \varepsilon c^R \wedge \neg c^f(x) = c^d\}| \geq \aleph_1.$$

The transition from φ to φ^* enables us to treat $\mathscr{L}$-satisfaction in models of ZFC (Zermelo–Fraenkel set theory with the axiom of choice). For technical reasons, we consider a system (ZFC) that differs from ZFC in having only finitely many instances of the axiom scheme of replacement, but that is strong enough to yield all set-theoretical facts we need. The reader should think of (ZFC) as ZFC and verify that at the end we have needed only finitely many axioms of replacement. The main reason for introducing (ZFC) is the following: In contrast to the situation with ZFC, one can prove that for (ZFC) there are cofinally many ordinals α for which $(V_\alpha, \in_{V_\alpha})$ is a model of (ZFC). (V_α denotes the set of all sets of rank $<\alpha$.)

Next, we call a structure $\mathfrak{M}$ *good*, if $\varepsilon \in \boldsymbol{\tau}_{\mathfrak{M}}$, $(M, \varepsilon^{\mathfrak{M}}) \models$ (ZFC), and

$$(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}}) = (\{a \in M \mid \mathfrak{M} \models a \varepsilon \aleph_1\}, \{(a, b) \in M \times M \mid \mathfrak{M} \models a \varepsilon b \wedge b \varepsilon \aleph_1\})$$

is an $\aleph_1$-like ordering. For good models $\mathfrak{M}$ (un-)countability in $\mathfrak{M}$ means (un-)countability in the real universe. This can be made precise in the following way. If $\mathfrak{A}$ is a $\boldsymbol{\tau}$-structure, there is a minimal ordinal $\alpha > \omega_1$ such that $\mathfrak{A} \in V_\alpha$ and $(V_\alpha, \in_{V_\alpha}) \models$ (ZFC). We expand $(V_\alpha, \in_{V_\alpha})$ to a good $\boldsymbol{\sigma}$-model $\mathfrak{M}(\mathfrak{A})$ of Γ such that c_0 and the $c^{\S}$ describe $\mathfrak{A}$ in $\mathfrak{M}(\mathfrak{A})$; that is,

$$A = \{a \in M = V_\alpha \mid \mathfrak{M}(\mathfrak{A}) \models a \varepsilon c_0\}$$

and, say, for unary $f \in \boldsymbol{\tau}$,

$$f^{\mathfrak{A}} = \{(a, b) \in M \times M \mid \mathfrak{M}(\mathfrak{A}) \models (a, b) \varepsilon c^f\}.$$

Conversely, any good $\boldsymbol{\sigma}$-model $\mathfrak{M}$ of Γ yields a $\boldsymbol{\tau}$-structure $\mathfrak{A}(\mathfrak{M})$ such that c_0 and the $c^{\S}$ describe $\mathfrak{A}(\mathfrak{M})$ in $\mathfrak{M}$. Using these notations, we have:

Lemma A. *For any* $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$, $\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau}]$ *and* $\mathfrak{M} \in \mathrm{Str}[\boldsymbol{\sigma}]$,

(i) *if* $\mathfrak{A} \models_{\mathscr{L}} \Phi$, *then* $\mathfrak{M}(\mathfrak{A})$ *is a good model of* $\Phi^* \cup \Gamma$; *and*
(ii) *if* $\mathfrak{M}$ *is a good model of* $\Phi^* \cup \Gamma$, *then* $\mathfrak{A}(\mathfrak{M}) \models \Phi$.

Proof. By induction one gets that for any $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ and any $\boldsymbol{\tau}$-structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ iff $\mathfrak{M}(\mathfrak{A}) \models \varphi^*$. Part (ii) is proved similarly. □

As stated above, we leave it to the reader to define a logic $\mathscr{L}^*$ that has as a standard part the class of good $\{\varepsilon\}$-structures and to show $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ (for ε-free sentences).

The next lemma yields $\aleph_0$-compactness of $\mathscr{L}^*$.

Lemma B. *For* $\Psi \subseteq \mathscr{L}[\boldsymbol{\sigma}]$, *the following are equivalent*:

(i) $\Psi \cup$ (ZFC) *has a model.*
(ii) $\Psi \cup$ (ZFC) *has a good model of power* $\aleph_1$. □

The direction from (ii) to (i) is trivial. For the other direction we invoke the so-called Keisler–Morley lemma (see Theorem IV.3.2.5(ii)), which is here stated for its own interest:

3.2.1 Lemma (Keisler, Morley). *Let* $\mathfrak{M}$ *be a countable* $\{\varepsilon\}$*-model of* (ZFC). *Then there exists a countable* $\{\varepsilon\}$*-structure* $\mathfrak{M}' \succ \mathfrak{M}$ *such that* $(\aleph_1^{\mathfrak{M}'}, \varepsilon_{\aleph_1}^{\mathfrak{M}'})$ *is a proper end extension of* $(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}})$.

Now, to prove the other implication in Lemma B, we start with a countable model $\mathfrak{M}$ of $\Psi \cup$ (ZFC) and build an elementary chain $(\mathfrak{M}_\alpha)_{\alpha \leq \aleph_1}$, taking unions at limit points and setting $\mathfrak{M}_0 = \mathfrak{M}$ and $\mathfrak{M}_{\alpha+1} = \mathfrak{M}'_\alpha$ in the sense of Lemma 3.2.1. (The additional constants in $\boldsymbol{\sigma}$ are not essential.) Then $\mathfrak{M}_{\aleph_1}$ satisfies (ii) of Lemma B.

We can now show that $\mathscr{L}$ is $\aleph_0$-compact. Assume $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$, and every finite subset of Φ has a model. Then, by part (i) of Lemma A, every finite subset of $\Phi^* \cup \Gamma$ has a model, and hence so does $\Phi^* \cup \Gamma$. Using Lemma B and part (ii) of Lemma A, we see that Φ has a model of power $\leq \aleph_1$. In particular, we also obtain the conclusion that $\mathscr{L}$ has the Löwenheim–Skolem property down to $\aleph_1$. Finally, to show that $\mathscr{L}$ is recursively enumerable for consequence, we observe that for any $\Phi \cup \{\varphi\} \subseteq \mathscr{L}[\boldsymbol{\tau}]$,

$$\Phi \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \Phi^* \cup \Gamma \cup (\mathrm{ZFC}) \models_{\mathscr{L}_{\omega\omega}} \varphi^*,$$

and that the operation * is effective.

In concluding this subsection, we digress to take a brief look at $\mathscr{L}_{\omega\omega}(\text{aa})$ for structures of power $\aleph_1$ (see Section IV.4.2 for the general case) as well as at $\mathscr{L}_{\omega\omega}(Q^{\text{cf}\,\omega})$.

3.2.2 Theorem. *$\mathscr{L}_{\omega\omega}(\text{aa})$, restricted to structures of power $\aleph_1$, is $\aleph_0$-compact and recursively enumerable for consequence.*

Proof. We proceed in a manner similar to that for $\mathscr{L}_{\omega\omega}(Q_1)$. A structure $\mathfrak{M}$ with $\varepsilon \in \boldsymbol{\tau}_{\mathfrak{M}}$ is called *good* if $\mathfrak{M} \models (\text{ZFC})$, $(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}})$ is an $\aleph_1$-like ordering that admits a continuous embedding π of the real $\aleph_1$, and further, for every $s \in M$ such that $\mathfrak{M} \models$ *s is a stationary subset of* $\aleph_1$, the set $\{a \in M \mid a\varepsilon^{\mathfrak{M}}s\} \cap \text{rg}(\pi)$ is stationary in $\text{rg}(\pi)$. The analogue of Lemma A is an exercise on closed unbounded subsets of $\aleph_1$. The analogue of Lemma B uses a stronger form of the Keisler–Morley lemma due to Hutchinson [1976a], according to which, given some $s \in M$ which is a stationary subset of $\aleph_1$ in $\mathfrak{M}$, the structure $\mathfrak{M}'$ can be chosen such that $\aleph_1^{\mathfrak{M}'}$ has a least new element, say p, and $p\varepsilon^{\mathfrak{M}'}s$.

Now, to obtain a good elementary extension of some countable model $\mathfrak{M}$ of (ZFC), one splits the real $\aleph_1$ into $\aleph_1$ disjoint stationary subsets S_α (for $\alpha < \aleph_1$) and builds an elementary chain $(\mathfrak{M}_\alpha)_{\alpha \le \aleph_1}$ over $\mathfrak{M} = \mathfrak{M}_0$ by Hutchinson's lemma such that for each $s \in M_{\aleph_1}$ which is a stationary subset of $\aleph_1$ in $\mathfrak{M}_{\aleph_1}$, there is some $\alpha < \aleph_1$ with $\pi(\beta) = p_\beta \varepsilon^{\mathfrak{M}_{\beta+1}} s$ for all sufficiently large $\beta \in S_\alpha$. We describe the successor step. For simplicity we assume that all M_α are chosen as subsets of some fixed set $\{a_\alpha \mid \alpha < \aleph_1\}$, where $a_\alpha \neq a_\beta$ for $\alpha < \beta < \aleph_1$. Suppose that $\beta < \aleph_1$ and $\mathfrak{M}_\beta$ has already been constructed and is a countable elementary extension of $\mathfrak{M}$. Let $\beta \in S_\alpha$. Define s to be a_α, if a_α is a stationary subset of $\aleph_1$ in $\mathfrak{M}_\beta$ and to be $\aleph_1^{\mathfrak{M}_\beta}$ else. Then choose $\mathfrak{M}_{\beta+1}$ according to Hutchinson's lemma with a least new countable ordinal $\pi(\beta) = p_\beta$, $p_\beta \varepsilon^{\mathfrak{M}_{\beta+1}} s$. $\square$

3.2.3 Theorem. *$\mathscr{L}_{\omega\omega}(Q^{\text{cf}\,\omega})$ is compact, recursively enumerable for consequence and has the Löwenheim–Skolem property down to $\aleph_1$.*

In this case $\boldsymbol{\tau}$ need not be countable. We call a structure $\mathfrak{M}$ with $\varepsilon \in \boldsymbol{\tau}_{\mathfrak{M}}$ *good*, if $\mathfrak{M} \models (\text{ZFC})$, $(\omega^{\mathfrak{M}}, \varepsilon_\omega^{\mathfrak{M}})$ has cofinality ω, and for all $b \in M$ that are uncountable regular cardinals in $\mathfrak{M}$, $(b, \varepsilon_b^{\mathfrak{M}})$ has cofinality $\geq \omega_1$. Then the analogue of Lemma A is routine. In the analogue of Lemma B, we have to cancel the limitation of power in part (ii), if $|\boldsymbol{\tau}| \geq \aleph_2$. The role of the Keisler–Morley lemma is taken over by:

3.2.4 Lemma. *Every* (ZFC)*-model $\mathfrak{M} = (M, \varepsilon^{\mathfrak{M}})$ has a good elementary extension.*

Proof. We start with a suitable chain construction that yields a structure $\mathfrak{M}' \succ \mathfrak{M}$ such that for all $b' \in M'$ that are uncountable regular cardinals in $\mathfrak{M}'$, $(b', \varepsilon_{b'}^{\mathfrak{M}'})$ has cofinality ω_1. A good extension $\mathfrak{M}'' \succ \mathfrak{M}'$ can now be constructed as the union of an elementary chain of length ω, where $\mathfrak{M}_0 = \mathfrak{M}'$ and for each i, $\mathfrak{M}_{i+1} \succ \mathfrak{M}_i$, $\omega^{\mathfrak{M}_i}$ gets longer in $\mathfrak{M}_{i+1}$ and no regular uncountable cardinal of $\mathfrak{M}_i$ gets longer in $\mathfrak{M}_{i+1}$. To obtain $\mathfrak{M}_{i+1}$ from $\mathfrak{M}_i$, one defines inside $\mathfrak{M}_i$ an ultrapower

$$\mathfrak{M}_i^{\omega^{\mathfrak{M}_i}}/\mathscr{U} = \mathfrak{M}_i',$$

where $\mathscr{U}$ is some free ultrafilter on $\omega^{\mathfrak{M}_i}$ in $\mathfrak{M}_i$. Then $\mathfrak{M}_{i+1}$ is chosen such that $\mathfrak{M}_{i+1} \cong \mathfrak{M}'_i$ via an extension of the canonical embedding of $\mathfrak{M}_i$ into $\mathfrak{M}'_i$ (in the real universe).

When checking the details, one sees that the proof of the ultrafilter theorem for $\mathfrak{M}'_i$ requires the instances

$$\forall \mathbf{z}(\forall x \in \omega\, \exists y\, \varphi(x, y, \mathbf{z}) \to \exists u\, \forall x \in \omega\, \exists y \in u \varphi(x, y, \mathbf{z}))$$

of the collection scheme. These can be added to (ZFC), since they are satisfied in $(V_\alpha, \in_{V_\alpha})$, if α is a limit ordinal of cofinality $\neq \omega$, and there are cofinally many such α, for which $(V_\alpha, \in_{V_\alpha})$ is a model of (ZFC). □

4. *Lindström Quantifiers*

Let $\mathfrak{K}$ be a class of structures of some fixed (finite) vocabulary closed under isomorphism. For a given logic $\mathscr{L}$, is there an extension of $\mathscr{L}$ more natural than $\mathscr{L}(\mathfrak{K})$, in which $\mathfrak{K}$ is characterizable? In the first part of this section, we will give an affirmative answer that uses the notion of a *Lindström quantifier* as developed by Lindström [1966a]. At the same time this notion enables us to systematize—at least to a certain extent—the variety of specific logics that we have considered up to now. The systematization not only assists in the representation of logics but can also be helpful from a methodological point of view. In the second part of this section, we will illustrate the latter aspect by proving a generalization of the back-and-forth characterization of elementary equivalence for logics with monotone quantifiers that covers several of the Ehrenfeucht–Fraissé type theorems for stronger logics. In order to avoid any cumbersome notation, we will confine ourselves to the one-sorted case and treat logics with free variables in the sense of 1.1.2.

4.1. *Definitions and Examples*

Let $\boldsymbol{\sigma}$ be a *finite* vocabulary and Q a quantifier symbol suitable for $\boldsymbol{\sigma}$ (in a sense that will become clear from Definition 4.1.1). Furthermore, let $\mathfrak{K}$ be a class of $\boldsymbol{\sigma}$-structures closed under isomorphism. We confine ourselves to the special case $\boldsymbol{\sigma} = \{R, f, c\}$ with binary R and unary f.

4.1.1 Definition. For any logic $\mathscr{L}$, the expanded logic $\mathscr{L}(Q_\mathfrak{K})$ is obtained as follows:

$\mathrm{Form}_{\mathscr{L}(Q_\mathfrak{K})}[\boldsymbol{\tau}]$ is taken as the smallest class containing $\mathrm{Form}_{\mathscr{L}}[\boldsymbol{\tau}]$ which is closed under boolean operations and particularizations (see Definition 1.2.1) and that with each φ, ψ, χ and for any variables $x_0 \neq x_1, y_0 \neq y_1, z_0$ also contains the new formula

$$\vartheta = Qx_0x_1y_0y_1z_0\varphi\psi\chi.$$

A variable u is *free* in ϑ, if it is free in φ or ψ or χ and different from x_0, x_1 or y_0, y_1 or z_0, respectively.

$\mathrm{Sent}_{\mathscr{L}(Q_{\mathfrak{K}})}[\tau]$ is the class of sentences from $\mathrm{Form}_{\mathscr{L}(Q_{\mathfrak{K}})}[\tau]$.

Finally, the meaning of Q is determined by the satisfaction condition:

$$\begin{aligned}
&\mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} Qx_0x_1y_0y_1z_0\, \varphi(x_0, x_1)\, \psi(y_0, y_1)\, \chi(z_0)\\
&\quad\text{iff there is a } \boldsymbol{\sigma}\text{-structure } \mathfrak{C} \in \mathfrak{K} \text{ such that } C = A,\\
&\quad R^{\mathfrak{C}} = \{(a, b) \in C \times C \mid \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \varphi[a, b]\},\\
&\quad \text{graph of } f^{\mathfrak{C}} = \{(a, b) \in C \times C \mid \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \psi[a, b]\},\\
&\quad \text{and } \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \chi[a] \text{ exactly for } a = c^{\mathfrak{C}}.
\end{aligned}$$

The quantifier Q with the interpretation by $\mathfrak{K}$ (for short, $Q_{\mathfrak{K}}$) is called a *Lindström quantifier.*

Let $\mathscr{L}$ be regular. As it is clear that

$$\mathfrak{K} = \mathrm{Mod}^{\sigma}_{\mathscr{L}(Q_{\mathfrak{K}})}(Qx_0x_1y_0y_1z_0Rx_0x_1f(y_0) = y_1z_0 = c),$$

we see that $\mathfrak{K}$ is EC in $\mathscr{L}(Q_{\mathfrak{K}})$, even in $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$. On the other hand, if $\mathfrak{K}$ is EC in $\mathscr{L}$, then $\mathscr{L}(Q_{\mathfrak{K}}) \leq \mathscr{L}$ and hence $\mathscr{L}(Q_{\mathfrak{K}}) \equiv \mathscr{L}$. To see the key fact, assume that $\mathfrak{K} = \mathrm{Mod}^{\sigma}_{\mathscr{L}}(\xi)$. Then the $\mathscr{L}(Q_{\mathfrak{K}})$-formula $Qx_0x_1y_0y_1z_0\, \varphi(x_0, x_1)\, \psi(y_0, y_1)\, \chi(z_0)$ (with $\mathscr{L}$-formulas φ, ψ, χ) has the same meaning in $\mathscr{L}(Q_{\mathfrak{K}})$ as the formula $\xi[R/\lambda x_0x_1\varphi(x_0, x_1), f/\lambda y_0y_1\psi(y_0, y_1)\; c/\lambda z_0\chi(z_0)]$ has in $\mathscr{L}$.

The definition of $\mathscr{L}(Q_{\mathfrak{K}})$ can easily be generalized to the case of more than one Lindström quantifier, and it is not difficult to see that for regular $\mathscr{L}$ the logic $\mathscr{L}(Q_{\mathfrak{K}_\iota} \mid \iota \in I)$ with Lindström quantifiers $Q_{\mathfrak{K}_\iota}$ is regular, possibly up to the relativization and the substitution property. However, the latter property holds, for example, in case $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$. A counter-example to relativization is provided by $\mathscr{L}_{\omega\omega}(Q^C)$ which is defined below. In Definition 4.1.4 we describe a variant of Lindström quantifiers that also guarantees the relativization property.

The following list demonstrates that it is possible to model numerous quantifiers on Lindström quantifiers and thus illustrates the scope of this notion.

4.1.2 Examples. In each of the following, a well-known quantifier becomes $Q_{\mathfrak{K}}$ for the class indicated:

(i) $\exists$ for $\mathfrak{K} = \{(A, C) \mid \varnothing \neq C \subseteq A\}$.
(ii) Q^n_α for $\mathfrak{K} = \{(A, M) \mid M \subseteq A^n$, there is $C \subseteq A$, $|C| \geq \aleph_\alpha$ and $C^n \subseteq M\}$;
(iii) $Q^{\mathrm{cf}\,\omega}$ for $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) \mid <^{\mathfrak{A}}$ is a linear ordering relation $\subseteq A \times A$ of cofinality $\omega\}$;
(iv) Q^{WO}, the so-called *well-ordering quantifier*, for $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) \mid <^{\mathfrak{A}}$ is a well-ordering relation $\subset A \times A\}$;
(v) Q^C, the so-called *Chang quantifier*, a specialization of the equicardinality quantifier I, for $\mathfrak{K} = \{(A, C) \mid C \subseteq A, |C| = |A|\}$.

In order to model higher-order quantifiers, one could introduce Lindström quantifiers of higher order. In principle, however, the present framework is universal in broad sense:

4.1.3 Theorem. *Let $\mathscr{L}$ be a regular logic which is finitary, that is for any τ,*

$$\mathrm{Form}_{\mathscr{L}}[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ \tau_0 \text{ finite}}} \mathrm{Form}_{\mathscr{L}}[\tau_0].$$

Then, $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}_\iota} \mid \iota \in I)$, where the $\mathfrak{K}_\iota$ run over all classes of finite vocabulary that are EC *in $\mathscr{L}$.*

Proof. For "$\leq$" note that each $\mathfrak{K}_\iota$ is EC in $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}_\iota})$. As for the other direction, use the fact that $\mathscr{L}(Q_{\mathfrak{K}_\iota}) \equiv \mathscr{L}$ for every $\iota \in I$. □

In particular, second-order logic $\mathscr{L}^2$ has a representation as in Theorem 4.1.3. Any such representation requires I to be infinite, that is, $\mathscr{L}^2$ is *not finitely generated*; for otherwise, according to a consideration in Section 7.3, we would get a contradiction, since (the one-sorted version of) $\mathscr{L}^2$ has the Beth property.

Returning now to the relativization property, we introduce a variant of $\mathscr{L}(Q_{\mathfrak{K}})$.

4.1.4 Definition. The logic $\mathscr{L}(Q^*_{\mathfrak{K}})$ is defined as follows. We change the definition of $\mathscr{L}(Q_{\mathfrak{K}})$ given in Definition 4.1.1 by allowing predicates for the domains of structures in $\mathfrak{K}$. Using a quantifier symbol Q^* instead of Q, we replace the quantifier clause for Q in Definition 4.1.1 by

$$\vartheta^* = Q^* u_0 x_0 x_1 y_0 y_1 z_0 \xi\varphi\psi\chi,$$

where the meaning of Q^* is now determined by

$$\mathfrak{A} \models_{\mathscr{L}(Q^*_{\mathfrak{K}})} Q^* u_0 x_0 x_1 y_0 y_1 z_0\, \xi(u_0)\, \varphi(x_0, x_1)\, \psi(y_0, y_1)\, \chi(z_0)$$

iff there is a σ-structure $\mathfrak{C} \in \mathfrak{K}$ such that $C = \{a \in A \mid \mathfrak{A} \models_{\mathscr{L}(Q^*_{\mathfrak{K}})} \xi[a]\}$ and $R^{\mathfrak{C}}, f^{\mathfrak{C}}$ and $c^{\mathfrak{C}}$ are as in Definition 4.1.1.

For regular $\mathscr{L}$, the logic $\mathscr{L}(Q^*_{\mathfrak{K}})$ is regular, possibly up to substitution, and really regular for instance in case $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$. Intuitively, relativization to some predicate P can be defined by induction on formulas with the essential clause for the relativization of a Q^*-formula being:

$$Q^* u_0 \ldots z_0 (Pu_0 \wedge \xi^P)(Px_0 \wedge Px_1 \wedge \varphi^P)(Py_0 \wedge Py_1 \wedge \psi^P)(Pz_0 \wedge \chi^P).$$

It is obvious that $\mathscr{L}(Q_{\mathfrak{K}}) \leq \mathscr{L}(Q^*_{\mathfrak{K}})$. For instance, the Q-formula ϑ from Definition 4.1.1 has the same meaning in $\mathscr{L}(Q_{\mathfrak{K}})$ as the Q^*-formula ϑ^* from above has in

$\mathscr{L}(Q^*_{\mathfrak{K}})$, if one takes $u_0 = u_0$ for ξ. Concerning the other direction we have the following fact:

4.1.5 Proposition. *With new unary U set*

$$\mathfrak{K}^* := \{\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\sigma} \cup \{U\}] \mid U^{\mathfrak{A}}\ \boldsymbol{\sigma}\text{-closed and } (\mathfrak{A} \upharpoonright \boldsymbol{\sigma}) \mid U^{\mathfrak{A}} \in \mathfrak{K}\}.$$

Then $\mathscr{L}(Q^*_{\mathfrak{K}}) \equiv \mathscr{L}(Q_{\mathfrak{K}^*})$.

Proof. The argument for "$\geq$" is trivial. For "$\leq$" observe for instance that

$$Q^*_{\mathfrak{K}} u_0 x_0 x_1 y_0 y_1 z_0\ \xi(u_0)\ \varphi(x_0, x_1)\ \psi(y_0, y_1)\ \chi(z_0)$$

has the same meaning as

$$\begin{aligned} &Q_{\mathfrak{K}^*} u_0 \ldots z_0\, \xi(u_0)(\xi(x_0) \wedge \xi(x_1) \wedge \varphi(x_0, x_1)) \\ &\qquad ((\xi(y_0) \wedge \xi(y_1) \wedge \psi(y_0, y_1)) \vee (\neg \xi(y_0) \wedge y_1 = y_0)) \\ &\qquad (\xi(z_0) \wedge \chi(z_0)), \end{aligned}$$

where $\xi(u_0)$ represents U. $\square$

Taking Proposition 4.1.5 into consideration it is not difficult to extend results about logics $\mathscr{L}(Q_{\mathfrak{K}})$ to logics $\mathscr{L}(Q^*_{\mathfrak{K}})$—at least in many cases (for example, Theorem 4.1.3 and the results in Section 4.2).

Let us now return to our introductory question. For numerous logics $\mathscr{L}$ such as $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$ or $\mathscr{L} = \mathscr{L}_{\kappa\lambda}(Q^*_{\mathfrak{K}_\iota} \mid \iota \in I)$, the logic $\mathscr{L}(Q^*_{\mathfrak{K}})$ is, with respect to elementary classes, the smallest regular extension of $\mathscr{L}$ in which $\mathfrak{K}$ is EC. In this sense the transition from $\mathscr{L}$ to $\mathscr{L}(Q^*_{\mathfrak{K}})$ is a natural closure operation. What can we say about the relationship to $\mathscr{L}(\mathfrak{K})$ as defined in Section 2.6? If, for instance, $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) \mid \mathfrak{A} \cong (\omega, <)\}$, then, of course, we have

$$(*) \qquad \mathscr{L}_{\omega\omega}(\mathfrak{K}) = \omega-\text{logic} \leq \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}^*}) \equiv \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}}).$$

Using a method like that in the proof of Proposition 3.1.7 one obtains for the other direction

$$(**) \qquad \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}}) \leq_{\mathrm{PC}} \mathscr{L}_{\omega\omega}(\mathfrak{K}) \text{ for vocabularies not containing } U, <.$$

Whereas the analogue of $(*)$ is true in general, the analogue of $(**)$ may fail. For instance, if $\mathfrak{K}$ is the class of all fields of characteristic zero, $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is compact, but $\mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}})$ is not.

4.2. Partial Isomorphisms and a Characterization of $\mathscr{L}$-Equivalence

The characterization of elementary equivalence in terms of partial isomorphisms or games by Fraissé and Ehrenfeucht (cf. Section IX.4 for a thorough treatment) has been extended to various stronger logics such as $\mathscr{L}_{\omega\omega}(Q_1)$, $\mathscr{L}_{\omega\omega}(Q_1^n)$, $\mathscr{L}_{\omega\omega}(\mathrm{aa})$. A generalization to extensions of $\mathscr{L}_{\omega\omega}$ by arbitrary Lindström quantifiers is given in Caicedo [1979]. The characterization becomes very natural for quantifiers $Q_{\mathfrak{K}}$ and $Q_{\mathfrak{K}}^*$, where $\mathfrak{K}$ is of finite relational vocabulary $\boldsymbol{\sigma}$ and monotone (Krawczyk–Krynicki [1976], Weese [1980]). The following considerations are devoted to this case. For reasons of readability we fix a relational vocabulary $\boldsymbol{\sigma} = \{S\}$, S l-ary, and a class $\mathfrak{K}$ of $\boldsymbol{\sigma}$-structures, $\mathfrak{K}$ closed under isomorphisms. We treat the quantifier $Q_{\mathfrak{K}}$.

4.2.1 Definition. For $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}[\boldsymbol{\tau}]$, p is a *partial isomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$, if p is a bijection from $\mathrm{dom}(p) \subseteq A$ onto $\mathrm{rg}(p) \subseteq B$ such that the following hold:

(i) for all $n \geq 1$, n-ary $R \in \boldsymbol{\tau}$ and $a_0, \ldots, a_{n-1} \in \mathrm{dom}(p)$:
$R^{\mathfrak{A}}\mathbf{a}$ iff $R^{\mathfrak{B}}p(\mathbf{a})$, where $p(\mathbf{a})$ stands for $(p(a_0), \ldots, p(a_{n-1}))$;
(ii) for all $n \geq 1$, n-ary $f \in \boldsymbol{\tau}$ and $a_0, \ldots, a_{n-1}, a \in \mathrm{dom}(p)$:
$f^{\mathfrak{A}}(\mathbf{a}) = a$ iff $f^{\mathfrak{B}}(p(\mathbf{a})) = p(a)$;
(iii) for all $c \in \boldsymbol{\tau}$ and $a \in \mathrm{dom}(p)$: $c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.

Part $(\mathfrak{A}, \mathfrak{B})$ denotes the set of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$.

Sometimes, one demands in addition that the domain of a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ be $\boldsymbol{\tau}$-closed in $\mathfrak{A}$ (or empty). However, the difference between the two variants involves only minor technicalities.

4.2.2 Definition. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\boldsymbol{\tau}$-structures, $0 \leq \alpha \leq \omega$, and $I = (I_\beta)_{\beta \leq \alpha}$ a sequence of subsets of Part$(\mathfrak{A}, \mathfrak{B})$.

We say that I has the *$\exists$-forth property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $a \in A$ there exists $q \in I_m$ such that $p \subseteq q$ and $a \in \mathrm{dom}(q)$.

Similarly, we say that I has the *$\exists$-back property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $b \in B$ there exists $q \in I_m$ such that $p \subseteq q$ and $b \in \mathrm{rg}(q)$.

Likewise I has the *$Q_{\mathfrak{K}}$-forth property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $\mathfrak{C} \in \mathfrak{K}$ with $C = A$ there is $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ such that for all $\mathbf{d} \in S^{\mathfrak{D}}$ there exists $q \in I_m$ with $p \subseteq q$, $d_0, \ldots, d_{l-1} \in \mathrm{rg}(q)$ and $q^{-1}(\mathbf{d}) \in S^{\mathfrak{C}}$.

Similarly, we say that I has the *$Q_{\mathfrak{K}}$-back property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ there is $\mathfrak{C} \in \mathfrak{K}$ with $C = A$ such that for all $\mathbf{c} \in S^{\mathfrak{C}}$ there exists $q \in I_m$ with $p \subseteq q$, $c_0, \ldots, c_{l-1} \in \mathrm{dom}(q)$ and $q(\mathbf{c}) \in S^{\mathfrak{D}}$.

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *α-isomorphic via I*, written $I: \mathfrak{A} \cong_\alpha \mathfrak{B}$, iff $I = (I_m)_{m \leq \alpha}$ is a sequence of length $(\alpha + 1)$ of non-empty subsets of Part$(\mathfrak{A}, \mathfrak{B})$ having the $\exists$-back and the $\exists$-forth property. $\mathfrak{A}$ and $\mathfrak{B}$ are *α-isomorphic*, written $\mathfrak{A} \cong_\alpha \mathfrak{B}$, iff there exists an I such that $I: \mathfrak{A} \cong_\alpha \mathfrak{B}$.

The notion of α, $\mathfrak{K}$-*isomorphic structures* is defined similarly, demanding in addition that the partial isomorphisms in question also meet the $Q_{\mathfrak{K}}$-back and the $Q_{\mathfrak{K}}$-forth property.

We call the class $\mathfrak{K}$ and also $Q_{\mathfrak{K}}$ *monotone*, if for all A, M, M' such that $(A, M) \in \mathfrak{K}$ and $M \subseteq M' \subseteq A^l$, we have $(A, M') \in \mathfrak{K}$.

The main result in this section can now be formulated as:

4.2.3. Theorem. *Let $\mathfrak{K}$ of finite relational vocabulary be monotone. Then for finite τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ the following are equivalent*:

(i) $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})} \mathfrak{B}$;
(ii) $\mathfrak{A} \cong_{n, \mathfrak{K}} \mathfrak{B}$ *for all* n;
(iii) $\mathfrak{A} \cong_{\omega, \mathfrak{K}} \mathfrak{B}$.

If we dispense with $Q_{\mathfrak{K}}$, the proof below will yield the analogous result for $\mathscr{L}_{\omega\omega}$, that is, the Ehrenfeucht–Fraissé characterization of elementary equivalence:

4.2.4 Corollary. *For finite τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ the following are equivalent*:

(i) $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}} \mathfrak{B}$;
(ii) $\mathfrak{A} \cong_n \mathfrak{B}$ *for all* n;
(iii) $\mathfrak{A} \cong_\omega \mathfrak{B}$. $\square$

Proof of Theorem 4.2.3. Let $\mathfrak{K}$ be as above. We set $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$ and fix some finite vocabulary τ. By $\varphi, \psi, \ldots$ we denote formulas from $\mathscr{L}[\tau]$. Each φ is equivalent to a so-called *term-reduced* formula—a formula where all atomic subformulas are of kinds $Rx_0 \ldots x_{n-1}$, $x = y$, $c = y$, or $f(x_0, \ldots, x_{n-1}) = y$. We can obviously confine ourselves to such formulas, which we do for technical convenience.

The implication from (iii) to (ii) is trivial. To prove that (ii) implies (i), we define the so-called *quantifier rank* of φ, $\mathrm{qrk}(\varphi)$, inductively by the following clauses:

$$\begin{aligned} \mathrm{qrk}(\varphi) &= 0, \quad \text{if } \varphi \text{ is atomic;} \\ \mathrm{qrk}(\neg\varphi) &= \mathrm{qrk}(\varphi); \\ \mathrm{qrk}(\varphi \wedge \psi) &= \max\{\mathrm{qrk}(\varphi), \mathrm{qrk}(\psi)\}; \\ \mathrm{qrk}(\exists x\varphi) &= \mathrm{qrk}(Q\mathrm{x}\varphi) = 1 + \mathrm{qrk}(\varphi). \end{aligned}$$

Next we write

$$\mathfrak{A} \equiv_{n,\mathfrak{K}} \mathfrak{B} \quad \text{iff} \quad \begin{array}{l}\text{for all (term-reduced) sentences } \varphi \text{ with } \mathrm{qrk}(\varphi) \leq n, \\ \text{we have } \mathfrak{A} \models \varphi \text{ iff } \mathfrak{B} \models \varphi.\end{array}$$

Then the implication we want follows from:

$(*)$ For all n, if $\mathfrak{A} \cong_{n,\mathfrak{K}} \mathfrak{B}$, then $\mathfrak{A} \equiv_{n,\mathfrak{K}} \mathfrak{B}$.

To prove (∗), let $I\colon \mathfrak{A} \cong_{n,\mathfrak{K}} \mathfrak{B}$ be given. One shows by induction on $\mathrm{qrk}(\varphi)$ that for all $m \le n$, $p \in I_m$, $\varphi(x_0, \ldots, x_{k-1})$ with $\mathrm{qrk}(\varphi) \le m$, and $a_0, \ldots, a_{k-1} \in \mathrm{dom}(p)$, $\mathfrak{A} \models \varphi[\mathbf{a}]$ iff $\mathfrak{B} \models \varphi[p(\mathbf{a})]$. For atomic φ one uses that φ is term-reduced. For the Q-step, let $m \le n$, $p \in I_m$, and $a_0, \ldots, a_{k-1} \in \mathrm{dom}(p)$ be given and assume $\varphi = Qy_0 \ldots y_{l-1}\, \psi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{l-1})$, $\mathrm{qrk}(\varphi) \le m$. If for instance $\mathfrak{A} \models \varphi[\mathbf{a}]$, then

$$\mathfrak{C} = (A, \{\mathbf{c} \in A^l \mid \mathfrak{A} \models \psi[\mathbf{a}, \mathbf{c}]\}) \in \mathfrak{K}.$$

For $\mathfrak{C}$ and p we take $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ as guaranteed by the $Q_{\mathfrak{K}}$-forth property and define $\mathfrak{D}'$ to be the structure

$$\mathfrak{D}' = (B, \{\mathbf{d} \in B^l \mid \mathfrak{B} \models \psi[p(\mathbf{a}), \mathbf{d}]\}).$$

As $\mathfrak{K}$ is monotone, we get $\mathfrak{B} \models \varphi[p(\mathbf{a})]$, if we have proved

$$(**) \qquad S^{\mathfrak{D}'} \supseteq S^{\mathfrak{D}}.$$

To see (∗∗), let $\mathbf{d} \in S^{\mathfrak{D}}$ be given. Choose $q \in I_{m-1}$, $q \supseteq p$, such that $d_0, \ldots, d_{l-1} \in \mathrm{rg}(q)$ and $q^{-1}(\mathbf{d}) \in S^{\mathfrak{C}}$. As $\mathrm{qrk}(\psi) \le m - 1$, the induction hypothesis yields $\mathfrak{A} \models \psi[\mathbf{a}, q^{-1}(\mathbf{d})]$ iff $\mathfrak{B} \models \psi[p(\mathbf{a}), \mathbf{d}]$, and hence $\mathbf{d} \in S^{\mathfrak{D}'}$.

Finally, we come to the implication from (i) to (iii). This is the only point where we need the finiteness of τ. To give a more systematic treatment, we insert a general definition which is modelled on the extension properties of partial isomorphisms that we want to realize.

4.2.5 Definition. For $\mathfrak{A} \in \mathrm{Str}[\tau]$, $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in A^k$ and $\mathbf{x} = (x_0, \ldots, x_{k-1})$ the formulas $\psi^m_{\mathfrak{K},\mathfrak{A},\mathbf{a}}(\mathbf{x})$ (or, shorter, $\psi^m_{\mathbf{a}}$) are given as follows:

(i) $$\psi^0_{\mathbf{a}} = \bigwedge \{\varphi(\mathbf{x}) \mid \varphi \text{ term-reduced, atomic or negated atomic, } \mathfrak{A} \models \varphi[\mathbf{a}]\};$$

(ii) $$\begin{aligned}\psi^{m+1}_{\mathbf{a}} = {} & \bigwedge_{c \in A} \exists y \psi^m_{\mathbf{a},c}(\mathbf{x}, y) \wedge \forall y \bigvee_{c \in A} \psi^m_{\mathbf{a},c}(\mathbf{x}, y) \\ & \wedge \bigwedge_{\substack{M \subseteq A^l \\ (A, M) \in \mathfrak{K}}} Q\mathbf{y} \bigvee_{\mathbf{c} \in M} \psi^m_{\mathbf{a},\mathbf{c}}(\mathbf{x}, \mathbf{y}) \\ & \wedge \bigwedge_{\substack{M \subseteq A^l \\ (A, M) \notin \mathfrak{K}}} \neg Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi^m_{\mathbf{a},\mathbf{c}}(\mathbf{x}, \mathbf{y}).\end{aligned}$$

As τ is finite, it can immediately be seen that in the definition of $\psi^m_{\mathbf{a}}$ all conjunctions and disjunctions can be chosen finite. Hence $\psi^m_{\mathbf{a}} \in \mathscr{L}[\tau]$. The following facts can easily be proved by induction on m.

4.2.6 Lemma. *For* $\mathfrak{A} \in \operatorname{Str}[\tau]$ *and* $a_0, \ldots, a_{k-1} \in A$ *we have*:

(i) $\operatorname{qrk}(\psi_{\mathbf{a}}^m) = m$;
(ii) $\mathfrak{A} \models \psi_{\mathbf{a}}^m[\mathbf{a}]$;
(iii) $\psi_{\mathbf{a},a}^m \models \psi_{\mathbf{a}}^m$ *for all* $a \in A$; *and hence*
(iv) $\psi_{\mathbf{a}}^{m+1} \models \psi_{\mathbf{a}}^m$. $\square$

The Proof of 4.2.3 *Concluded.* Assume $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and define

$$I_m = \{p \in \operatorname{Part}(\mathfrak{A}, \mathfrak{B}) \mid \operatorname{dom}(p) = \{a_0, \ldots, a_{k-1}\} \text{ for distinct } a_i \text{ and } \mathfrak{B} \models \psi_{\mathbf{a}}^m[p(\mathbf{a})]\}, \quad \text{and}$$

$$I_\omega = \{\varnothing\}.$$

Then the assertion follows from

$$(+) \qquad (I_\alpha)_{\alpha \le \omega} \colon \mathfrak{A} \cong_{\omega, \mathfrak{K}} \mathfrak{B}.$$

We now argue for $(+)$. First, because of $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and Lemma 4.2.6(ii), we have $\varnothing \in I_m$ for all m. Let us, for example, check the $Q_{\mathfrak{K}}$-back property. Assume $p \in I_{m+1}$, $\operatorname{dom}(p) = \{a_0, \ldots, a_{k-1}\}$, and $(B, N) \in \mathfrak{K}$. We have to find $M \subseteq A^l$ such that $(A, M) \in \mathfrak{K}$ and (A, M) meets the further requirements of the $Q_{\mathfrak{K}}$-back property. We set

$$M = \left\{\mathbf{c} \in A^l \mid \mathfrak{B} \models \bigvee_{\mathbf{d} \in N} \psi_{\mathbf{a},\mathbf{c}}^m[p(\mathbf{a}), \mathbf{d}]\right\}.$$

First, we see that for each $\mathbf{c} \in M$ there is $\mathbf{d} \in N$ such that $\mathfrak{B} = \psi_{\mathbf{a},\mathbf{c}}^m[p(\mathbf{a}), \mathbf{d}]$. Hence, by definition of I_m and Lemma 4.2.6(iii), if $\mathbf{c}$ is given, we can choose

$$q = p \cup \{(c_0, d_0), \ldots, (c_{l-1}, d_{l-1})\} \in I_m.$$

Obviously $q \in \operatorname{Part}(\mathfrak{A}, \mathfrak{B})$, because by 4.2.6(iv) we have $\mathfrak{B} = \psi_{\mathbf{a},\mathbf{c}}^0[p(\mathbf{a}), \mathbf{d}]$.

It remains to show that $(A, M) \in \mathfrak{K}$. By definition of M,

$$N' = \left\{\mathbf{d} \in B^l \mid \mathfrak{B} \models \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi_{\mathbf{a},\mathbf{c}}^m[p(\mathbf{a}), \mathbf{d}]\right\} \supseteq N,$$

and as $\mathfrak{K}$ is monotone, we obtain that $(B, N') \in \mathfrak{K}$; that is,

$$\mathfrak{B} \models Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi_{\mathbf{a},\mathbf{c}}^m(p(\mathbf{a}), \mathbf{y}).$$

As $\mathfrak{B} \models \psi_{\mathbf{a}}^{m+1}[p(\mathbf{a})]$, the formula

$$\neg Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi_{\mathbf{a},\mathbf{c}}^m(\mathbf{x}, \mathbf{y})$$

cannot be a conjunct of $\psi_{\mathbf{a}}^{m+1}(\mathbf{x})$. Hence, $(A, M) \in \mathfrak{K}$. $\square$

Remarks. (a) In the preceding proof one can avoid the restriction to term-reduced formulas if one replaces the quantifier rank by a notion of rank that also takes into consideration the complexity of terms.

(b) Theorem 4.2.3 can be extended without difficulty to the case of finitely many monotone Lindström quantifiers.

(c) As for first-order logic, the algebraic characterization of $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$-equivalence can be reformulated in terms of game-theoretical notions; see, for example, Weese [1980]. If we translate Theorem 4.2.3, say for $\mathscr{L}_{\omega\omega}(Q_1)$—note that Q_1 is monotone!—into the game-theoretical version, we get the following characterization of $\mathscr{L}_{\omega\omega}(Q_1)$-equivalence:

> *For any finite τ, two τ-structures $\mathfrak{A}$, $\mathfrak{B}$ are $\mathscr{L}_{\omega\omega}(Q_1)$-equivalent iff player* II *has a winning strategy in the game $G_n(\mathfrak{A}, \mathfrak{B})$ for all $n \in \omega$.*

The game $G_n(\mathfrak{A}, \mathfrak{B})$ is defined as follows: A play in $G_n(\mathfrak{A}, \mathfrak{B})$ takes place between two players I, II and consists of n consecutive moves which are either $\exists$-moves or Q_1-moves. Furthermore, at the beginning of each move player I is free to choose the kind of move he wants. The moves run as follows: $\exists$-move: Player I chooses an element $a \in A$ or an element $b \in B$. This done, player II then chooses some $b \in B$ or some $a \in A$ respectively. Q_1-move: Player I chooses a subset $M \subseteq A$ (or a subset $N \subseteq B$) of power $\geq \aleph_1$. Player II then chooses some $N \subseteq B$ (or some $M \subseteq A$) of power $\geq \aleph_1$. Subsequently, player I chooses some $b \in N$ (or some $a \in M$), and finally player II chooses some $a \in M$ (or some $b \in N$, respectively). Player II wins the play iff the set $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ of pairs from $A \times B$ chosen in the play is a partial isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

4.2.7 Application. As an easy application of Theorem 4.2.3 we complete the argument for Keisler's *counterexample to interpolation in $\mathscr{L}_{\omega\omega}(Q_1)$* from Example 1 of Section 2.2. For $i = 0, 1$, let $\mathfrak{A}_i = (A_i, E^{\mathfrak{A}_i})$, where $E^{\mathfrak{A}_i}$ is an equivalence relation with only uncountable equivalence classes and $A_i/E^{\mathfrak{A}_i}$ is countably infinite for $i = 0$ and uncountable for $i = 1$. It is easy to see that $(I_\alpha)_{\alpha \leq \omega}$: $\mathfrak{A}_0 \cong_{\omega, Q_1} \mathfrak{A}_1$, where for $\alpha \leq \omega$ the set I_α consists of all partial isomorphisms from $\mathfrak{A}_0$ into $\mathfrak{A}_1$ which have a finite domain. By Theorem 4.2.3, $\mathfrak{A}_0 \equiv_{\mathscr{L}_{\omega\omega}(Q_1)} \mathfrak{A}_1$, and hence by Proposition 3.1.3 interpolation fails for $(*)$ in Example 1. (As $\mathfrak{A}_i \in \mathrm{Mod}(\exists R\ \varphi_i(E, R))$ and $\mathfrak{A}_0 \equiv_{\mathscr{L}_{\omega\omega}(Q_1)} \mathfrak{A}_1$, the classes $\mathrm{Mod}(\exists R\ \varphi_0(E, R))$ and $\mathrm{Mod}(\exists R\ \varphi_1(E, R))$ cannot be separated by a class EC in $\mathscr{L}_{\omega\omega}(Q_1)$.) $\square$

4.3. *Partially Isomorphic Structures*

In the last paragraph $\mathfrak{A}_0$ and $\mathfrak{A}_1$ were seen to be ω, Q_1-isomorphic in a strong sense, as all I_α are equal: they are *ω, Q_1-partially isomorphic.* To give a definition, let $\mathfrak{A}$, $\mathfrak{B}$ be τ-structures and $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$. We say that I has the *$\exists$-forth* (*$\exists$-back*) *property*, if for all $p \in I$ and $a \in A (b \in B)$ there is $q \in I$, $q \supseteq p$ with $a \in \mathrm{def}(q)$ (or $b \in \mathrm{rg}(q)$, respectively). $\mathfrak{A}$ and $\mathfrak{B}$ are called *partially isomorphic*, $\mathfrak{A} \cong_p \mathfrak{B}$, if there is I such that I: $\mathfrak{A} \cong_p \mathfrak{B}$, that is, if $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$, I is not empty and has the $\exists$-forth and the $\exists$-back property. The notions $I: \mathfrak{A} \cong_{p,\mathfrak{K}} \mathfrak{B}$ and $\mathfrak{A} \cong_{p,\mathfrak{K}} \mathfrak{B}$ are defined similarly, also incorporating the $Q_{\mathfrak{K}}$-forth and the $Q_{\mathfrak{K}}$-back property into the definition.

Looking first at the $Q_{\mathfrak{K}}$-free version, *a fortiori*, the structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ given in the argument of 4.2.7 are partially isomorphic. Furthermore, any two dense open orderings are partially isomorphic—also via the set of partial isomorphisms with finite domain.

The relation $\cong_\omega$ can be considered as a finite approximation of the isomorphism relation. In good accordance with this view, ω-isomorphic structures are isomorphic in case they are finite. Similarly, the stronger notion of $\cong_p$ embodies countable approximations of isomorphisms:

4.3.1 Theorem. *Countable partially isomorphic structures are isomorphic.*

Proof. Assume $I: \mathfrak{A} \cong_p \mathfrak{B}$, $A = \{a_i \mid i \in \omega\}$, and $B = \{b_i \mid i \in \omega\}$. By induction on i one can define $p_i \in \operatorname{Part}(\mathfrak{A}, \mathfrak{B})$ such that for all i: $p_i \subseteq p_{i+1}$, $a_i \in \operatorname{dom}(p_{2i})$, $b_i \in \operatorname{rg}(p_{2i+1})$. Then $\bigcup_i p_i: \mathfrak{A} \cong \mathfrak{B}$. □

The theorem generalizes a well-known result of Cantor according to which any two countable dense open orderings are isomorphic. However, it is not valid for uncountable structures: As mentioned above, any two dense open orderings are partially isomorphic, and there are easy examples of non-isomorphic dense open orderings even of the same cardinality $\aleph_\alpha$, for every $\alpha \geq 1$. Take, for instance, $\aleph_\alpha$ many copies of the rationals and order them either according to $\aleph_\alpha$ or inversely. Moreover, any two infinite sets or any two algebraically closed fields of infinite degree of transcendence (so-called universal domains) of the same characteristic are partially isomorphic.

We see from Theorem 4.3.1 that $\cong_p$ is strictly stronger than elementary equivalence. Hence, from a model-theoretical point of view, we may ask whether there is some logic $\mathscr{L}$ (necessarily) stronger than first-order logic, such that $\cong_p$ equals $\mathscr{L}$-equivalence. The answer is affirmative.

4.3.2 Theorem (Karp [1965]). *For all structures* $\mathfrak{A}$ *and* $\mathfrak{B}$, $\mathfrak{A} \cong_p \mathfrak{B}$ *iff* $\mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$.

From an algebraic point of view, any two universal domains of the same characteristic—even if they are not isomorphic—are not essentially different. The fact that they are partially isomorphic demonstrates that $\cong_p$ can be considered as a methodologically interesting weakening of the isomorphism relation (see also Barwise [1973b]).

The direction from right to left in Theorem 4.3.2 tells us that $\mathscr{L}_{\infty\omega}$ is weak enough not to distinguish between structures that are "weakly identical" in the sense of being partially isomorphic. This feature leads us to a new notion: For any logic $\mathscr{L}$, define $\mathscr{L}$ to have the *Karp property* iff any two partially isomorphic structures are $\mathscr{L}$-equivalent. The direction from right to left in Theorem 4.3.2 now yields that $\mathscr{L}_{\infty\omega}$ is a strongest logic with this property, in the sense that if a logic $\mathscr{L}$ has the Karp property, then any two $\mathscr{L}_{\infty\omega}$-equivalent structures are also $\mathscr{L}$-equivalent (that is, $\mathscr{L} \leq_{\equiv} \mathscr{L}_{\infty\omega}$).

A proof of Theorem 4.3.2 (see Theorem IX.4.3.1 or Barwise [1973b, 1975]) can be given as a suitable "infinitary" version of the corresponding proof for $\mathscr{L}_{\omega\omega}$ and $\cong_{\omega}$, that is, for Corollary 4.2.4. Returning now to partial isomorphisms including Lindström quantifiers, we can proceed similarly with the proof of Theorem 4.2.3, thus verifying the following generalization of Theorem 4.3.2.

4.3.3 Theorem. *Let $Q_{\mathfrak{K}_\iota}$, for $\iota \in I$, be monotone relational Lindström quantifiers. Then for any τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ we have:*

$$\mathfrak{A} \cong_{p,(\mathfrak{K}_\iota|\iota\in I)} \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}(Q_{\mathfrak{K}_\iota}|\iota\in I)} \mathfrak{B}. \qquad \square$$

5. Compactness and Its Neighbourhood

Up to now we have described important examples in the framework of general logics and we have tried to isolate some systematizing aspects such as Lindström quantifiers and (R)PC-reducibility. In this and the concluding sections we will try to provide an insight into some basic features of essential model-theoretic notions. Our considerations are grouped around compactness, Löwenheim-Skolem properties and interpolation. Later chapters will exhibit interesting bridges between these concepts which constitute some of the main achievements of abstract model theory. For the remainder of this chapter, we will assume that the logics under consideration are regular.

5.1. Notions of Compactness

In Definition 1.2.4 we introduced the notions of compactness and κ-compactness. The following generalization, which deprives finiteness of its designated role, is important for instance, with infinitary languages.

5.1.1 Definition. For $\kappa \geq \lambda \geq \aleph_0$, $\mathscr{L}$ is *(κ, λ)-compact* iff for all τ and $\Phi \subseteq \mathscr{L}[\tau]$ of power $\leq \kappa$, if each subset of Φ of power $< \lambda$ has a model, then Φ has a model.

The notion "compact" stems from a connection with topology. Given $\mathscr{L}$ and τ, where $\mathscr{L}[\tau]$ is a set, define a topological space $\mathfrak{X}_{\mathscr{L}}[\tau]$ in the following way. The domain $X_{\mathscr{L}}[\tau]$ of $\mathfrak{X}_{\mathscr{L}}[\tau]$ forms a set of representatives of $\mathrm{Str}[\tau]$ modulo $\mathscr{L}$-equivalence, and a basis of (clopen) sets is given by the sets $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\varphi) \cap X_{\mathscr{L}}[\tau]$ for $\varphi \in \mathscr{L}[\tau]$. $\mathfrak{X}_{\mathscr{L}}[\tau]$ is a Hausdorff space, and it is easy to prove

(*) $\quad$ $\mathscr{L}$ is compact iff all $\mathfrak{X}_{\mathscr{L}}[\tau]$ are compact.

Call a topological space $\mathfrak{X}$ (κ, λ)-compact if for all sets C of closed subsets of $\mathfrak{X}$ with $|C| \leq \kappa$ and $\bigcap C = \emptyset$ there exists $C' \subseteq C$ with $|C'| < \lambda$ and $\bigcap C' = \emptyset$. Then, according to an observation of Mannila [1983], topological (κ, λ)-compactness does not correspond—in the sense of $(*)$—to (κ, λ)-compactness of logics, but to a stronger compactness property, the so-called $(\kappa, \lambda)^*$-compactness, which will play a central role in Chapter XVIII.

Compactness properties have an influence on the number of symbols in a sentence φ that are essential for the meaning of φ. We make this precise by use of the following notion. Let φ be from $\mathscr{L}[\boldsymbol{\tau}]$ and $\boldsymbol{\sigma} \subseteq \boldsymbol{\tau}$. We say that φ *depends only on the symbols in* $\boldsymbol{\sigma}$, if for all $\boldsymbol{\tau}$-structures $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} \cong \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$ we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. For $\mathscr{L}_{\infty\omega}$ there does not exist a uniform bound for the number of symbols that are essential for the meaning of a sentence. According to the following proposition compactness properties lead to a dual situation.

5.1.2 Proposition. *If $\mathscr{L}$ is (κ, λ)-compact and $|\boldsymbol{\tau}| \leq \kappa$, then any $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ depends on less than λ symbols. Hence, any sentence of a compact logic depends only on finitely many symbols.*

Proof. Assume $|\boldsymbol{\tau}| \leq \kappa$ and $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$. We take a renaming $\rho: \boldsymbol{\tau} \to \boldsymbol{\tau}'$, where $\boldsymbol{\tau}' \cap \boldsymbol{\tau} = \emptyset$, and set

$$\Phi = \{\forall \mathbf{x}(R\mathbf{x} \leftrightarrow \rho(R)\mathbf{x}) \mid R \in \boldsymbol{\tau}\}$$
$$\cup \{\forall \mathbf{x}\, f(\mathbf{x}) = \rho(f)(\mathbf{x}) \mid f \in \boldsymbol{\tau}\} \cup \{c = \rho(c) \mid c \in \boldsymbol{\tau}\}.$$

Then $\Phi \models \varphi \leftrightarrow \varphi^{\rho}$. As $|\Phi| \leq \kappa$, (κ, λ)-compactness yields a subset $\Phi_0 \subseteq \Phi$ with $|\Phi|_0 < \lambda$ and $\Phi_0 \models \varphi \leftrightarrow \varphi^{\rho}$. Let $\boldsymbol{\sigma}$ be the set of symbols of $\boldsymbol{\tau}$ which occur in Φ_0. Then $|\boldsymbol{\sigma}| < \lambda$, and if $\mathfrak{A}$, $\mathfrak{B}$ are $\boldsymbol{\tau}$-structures with $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} \cong \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$, say $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} = \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$, we have $(\mathfrak{A}, (\rho(\S)^{\mathfrak{B}^{\rho}})_{\S \in \boldsymbol{\tau}}) \models \Phi_0$ and therefore $\mathfrak{A} \models \varphi$ iff $\mathfrak{B}^{\rho} \models \varphi^{\rho}$ iff $\mathfrak{B} \models \varphi$. $\square$

5.2. *Well-Ordering Numbers*

Compactness properties provide a powerful tool for constructing non-standard models. For instance, $\aleph_0$-compactness implies the non-characterizability of infinite well-orderings. On the other hand, the logic $\mathscr{L}_{\omega_1\omega}$, which is not $\aleph_0$-compact, admits characterizations of all countable well-orderings. By the following definitions we create the appropriate terminology to exhibit precise relations between compactness properties and the characterizability of well-orderings. For technical convenience we introduce a number ∞ with $\alpha < \infty$ for all ordinals α.

5.2.1 Definition. Let be $< \in \boldsymbol{\tau}$ and $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$. We say that Φ *pins down the ordinal* α (*via* $<$), if

(i) for all models $\mathfrak{A}$ of Φ, $<^{\mathfrak{A}}$ is a well-ordering of its field;
(ii) there is a model $\mathfrak{A}$ of Φ such that $<^{\mathfrak{A}}$ is a well-ordering of order type α.

We define $w_\kappa(\mathscr{L})$ to be the supremum of all ordinals that can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$ and call $w(\mathscr{L}) = w_1(\mathscr{L})$ the *well-ordering number* of $\mathscr{L}$. A logic $\mathscr{L}$ is *bounded*, if there is no sentence that pins down arbitrarily large ordinals.

By regularity of $\mathscr{L}$ we have $w(\mathscr{L}) \geq \omega$. If Φ pins down α via $<$, then any $\beta \leq \alpha$ is pinned down by $\Phi \cup \{\prec$ *is an initial segment* of $<\}$ via $\prec$, and $\alpha + 1$ is pinned down via $\prec$ by Φ together with $\prec$ *equals* $<$ *with the least element put at the end* (assumed $\alpha \geq \omega$). Hence $w_\kappa(\mathscr{L}) = \infty$ or $w_\kappa(\mathscr{L})$ is a limit ordinal, and an ordinal α can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$ iff $\alpha < w_\kappa(\mathscr{L})$. Similar arguments yield that $w_\kappa(\mathscr{L})$ is closed under the ordinal operations of addition, multiplication and exponentiation.

There is a useful characterization of well-ordering numbers:

5.2.2 Proposition. *Suppose $\kappa \geq 1$ and $w_\kappa(\mathscr{L}) < \infty$. Then $w_\kappa(\mathscr{L})$ is the least ordinal α such that for all $\Phi \subseteq \mathscr{L}[\tau]$ with $< \in \tau$ and $|\Phi| \leq \kappa$ it is the case that if for arbitrarily large $\beta < \alpha$, Φ has a model $\mathfrak{A}$ where $<^{\mathfrak{A}}$ is a well-ordering of order type β, then Φ has a model $\mathfrak{B}$, where $<^{\mathfrak{B}}$ is not a well-ordering.*

Proof. Assume $w_\kappa(\mathscr{L}) < \infty$ and let α be the ordinal in question. By constructions such as in the preceding paragraph one can easily see that $w_\kappa(\mathscr{L}) \leq \alpha$. For the other direction, it is sufficient to show: If $< \in \tau$, $\Phi \subseteq \mathscr{L}[\tau]$, $|\Phi| \leq \kappa$, and if for arbitrarily large $\beta < w_\kappa(\mathscr{L})$, Φ has a model $\mathfrak{A}$ with $<^{\mathfrak{A}}$ a well-ordering of order type β, then Φ does not pin down ordinals via $<$. In order to establish this, let Φ be given such that Φ satisfies the hypothesis and pins down ordinals via $<$. As $\mathscr{L}$ allows elimination of function symbols, we may assume that τ is relational. With new binary R, $\prec$, and f let Ψ consist of the following sentences:

(1) $\prec$ *is a linear ordering* $\wedge\ \forall x \in$ *field*$(\prec)\ \exists z\ Rxz$;

(2) $\forall x \in$ *field*$(\prec)\ \varphi^{\{z \mid Rxz\}}$ *for* $\varphi \in \Phi$;

(3) $\forall y \in$ *field*$(\prec)\ \exists x \succ y$: $\lambda z f(x, z)$ *is an isomorphism from* (*field*$(< \restriction \{z \mid Rxz\})$, $< \restriction \{z \mid Rxz\}$) *onto* $(\{z \mid z \prec x\}, \prec \restriction \{z \restriction z \prec x\})$.

Then Ψ pins down $w_\kappa(\mathscr{L})$ and is of power $\leq \kappa$—a contradiction. ☐

5.2.3 Examples. (a) $w_\kappa(\mathscr{L}_{\omega\omega}) = \omega$ for all $\kappa \geq 1$.

(b) For $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$ we have $w(\mathscr{L}) = w_{\aleph_0}(\mathscr{L}) = \omega$, but for instance $w_{2^{\aleph_0}}(\mathscr{L}) \geq (2^{\aleph_0})^+$. (Note that for any well-ordering $\prec$ of the reals the structure $(\mathbb{R}, +, \cdot, <, \mathbb{Q}, \prec, (r)_{r \in \mathbb{R}})$ is characterizable up to isomorphism by its $\mathscr{L}$-theory, because $(\mathbb{R}, +, \cdot, <, \mathbb{Q}, (r)_{r \in \mathbb{R}})$ is $\mathscr{L}$-maximal, that is, it has no strict extension in the sense of $\prec_{\mathscr{L}}$ (Exercise!).) For further results see Fuhrken [1965].

(c) $w(\mathscr{L}_{\omega_1\omega}) = \omega_1$. We have $w(\mathscr{L}_{\omega_1\omega}) \geq \omega_1$, because a countable ordinal $\alpha \neq 0$ is pinned down by the $\mathscr{L}_{\omega_1\omega}$-sentence

$$\text{“} < \textit{is a linear ordering} \text{”} \wedge \forall x \bigvee \{\mu_\beta(x) \mid \beta < \alpha\},$$

where μ_β is defined inductively by

$$\mu_\beta(x) = \forall y(y < x \leftrightarrow \bigvee \{\mu_\gamma(y) | \gamma < \beta\}).$$

A similar argument works for all admissible fragments $\mathscr{L}_{\mathscr{A}}$, showing us that $w(\mathscr{L}_{\mathscr{A}}) \geq o(\mathscr{A})$, the least ordinal not in $\mathscr{A}$. The converse inequality is true for countable $\mathscr{A}$ and yields $w(\mathscr{L}_{\omega_1\omega}) \leq \omega_1$.

(d) If $\mathscr{L} \leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}^*$, then $w_\kappa(\mathscr{L}) \leq w_\kappa(\mathscr{L}^*)$. Using this fact and the remark on countable admissible sets in (c), one can deduce that

$$w(\mathscr{L}^{w2}) = w(\mathscr{L}(Q_0)) = w(\mathscr{L}(\omega, <)) = \omega_1^{\mathrm{CK}},$$

the least non-recursive ordinal (the "Church–Kleene ω_1").

(e) The argument from (c) can be extended to arbitrary ordinals α, if we admit sentences from $\mathscr{L}_{\infty\omega}$. Hence, $w(\mathscr{L}_{\infty\omega}) = \infty$. On the other hand, $\mathscr{L}_{\infty\omega}$ is bounded (López-Escobar [1966]).

(f) The logics $\mathscr{L}^2$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$, $\mathscr{L}_{\omega\omega}(I)$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{WO}})$, $\mathscr{L}_{\omega_1\omega_1}$ are not bounded as they admit a definition of well-orderings, at least as a projective or a relativized projective class (see Sections 2.3, 2.5 and Example 4.1.2(iv)).

We now return to our introductory remark and state a precise relation between compactness and the characterizability of well-orderings. A stronger form is implicit in Theorem III.2.1.4 in the equivalence of (i) and (iii).

5.2.4 Proposition. *$\mathscr{L}$ is $\aleph_0$-compact iff $w_{\aleph_0}(\mathscr{L}) = \omega$.*

Proof. For the interesting direction, assume $\mathscr{L}$ to be not $\aleph_0$-compact and $\Phi = \{\varphi_n | n \in \omega\}$ to be a countable set of sentences of some vocabulary τ such that any finite subset of Φ has a model, but Φ itself does not. Since $\mathscr{L}$ allows elimination of function symbols, we can assume that τ is relational. Then, with new binary relation symbols R and $<$, the set Φ' pins down ω, where Φ' consists of

(1) $<$ *is a linear ordering*;

(2) $\forall x \in \mathit{field}(<)\, \exists z\, Rxz$;

(3) $\forall x \in \mathit{field}(<)(|\{y | y \leq x\}| \geq n \rightarrow \varphi_n^{\{z | Rxz\}})$ for $n \in \omega$. ☐

At this point we can make another idea precise. Often compactness of a logic can be proved by defining a calculus and showing its completeness. In the framework of our precise notions we can extract the following general fact:

5.2.5 Theorem. *Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}_0}, \ldots, Q^*_{\mathfrak{K}_{n-1}})$ be a logic with Lindström quantifiers (in the sense of Definition 4.1.4), where $\mathscr{L}$ is recursively enumerable for validity. Then, for any $\tau \in \mathrm{HF}$, $\mathscr{L}$ satisfies the compactness property for recursive sets of sentences from $\mathscr{L}[\tau]$.*

Proof. First, we treat the special case where $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\aleph_i} | i < n)$ is recursively enumerable for consequence. Let $\Phi \subseteq \mathscr{L}[\tau]$ be a recursive set of sentences such that any finite subset has a model. If Φ had no model, we could pass from Φ to a recursive (!) set Φ' as defined in the preceding proof. Adding recursive definitions of addition and multiplication on field($<$) to Φ' would lead to a recursive set Φ'' characterizing the set of natural numbers with addition and multiplication. Hence, the consequences of Φ'' could not be recursively enumerable. Contradiction. By a technique that goes back to Kleene (see Craig–Vaught [1958]) one can give a *finite* axiomatization of Φ'' by use of additional predicates. Hence, the assumption that $\mathscr{L}$ is recursively enumerable for validity is sufficient for the preceding argument. ☐

5.3. Substitutes

There are extensions of first-order logic—and $\mathscr{L}_{\omega_1\omega}$ is one of the best examples—that admit an interesting model theory despite the fact that essential properties such as compactness fail. They illustrate that the value of a logical system should not only be measured by the number of significant properties of first-order logic that are preserved. For instance, $\mathscr{L}_{\omega_1\omega}$ compensates missing compactness by other properties that are well adapted to its specific syntax and its expressive power, such as that of having the "small" well-ordering number ω_1, or the interpolation property. Guided by such experience and moreover by results such as Proposition 5.2.4, we may arrive at the idea of considering compactness not only in the "crude" sense of κ-compactness or its variants, but of measuring it, for instance, by the size of the well-ordering number. In this sense, the logic $\mathscr{L}_{\infty\omega}$, having well-ordering number ∞, but being bounded, has preserved a vestige of compactness.

Taking these aspects seriously, we are led to the following way of exploring the value of some logic $\mathscr{L}$. Instead of asking for the preservation of properties of $\mathscr{L}_{\omega\omega}$, we try to isolate properties of $\mathscr{L}$ that are able to *replace* missing properties of $\mathscr{L}_{\omega\omega}$ or are useful in connection with the special features of $\mathscr{L}$. Properties of the first kind could be called *substitutes* (for the corresponding properties of $\mathscr{L}_{\omega\omega}$). Adhering to compactness we try to give an illustration by some examples. When doing so, however, we should bear in mind that we are not searching for some technical means, but rather are on the trace of some kind of "methodological ferment".

Example 1. *Barwise compactness*, based on a suitable generalization of finiteness, may be considered as the most convincing example. (For details see Barwise [1975] or Chapter VIII.)

Example 2. *Small well-ordering numbers and boundedness.* We have already mentioned $\mathscr{L}_{\omega_1\omega}$ and the role of its well-ordering number being ω_1 (see also Flum [1975b]). A further illustration will be treated in Theorem III.3.6: If we combine boundedness as a substitute for compactness with the so-called *countable* *approximation property* (see Kueker [1977]) as a substitute for the Löwenheim–Skolem property down to $\aleph_0$, we get a "substitute" for Lindström's first theorem with $\mathcal{L}_{\infty\omega}$ as a "substitute" for $\mathcal{L}_{\omega\omega}$.

The reader who watches carefully for methodological aspects, will meet further examples at various points. Certainly he will do so when he recognizes the role of *indiscernibles* (instead of compactness properties) as a means of obtaining upper bounds for Hanf numbers ("stretching method", see the examples following Theorem 6.1.6).

6. Löwenheim–Skolem Properties

The well-ordering number $w(\mathcal{L})$ and its generalizations $w_\kappa(\mathcal{L})$ center around the characterization of well-orderings. Löwenheim–Skolem phenomena refer to analogous questions concerning the cardinality of models. There are two dual aspects: one deals with Hanf numbers (as a counterpart of well-ordering numbers), the other one with Löwenheim numbers.

The following definitions and results can be restated for the many-sorted case, if one defines the cardinality of a many-sorted τ-structure $\mathfrak{A}$ as $\sum_{s\in\tau}|A_s|$ (see Definition 1.2.4(vii)).

6.1. Hanf Numbers

For any logic $\mathcal{L}$, compactness yields the upward Löwenheim–Skolem theorem in the following form: If Φ is a set of sentences of $\mathcal{L}$ of power $\leq\kappa$ that has an infinite model, then Φ has models of arbitrarily high cardinality. In the terminology to come this means that $h_\kappa(\mathcal{L}) = \aleph_0$ for all κ.

6.1.1. Definition. We say that $\Phi \subseteq \mathcal{L}[\tau]$ *pins down the cardinal* κ iff Φ has a model of cardinality κ, but Φ does not have models of arbitrarily high cardinalities. We let $h_\kappa(\mathcal{L})$ be the supremum of all cardinals that can be pinned down by a set of $\mathcal{L}$-sentences of power $\leq\kappa$ and call $h(\mathcal{L}) := h_1(\mathcal{L})$ the *Hanf number* of $\mathcal{L}$.

By regularity, $h(\mathcal{L}) \geq \aleph_0$. To get more information, let $\Phi \subseteq \mathcal{L}[\tau]$ pin down arbitrarily high cardinals below μ, $\mu \geq \aleph_0$. Assume without loss of generality that τ is relational. Then Ψ pins down μ, where Ψ consists of

(1) $<$ *is a linear ordering of the universe*;

(2) $\forall x\, \varphi^{\{z \mid Rxz\}}$ *for* $\varphi \in \Phi$;

(3) $\forall x\, \lambda u f(x, u) \restriction \{y \mid y \leq x\}$ *is an injection into* $\{z \mid Rxz\}$.

From this we see (taking μ^+ instead of μ) that $h_\kappa(\mathcal{L}) = \infty$ or $h_\kappa(\mathcal{L})$ is a limit cardinal that cannot be pinned down by a set of $\mathcal{L}$-sentences of power $\leq\kappa$. Hence,

$h_\kappa(\mathscr{L}) = \infty$ or $h_\kappa(\mathscr{L})$ is the least cardinal μ such that every set of $\mathscr{L}$-sentences of power $\leq \kappa$ that has a model of cardinality μ has arbitrarily large models. Moreover, we obtain as a weak analogue of Proposition 5.2.2:

6.1.2 Proposition. *If $\Phi \subseteq \mathscr{L}[\tau]$, $|\Phi| \leq \kappa$, and Φ has models of arbitrarily high cardinality below $h_\kappa(\mathscr{L})$, then Φ has models of arbitrarily high cardinality.* ☐

We have $h(\mathscr{L}_{\infty\omega}) = \infty$ even if we restrict ourselves to finite vocabularies (for instance to $\{<\}$, as can be obtained from Examples 5.2.3(c), (e)). On the other hand, logics with "few" sentences should have Hanf numbers $< \infty$. To make this precise, we introduce a new notion.

6.1.3 Definition. Occ($\mathscr{L}$), *the occurrence number of* $\mathscr{L}$, is the least cardinal μ such that for all τ,

$$\mathscr{L}[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ |\tau_0| < \mu}} \mathscr{L}[\tau_0],$$

if such a cardinal exists; otherwise Occ($\mathscr{L}$) $= \infty$.[3]

The following theorem can be considered as one of the earliest results of what is now called abstract model theory.

6.1.4 Theorem (Hanf [1960]). *Let $\mathscr{L}$ be small (that is, for all τ, $\mathscr{L}[\tau]$ is a set) and assume that* Occ($\mathscr{L}$) $< \infty$. *Then for all κ, $h_\kappa(\mathscr{L}) < \infty$.*

Proof. Set $\mu = \kappa \cdot \mathrm{Occ}(\mathscr{L})$ and let τ be a "universal" vocabulary of power μ; that is, τ contains μ many relation and function symbols of each arity and μ many constants. In order to investigate $h_\kappa(\mathscr{L})$, we can confine ourselves to τ-sentences of $\mathscr{L}$. As $\mathscr{L}[\tau]$ is a set, we have

$$h_\kappa(\mathscr{L}) = \sup\{|A| \mid \mathfrak{A} \models \Phi, \Phi \subseteq \mathscr{L}[\tau], |\Phi| \leq \kappa \text{ and } \Phi \text{ does not have arbitrarily large models}\} < \infty$$

(Axiom of Replacement!). ☐

The use of the Axiom of Replacement in the argument above is quite essential. This can already be illustrated in case $\mathscr{L} = \mathscr{L}^2$ (see Barwise [1972b]).

[3] Just as one defines the occurrence number Occ($\mathscr{L}$) one can introduce a so-called *dependence number* $\mathbf{o}(\mathscr{L})$, as is done in Chapter XVIII, 2.1.4: $\mathbf{o}(\mathscr{L})$ is the smallest cardinal κ such that for all τ and $\varphi \in \mathscr{L}[\tau]$ there is a vocabulary $\sigma \subset \tau$ of cardinality $< \kappa$ such that φ depends only on σ, and $\mathbf{o}(\mathscr{L}) = \infty$ if no such κ exists. Intuitively, the dependence number is the semantic side and the occurrence number the syntactic side of one and the same coin. Indeed, using the substitution property to remove dummy relation symbols, function symbols, and constants, one can easily see that $\mathbf{o}(\mathscr{L})$ and Occ($\mathscr{L}$) can play the same role in the one-sorted case. In the many-sorted case this may not be true because the substitution property as we have stated it in 1.2.3 does not enable us to remove dummy sort symbols; however, it can be guaranteed by a suitable reformulation of 1.2.3 which we leave to the reader.

Compactness properties yield small Hanf numbers. For example, if $\mathscr{L}$ is (κ, λ)-compact for all κ, then $h_\kappa(\mathscr{L}) \leq \lambda$ for all κ. On the other hand, compactness fades away with growing well-ordering numbers. Hence the question: Do large well-ordering numbers come along with large Hanf numbers? For a precise answer we introduce the *beth numbers* from classical set theory:

6.1.5 Definition. We define by recursion:

(i) $\beth_0(\kappa) = \kappa$;
(ii) $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$;
(iii) $\beth_\beta(\kappa) = \sup\{\beth_\alpha(\kappa) \mid \alpha < \beta\}$ for limit β.

To illustrate the size of beth numbers, let A be a set of power κ and define $V_\alpha^*(A)$, a variant of the von Neumann hierarchy over A, by the following equations:

(i′) $V_0^*(A) = A$;
(ii′) $V_{\alpha+1}^*(A) =$ power set of $V_\alpha^*(A)$,
(iii′) $V_\beta^*(A) = \bigcup \{V_\alpha^*(A) \mid \alpha < \beta\}$ for limit β.

Then for all α we have $|V_\alpha^*(A)| = \beth_\alpha(\kappa)$.

Now assume that $\lambda < h_\kappa(\mathscr{L})$ is pinned down by a set $\Phi \subseteq \mathscr{L}[\tau]$ of power $\leq \kappa$, where τ can be chosen relational ($\mathscr{L}$ allows elimination of function symbols!). With new binary relation symbols V, ε and new constants c_0, c_1 let Φ' consist of

(1) $\exists z\, Vc_0 z \wedge \forall z(Vc_0 z \vee Vc_1 z)$;

(2) $\varphi^{\{z \mid Vc_0 z\}}$ for $\varphi \in \Phi$;

(3) $\forall xy(\forall z(z\varepsilon x \leftrightarrow z\varepsilon y) \rightarrow x = y)$; that is, "$\varepsilon$ *is extensional*";

(4) $\forall z(Vc_1 z \leftrightarrow \forall u \varepsilon z\, Vc_0 z)$.

Then for any model $\mathfrak{A}$ of Φ' we have with $\mu_i = |\{a \in A \mid (c_i^{\mathfrak{A}}, a) \in V^{\mathfrak{A}}\}|$ that $|A| \leq \mu_0 + \mu_1$, where $\mu_0 < h_\kappa(\mathscr{L})$ and $\mu_1 \leq \beth_1(\mu_0)$. Hence Φ' pins down cardinals, and obviously $h_\kappa(\mathscr{L}) > \beth_1(\lambda)$.

Φ' can be considered as a description of the first two steps of the modified von Neumann hierarchy over the domain of models of Φ, where Φ pins down λ. The construction can be easily generalized in a natural way to describe the hierarchy along well-orderings that can be pinned down in $\mathscr{L}$. Thus, one can prove:

6.1.6 Theorem. *Assume that each ordinal $\alpha < w_\kappa(\mathscr{L})$ can be pinned down by a set Ψ_α of sentences, $|\Psi_\alpha| \leq \kappa$, having a model $\mathfrak{A}$ of power $< h_\kappa(\mathscr{L})$ where $<^{\mathfrak{A}}$ is of order type α. Then for every $\lambda < h_\kappa(\mathscr{L})$, $h_\kappa(\mathscr{L}) \geq \beth_{w_\kappa(\mathscr{L})}(\lambda)$.*

As an application we obtain, for instance, that

$$h(\mathscr{L}_{\omega\omega}(Q_1)) \geq \beth_\omega(\aleph_0) = \beth_\omega(\aleph_1);$$

$$h(\mathscr{L}_{\omega_1\omega}) \geq \beth_{\omega_1}(\aleph_0);$$

$$h(\mathscr{L}) \geq \beth_{\omega_1^{CK}}(\aleph_0) \quad \text{for } \mathscr{L} = \mathscr{L}^{w2}, \mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}_{\omega\omega}(\omega, <).$$

What about the other direction in these examples? It is valid, too. Thus, in each case we get equality. The corresponding proofs are based on partition theorems and indiscernibles. These techniques can also be used to get further strong results in the same direction (see, for example, Barwise [1975]).

If a logic is weak in pinning down ordinals, it may happen that we are unable to give satisfactory information about Hanf numbers. For example, for $\mathscr{L} = \mathscr{L}_{\omega\omega}(I)$, the size of $h(\mathscr{L})$ depends on set theory: If $V = L$, then $h(\mathscr{L}) = h(\mathscr{L}^2)$. On the other hand, $h(\mathscr{L})$ may be smaller than the Löwenheim number $l(\mathscr{L})$ as defined below, which may itself be smaller than $2^{\aleph_0}$ (see Section VI.2.1 and Väänänen [1982a]).

Warning. We have become accustomed to numerous preservation facts for (R)PC-reducibility. For instance, we obviously have

$$(*) \qquad \text{If } \mathscr{L} \leq_{\mathrm{PC}} \mathscr{L}^*, \quad \text{then for all } \kappa, \quad h_\kappa(\mathscr{L}) \leq h_\kappa(\mathscr{L}^*).$$

However, it is plausible that we would meet difficulties if we were to try to prove $(*)$ for $\leq_{\mathrm{RPC}}$. Indeed, in the remark preceding Proposition 7.2.5 we will see that there are counterexamples.

6.2. *Löwenheim Numbers*

Löwenheim numbers measure the strength of downward Löwenheim–Skolem theorems.

6.2.1 Definition. $l_\kappa(\mathscr{L})$ is the least cardinal μ such that any satisfiable set of $\mathscr{L}$-sentences of power $\leq \kappa$ has a model of power $\leq \mu$, provided there is such a cardinal; otherwise, $l_\kappa(\mathscr{L}) = \infty$. We call $l(\mathscr{L}) := l_1(\mathscr{L})$ the *Löwenheim number* of $\mathscr{L}$.

Obviously, $\mathscr{L}$ has the Löwenheim–Skolem property down to λ iff $l(\mathscr{L}) \leq \lambda$. By taking inequalities between κ many constants we see that $l_\kappa(\mathscr{L}) \geq \max\{\kappa, \aleph_0\}$. The proof of the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega\omega}(Q_1)$ as mentioned in Example 1 of Section 2.2 can be generalized and yields $l(\mathscr{L}_{\omega\omega}(Q_\alpha)) = l_{\aleph_\alpha}(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \aleph_\alpha$. Clearly, $l(\mathscr{L}_{\infty\omega}) = \infty$. But if $\mathscr{L}$ is small (that is, if all $\mathscr{L}[\tau]$ are sets) and $\mathrm{Occ}(\mathscr{L}) < \infty$, then by an argument like that for Hanf's theorem (6.1.4), we have $l_\kappa(\mathscr{L}) < \infty$ for all κ.

Numerous results such as $l_{\aleph_\alpha}(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \aleph_\alpha$ can be strengthened by showing that structures possess small elementary substructures; however, this possibility may fail already with familiar logics. For instance, $l(\mathscr{L}_{\omega\omega}(\mathrm{aa})) = \aleph_1$, but the existence of $\mathscr{L}_{\omega\omega}(\mathrm{aa})$-elementary substructures of power $\leq \aleph_1$ is independent from ZFC (see remark after IV.4.2.5). For a closer look at Löwenheim–Skolem properties and substitutes the reader is referred to Section III.3.

7. *Interpolation and Definability*

In this final section we return to central notions of a more "logical" character. The main topics we shall touch concern interpolation and a generalization of Robinson's consistency theorem in Section 7.1, Δ-interpolation in Section 7.2 and variations of Beth's definability theorem in Section 7.3. Again we confine ourselves to regular logics. *However, we explicitly include the many-sorted case.* As the reformulation of the usual interpolation property given in Definition 1.2.4(viii) by separability of projective classes as in Proposition 3.1.3 splits into cases—referring to "PC" in the one-sorted version and to "RPC" in the many sorted version—we use "(R)PC" to stand for "PC" in the first and for "RPC" in the second case.

7.1. Interpolation and the Robinson Property

As a generalization of the interpolation property, we state

7.1.1 Definition. Let $\mathscr{L}$, $\mathscr{L}^*$ be logics. *$\mathscr{L}^*$ has the interpolation property for $\mathscr{L}$* or *$\mathscr{L}^*$ allows interpolation for $\mathscr{L}$* iff any two disjoint classes of the same vocabulary that are (R)PC in $\mathscr{L}$ can be separated by a class EC in $\mathscr{L}^*$.

Interpolation is indeed rare. The positive examples among the logics we have mentioned up to now can be listed very quickly:

7.1.2 Examples. (a) $\mathscr{L}_{\omega\omega}$. The one-sorted case is due to Craig [1957a], the many-sorted one is proved in Feferman [1968a]. The one-sorted version follows from the many-sorted one, even in the stronger form with "RPC" instead of "PC", because relativized reducts can be rewritten as simple reducts of many-sorted structures (see Barwise [1973a]). It is especially with interpolation that many-sortedness pays. As seen in Feferman [1974a], the many-sorted version of the interpolation theorem together with its possible refinements is a powerful tool even for one-sorted model theory, offering for instance elegant proofs of various preservation theorems. For a proof of a strong version of $\mathscr{L}_{\omega\omega}$-interpolation the reader is referred to Theorem X.2.2.9.

(b) $\mathscr{L}_{\omega_1\omega}$ (Lopez-Escobar [1965b]) and countable admissible fragments (Barwise [1969b]).

Interpolation properties seem to indicate some kind of balance between syntax and semantics. This can be seen, for instance, from the work of Zucker [1978] or from the fact that interpolation implies Beth's definability theorem, according to which implicit definitions can be made explicit. Last but not least it is illustrated by a result of Feferman [1974a] according to which Δ-interpolation is equivalent to truth maximality (see Corollary XVII.1.1.17). Hence we may expect that interpolation properties (or definability properties, see Section 7.3) fail if syntax and semantics are not in an equilibrium. The counterexamples to interpolation that we have mentioned up to now (such as $\mathscr{L}^{w2}$, being able to code its own truth, or $\mathscr{L}_{\omega\omega}(Q_1)$, being able to characterize uncountability) are not astonishing if seen in the light of these heuristics.

7.1.3 Further Counterexamples. (a) In the case of *large infinitary languages*, the main fact is that $\mathscr{L}_{\infty\omega}$ *does not allow interpolation for* $\mathscr{L}_{\omega_2\omega}$. For a proof we consider the classes

$$\mathfrak{K}_{\aleph_0} = \{A \mid A \neq \emptyset, |A| \leq \aleph_0\}, \qquad \mathfrak{K}^{\aleph_1} = \{A \mid |A| \geq \aleph_1\}.$$

$\mathfrak{K}_{\aleph_0}$ and $\mathfrak{K}^{\aleph_1}$ are PC in $\mathscr{L}_{\omega_2\omega}$ (for $\mathfrak{K}^{\aleph_1}$ we can use the sentence

$$\bigwedge \{c_\alpha \neq c_\beta \mid \alpha < \beta < \aleph_1\}).$$

But $\mathfrak{K}_{\aleph_0}$ and $\mathfrak{K}^{\aleph_1}$ cannot be separated by a class EC in $\mathscr{L}_{\infty\omega}$, as all infinite sets are partially isomorphic and, hence, $\mathscr{L}_{\infty\omega}$-equivalent by Karp's theorem (4.3.2). (For further results see Example IX.2.3.1 and Theorem IX.2.3.2.)

(b) For extensions of $\mathscr{L}_{\omega\omega}(Q_1)$, we find that $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ does not allow interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$, and $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ does not allow interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$. Hence, none of the logics $\mathscr{L}_{\omega\omega}(Q_1^n)$ for $n \geq 1$, $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ or $\mathscr{L}_{\omega\omega}(\mathrm{pos})$ has the interpolation property.

To argue for the first assertion, let $\mathfrak{K}_{\mathrm{cf}\,\omega}$, $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ be the classes of orderings of cofinality ω, ω_1, respectively. Both are PC in $\mathscr{L}_{\omega\omega}(Q_1)$: $\mathfrak{K}_{\mathrm{cf}\,\omega}$ via a sentence $\varphi_0(<, U_0)$ saying that $<$ is an ordering of the universe without last element and U_0 of power $\leq \aleph_0$ a cofinal subset, and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ via a sentence $\varphi_1(<, U_1)$ saying that $<$ is an ordering of the universe and U_1 a cofinal subset such that $< \restriction U_1 \times U_1$ is an $\aleph_1$-like ordering. $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ cannot be separated by a class of orderings EC in $\mathscr{L}_{\omega\omega}(Q^n \mid n \geq 1)$. For let $\mathfrak{A} = (\mathbb{R}, <^{\mathbb{R}})$ be the ordering of the reals and $\mathfrak{B} = (\mathfrak{B}, <^{\mathfrak{B}})$ the result of replacing each ordinal in $\aleph_1$ by a copy of $\mathfrak{A}$. Then $\mathfrak{A} \in \mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{B} \in \mathfrak{K}_{\mathrm{cf}\,\omega_1}$. On the other hand, we have $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(Q_1^n)} \mathfrak{B}$ for all $n \geq 1$, as $(I)_{\alpha \leq \omega}\colon \mathfrak{A} \cong_{\omega, Q_1^n} \mathfrak{B}$, where I is the set of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ with finite domain. (For the second assertion and further material, see Section IV.6.3).

(c) $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$, the fully compact extension of $\mathscr{L}_{\omega\omega}$, does not have the interpolation property (alas!). To sketch a counterexample, call a tree $(T, <, E)$ with an equivalence relation E on T whose equivalence classes are maximal antichains ("levelled tree") *rankable* by a linear ordering $(R, \prec)$, if there exists a homomorphism π from $(T, <)$ onto $(R, \prec)$ such that the equivalence classes of E are the preimages of π. Define $\mathfrak{K}_0, \mathfrak{K}_1$ to be the class of levelled trees rankable by some ordering of cofinality ω, $\geq \omega_1$, respectively. Then $\mathfrak{K}_0$ and $\mathfrak{K}_1$ are disjoint and PC in $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$. Define $\mathfrak{T}_0$ to be the set $\{t \mid t\colon \{a \in \mathbb{Q} \mid a <^{\mathbb{Q}} b\} \to \{0, 1\}, b \in \mathbb{Q}\}$ ordered by inclusion where two points are equivalent if they have the same domain, and define $\mathfrak{T}_1$ similarly, using a dense $\aleph_1$-like end extension of $(\mathbb{Q}, <^{\mathbb{Q}})$. Then $\mathfrak{T}_i \in \mathfrak{K}_i$ $(i = 0, 1)$, but $\mathfrak{T}_0 \prec_{\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})} \mathfrak{T}_1$. See also Mekler–Shelah [1983, Theorem 3.5].

(d) For a general class of counterexamples the reader can refer to Proposition VI.2.3.1.

In first order logic there is access to interpolation via Robinson's consistency theorem. This possibility can be generalized.

7.1.4 Definition. $\mathscr{L}$ has the *Robinson property* iff for any vocabularies τ_0, τ_1 and $\tau = \tau_0 \cap \tau_1$ and for all classes (!) $\underset{\sim}{\Phi} \subseteq \mathscr{L}[\tau]$ and $\underset{\sim}{\Phi}_i \subseteq \mathscr{L}[\tau_i]$ $(i = 0, 1)$, if $\underset{\sim}{\Phi}$ is complete (i.e. all τ-models of $\underset{\sim}{\Phi}$ are $\mathscr{L}$-equivalent) and if $\underset{\sim}{\Phi} \cup \underset{\sim}{\Phi}_i$ has a model for $i = 0, 1$, then $\underset{\sim}{\Phi} \cup \underset{\sim}{\Phi}_0 \cup \underset{\sim}{\Phi}_1$ has a model.

7.1.5 Proposition. *Let $\mathscr{L}$ be small (i.e. all $\mathscr{L}[\tau]$ are sets). Then, if $\mathscr{L}$ is compact, $\mathscr{L}$ has the interpolation property iff $\mathscr{L}$ has the Robinson property.*

Proof. Let $\mathscr{L}$ be compact and τ_0, τ_1 and τ be given as in Definition 7.1.4. Since $\mathscr{L}$ is small, all classes of sentences defined below are sets so that the compactness property is applicable. Assume first that $\mathscr{L}$ has the Robinson property and let $\varphi_i \in \mathscr{L}[\tau_i]$ $(i = 0, 1)$ be given such that

$$(*) \qquad \varphi_0 \models \varphi_1.$$

Setting $\Phi' = \{\varphi \in \mathscr{L}[\tau] \mid \varphi_0 \models \varphi\}$, we have $\Phi' \models \varphi_1$. (Otherwise, if $\mathfrak{B} \in \mathrm{Str}[\tau]$ has an expansion satisfying $\Phi' \cup \{\neg\varphi_1\}$, then $\mathrm{Th}_{\mathscr{L}}(\mathfrak{B}) \cup \{\neg\varphi_1\}$ has a model, and by a compactness argument, so does $\mathrm{Th}_{\mathscr{L}}(\mathfrak{B}) \cup \{\varphi_0\}$. Hence the Robinson property yields a model of $\{\varphi_0, \neg\varphi_1\}$—a contradiction to $(*)$.)

Now, by compactness, there is some finite subset of Φ', say Φ'', such that $\Phi'' \models \varphi_1$. Obviously, $\bigwedge \Phi''$ is an interpolant for $(*)$.

For the other direction let Φ, Φ_0, Φ_1 be given as in Definition 7.1.4, Φ complete, $\Phi \cup \Phi_i$ satisfiable for $i = 0, 1$ and without loss of generality $\Phi \subseteq \Phi_0$. As $\mathscr{L}$ is compact it suffices to show that for any finite conjunction φ_i over Φ_i $(i = 0, 1)$ the set $\{\varphi_0, \varphi_1\}$ is satisfiable.

Assume for contradiction that $\{\varphi_0, \varphi_1\}$ has no model. Then the interpolation property yields a sentence $\varphi \in \mathscr{L}[\tau]$ such that $\varphi_0 \models \varphi$ and $\varphi \models \neg\varphi_1$. As $\Phi \cup \{\varphi_0\}$ has a model and Φ is complete, we have $\Phi \models \varphi$. But then $\Phi \cup \{\varphi_1\}$ has no model, a contradiction to the satisfiability of $\Phi \cup \Phi_1$. $\square$

Proposition 7.1.5 can be strengthened considerably: For logics with sufficiently small occurrence number, the Robinson property yields compactness (see Theorem XIX.1.3 and Chapter XVIII).

7.2. *Δ-interpolation and Δ-closure*

The following notions have proved to be very fruitful.

7.2.1 Definition. A class $\mathfrak{K}$ of τ-structures is said to be Δ in $\mathscr{L}$ (in symbols $\mathfrak{K} \in \Delta_{\mathscr{L}}$) iff $\mathfrak{K}$ and $\bar{\mathfrak{K}} = \mathrm{Str}[\tau] \backslash \mathfrak{K}$ are (R)PC in $\mathscr{L}$. A logic $\mathscr{L}$ has the *Δ-interpolation property* iff every Δ class of $\mathscr{L}$ is EC in $\mathscr{L}$. A logic $\mathscr{L}^*$ has the *Δ-interpolation property for $\mathscr{L}$* (or *$\mathscr{L}^*$ allows Δ-interpolation for $\mathscr{L}$*) iff every Δ class of $\mathscr{L}$ is EC in $\mathscr{L}^*$.

As we have already observed in Section 3.1, Δ-interpolation is a weakening of interpolation. Moreover, Theorem 7.2.6 will show us that it is a strict one. For several reasons, however, it is an interesting one, one that is able to compete seriously with the perhaps too strong notion of interpolation:

(1) According to a remark after Example 7.1.2(b), Δ-interpolation is equivalent to truth-maximality and thus, in a precise sense, embodying a balance between syntax and semantics.
(2) Δ-interpolation is equivalent to a certain variant of Beth's definability theorem, see Proposition 7.3.3.
(3) Δ-interpolation is by far not as rare as interpolation. This will become clear from the notion of Δ-closure given below.

7.2.2 Examples and Counterexamples. (a) $\mathscr{L}_{\omega\omega}(Q_1)$ does not allow Δ-interpolation as the classes corresponding to Keisler's counterexample to interpolation (see ($*$) in Example 1 of Section 2.2) are Δ in $\mathscr{L}_{\omega\omega}(Q_1)$.

(b) Even sharper: $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ does not allow Δ-interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$. (For a proof see Theorem IV.6.3.3.)

(c) Similar to (a), the counterexample to interpolation for $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$ as given in 7.1.3(c) is also a counterexample to Δ-interpolation.

In contrast to the interpolation property, the Δ-interpolation property guarantees the existence only of such elementary classes as are uniquely determined. Hence, unlike interpolation, Δ-interpolation leads to a natural closure operation which we now examine.

7.2.3 Definition. The Δ-*closure* of $\mathscr{L}$, $\Delta(\mathscr{L})$, is the logic that has as elementary classes just the classes that are Δ in $\mathscr{L}$. To develop a more precise description, let $\Delta(\mathscr{L})[\tau]$ consist of all pairs

$$\varphi = (\exists_{\tau_0 \setminus \tau}\varphi_0, \exists_{\tau_1 \setminus \tau}\varphi_1),$$

where $\tau_i \supseteq \tau$, $\varphi_i \in \mathscr{L}[\tau_i]$ $(i = 0, 1)$, and $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_0 \setminus \tau}\varphi_0)$ and $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_1 \setminus \tau}\varphi_1)$ are complementary, and set

$$\mathrm{Mod}_{\Delta(\mathscr{L})}^{\tau}(\varphi) = \mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_0 \setminus \tau}\varphi_0).$$

7.2.4 Theorem (Properties of the Δ-Closure). *Assume that* $\mathrm{Occ}(\mathscr{L}) = \aleph_0$. *Then*

(i) $\Delta(\mathscr{L})$ *is a regular logic with occurrence number* $\aleph_0$.
(ii) Δ *is a closure operation on the logics under consideration, that is,*
   (1) $\mathscr{L} \leq \Delta(\mathscr{L})$;
   (2) *If* $\mathscr{L} \leq \mathscr{L}^*$, *then* $\Delta(\mathscr{L}) \leq \Delta(\mathscr{L}^*)$;
   (3) $\Delta(\Delta(\mathscr{L})) \equiv \Delta(\mathscr{L})$.
(iii) $\Delta(\mathscr{L})$ *has the* Δ-*interpolation property and* $\mathscr{L} \equiv_{(\mathrm{R})\mathrm{PC}} \Delta(\mathscr{L})$.
(iv) $\Delta(\mathscr{L})$ *is modulo equality via elementary classes the strongest logic* $\leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}$ *and the smallest* $\leq_{(\mathrm{R})\mathrm{PC}}$-*extension of* $\mathscr{L}$ *having the* Δ-*interpolation property.*

Remarks. The first statement of (iv) says that if $\mathscr{L}^* \leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}$, then $\mathscr{L}^* \leq \Delta(\mathscr{L})$. Thus it makes precise the range of (R)PC-reducibility: There is a unique borderline realized by $\Delta(\mathscr{L})$.

As the proof below will show the condition on $\mathrm{Occ}(\mathscr{L})$ is used for instance to formulate $\boldsymbol{\tau}$-closedness of predicates. In infinitary languages this can be done even for infinite vocabularies. Hence Theorem 7.2.4 is also valid for logics such as $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}_{\infty\omega}$.

Sketch of Proof of Theorem 7.2.4. We show some parts of (i) and (ii)(3), confining ourselves to the one-sorted case and considering typical examples. If S is, say, unary and $\varphi(c) = (\exists R\varphi_0(R, c), \exists S\varphi_1(S, c))$, then one can take $(\exists S\varphi_1(S, c), \exists R\varphi_0(R, c))$ for $\neg\varphi(c)$ and $(\exists R\ \exists c\varphi_0(R, c), \exists S'\ \forall c\varphi_1(\lambda z S'cz, c))$ for $\exists c\varphi(c)$ where S' is a new binary relation symbol. To show the relativization property, let $\chi = (\exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}\chi_0, \exists_{\boldsymbol{\sigma}_1\setminus\boldsymbol{\sigma}}\chi_1) \in \Delta(\mathscr{L})[\boldsymbol{\sigma}]$ (with $\boldsymbol{\sigma} \subseteq \boldsymbol{\sigma}_i$, $\boldsymbol{\sigma}_i$ finite, $\chi_i \in \mathscr{L}[\boldsymbol{\sigma}_i]$ and, say, $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_1$) and let $\vartheta = (\vartheta_0, \vartheta_1)$ be some $\Delta(\mathscr{L})$-sentence of meaning $\forall c(Uc \leftrightarrow \varphi(c))$, U new and, say, $\vartheta \in \Delta(\mathscr{L})[\boldsymbol{\tau} \cup \{U\}]$. Then one can obtain a $\Delta(\mathscr{L})$-sentence of meaning $\chi^{\{c\,|\,\varphi(c)\}}$ by suitably rephrasing the equivalent statement

$$(\exists U\ \exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}(\vartheta_0 \wedge U\boldsymbol{\sigma}_0\text{-}closed \wedge \chi_0^U), \exists U\ \exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}(\vartheta_0 \wedge (\neg U\boldsymbol{\sigma}_0\text{-}closed \vee \chi_1^U))).$$

In order to prove $\Delta(\Delta(\mathscr{L})) \equiv \Delta(\mathscr{L})$ one observes that a typical sentence of $\Delta(\Delta(\mathscr{L}))$ such as

$$(\exists R(\exists S\varphi_0(R, S, \ldots), \ldots), \exists R'(\exists S'\varphi_0'(R', S', \ldots), \ldots))$$

has the same meaning as $(\exists RS\varphi_0(R, S, \ldots), \exists R'S'\varphi_0'(R', S', \ldots))$. ☐

Remark. Properties that are transferred from $\mathscr{L}$ to $\Delta(\mathscr{L})$ include those which are inherited by $\equiv_{(\mathrm{R})\mathrm{PC}}$. Therefore the Δ-closure preserves (κ, λ)-compactness, well-ordering numbers, Löwenheim numbers and boundedness. On the other hand it does not necessarily preserve Hanf numbers in the many-sorted case. For example, as shown in Väänänen [1983], it is consistent to assume in this case that $h(\mathscr{L}_{\omega\omega}(I)) < h(\Delta(\mathscr{L}_{\omega\omega}(I)))$. However, if Δ classes and the Δ-closure are defined via PC as in the one-sorted case, Hanf numbers are preserved also.

7.2.5 Proposition. *The Δ-closure does not preserve the Karp property.*

Proof. $\mathscr{L}_{\omega_2\omega}$ has the Karp property (see the remarks following Theorem 4.3.2). According to Counterexample 7.1.3(a) the classes $\mathfrak{K}_{\aleph_0} = \{A \neq \emptyset \,|\, |A| \leq \aleph_0\}$ and $\mathfrak{K}^{\aleph_1} = \{A \,|\, |A| \geq \aleph_1\}$ are Δ in $\mathscr{L}_{\omega_2\omega}$ and so are EC in $\Delta(\mathscr{L}_{\omega_2\omega})$. Therefore, $\aleph_0 \not\equiv_{\Delta(\mathscr{L}_{\omega_2\omega})} \aleph_1$. But on the other hand we have $\aleph_0 \cong_p \aleph_1$. Thus, the Karp property fails for $\Delta(\mathscr{L}_{\omega_2\omega})$. ☐

We can say even more. By the proof of Proposition 7.2.5 we have $\mathscr{L}_{\omega\omega}(Q_1) \leq \Delta(\mathscr{L}_{\omega_2\omega})$. Hence the classes $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ of orderings of cofinality ω, ω_1, respectively, which are PC in $\mathscr{L}_{\omega\omega}(Q_1)$ (compare the proof of Counterexample 7.1.3(b)), are PC in $\Delta(\mathscr{L}_{\omega_2\omega})$. Assume that there is some class $\mathfrak{K}$, $\mathfrak{K}$ EC in $\Delta(\mathscr{L}_{\omega_2\omega})$, that separates $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$. Then for suitable $\tau \supseteq \{<\}$ and $\varphi \in \mathscr{L}_{\omega_2\omega}[\tau]$, we have $\mathfrak{K} = \mathrm{Mod}(\varphi) \upharpoonright \{<\}$. Let $\mathscr{L}$ be the smallest fragment of $\mathscr{L}_{\omega_2\omega}$ containing φ. We take some $\mathfrak{A} \in \mathrm{Str}[\tau]$ with $(A, <^{\mathfrak{A}})$ an ordering of cofinality ω_2 and build into $\mathfrak{A}$ a chain $(\mathfrak{A}_\alpha)_{\alpha \leq \aleph_1}$, forming unions at limit points, such that for all $\alpha < \aleph_1$, $\mathfrak{A}_\alpha \prec_{\mathscr{L}} \mathfrak{A}_{\alpha+1} \prec_{\mathscr{L}} \mathfrak{A}$, and $<^{\mathfrak{A}_\alpha}$ is not cofinal in $<^{\mathfrak{A}_{\alpha+1}}$. Then $\mathfrak{A}_\omega \upharpoonright \{<\} \in \mathfrak{K}$ iff $\mathfrak{A}_{\aleph_1} \upharpoonright \{<\} \in \mathfrak{K}$. This, however, is a contradiction. Thus we have proved

7.2.6 Theorem (H. Friedman). *Δ-interpolation is strictly weaker than interpolation. For instance, $\Delta(\mathscr{L}_{\omega_2\omega})$ does not allow interpolation.* $\square$

The reader should consult Theorem IV.6.3.5 for another example.

Concluding Remarks. (a) Our definition of $\Delta(\mathscr{L})$ as sketched in Definition 7.2.3 is useful for technical purposes. But it does have a remarkable disadvantage: even the $\mathscr{L}_{\omega\omega}$-part of $\Delta(\mathscr{L}_{\omega\omega})(\equiv \mathscr{L}_{\omega\omega}(!))$ is not effective, since for sufficiently rich τ the $\Delta(\mathscr{L}_{\omega\omega})$-sentences of the form $(\varphi, \exists x\, x \neq x)$ (that is, those with $\varphi \in \mathscr{L}_{\omega\omega}[\tau]$ and $\models \varphi$) do not form a recursive set. A more significant example illustrating the task of giving an informative description of Δ-closures is due to Barwise [1974a] (see Theorem XVII.3.2.2):

$$\Delta(\mathscr{L}^{w2}) \equiv \Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_{\omega_1^{\mathrm{CK}}\omega} \quad \text{(for finite vocabularies).}$$

(b) If $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}_0}, \ldots, Q^*_{\mathfrak{K}_n})$ with Lindström quantifiers $Q^*_{\mathfrak{K}_i}$ and if $\mathbb{K}$ consists of those classes which are Δ in $\mathscr{L}$ and of finite vocabulary, then obviously $\Delta(\mathscr{L}) = \mathscr{L}(Q^*_{\mathfrak{K}} \mid \mathfrak{K} \in \mathbb{K})$. Now, if $\mathbb{K}_0$ is a finite subset of $\mathbb{K}$, then one can prove by a slight variation of the technique used in the proof of Proposition 3.1.7 that, if $\mathscr{L}$ is recursively enumerable for consequence then so is $\mathscr{L}(Q^*_{\mathfrak{K}} \mid \mathfrak{K} \in \mathbb{K}_0)$. It is in this sense that the Δ-closure preserves axiomatizability *locally*.

7.3. Definability Properties

In Definition 1.2.4(ix) we formulated the *Beth property* by a natural translation of Beth's definability theorem into the framework of abstract model theory. The following definitions introduce some variants.

7.3.1 Definition. Assume $\S \in \tau$ and $\varphi \in \mathscr{L}[\tau]$. We say that φ *defines* $\S$ *strongly implicitly* iff for each $\mathfrak{A} \in \mathrm{Str}[\tau \setminus \{\S\}]$ there is *exactly one* expansion $(\mathfrak{A}, \S^{\mathfrak{A}})$ of $\mathfrak{A}$ which is a model of φ. The logic $\mathscr{L}$ has the *weak Beth property* iff for each τ, $\S$ and φ as above, if φ defines $\S$ strongly implicitly, then $\S$ is explicitly definable relative to φ.

Like Δ-interpolation which guarantees the existence of uniquely determined elementary classes, the weak Beth property ensures the existence of uniquely determined explicit definitions. Hence it induces a natural closure operation on logics yielding the so-called weak Beth closure WB($\mathscr{L}$) of a logic $\mathscr{L}$ (it is treated, for example, in Sections XVII.1.2 and 4.1). As can be shown by examples (see Chapter XVIII, 4.2.2) the weak Beth property is strictly weaker than the Beth property. According to H. Friedman [1973] and Badger [1980], $\mathscr{L}_{\omega\omega}(Q_1^n)$ does not have the Beth property for $n \geq 1$. It is open as to whether or not it has the weak Beth property. (For $n = 1$ see also Mekler–Shelah [198?].)

Failure of the Weak Beth Property. We have already mentioned after Definition 2.1.2 that there is a fairly general method of disproving the (weak) Beth property by a codification of truth. The method goes back to Craig [1965] and is explicitly used in Mostowski [1968] and Lindström [1969]. It applies to logics such as $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^{w2}$, or $\mathscr{L}_{\omega\omega}$ enlarged with finitely many Lindström quantifiers in which, for example, the standard model of arithmetic is characterizable and which allow an arithmetization of their semantics. A systematic treatment can be found in Section XVII.11.2. For illustration we give an example for the one-sorted case. We assume that $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$ with $\mathfrak{K}$ of vocabulary $\{R\}$, R binary, and that for some finite $\tau \supseteq \{+, \cdot, <, 0, 1\}$ and some $\varphi \in \mathrm{Sent}_{\mathscr{L}}[\tau]$ the sentence $\exists_{\tau \setminus \{+, \cdot, <, 0, 1\}} \varphi$ characterizes the standard model of arithmetic.

For our procedure we use an effective Gödel numbering

$$\gamma\colon \mathrm{Form}_{\mathscr{L}}[\tau] \xrightarrow[\text{onto}]{1-1} \omega,$$

and we code assignments of finitely many variables over ω—the case we are interested in—by elements of ω in some natural manner, identifying variables with natural numbers. Then, with a binary relation symbol Sat, we construct a ($\tau \cup \{\mathrm{Sat}\}$)-sentence σ of $\mathscr{L}$ such that (abbreviating $\underbrace{1 + \cdots + 1}_{m\text{-times}}$ by $\underline{m}$)

(*) Sat is defined strongly implicitly by the sentence

$$\hat{\sigma} = (\varphi \wedge \sigma) \vee (\neg \varphi \wedge \forall xy \neg \mathrm{Sat}\, xy).$$

(**) If $(\mathfrak{A}, \mathrm{Sat}^{\mathfrak{A}}) \models \varphi \wedge \sigma$, then, for all $m, n \in \omega$, we have

$\mathfrak{A} \models \mathrm{Sat}\, \underline{m}^{\mathfrak{A}} \underline{n}^{\mathfrak{A}}$ iff $\underline{m}^{\mathfrak{A}}$ codes an assignment π over A the domain of which contains all variables occurring free in $\bar{\gamma}^1(n)$ such that $\bar{\gamma}^1(n)$ is true under π in $\mathfrak{A}$.

To obtain σ, one describes the inductive definition of satisfaction for $\mathscr{L}[\tau]$-formulas. For example, the Q-step can be treated as follows: Let $f\colon \omega^3 \to \omega$ be a recursive function such that for all $l, m, n \in \omega$,

$$f(l, m, n) = \gamma(Q v_l v_m \bar{\gamma}^1(n)).$$

Then, writing "Sat(x, y)" for "Sat xy", the following sentence becomes a conjunct of σ:

$$\forall x\, \forall uvw\, (\mathrm{Sat}(w, \text{“} f(u, v, x)\text{”}) \leftrightarrow (\text{“}w \textit{ assignment for } f(u, v, x)\text{”} \\ \wedge\, Qyz\, \mathrm{Sat}(\text{“}w \restriction (\mathrm{dom}(w)\setminus\{u, v\}) \cup \{(u, y), (v, z)\}\text{”}, x))),$$

where the parts in quotation marks have to be replaced by an arithmetical definition.

Proof of the Failure. Now, assume $\mathscr{L}$ to have the weak Beth property. Then, by $(*)$, there is $\psi(v_0, v_1) \in \mathrm{Form}_{\mathscr{L}}[\tau]$ defining Sat explicitly relative to $\hat{\sigma}$. Let n be the Gödel number of $\neg\psi(\text{“}\{(0, v_0)\}\text{”}, v_0)$ and assume $\mathfrak{A}$ to be a model of $\varphi \wedge \sigma$. Then, by $(**)$, we obtain

$$\mathfrak{A} \models \psi(\text{“}\{(0, \underline{n})\}\text{”}, \underline{n}) \text{ iff } \mathfrak{A} \models \neg\psi(\text{“}\{(0, \underline{n})\}\text{”}, \underline{n}).$$

This is a contradiction. $\square$

As the interpolation property yields the definability property, counterexamples to the latter are, in effect, counterexamples to the former. Positive results concerning the other direction are described in Chapter XVIII.4.

We conclude with a link between interpolation and definability which goes back to Feferman [1974a]. For this purpose, we strengthen the weak Beth property in a new direction.

7.3.2 Definition. The logic $\mathscr{L}$ has the *projective weak Beth property* iff for all τ, τ' with $\tau \subseteq \tau'$ and for all $\S \in \tau$ and $\varphi \in \mathscr{L}[\tau']$, if $\exists_{\tau'\setminus\tau}\varphi$ defines $\S$ strongly implicitly, then $\S$ is explicitly definable relative to $\exists_{\tau'\setminus\tau}\varphi$.

7.3.3 Proposition. *$\mathscr{L}$ allows Δ-interpolation iff $\mathscr{L}$ has the projective weak Beth property.*

Proof. Assume first that $\mathscr{L}$ has the projective weak Beth property, and let $\mathfrak{K}_i = \mathrm{Mod}(\exists_{\tau_i\setminus\tau}\varphi_i)$ $(i = 0, 1)$ be two disjoint complementary classes of τ-structures. With new unary P (in the many-sorted case this P will be equipped with some sort symbol $s \in \tau$) we set

$$\chi = (\exists_{\tau_0\setminus\tau}\varphi_0 \wedge \forall x P x) \vee (\exists_{\tau_1\setminus\tau}\varphi_1 \wedge \forall x \neg P x).$$

Obviously, χ strongly implicitly defines P and the projective weak Beth property applies. Let ψ be an explicit definition of P relative to χ. Then

$$\{(\mathfrak{A}, a) \mid \mathfrak{A} \in \mathrm{Str}[\tau], \mathfrak{A} \models \exists_{\tau_0\setminus\tau}\varphi_0, a \in A_{(s)}\} = \mathrm{Mod}(\psi)$$

is EC in $\mathscr{L}$ and hence, by particularization, so is $\mathfrak{K}_0$.

For the other direction, assume $\mathscr{L}$ to allow Δ-interpolation and $\exists_{\tau'\setminus\tau}\varphi$ to strongly implicitly define $\S \in \tau$, say $\S = P$, a unary relation symbol. Then

$$\mathfrak{K} = \{(\mathfrak{A} \upharpoonright (\tau\setminus\{P\}), a) \mid \mathfrak{A} \models \exists_{\tau'\setminus\tau}\varphi, a \in P^{\mathfrak{A}}\}$$

is (R)PC in $\mathscr{L}$. The complementary class $\bar{\mathfrak{K}}$ can be written as

$$\bar{\mathfrak{K}} = \{(\mathfrak{A} \upharpoonright (\tau\setminus\{P\}), a) \mid \mathfrak{A} \models \exists_{\tau'\setminus\tau}\varphi, a \notin P^{\mathfrak{A}}\}$$

and hence is (R)PC in $\mathscr{L}$, too. Therefore, $\mathfrak{K}$ is EC in $\mathscr{L}$. Now take ψ such that $\mathfrak{K} = \mathrm{Mod}(\psi)$. Then ψ is an explicit definition of P relative to $\exists_{\tau'\setminus\tau}\varphi$. □

Following the pattern of Definition 7.3.2, the reader may define the so-called *projective* Beth property. For instance, suppose a sentence $\varphi(\bar{R}, \bar{S}, P)$ defines P implicitly relative to $\bar{S}$ in the sense that, with new symbols $\bar{R}'$, P',

$$\varphi(\bar{R}, \bar{S}, P) \wedge \varphi(\bar{R}', \bar{S}, P') \models \forall\bar{x}\,(P\bar{x} \leftrightarrow P'\bar{x}).$$

Then the projective Beth property implies that $\varphi(\bar{R}, \bar{S}, P)$ admits an explicit definition of P relative to $\bar{S}$, that is, a formula $\psi(\bar{S}, \bar{x})$ such that

$$\varphi(\bar{R}, \bar{S}, P) \models \forall\bar{x}(P\bar{x} \leftrightarrow \psi(\bar{S}, \bar{x})).$$

The usual proofs of Beth's definability theorem—including the original proof in Beth [1953]—extend immediately to the projective Beth property. Even more: A slight modification of the preceding argument shows that the interpolation property and the projective Beth property are equivalent for all regular logics (Rowlands-Hughes [1979]). Thus we get an alternative answer to the question about the relationship between interpolation and definability as posed in the remarks following Definition 1.2.5.