Chapter VIII
Recursion on Ordinals

In the preceding two chapters we have put our attention on the generalized recursion theories which arise from ordinary recursion theory by the introduction of functionals of types 2 and higher. As we saw in § VI.7, these theories can equally well be viewed as theories of effective computability over domains which include such functionals. In this chapter, we shall study theories of effective computability over (initial segments of) the class of ordinal numbers.

The intuitive notion of effective computability for functions from ordinals to ordinals is again based on an idealized computing machine M which is equipped to store and manipulate ordinals according to a given program and produce an ordinal as answer. Many features of the "language" in which the programs for M are to be written are unchanged from earlier notions of computability, but there are two new basic instructions. These ensure that if H is a partial computable functions, then so are F and G defined by:

$$F(\rho, \boldsymbol{\mu}) \simeq \sup^{+}_{\pi<\rho} H(\pi, \boldsymbol{\mu}),$$

and

$$G(\boldsymbol{\mu}) \simeq \text{"least" } \pi \,.\, H(\pi, \boldsymbol{\mu}) \simeq 0.$$

The justification for including these schemes lies in a generalization of the notion of finiteness, which we call *metafiniteness.*

In ordinary recursion theory an object is finite iff it is in a one-to-one correspondence with a natural number — that is, with an element of the fundamental domain. If this were the only property of finite objects considered, we might say that an object is metafinite iff it is in a one-to-one correspondence with an ordinal. Of course, given the Axiom of Choice, this would make every set metafinite! Instead, we observe that in addition, every finite set of natural numbers is computable and indeed is in a *computable* one-to-one correspondence with a natural number. Hence we shall call an object *metafinite* iff it is in a computable one-to-one relationship with an ordinal.

Our basic principle of intuitive calculability over the ordinals is that a metafinite sequence of computations may be regarded as a completed totality and an answer drawn from the sequence of results of these computations. Thus, from the metafinite sequence of values $\langle H(\pi, \boldsymbol{\mu}) : \pi < \rho\rangle$, we can compute the least ordinal greater than all members of the sequence. Similarly, if for some ν, $H(\nu, \boldsymbol{\mu}) \simeq 0$ but for all $\pi < \nu$, $H(\pi, \boldsymbol{\mu})$ is defined with a non-zero value, then from the metafinite sequence $\langle H(\pi, \boldsymbol{\mu}) : \pi \leqslant \nu\rangle$ we can compute that $G(\boldsymbol{\mu}) \simeq \nu$.

In fact, our main concern will be with various refinements of this notion of computability. The most important of these consists simply in restricting attention to ordinals less than some fixed κ. For $\kappa = \omega$ we get a new characterization of ordinary recursion theory (on numbers) — in this case, the two new schemes are superfluous. In general, we shall study the class of κ-partial recursive functions. Many of the simplest properties of ordinary recursion theory are shared by κ-recursion theory for all κ, but it soon becomes evident that not all κ are equally well suited to support a recursion theory. Those that are suited are called *recursively regular* or *admissible* and studied in § 2.

In § 3 we show, among other things, that the ω_1-(semi-) recursive relations over ω are exactly the (Π^1_1) Δ^1_1 relations, and the $\aleph_1$-(semi-) recursive relations over ω are exactly the (Σ^1_2) Δ^1_2 relations. From one point of view these results reinforce the analogies and similarities of structure we have already observed; from another, they support the naturalness of the definition of κ-recursiveness. Both of these conclusions are further confirmed in § 4, where we show that for any type-2 functional I such that E is recursive in I, every $\omega_1[\mathsf{I}]$-(semi-) recursive relation on numbers is (semi-) recursive in I and, under certain conditions, which are satisfied by $\mathsf{E}, \mathsf{E}_1, \mathsf{E}_2, \ldots$, the converse holds as well.

In the next two sections, we investigate the general structure of the ordinals using measures of complexity defined in terms of ordinal recursion. Some of these are closely analogous to so-called "large cardinals" of set theory. Here again there are interesting connections with the analytical hierarchy and type-2 recursion. For example, the least stable ordinal is δ^1_2, the least ordinal not the order type of a Δ^1_2 well-ordering of ω, and the least recursively inaccessible ordinal is $\omega_1[\mathsf{E}_1]$.

In § 7 we explore the close connection between ordinal recursion and the hierarchy of constructible sets. Most of the recursion-theoretic properties of an ordinal κ correspond to model-theoretic properties of L_κ, the class of sets constructible in fewer than κ steps.

Some of our previous notational conventions are changed for this chapter. The letters f, g, h and F, G, H, I now denote k-ary partial functions from Or into Or; $R, S, T, \ldots$ are k-ary relations over Or.

1. *Recursive Ordinal Functions*

When we expand our fundamental domain from ω to a larger ordinal κ or to all of Or, some notions of ordinary recursion theory have obvious counterparts whereas others may have either more or fewer than one natural version. Such things as the successor and projection functions need no explanation and we assume that the reader is familiar with the operations of ordinal arithmetic: for all ordinals μ and ρ,

$$\mu + 0 = \mu, \qquad \mu + (\rho + 1) = (\mu + \rho) + 1,$$

$$\text{and if } \rho \text{ is a limit ordinal, } \mu + \rho = \sup^{+}_{\pi<\rho}(\mu + \pi);$$

$$\mu \cdot 0 = 0, \qquad \mu \cdot (\rho + 1) = (\mu \cdot \rho) + \mu,$$

$$\text{and if } \rho \text{ is a limit ordinal, } \mu \cdot \rho = \sup^{+}_{\pi<\rho}(\mu \cdot \pi);$$

$$\mu^0 = 1, \qquad \mu^{\rho+1} = \mu^{\rho} \cdot \mu,$$

$$\text{and if } \rho \text{ is a limit ordinal, } \mu^{\rho} = \sup^{+}_{\pi<\rho}(\mu^{\pi});$$

$$\mu \dot{-} \rho = \begin{cases} \text{least } \nu(\mu = \rho + \nu), & \text{if } \quad \rho \leqslant \mu; \\ 0, & \text{otherwise.} \end{cases}$$

We shall say that F arises from G, H, and I by ***Primitive Recursion*** iff for all $\boldsymbol{\mu}$ and ρ,

$$F(0, \boldsymbol{\mu}) \simeq G(\boldsymbol{\mu});$$

$$F(\rho + 1, \boldsymbol{\mu}) \simeq H(F(\rho, \boldsymbol{\mu}), \rho, \boldsymbol{\mu});$$

and if ρ is a limit ordinal,

$$F(\rho, \boldsymbol{\mu}) \simeq I(\sup^{+}_{\pi<\rho} F(\pi, \boldsymbol{\mu}), \rho, \boldsymbol{\mu}).$$

Ordinal addition, multiplication, and exponentiation are examples as is the *predecessor function*:

$$\mathrm{Pd}(0) = 0, \qquad \mathrm{Pd}(\rho + 1) = \rho, \quad \text{and if } \rho \text{ is a limit ordinal,} \quad \mathrm{Pd}(\rho) = \rho.$$

The notions of expansion, bounded search, definition by cases, relational and functional composition, and bounded quantification all have natural extensions to functions and relations on ordinals; we encourage the reader to verify this for himself. For example, F arises from G by *bounded search* iff

$$\begin{aligned} F(\rho, \boldsymbol{\mu}) &\simeq \text{“least”}\ \pi < \rho\,.\, G(\pi, \boldsymbol{\mu}) \simeq 0 \\ &\simeq \begin{cases} \pi, \text{ if } G(\pi, \boldsymbol{\mu}) \simeq 0 \text{ and } (\forall \sigma < \pi)(\exists \nu > 0)\,.\, G(\sigma, \boldsymbol{\mu}) \simeq \nu; \\ \rho, \quad \text{if} \quad (\forall \pi < \rho)(\exists \nu > 0)\,.\, G(\pi, \boldsymbol{\mu}) \simeq \nu; \\ \text{undefined}, \quad \text{otherwise.} \end{cases} \end{aligned}$$

In ordinary recursion theory, by unbounded search we mean a search through all ordinals less than ω. In the present context it will be useful to provide for varying "lengths" of searches. We say that F arises from G by *λ-search* and write

$$F(\boldsymbol{\mu}) \simeq \lambda\text{-“least”}\ \pi\,.\, G(\pi, \boldsymbol{\mu}) \simeq 0$$

iff

$$F(\boldsymbol{\mu}) \simeq \begin{cases} \pi, \text{ if } \pi < \lambda,\ G(\pi, \boldsymbol{\mu}) \simeq 0, \quad \text{and} \quad (\forall \sigma < \pi)(\exists \nu > 0)\, G(\sigma, \boldsymbol{\mu}) \simeq \nu; \\ \text{undefined}, \quad \text{if there is no such } \pi < \lambda. \end{cases}$$

Similarly, F arises from G by *unbounded search,*

$$F(\boldsymbol{\mu}) \simeq \text{“least”}\ \pi\,.\, G(\pi, \boldsymbol{\mu}) \simeq 0$$

when the same condition holds without the restriction to $\pi < \lambda$.

We shall need simple "coding" functions for finite sequences of ordinals. Since ordinals do not have unique prime power decompositions ($2^\omega = 3^\omega = \omega$), we must use a different technique. Many approaches are possible here, but the following is one of the simplest. We first define a coding function $\langle\langle\ \ \rangle\rangle$ for pairs of ordinals and then treat k-tuples by iterated pairing. Let

$$\langle\langle \mu, \nu \rangle\rangle = 3^{\mu+\nu} + 3^{\mu}.$$

Since for any μ and ν, $3^{\mu+\nu} < 3^{\mu+\nu} + 3^{\mu} < 3^{\mu+\nu+1}$, if we set

$$f(\sigma) \simeq \text{least } \pi < \sigma\,.\, \sigma < 3^{\pi+1},$$

then $f(\langle\langle \mu, \nu \rangle\rangle) = \mu + \nu$. Let

$$(\sigma)^0 = f(\sigma \dot{-} 3^{f(\sigma)}) \quad \text{and} \quad (\sigma)^1 = f(\sigma) \dot{-} (\sigma)^0.$$

Then

$$(\langle\langle \mu, \nu \rangle\rangle)^0 = f(3^{\mu+\nu} + 3^{\mu}) \dot{-} 3^{\mu+\nu} = f(3^{\mu}) = \mu$$

and

$$(\langle\langle \mu, \nu \rangle\rangle)^1 = (\mu + \nu) \dot{-} \mu = \nu.$$

It follows that $\langle\langle\ \ \rangle\rangle$ is one-one. Now set

$$\langle\ \ \rangle^0 = 1, \quad \text{and for all } k \text{ and all } \boldsymbol{\mu} = (\mu_0, \ldots, \mu_k),$$
$$\langle \boldsymbol{\mu} \rangle^{k+1} = \langle\langle \mu_0, \langle \mu_1, \ldots, \mu_k \rangle^k \rangle\rangle.$$

Clearly each $\langle\ \ \rangle^k$ is one-one. To see that the different $\langle\ \ \rangle^k$ have disjoint images, let g be the function defined for $\rho < \omega$ by:

$$g(0,\sigma) = \sigma \quad \text{and} \quad g(\rho+1,\sigma) = (g(\rho,\sigma))^1.$$

Then if we set

$$\lg(\sigma) = \text{least } \rho < \sigma \,.\, g(\rho,\sigma) = 1,$$

then $\lg(\langle\mu_0,\dots,\mu_{k-1}\rangle^k) = k$. We may thus omit the superscript k without ambiguity. In terms of g we have also decoding functions:

$$(\sigma)_i = (g(i,\sigma))^0.$$

To define a concatenation function $*$, let, for $\rho < \omega$,

$$h(0,\sigma,\tau) = \tau, \quad \text{and}$$
$$h(\rho+1,\sigma,\tau) = \langle\langle(\sigma)_{\lg(\sigma)\dot{-}(\rho+1)}, h(\rho,\sigma,\tau)\rangle\rangle.$$

Then if

$$\sigma * \tau = h(\lg(\sigma),\sigma,\tau),$$

we have easily

$$\langle\boldsymbol{\mu}\rangle * \langle\boldsymbol{\nu}\rangle = \langle\boldsymbol{\mu},\boldsymbol{\nu}\rangle.$$

Finally, if we set

$$\mathrm{Sq}(0,\sigma) \leftrightarrow \sigma = 1;$$
$$\mathrm{Sq}(\rho+1,\sigma) \leftrightarrow (\exists\pi < \sigma)(\exists\tau < \sigma)[\mathrm{Sq}(\rho,\tau) \wedge \sigma = \langle\langle\pi,\tau\rangle\rangle];$$
$$\mathrm{Sq}(\rho,\sigma) \leftrightarrow 0 = 1, \quad \text{if} \quad \rho \text{ is a limit ordinal};$$

and

$$\mathrm{Sq}(\sigma) \leftrightarrow (\exists\rho < \sigma)\mathrm{Sq}(\rho,\sigma);$$

then $\mathrm{Sq}(\sigma)$ iff for some $\boldsymbol{\mu}$, $\sigma = \langle\boldsymbol{\mu}\rangle$.

The formal definition of the classes of recursive ordinal functions that we shall study proceeds similarly as in previous cases. For each κ and λ we define a set $\Omega_{\kappa\lambda}$ of sequences of the form $(a,\boldsymbol{\mu},\nu)$, where a is a natural number and $\mu_0,\dots,\mu_{k-1},\nu$ are ordinals less than κ (the role of λ is explained in the next paragraph). Clauses (0)–(2) of the definition are the natural extensions of the corresponding clauses of Definition II.2.1 to ordinal arguments and values with two exceptions. First, there is no clause for applying function arguments, as we have none. Second, only the constant functions with natural number values are included. This restriction is unavoidable if we want to preserve the property that indices (programs) be finite objects.

Clauses (3) and (4) will introduce the operations of $\sup^+$ and λ-search and we shall have:

$$\{\langle 3, k+1, b\rangle\}_{\kappa\lambda}(\rho, \boldsymbol{\mu}) \simeq \sup^+_{\pi<\rho}\{b\}_{\kappa\lambda}(\pi, \boldsymbol{\mu}),$$

and

$$\{\langle 4, k, b\rangle\}_{\kappa\lambda}(\boldsymbol{\mu}) \simeq \lambda\text{-“least”}\ \pi.\{b\}_{\kappa\lambda}(\pi, \boldsymbol{\mu}) \simeq 0.$$

To avoid uninteresting pathological cases, we shall always assume that κ is a limit ordinal.

1.1 Definition. For any κ and λ, $\Omega_{\kappa\lambda}$ is the smallest set such that for all $k, n \in \omega$, all $i < k$, all $\boldsymbol{\mu} \in {}^k\kappa$, and all ν, π, ρ, σ, and $\tau < \kappa$,

(0) $(\langle 0, k, 0, n\rangle, \boldsymbol{\mu}, n) \in \Omega_{\kappa\lambda}$;

$(\langle 0, k, 1, i\rangle, \boldsymbol{\mu}, \mu_i) \in \Omega_{\kappa\lambda}$;

$(\langle 0, k, 2, i\rangle, \boldsymbol{\mu}, \mu_i + 1) \in \Omega_{\kappa\lambda}$;

$(\langle 0, k+4, 4\rangle, \pi, \rho, \sigma, \tau, \boldsymbol{\mu}, \pi) \in \Omega_{\kappa\lambda}$, if $\sigma = \tau$;

$(\langle 0, k+4, 4\rangle, \pi, \rho, \sigma, \tau, \boldsymbol{\mu}, \rho) \in \Omega_{\kappa\lambda}$, if $\sigma \neq \tau$;

$(\langle 0, k+2, 5\rangle, p, q, \boldsymbol{\mu}, \mathrm{Sb}_0(p, q)) \in \Omega_{\kappa\lambda}$ for all $p, q \in \omega$;

(1) for any $k', b, c_0, \ldots, c_{k'-1} < \omega$ and any $\xi_0, \ldots, \xi_{k'-1} < \kappa$, if for all $i < k'$, $(c_i, \boldsymbol{\mu}, \xi_i) \in \Omega_{\kappa\lambda}$ and $(b, \boldsymbol{\xi}, \nu) \in \Omega_{\kappa\lambda}$, then $(\langle 1, k, b, c_0, \ldots, c_{k'-1}\rangle, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$;

(2) for any b, if $(b, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$, then

$$(\langle 2, k+1\rangle, b, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda};$$

(3) if ν is the least ordinal such that $(\forall \pi < \rho)(\exists \xi < \nu).(b, \pi, \boldsymbol{\mu}, \xi) \in \Omega_{\kappa\lambda}$, then $(\langle 3, k+1, b\rangle, \rho, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$;

(4) if $\nu < \lambda$, $(b, \nu, \boldsymbol{\mu}, 0) \in \Omega_{\kappa\lambda}$, and $(\forall \pi < \nu)(\exists \xi > 0).(b, \pi, \boldsymbol{\mu}, \xi) \in \Omega_{\kappa\lambda}$, then $(\langle 4, k, b\rangle, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$.

As in previous cases, the definition of $\Omega_{\kappa\lambda}$ may be interpreted as a closure under certain functions. All of these functions have rank at most $\mathrm{Card}(\rho)$ for some $\rho < \kappa$ and it follows that the closure ordinal of the inductive definition is at most the least regular cardinal $\geqslant \kappa$ (this is what the reader was supposed to discover in Exercise I.3.10). We shall see that in general it is much smaller than this.

Since ν in clause (4) is initially restricted to κ, $\Omega_{\kappa\lambda} = \Omega_{\kappa\kappa}$ for all $\lambda \geqslant \kappa$. The hypothesis of clause (4) is never satisfied if $\lambda = 0$ so $\Omega_{\kappa 0}$ is in fact defined by clauses (0)–(3). If $\kappa \leqslant \kappa'$ and $\lambda \leqslant \lambda'$, then clearly $\Omega_{\kappa\lambda} \subseteq \Omega_{\kappa'\lambda'}$.

1.2 Lemma. *For all κ, λ, a, and $\boldsymbol{\mu}$, there is at most one ν such that $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$.*

Proof. Similar to that of Lemma II.2.2 with a transfinite induction replacing the simple induction. □

Thus we may set

$$\{a\}_{\kappa\lambda}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}.$$

In fact, we shall not be concerned with quite this degree of generality.

1.3 Definition. For all κ, λ, a, $\boldsymbol{\mu}$, and ν,
(i) $\{a\}_{\kappa}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}$;
(ii) $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \exists\kappa\,.\,(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$;
(iii) $\{a\}_{\infty}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \exists\kappa\ \exists\lambda\,.\,(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}$.

A partial function $F: {}^{k}\kappa \to \kappa$ is *κ-partial recursive* iff $F = \{a\}_{\kappa}$ for some $a \in \omega$ and *κ-recursive* iff it is κ-partial recursive and total ($\mathrm{Dm}\, F = {}^{k}\kappa$). A relation $R \subseteq {}^{k}\kappa$ is *κ-recursive* iff its characteristic function K_R is κ-recursive. R is *κ-semi-recursive* iff R is the domain of some κ-partial recursive function and *κ-co-semi-recursive* iff $({}^{k}\kappa \sim R)$ is κ-semi-recursive. There is a slight technical problem in making parallel definitions for the other two notions of Definition 1.3; for some a, $\{a\}_{\infty\lambda}$ and $\{a\}_{\infty}$ are not sets because their domains are proper classes of ordinals. We shall in practice ignore this point and proceed as if they were sets. The worried reader may either think of this chapter as formulated in a set theory which admits proper classes or verify that each use of a symbol $\{a\}_{\infty\lambda}$ or $\{a\}_{\infty}$ may be replaced by its definition in terms of sets. Thus we say that a partial function $F: {}^{k}\mathrm{Or} \to \mathrm{Or}$ is *(∞, λ)-partial recursive* [*∞-partial recursive*] iff $F = \{a\}_{\infty\lambda}$ [$F = \{a\}_{\infty}$] for some $a \in \omega$, etc.

In each of these cases we say that F is **-partial recursive in parameters* iff for some *-partial recursive function G and some $\boldsymbol{\rho}$, $F(\boldsymbol{\mu}) \simeq G(\boldsymbol{\mu}, \boldsymbol{\rho})$. If F (but not necessarily G) is total, then F is **-recursive in parameters.* When it is relevant, we say that F is **-(partial) recursive in the parameters $\boldsymbol{\rho}$.* Similar terminology applies to relations. Since indices are natural numbers, the constant function with value ρ is *-recursive for only denumerably many ρ, so that in general allowing parameters properly enlarges the collection of *-recursive functions (cf. Exercise 1.17).

The classes of κ-partial recursive functions and (∞, κ)-partial recursive functions are closed under κ-search and the class of ∞-partial recursive functions is closed under unbounded search. If $\kappa \leqslant \lambda$, then $\{a\}_\kappa \subseteq \{a\}_\lambda \subseteq \{a\}_{\infty\lambda} \subseteq \{a\}_\infty$, so each κ-partial recursive function F has natural extensions to λ-, (∞, λ)-, and ∞-partial recursive functions. Of course, different indices for F lead in general to different extensions. If $\{a\}_\kappa$ is total on ${}^k\kappa$, then no new computations with arguments less than κ are added in the extensions, but this is not in general true if $\{a\}_\kappa$ is partial. New computations with arguments and values from κ may be made possible by the availability to $\{a\}_\lambda$, $\{a\}_{\infty\lambda}$, and $\{a\}_\infty$ of intermediate arguments and values greater than κ. The $(\infty, 0)$-recursive functions will play somewhat the role here of the primitive recursive functions of ordinary recursion theory. They form a convenient class sufficient for many elementary calculations. Although it is not in general true that the restriction of an $(\infty, 0)$ recursive function to arguments from κ is κ-recursive, this will be true whenever κ is recursively regular (Definition 2.1).

1.4 Theorem. *For any k and any $F: {}^k\omega \to \omega$, F is ω-partial recursive iff F is $(\omega, 0)$-partial recursive iff F is (ordinary) partial recursive.*

Proof. Exercise 1.15. □

In the following we shall use only the obvious part of this theorem:if F is (ordinary) recursive, then F is $(\omega, 0)$-recursive and thus has an $(\infty, 0)$-partial recursive extension.

1.5 Theorem. *For all κ and λ, the classes of κ-partial recursive $[(\infty, \lambda)$-partial recursive, ∞-partial recursive$]$ functions and κ-recursive $[(\infty, \lambda)$-recursive, ∞-recursive$]$ relations are closed under definition by cases.*

Proof. We shall prove the result for κ-recursion. Suppose

$$F(\boldsymbol{\mu}) \simeq \begin{cases} G_i(\boldsymbol{\mu}), & \text{if} \quad R_i(\boldsymbol{\mu}) \quad \text{for } i < k'; \\ G_{k'}(\boldsymbol{\mu}), & \text{otherwise}; \end{cases}$$

where the R_i $(i < k')$ are pairwise disjoint κ-recursive relations and the G_i $(i \leqslant k')$ are κ-partial recursive functions. We proceed by induction on k'. If $k' = 0$ there is nothing to prove, so suppose $k' > 0$ and the result is true for smaller values of k'. Then there exists a κ-partial recursive function H such that

$$H(\boldsymbol{\mu}) \simeq \begin{cases} G_i(\boldsymbol{\mu}), & \text{if} \quad R_i(\boldsymbol{\mu}) \quad \text{for } 0 < i < k'; \\ G_{k'}(\boldsymbol{\mu}), & \text{otherwise}; \end{cases}$$

so that

$$F(\boldsymbol{\mu}) \simeq \begin{cases} G_0(\boldsymbol{\mu}), & \text{if} \quad R_0(\boldsymbol{\mu}); \\ H(\boldsymbol{\mu}), & \text{otherwise.} \end{cases}$$

Let b and c be κ-indices of G_0 and H, respectively. Then

$$F(\boldsymbol{\mu}) \simeq \{\{\langle 0,4,4\rangle\}_\kappa(b, c, K_{R_0}(\boldsymbol{\mu}), 0)\}_\kappa(\boldsymbol{\mu})$$

and is thus κ-partial recursive. □

The equality relation on ordinals is $(\infty, 0)$-recursive, since

$$K_=(\sigma, \tau) = \{\langle 0,4,4\rangle\}_{\infty 0}(0, 1, \sigma, \tau).$$

Then the functions sg^+ and sg^- are $(\infty, 0)$-recursive and κ-recursive for all κ, and it follows as in § II.1 that the classes of κ-recursive, (∞, λ)-recursive, and ∞-recursive relations are Boolean Algebras (the arithmetic operations used there are applied only to natural numbers).

Suppose that R is κ-recursive,

$$P(\rho, \boldsymbol{\mu}) \leftrightarrow (\forall \pi < \rho) R(\pi, \boldsymbol{\mu}),$$

and

$$Q(\rho, \boldsymbol{\mu}) \leftrightarrow (\exists \pi < \rho) R(\pi, \boldsymbol{\mu}).$$

Then

$$K_P(\rho, \boldsymbol{\mu}) = (\sup^+_{\pi<\rho} K_R(\pi, \boldsymbol{\mu})) \dot{-} 1$$

so P is also κ-recursive. That Q is κ-recursive follows by closure under complementation. The same applies to (∞, λ)- and ∞-recursion. We now have

$$\sigma < \tau \leftrightarrow (\exists \pi < \tau) \sigma = \pi$$

so that the relations, $<$, $\leqslant$, $>$, and $\geqslant$ are all $(\infty, 0)$-recursive.

1.6 Lemma. *The following relations are* $(\infty, 0)$-*recursive*:

(i) $\mathrm{Lm}(\rho) \leftrightarrow \rho$ *is a limit ordinal*;

(ii) $\mathrm{Suc}(\rho) \leftrightarrow \rho$ *is a successor ordinal*;

(iii) $\{\rho : \rho = \omega\}$;

(iv) $\{\rho : \rho < \omega\}$.

Proof. These are immediate from the preceding remarks and the following relationships:

$$\mathrm{Lm}(\rho) \leftrightarrow \rho \neq 0 \wedge (\forall \pi < \rho) \pi + 1 < \rho;$$

$$\mathrm{Suc}(\rho) \leftrightarrow (\exists \pi < \rho) \rho = \pi + 1;$$

$$\rho = \omega \leftrightarrow \mathrm{Lm}(\rho) \wedge (\forall \pi < \rho) \neg \mathrm{Lm}(\pi);$$

$$\rho < \omega \leftrightarrow \neg \mathrm{Lm}(\rho) \wedge (\forall \pi < \rho) \neg \mathrm{Lm}(\pi). \quad \square$$

Note that we are not asserting that the constant function with value ω is $(\infty, 0)$-recursive — it is not (Corollary 2.3). However, for any $\kappa > \omega$,

$$\omega = \kappa\text{-least } \pi . \mathrm{Lm}(\pi)$$

so this function is κ-recursive.

1.7 Lemma. *The predecessor function* Pd *is* $(\infty, 0)$-*recursive.*

Proof. Let

$$f(\pi, \rho) = \begin{cases} \pi, & \text{if } \quad \pi + 1 < \rho; \\ 0, & \text{otherwise.} \end{cases}$$

Then f is $(\infty, 0)$-recursive and $\mathrm{Pd}(\rho) = \sup^{+}_{\pi < \rho} f(\pi, \rho)$. $\square$

1.8 Recursion Theorem. *For any κ and any κ-partial recursive function F, there exists an $\bar{e} < \omega$ such that*

$$\{\bar{e}\}_{\kappa}(\boldsymbol{\mu}) \simeq F(\bar{e}, \boldsymbol{\mu}).$$

The same holds for (∞, λ)- and ∞-recursion.

Proof. As for Theorem II.2.6. $\square$

1.9 Theorem. *For any κ and λ, the classes of κ-partial recursive, (∞, λ)-partial recursive, and ∞-partial recursive functions are closed under primitive recursion.*

Proof. Suppose that F arises from κ-partial recursive functions G, H, and I by primitive recursion. Let f be defined by:

$$f(e, \rho, \boldsymbol{\mu}) \simeq \begin{cases} G(\boldsymbol{\mu}), & \text{if } \quad \rho = 0; \\ H(\{e\}_{\kappa}(\mathrm{Pd}(\rho), \boldsymbol{\mu}), \mathrm{Pd}(\rho), \boldsymbol{\mu}), & \text{if } \quad \mathrm{Suc}(\rho); \\ I(\sup^{+}_{\pi < \rho} \{e\}_{\kappa}(\pi, \boldsymbol{\mu}), \rho, \boldsymbol{\mu}), & \text{if } \quad \mathrm{Lm}(\rho). \end{cases}$$

Then f is κ-partial recursive so by the κ-Recursion Theorem there exists an index $\bar{e}$ such that $f(\bar{e}, \rho, \boldsymbol{\mu}) \simeq \{\bar{e}\}_\kappa(\rho, \boldsymbol{\mu})$. It is straightforward to prove by induction that $F = \{\bar{e}\}_\kappa$. □

1.10 Corollary. *The functions of ordinal addition, multiplication, and exponentiation are all* $(\infty, 0)$*-recursive.*

Proof. The ordinal addition function $+$ arises from three $(\infty, 0)$-recursive functions G, H, and I by primitive recursion, where

$$G(\rho, \boldsymbol{\mu}) = \mu, \qquad H(\nu, \rho, \boldsymbol{\mu}) = \nu + 1, \quad \text{and} \quad I(\sigma, \rho, \boldsymbol{\mu}) = \sigma.$$

It follows from Theorem 1.9 that $+$ is $(\infty, 0)$-recursive. Similar arguments apply to multiplication and exponentiation. □

1.11 Theorem. *For all κ and λ, the classes of κ-partial recursive functions, (∞, λ)-partial recursive functions, and ∞-partial recursive functions are closed under* sup *and* inf.

Proof. Suppose H is κ-partial recursive,

$$F(\rho, \boldsymbol{\mu}) \simeq \sup\nolimits_{\pi<\rho} H(\pi, \boldsymbol{\mu}),$$

and

$$G(\rho, \boldsymbol{\mu}) \simeq \inf\nolimits_{\pi<\rho} H(\pi, \boldsymbol{\mu}).$$

Then $F(\rho, \boldsymbol{\mu}) \simeq \mathrm{Pd}(\overset{+}{\sup}_{\pi<\rho} H(\pi, \boldsymbol{\mu}))$ and $G(\rho, \boldsymbol{\mu}) \simeq H(I(\rho, \boldsymbol{\mu}), \boldsymbol{\mu})$, where I satisfies the primitive recursion:

$$I(0, \boldsymbol{\mu}) = 0$$

$$I(\rho+1, \boldsymbol{\mu}) \simeq \begin{cases} \rho, & \text{if} \quad H(\rho, \boldsymbol{\mu}) < H(I(\rho, \boldsymbol{\mu}), \boldsymbol{\mu}); \\ I(\rho, \boldsymbol{\mu}), & \text{otherwise}; \end{cases}$$

$$I(\rho, \boldsymbol{\mu}) \simeq \sup\nolimits_{\pi<\rho} I(\pi, \boldsymbol{\mu}), \quad \text{if} \quad \rho \text{ is a limit ordinal.} \quad \square$$

The κ-partial recursive functions are closed under κ-search and the ∞-partial recursive functions under unbounded search, so it follows that both classes are closed under bounded search. The next theorem shows that this is also true for the (∞, λ)-partial recursive functions, even when $\lambda = 0$.

1.12 Theorem. *For all λ, the class of (∞, λ)-partial recursive functions is closed under bounded search.*

Proof. The proof is an extension of that of II.1.5 (ii). If G is (∞, λ)-partial recursive and

$$F(\rho, \boldsymbol{\mu}) \simeq \text{“least”}\ \pi < \rho\,.\, G(\pi, \boldsymbol{\mu}) \simeq 0,$$

then F satisfies the primitive recursion:

$$F(0, \boldsymbol{\mu}) \simeq 0;$$

$$F(\rho + 1, \boldsymbol{\mu}) \simeq F(\rho, \boldsymbol{\mu}) + \mathrm{sg}^{+}(G(F(\rho, \boldsymbol{\mu}), \boldsymbol{\mu}));$$

$$F(\rho, \boldsymbol{\mu}) \simeq \sup\nolimits_{\pi<\rho} F(\pi, \boldsymbol{\mu}), \quad \text{if} \quad \rho \text{ is a limit ordinal.} \quad \square$$

1.13 Corollary. *The ordinal subtraction function* $\dot{-}$ *and the sequence-manipulation functions and relation,* $\langle \quad \rangle^k$, $(\quad)$, $*$, *and* Sq *are all* $(\infty, 0)$-*recursive.*

Proof. Immediate from the definitions and preceding results. $\square$

1.14–1.17 Exercises

1.14. Describe the computation trees for (κ, λ)-recursion.

1.15. Prove Theorem 1.4.

1.16. Compare the class of arithmetical relations on numbers with the class of relations on numbers which are $(\infty, 0)$-recursive in the parameter ω.

1.17. Show that there exists a countable ordinal κ such that properly more functions are κ-recursive in parameters than are simply κ-recursive.

1.18 Notes. The notion of recursiveness for functions on ordinals is a good example of an idea whose time had come and was discovered independently in one form or another by several people between 1960 and 1966. The idea was natural from several points of view, and this is reflected in the variety of motivations of those who made the key discoveries. From the side of set theory we have Takeuti's work of the 1950's in reformulating Gödel's constructibility in terms of a theory of ordinals. Takeuti [1960] and Kino–Takeuti [1962, – a] develop a theory of recursive ordinal functions essentially equivalent to ∞-recursiveness as described above and show that it behaves nicely when restricted to the ordinals less than any cardinal. Machover [1961] and Lévy [1963] arrived at essentially the same theory with the aim of using it to study the infinitary languages $\mathscr{L}_{\kappa\lambda}$.

Meanwhile Kreisel [1961] observed that in the analogy of Π_1^1 to Σ_1^0, the appropriate counterpart of Δ_1^1 is the class of *finite* sets, not the class of recursive sets. This is essentially because of the boundedness phenomenon — a Δ_1^1 subset of W is bounded (metafinite) whereas a recursive subset of ω is not bounded (finite). In a similar vein, Kreisel also proposed that the cardinality criteria, both in recursion theories and infinitary languages, should be replaced by more intrinsic definability criteria. These ideas led more or less directly to the metarecursion theory of Kreisel–Sacks [1965] in which the fundamental domain is the set of recursive ordinals as represented by a Π_1^1 set of unique notations for them. This is equivalent to ω_1-recursion as defined above.

The pieces were all fitted together independently by Kripke [1964, – a] and Platek [1966], who approached the area with the aim of generalizing ordinary recursion theory. Kripke expanded Kleene's equation calculus to serve for defining recursive functions over any initial segment of the ordinals, whereas Platek used an approach via primitive recursion and the search operator. These theories include all of the others and are equivalent to each other and in almost all respects to the one presented above.

Further discussion of some of these points may be found following Theorem 3.4 below and in the introduction to Barwise [1975]. The particular formulation of the theory that we have used is new here, but similar formulations have been used by several people in lectures.

2. Recursively Regular Ordinals

It is obvious that for many ordinals κ, the class of κ-partial recursive functions is not a very natural class. If κ is a successor ordinal, the successor function on κ is not total, hence not κ-recursive. Similarly, if κ is not closed under ordinal addition, multiplication, etc., then the restrictions of these functions to arguments from κ are only κ-partial recursive. Among other difficulties, this means that the sequence coding functions may not be everywhere defined. A natural "regularity" condition on κ is that it be closed under sufficiently many functions:

2.1 Definition. An ordinal κ is *recursively regular* iff κ is closed under all (∞, κ)-partial recursive functions.

An apparently stronger condition is
(a) for all all $a < \omega$, all $\boldsymbol{\mu} < \kappa$, and all ν

$$\{a\}_{\infty\kappa}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \{a\}_{\kappa}(\boldsymbol{\mu}) \simeq \nu.$$

Since the values of κ-partial recursive functions are by definition less than κ, (a) implies that κ is recursively regular. The converse is not obvious, however, since

$\{a\}_\kappa$ is restricted to using ordinals from κ in the middle of computations, while $\{a\}_{\infty\kappa}$ may use larger ordinals even to arrive at a value less than κ.

If we examine the schemas for computing (∞, κ)-partial recursive functions, it is intuitively clear that an (∞, κ) computation with arguments less than κ can arrive at a value $\geqslant \kappa$ only by means of the $\sup^+$ schema (3). The first time this happens, the function whose $\sup^+$ is $\geqslant \kappa$ is in fact κ-partial recursive, and we are led to the following condition:

(b) for all κ-partial recursive functions F and all ρ, $\boldsymbol{\mu} < \kappa$, if $F(\pi, \boldsymbol{\mu})$ is defined for all $\pi < \rho$, then $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) < \kappa$.

2.2 Theorem. *For all κ, κ is recursively regular iff* (a) *iff* (b).

Proof. Suppose first that κ is recursively regular and let F be a κ-partial recursive function with κ-index a. Then $\{a\}_{\infty\kappa}$ is an (∞, κ)-partial recursive function which extends F, and the function G defined by

$$G(\rho, \boldsymbol{\mu}) \simeq \sup^+_{\pi<\rho} \{a\}_{\infty\kappa}(\pi, \boldsymbol{\mu})$$

is likewise (∞, κ)-partial recursive. Then for ρ, $\boldsymbol{\mu} < \kappa$, since κ is closed under G, if for all $\pi < \rho$, $F(\pi, \boldsymbol{\mu})$ is defined, then

$$\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) = G(\rho, \boldsymbol{\mu}) < \kappa.$$

We have thus deduced that condition (b) holds.

We observed above that (a) implies that κ is recursively regular, so it remains to prove that (b) implies (a). Given (b), we first observe that κ is a limit ordinal, since if $\mu < \kappa$ and $F(\pi, \mu) = \mu$ for all π, then

$$\mu + 1 = \sup^+_{\pi<1} F(\pi, \mu) < \kappa.$$

We shall prove for all τ by induction over $\Omega_{\tau\kappa}$ that for all a, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$,

$$(*) \qquad \boldsymbol{\mu} < \kappa \wedge (a, \boldsymbol{\mu}, \nu) \in \Omega_{\tau\kappa} \rightarrow (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}.$$

For $a = \langle 0, k, \dots \rangle$ this is immediate (that κ is a limit ordinal is needed for the successor function). Suppose that $a = \langle 1, k, b, c_0, \dots, c_{k'-1} \rangle$ and ξ_i $(i < k')$ are such that $(c_i, \boldsymbol{\mu}, \xi_i) \in \Omega_{\tau\kappa}$ and $(b, \boldsymbol{\xi}, \nu) \in \Omega_{\tau\kappa}$. Since $\boldsymbol{\mu} < \kappa$, the induction hypothesis yields that each $(c_i, \boldsymbol{\mu}, \xi_i) \in \Omega_{\kappa\kappa}$. Hence each $\xi_i < \kappa$, so again by the induction hypothesis, $(b, \boldsymbol{\xi}, \nu) \in \Omega_{\kappa\kappa}$. Thus also $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}$.

The argument for clause (2) is even simpler and we omit it. Suppose that $a = \langle 3, k+1, b \rangle$, ρ and $\boldsymbol{\mu} < \kappa$, and $(a, \rho, \boldsymbol{\mu}, \nu) \in \Omega_{\tau\kappa}$. Then ν is the least ordinal such that for every $\pi < \rho$ there exists $\xi_\pi < \nu$ such that $(b, \pi, \boldsymbol{\mu}, \xi_\pi) \in \Omega_{\tau\kappa}$. By the induction hypothesis, for all $\pi < \rho$, $(b, \pi, \boldsymbol{\mu}, \xi_\pi) \in \Omega_{\kappa\kappa}$ — that is, $\{b\}_\kappa(\pi, \boldsymbol{\mu}) \simeq \xi_\pi$. Then by (b),

$$\nu = \sup^+\{\xi_\pi : \pi < \rho\} = \sup^+_{\pi<\rho}\{b\}_\kappa(\pi, \boldsymbol{\mu}) < \kappa.$$

It follows that $(a, \rho, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}$.

Finally, suppose that $a = \langle 4, k, b\rangle$, $\boldsymbol{\mu} < \kappa$, and $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\tau\kappa}$. Then $\nu < \kappa$, $(b, \nu, \boldsymbol{\mu}, 0) \in \Omega_{\tau\kappa}$, and for all $\pi < \nu$, $(b, \pi, \boldsymbol{\mu}, \xi_\pi) \in \Omega_{\tau\kappa}$ for some $\xi_\pi > 0$. By the induction hypothesis (since if $\pi < \nu$, also $\pi < \kappa$), $(b, \nu, \boldsymbol{\mu}, 0) \in \Omega_{\kappa\kappa}$ and all $(b, \pi, \boldsymbol{\mu}, \xi_\pi) \in \Omega_{\kappa\kappa}$. Hence also $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}$. □

The term "recursively regular" is derived from condition (b). Indeed, if the modifier "κ-partial recursive" is removed from (b), it becomes just the condition that κ be a regular cardinal. In particular, every regular cardinal is recursively regular.

Of course, every recursively regular κ is closed under all $(\infty, 0)$-partial recursive functions and the restriction of any $(\infty, 0)$-partial recursive function to a recursively regular κ is κ-partial recursive.

2.3 Corollary. *The constant function with value ω is not $(\infty, 0)$-recursive. In fact, no constant function with value $\geq \omega$ is $(\infty, 0)$-recursive.*

Proof. Suppose that $F(\mu) = \nu \geq \omega$ for all μ and F is $(\infty, 0)$-recursive. ω is a regular cardinal, hence recursively regular, so as $0 < \omega$, also $F(0) < \omega$, a contradiction. □

Although we have argued that some condition such as recursive regularity is *necessary* for an ordinal κ to support a reasonable recursion theory, we have not as yet given any reason to believe that such a condition is *sufficient*. To establish this, we shall show that there exists an $(\infty, 0)$-recursive relation T such that for all a, $\boldsymbol{\mu}$, and ν, and any recursively regular κ,

$$\{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu \leftrightarrow (\exists \upsilon < \kappa)[T(a, \langle\boldsymbol{\mu}\rangle, \upsilon) \wedge (\upsilon)_0 = \nu].$$

In other words, for recursively regular κ, κ-recursion theory has a normal form very similar to that of § II.3 for ordinary recursion theory.

The proof is in two main steps. Let

$$T_0(a, \langle\boldsymbol{\mu}\rangle, \nu, \kappa, \lambda, \sigma) \leftrightarrow (a, \boldsymbol{\mu}, \nu) \in \Omega^\sigma_{\kappa\lambda},$$

where, as usual, $\Omega^\sigma_{\kappa\lambda}$ denotes the σ-th stage of the inductive definition of $\Omega_{\kappa\lambda}$. We show first that T_0 is $(\infty, 0)$-recursive. Next we prove that for recursively regular κ,

$$(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa} \leftrightarrow (\exists \lambda < \kappa)(\exists \sigma < \kappa) . (a, \boldsymbol{\mu}, \nu) \in \Omega^\sigma_{\lambda\lambda}.$$

It follows that if we set

$$T(a, \langle \boldsymbol{\mu} \rangle, \upsilon) \leftrightarrow T_0(a, \langle \boldsymbol{\mu} \rangle, (\upsilon)_0, (\upsilon)_1, (\upsilon)_2, (\upsilon)_3),$$

then T has the required property.

Let H be an $(\infty, 0)$-recursive function such that for all k, κ and all $\mu_0, \ldots, \mu_{k-1} < \kappa$, $\langle \boldsymbol{\mu} \rangle < H(k, \kappa)$, say $H(k, \kappa) = \langle \kappa_0, \ldots, \kappa_{k-1} \rangle$ with all $\kappa_i = \kappa$.

2.4 Lemma. T_0 *is* $(\infty, 0)$-*recursive.*

Proof. We shall define an $(\infty, 0)$-recursive function F and prove by induction that for all σ,

$$(*) \qquad F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = \begin{cases} 0, & \text{if } (a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\kappa\lambda}; \\ 1, & \text{otherwise.} \end{cases}$$

If $\sigma = 0$, we set

$$(0) \quad G(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = \begin{cases} 0, & \text{if } a = \langle 0, k, \ldots \rangle \text{ and } \{a\}_{\kappa\lambda}(\boldsymbol{\mu}) \simeq \nu; \\ 1, & \text{otherwise.} \end{cases}$$

If $\sigma > 0$, then F should satisfy the following conditions:

(1) $F(\langle 1, k, b, c_0, \ldots, c_{k'-1} \rangle, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 0$ iff for some $\xi < H(k', \kappa)$,
(i) $(\forall i < k')(\exists \tau < \sigma) \,.\, F(c_i, \langle \boldsymbol{\mu} \rangle, (\xi)_i, \kappa, \lambda, \tau) = 0$,
and
(ii) $(\exists \tau < \sigma) \,.\, F(b, \xi, \nu, \kappa, \lambda, \tau) = 0$;

(2) $F(\langle 2, k+1 \rangle, \langle b, \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 0$ iff

$$(\exists \tau < \sigma) F(b, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \tau) = 0;$$

(3) $F(\langle 3, k+1, b \rangle, \langle \rho, \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 0$ iff ν, $\rho < \kappa$,
(i) $(\forall \pi < \rho)(\exists \xi < \nu)(\exists \tau < \sigma) \,.\, F(b, \langle \pi, \boldsymbol{\mu} \rangle, \xi, \kappa, \lambda, \tau) = 0$,
and
(ii) $\neg (\exists \nu' < \nu)(\forall \pi < \rho)(\exists \xi < \nu')(\exists \tau < \sigma) \,.\, F(b, \langle \pi, \boldsymbol{\mu} \rangle, \xi, \kappa, \lambda, \tau) = 0$;

(4) $F(\langle 4, k, b \rangle, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 0$ iff $\nu < \lambda$,
(i) $(\forall \pi < \nu)(\exists \xi < \kappa)(\exists \tau < \sigma)[\xi > 0 \wedge F(b, \langle \pi, \boldsymbol{\mu} \rangle, \xi, \kappa, \lambda, \tau) = 0]$,
and
(ii) $(\exists \tau < \sigma) \,.\, F(b, \langle \nu, \boldsymbol{\mu} \rangle, 0, \kappa, \lambda, \tau) = 0$;

(5) in all other cases, $F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 1$.

It is clear that $(*)$ holds if $\sigma = 0$. We assume as induction hypothesis that $\sigma > 0$ and $(*)$ with σ replaced by τ holds for all $\tau < \sigma$, and prove that $(*)$ itself holds. If $(a, \boldsymbol{\mu}, \nu) \notin \Omega^{\sigma}_{\kappa\lambda}$ by virtue of being of the wrong form, then $(*)$ holds by (5); otherwise one of (1)–(4) applies. Suppose first that $a = \langle 1, k, b, c_0, \ldots, c_{k'-1} \rangle$. Then by the induction hypothesis

$$F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa, \lambda, \sigma) = 0 \leftrightarrow \text{for some } \xi_0, \ldots, \xi_{k'-1},$$

(i) $(\forall i < k')(\exists \tau < \sigma).(c_i, \boldsymbol{\mu}, \xi_i) \in \Omega^{\tau}_{\kappa\lambda}$,
and
(ii) $(\exists \tau < \sigma).(b, \boldsymbol{\xi}, \nu) \in \Omega^{\tau}_{\kappa\lambda}$.

These are exactly the conditions which guarantee that $(a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\kappa\lambda}$. The other cases are similar. □

2.5 Lemma. *For any recursively regular κ and all a, $\boldsymbol{\mu}$, and ν,*

$$(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa} \leftrightarrow (\exists \lambda < \kappa)(\exists \sigma < \kappa).(a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}.$$

Proof. The implication $(\leftarrow)$ is immediate from the monotonicity of $\Omega_{\kappa\lambda}$ with respect to κ and λ. For the implication $(\rightarrow)$ we show that

$$\{(a, \boldsymbol{\mu}, \nu) : (\exists \lambda < \kappa)(\exists \sigma < \kappa).(a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}\}$$

is closed under the clauses (0)–(4) which define the monotone operator whose closure is $\Omega_{\kappa\kappa}$. Since κ is a limit ordinal, this is trivial for clause (0).

(1) Suppose that for all $i < k'$, $(c_i, \boldsymbol{\mu}, \xi_i) \in \Omega^{\sigma_i}_{\lambda_i\lambda_i}$ and $(b, \boldsymbol{\xi}, \nu) \in \Omega^{\sigma_{k'}}_{\lambda_{k'}\lambda_{k'}}$ with all λ_i, $\sigma_i < \kappa$. Since κ is a limit ordinal, there exist λ and $\sigma < \kappa$ such that for all $i \leq k'$, $\lambda_i < \lambda$ and $\sigma_i < \sigma$. But then clearly, $(\langle 1, k, b, c_0, \ldots, c_{k'-1} \rangle, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}$.

(2) If for some $\lambda, \sigma < \kappa$, $(b, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}$, then $(\langle 2, k+1 \rangle, b, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma+1}_{\lambda\lambda}$ and $\sigma + 1 < \kappa$.

(3) Suppose that $a = \langle 3, k+1, b \rangle$ and

$$(\forall \pi < \rho)(\exists \lambda < \kappa)(\exists \sigma < \kappa)(\exists \xi < \lambda).(b, \pi, \boldsymbol{\mu}, \xi) \in \Omega^{\sigma}_{\lambda\lambda}.$$

Then

$$(\forall \pi < \rho)(\exists \zeta < \kappa)(\exists \lambda < \zeta)(\exists \sigma < \zeta)(\exists \xi < \lambda)\, T_0(b, \langle \pi, \boldsymbol{\mu} \rangle, \xi, \lambda, \lambda, \sigma).$$

By Theorem 2.4, the relation R defined by

$$R(\zeta, \pi, b, \boldsymbol{\mu}) \leftrightarrow (\exists \lambda < \zeta)(\exists \sigma < \zeta)(\exists \xi < \lambda)\, T_0(b, \langle \pi, \boldsymbol{\mu} \rangle, \xi, \lambda, \lambda, \sigma)$$

is $(\infty, 0)$-recursive, hence κ-recursive, and we have

$$(\forall \pi < \rho)(\exists \zeta < \kappa) R(\zeta, \pi, b, \boldsymbol{\mu}).$$

Let F be the κ-recursive function such that

$$F(\pi, b, \boldsymbol{\mu}) = \kappa\text{-least } \zeta . R(\zeta, \pi, b, \boldsymbol{\mu}).$$

Because κ is recursively regular,

$$\tau = \sup^{+}_{\pi<\rho} F(\pi, b, \boldsymbol{\mu}) < \kappa.$$

It follows that

$$(\forall \pi < \rho)(\exists \lambda < \tau)(\exists \sigma < \tau)(\exists \xi < \lambda).(b, \pi, \boldsymbol{\mu}, \xi) \in \Omega^{\sigma}_{\lambda\lambda}$$

and thus

$$(\forall \pi < \rho)(\exists \lambda < \tau)(\exists \xi < \lambda).(b, \pi, \boldsymbol{\mu}, \xi) \in \Omega^{(\tau)}_{\tau\tau},$$

hence

$$(a, \rho, \boldsymbol{\mu}, \nu) \in \Omega^{\tau}_{\tau\tau},$$

where

$$\nu = \sup^{+}_{\pi<\rho} \cdot \{b\}_{\kappa}(\pi, \boldsymbol{\mu}).$$

Clause (4) is treated similarly. □

2.6 Theorem. *For any recursively regular κ and all a, $\boldsymbol{\mu}$, and ν,*

$$\{a\}_{\kappa}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow (\exists \upsilon < \kappa)[T(a, \langle \boldsymbol{\mu} \rangle, \upsilon) \wedge (\upsilon)_0 = \nu].$$

Proof. The implication ($\leftarrow$) is immediate. Suppose that $\{a\}_{\kappa}(\boldsymbol{\mu}) \simeq \nu$. Then $\nu < \kappa$ and by Lemma 2.5 there are $\sigma, \lambda < \kappa$ such that $(a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}$. If $\upsilon = \langle \nu, \lambda, \lambda, \sigma \rangle$, then the recursive regularity of κ implies that $\upsilon < \kappa$ and clearly $T(a, \langle \boldsymbol{\mu} \rangle, \upsilon)$ and $(\upsilon)_0 = \nu$. □

We have a similar representation for (∞, λ)- and ∞-partial recursive functions:

2.7 Theorem. *For all λ, a, $\boldsymbol{\mu}$, and ν,*

(i) $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \exists \upsilon\, [T(a, \langle \boldsymbol{\mu} \rangle, \upsilon) \wedge (\upsilon)_0 = \nu \wedge (\upsilon)_2 = \lambda]$;

(ii) $\{a\}_{\infty}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \exists \upsilon\, [T(a, \langle \boldsymbol{\mu} \rangle, \upsilon) \wedge (\upsilon)_0 = \nu]$. □

The theory of κ-(partial)-recursive functions and κ-(semi)-recursive relations can now be developed according to the pattern of § II.4. We list the facts below but omit proofs in most cases. For the next few paragraphs, κ is a fixed recursively regular ordinal.

2.8 Normal Form Corollary. *For any κ-partial recursive function F, there exists an $a \in \omega$ such that*

$$F(\boldsymbol{\mu}) \simeq (\kappa\text{-least } \upsilon \,.\, T(a, \langle \boldsymbol{\mu} \rangle, \upsilon))_0. \quad \square$$

From Theorem 2.6 it follows that a relation R is κ-semi-recursive iff for some $a \in \omega$

$$R(\boldsymbol{\mu}) \leftrightarrow (\exists \upsilon < \kappa)\, T(a, \langle \boldsymbol{\mu} \rangle, \upsilon).$$

Equivalently, R is κ-semi-recursive iff for some κ-recursive (or even $(\infty, 0)$-recursive) relation S,

$$R(\boldsymbol{\mu}) \leftrightarrow (\exists \sigma < \kappa)\, S(\sigma, \boldsymbol{\mu}).$$

One proves just as in § II.4 that the class of κ-semi-recursive relations properly includes the class of κ-recursive relations and is closed under finite intersection, bounded universal quantification, and relational composition with κ-partial recursive functions.

It is instructive to examine an alternative proof for closure under bounded quantification (which is also available in ordinary recursion theory). Suppose that R is κ-semi-recursive and S is a κ-recursive relation such that

$$R(\rho, \boldsymbol{\mu}) \leftrightarrow (\exists \sigma < \kappa)\, S(\sigma, \rho, \boldsymbol{\mu}).$$

Then

$$(\forall \pi < \rho)\, R(\pi, \boldsymbol{\mu}) \leftrightarrow (\forall \pi < \rho)(\exists \sigma < \kappa)\, S(\sigma, \pi, \boldsymbol{\mu}).$$

Let

$$V(\tau, \rho, \boldsymbol{\mu}) \leftrightarrow (\forall \pi < \rho)(\exists \sigma < \tau)\, S(\sigma, \pi, \boldsymbol{\mu}).$$

Then V is κ-recursive and

$$(\forall \pi < \rho)\, R(\pi, \boldsymbol{\mu}) \leftrightarrow (\exists \tau < \kappa)\, V(\tau, \rho, \boldsymbol{\mu}).$$

The implication $(\leftarrow)$ is obvious. On the other hand, if $(\forall \pi < \rho)\, R(\pi, \boldsymbol{\mu})$, let

$$F(\pi, \boldsymbol{\mu}) \simeq \kappa\text{-least } \sigma \,.\, S(\sigma, \pi, \boldsymbol{\mu})$$

and

$$\tau = \sup\nolimits^{+}_{\pi < \rho} F(\pi, \boldsymbol{\mu}).$$

By recursive regularity, $\tau < \kappa$ and clearly $V(\tau, \rho, \boldsymbol{\mu})$. Thus $(\forall \pi < \rho) R(\pi, \boldsymbol{\mu})$ defines a κ-semi-recursive relation. The reader versed in axiomatic set theory will observe here a similarity with the axiom of replacement. This analogy forms the basis for many of the results of § 7 (cf. Lemma 7.5).

2.9 Selection Theorem. *There exists a κ-partial recursive function* Sel_κ *such that for all a and $\boldsymbol{\mu}$,*

(i) $\exists \pi . \{a\}_\kappa(\pi, \boldsymbol{\mu}) \downarrow$ *iff* $\mathrm{Sel}_\kappa(a, \langle \boldsymbol{\mu} \rangle) \downarrow$, *and*

(ii) *if* $\exists \pi . \{a\}_\kappa(\pi, \boldsymbol{\mu}) \downarrow$, *then* $\{a\}_\kappa(\mathrm{Sel}_\kappa(a, \langle \boldsymbol{\mu} \rangle), \boldsymbol{\mu}) \downarrow$. $\square$

2.10 Corollary. *For any κ-semi-recursive relation R, there exists a κ-partial recursive function* Sel_R *such that for all $\boldsymbol{\mu}$,*

$$\exists \pi R(\pi, \boldsymbol{\mu}) \leftrightarrow R(\mathrm{Sel}_R(\boldsymbol{\mu}), \boldsymbol{\mu}) \leftrightarrow \mathrm{Sel}_R(\boldsymbol{\mu}) \downarrow . \quad \square$$

2.11 Corollary. *The class of κ-semi-recursive relations is closed under bounded quantification, finite union, and $\exists^0_\kappa$ (existential quantification over κ).* $\square$

2.12 Corollary. *The class of κ-partial recursive functions and κ-semi-recursive relations is closed under definition by positive cases.* $\square$

2.13 Corollary. *A relation is κ-recursive iff it is both κ-semi-recursive and κ-co-semi-recursive (complement of a κ-semi-recursive relation).* $\square$

2.14 Corollary. *For any partial function $F : {}^k\kappa \to \kappa$, F is κ-partial recursive iff its graph* Gr_F *is κ-semi-recursive and F is κ-recursive iff F is total and* Gr_F *is κ-recursive.* $\square$

2.15 Corollary. *For any non-empty set $A \subseteq \kappa$, the following are equivalent*:

(i) *A is κ-semi-recursive*;

(ii) $A = \mathrm{Im}(F)$ *for some κ-recursive function F*;

(iii) $A = \mathrm{Im}(F)$ *for some κ-partial recursive function F.* $\square$

2.16 Corollary. (i) *The class of κ-semi-recursive relations has the reduction property but not the separation property*;

(ii) *the class of κ-co-semi-recursive relations has the separation property but not the reduction property.* $\square$

If $\lambda < \kappa$ is also recursively regular, one would in general expect that $\{a\}_\lambda \subsetneq \{a\}_\kappa$. Hence, although every λ-partial recursive function has natural κ-partial recursive extensions, it is by no means obvious that every λ-partial recursive function is itself κ-partial recursive. Indeed, this cannot always be so. Since every regular cardinal is recursively regular, there exists a recursively regular ordinal κ such that there are uncountably many recursively regular $\lambda < \kappa$ (by the results of § 3 this is true for $\kappa = \aleph_1$). For each such λ, let F_λ be the λ-recursive function with constant value 0 (and domain λ). Since there are uncountably many such F_λ's, not all can be κ-partial recursive.

2.17 Theorem. *For any recursively regular $\lambda < \kappa$, if $\{\sigma : \sigma < \lambda\}$ is κ-recursive, then every λ-partial recursive function is also κ-partial recursive and every λ-semi-recursive relation is κ-recursive.*

Proof. Under the condition given, for any λ-partial recursive G, if F is defined by:

$$F(\boldsymbol{\mu}) \simeq \begin{cases} G(\boldsymbol{\mu}), & \text{if } \mu_0, \ldots, \mu_{k-1} < \lambda; \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

then $F = G$ and F is κ-partial recursive.

The constant function with value λ is κ-recursive since

$$\lambda = \kappa\text{-least } \tau\,.\, \tau \not< \lambda.$$

Then if R is any λ-semi-recursive relation there is an $a < \omega$ such that

$$R(\boldsymbol{\mu}) \leftrightarrow (\exists \upsilon < \lambda)\, T(a, \langle \boldsymbol{\mu} \rangle, \upsilon).$$

With respect to κ, the quantifier is bounded and the relation T is $(\infty, 0)$-recursive, so R is κ-recursive. □

We conclude this section with another characterization of the recursively regular ordinals.

2.18 Theorem. *For any κ, the following are equivalent:*

(i) *κ is recursively regular;*
(ii) $\Omega_{\kappa\kappa} = \bigcup\{\Omega^\sigma_{\lambda\lambda} : \lambda, \sigma < \kappa\}$;
(iii) $\Omega_{\kappa\kappa} = \Omega^{(\kappa)}_{\kappa\kappa}$.

Proof. That (i)→(ii) is Lemma 2.5. That (ii)→(iii) is obvious. Suppose that κ is *not* recursively regular. Then by Lemma 2.2 there exists a κ-partial recursive function F and ordinals ρ and $\boldsymbol{\mu} < \kappa$ such that $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) = \kappa$. Let G be the κ-partial recursive function defined by

$$G(e, \sigma) \simeq 0 \cdot \sup\nolimits^+_{\tau<\sigma} \{e\}_\kappa(\tau).$$

By the Recursion Theorem applied to a natural index for G, there exists an index $\bar{e}$ such that

$$G(\bar{e},\sigma) \simeq \{\bar{e}\}_\kappa(\sigma)$$

and for all $\sigma<\kappa$, $(\bar{e},\sigma,0)\in\Omega_{\kappa\kappa}\sim\Omega^{(\sigma)}_{\kappa\kappa}$ — that is, $\{\bar{e}\}_\kappa(\sigma)$ requires at least σ steps to compute its value 0. Let a be a natural index such that

$$\{a\}_\kappa(\tau,\boldsymbol{\mu}) \simeq 0\cdot \sup^+_{\pi<\tau}\{\bar{e}\}_\kappa(F(\pi,\boldsymbol{\mu})).$$

Then $(a,\rho,\boldsymbol{\mu},0)\in\Omega_{\kappa\kappa}\sim\Omega^{(\kappa)}_{\kappa\kappa}$, so condition (iii) fails. □

2.19 Corollary. $\{\kappa : \kappa$ is recursively regular$\}$ *is* $(\infty,0)$ *recursive.*

Proof. Let H be the $(\infty,0)$-recursive function defined just preceding Lemma 2.4 with the property that if $\boldsymbol{\mu}\in{}^k\kappa$, then $\langle\boldsymbol{\mu}\rangle<H(k,\kappa)$ and let $G(\kappa)=\sup^+_{k<\omega}H(k,\kappa)$. Then G is $(\infty,0)$-recursive and for any κ, κ is recursively regular iff

$$(\forall a<\omega)(\forall\zeta<G(\kappa))(\forall\nu<\kappa)$$
$$[T_0(a,\zeta,\nu,\kappa,\kappa,\kappa)\leftrightarrow(\exists\sigma<\kappa)\,T_0(a,\zeta,\nu,\kappa,\kappa,\sigma)].\ \square$$

2.20–2.25 Exercises

2.20. Is condition (b) changed if the word "partial" is removed?

2.21. Show that for recursively regular κ, the class of κ-partial recursive functions is the smallest class of partial functions from κ to κ which contains the restriction of every $(\infty,0)$-partial recursive function and is closed under composition and κ-search.

2.22. Show that if $\kappa>\omega$ is recursively regular and $R\subseteq{}^k\omega$ is κ-recursive, then R is also λ-recursive for some $\lambda<\kappa$.

2.23. Show that κ is recursively regular iff for every $F:\kappa\to\kappa$ which is κ-recursive in parameters, there exists a $\lambda<\kappa$ which is closed under F — that is, $(\forall\pi<\lambda).F(\pi)<\lambda$.

2.24. A subset A of κ is *κ-finite* iff A is κ-recursive and for some $\rho<\kappa$, $A\subseteq\rho$. Show that κ is recursively regular iff the image of any κ-finite set under a κ-recursive function is κ-finite.

2.25. Under what conditions is the class of κ-(semi-) recursive relations closed under $\exists^0_\lambda$?

2.26 Notes. The term "recursively regular" is due to Platek [1966], who viewed the concept as a weakening of regularity. Kripke [1964] uses the term "admissible" for the same notion, presumably to indicate that the ordinals so named admit a recursion theory, and this term has become more widely used in the literature. We have used "recursively regular" both because it seems to us a more descriptive term and to save "admissible" for a different but closely related meaning in § 7.

The reader may be wondering why we have defined only recursive ordinal *functions* rather than *functionals*. Certainly there is no difficulty in adjusting the definition of $\Omega_{\kappa\lambda}$ to allow for functions from κ to κ as arguments and most of § 1 would go through unchanged. Things become more complicated, however, when we come to discuss recursive regularity. It is not hard to see that the only ordinals κ which would satisfy the condition corresponding to Definition 2.1:

$$\forall a\,(\forall \boldsymbol{\mu} < \kappa)(\forall \mathbf{f}: \kappa \to \kappa)\forall \nu\,[\{a\}_{\infty\kappa}(\boldsymbol{\mu}, \mathbf{f}) \simeq \nu \to \nu < \kappa]$$

are the regular cardinals. All would be well if the f were restricted to be κ-recursive, but then they could be replaced by their κ-indices, and it is not clear what would be gained. Recursive ordinal functionals are useful for some purposes but not for any in this book.

3. *Ordinal Recursion and the Analytical Hierarchy*

To this point our only examples of recursively regular ordinals are the infinite regular cardinals. The first goal of this section is to establish the recursive regularity of many countable ordinals (uncountably many, in fact). The existence of many recursively regular ordinals of each uncountable power will be established in § 5. Our other objective here is to explore the intimate connection between ω_1-recursion and the classes Π^1_1 and Δ^1_1, on the one hand, and between $\aleph_1$-recursion and the classes Σ^1_2 and Δ^1_2, on the other.

Recall that we observed following Lemma 1.6 that for any $\kappa > \omega$, the constant function with value ω is κ-recursive. It follows that operators such as $\sup^+_{\pi<\omega}$ and $(\exists \pi < \omega)$ may be freely used in defining κ-partial recursive functions and κ-recursive relations.

3.1 Lemma. *For any recursively regular $\kappa > \omega$ and any $\gamma \in \mathsf{W}$, if γ is κ-recursive, then $\|\gamma\| < \kappa$.*

Proof. Suppose $\kappa > \omega$ is recursively regular and there exists $\gamma \in \mathsf{W}$ such that γ is κ-recursive and $\|\gamma\| \geqslant \kappa$. Without loss of generality we may assume $\|\gamma\| = \kappa$. By use of the κ-Recursion Theorem we may define a κ-partial recursive function F such that for all $p \in \mathrm{Fld}(\gamma)$,

$$F(p) \simeq \kappa\text{-least } \pi . (\forall q < \omega)[q <_\gamma p \to F(q) < \pi],$$

and otherwise, $F(p) = 0$. For all $p \in \mathrm{Fld}(\gamma)$, $|p|_\gamma < \kappa$ and by induction on $|p|_\gamma$ it follows that $F(p)$ is defined for all $p < \omega$. But then by the recursive regularity of κ,

$$\kappa = \|\gamma\| = \sup{}^+\{|p|_\gamma : p \in \mathrm{Fld}(\gamma)\} = \sup{}^+_{p<\omega} F(p) < \kappa,$$

a contradiction. $\square$

It follows immediately that no ordinal κ such that $\omega < \kappa < \omega_1$ is recursively regular, as for any such κ there exists an (ordinary) recursive function $\gamma \in \mathrm{W}$ such that $\|\gamma\| = \kappa$. γ is also κ-recursive and if κ were recursively regular, this would contradict Lemma 3.1.

We show next that ω_1 is recursively regular. We shall code ω_1-computations using elements of W as notations for the ordinals less than ω_1. The boundedness property (IV.2.1 (iii)) is the main tool for completing the proof.

Recall that for $m \in W$, $\|m\|$ is the order-type of the well-ordering $\leqslant_{\{m\}} = \{(p,q) : \{m\}(\langle p,q\rangle) = 0\}$, and for $m \notin W$, $\|m\| = \aleph_1$. There are Σ^1_1 relations $\leqslant_\Sigma$ and $<_\Sigma$ and Π^1_1 relations $\leqslant_\Pi$ and $<_\Pi$ such that whenever at least one of m and n belongs to W,

$$m \leqslant_\Sigma n \leftrightarrow \|m\| \leqslant \|n\| \leftrightarrow m \leqslant_\Pi n,$$

and

$$m <_\Sigma n \leftrightarrow \|m\| < \|n\| \leftrightarrow m <_\Pi n.$$

We shall write $\|\mathbf{m}\|$ for $(\|m_0\|, \ldots, \|m_{k-1}\|)$. Let

$$\Omega_W = \{\langle a, \mathbf{m}, n\rangle : \mathbf{m}, n \in W \wedge \{a\}_{\omega_1}(\|\mathbf{m}\|) \simeq \|n\|\}.$$

3.2 Lemma. $\Omega_W \in \Pi^1_1$.

Proof. We shall indicate how to give a direct Π^1_1 monotone inductive characterization of Ω_W — that is, we define a Π^1_1 monotone operator Γ such that $\bar{\Gamma} = \Omega_W$. It follows by Theorem II.3.1 that $\Omega_W \in \Pi^1_1$. The clauses which define Γ are translations of those which occur in the inductive definition of $\Omega_{\omega_1\omega_1}$ with ordinals less than ω_1 (the only ordinals that occur) replaced by their codes from W, and inequalities among these ordinals interpreted by $\leqslant_\Sigma$, etc.

Thus, Ω_W is the smallest set such that $\langle a, \mathbf{m}, n\rangle \notin \Omega_W$ unless $\mathbf{m}$ and n beong to W and for all k, n, $i < k$, and all $m_0, \ldots, m_{k-1}$, p, r, s, and $t \in W$,

(0) $\langle\langle 0, k, 0, n\rangle, \mathbf{m}, n'\rangle \in \Omega_W$ for all n' such that $\|n'\| = n$;

$\langle\langle 0, k, 1, i\rangle, \mathbf{m}, n\rangle \in \Omega_W$ for all n such that $n \preccurlyeq_\Pi m_i$ and $m_i \preccurlyeq_\Pi n$; $\langle\langle 0, k, 2, i\rangle, \mathbf{m}, n\rangle \in \Omega_W$ for all n such that $m_i \prec_\Pi n$ and $\neg \exists q\, [m_i \prec_\Sigma q \wedge q \prec_\Sigma n]$;

etc.;

(1)
(2) $\Big\}$ Similarly;

(3) if $\forall p\, [p \prec_\Sigma r \rightarrow \exists q\, (q \prec_\Pi n \wedge \langle b, p, \mathbf{m}, q\rangle \in \Omega_W)]$, and $\neg \exists n'(n' \prec_\Sigma n \wedge \forall p\, [p \prec_\Pi r \rightarrow \exists q\, (q \prec_\Sigma n' \wedge \langle b, p, \mathbf{m}, q\rangle \in \Omega_W)])$, then $\langle\langle 3, k+1, b\rangle, r, \mathbf{m}, n\rangle \in \Omega_W$;

(4) if $\forall p\, [p \prec_\Sigma n \rightarrow \exists q\, (\|q\| > 0 \wedge \langle b, p, \mathbf{m}, q\rangle \in \Omega_W)]$, and $\exists q\, [\|q\| = 0 \wedge \langle b, n, \mathbf{m}, q\rangle \in \Omega_W$, then $\langle\langle 4, k, b\rangle, \mathbf{m}, n\rangle \in \Omega_W$.

We leave it to the reader to verify that this is in fact a Π^1_1 monotone inductive definition of Ω_W. □

3.3 Theorem. *ω_1 is the least recursively regular ordinal greater than ω.*

Proof. If remains to show that ω_1 is recursively regular; we use criterion (b) of Theorem 2.2. Let F be any ω_1-partial recursive function and ρ and $\boldsymbol{\mu} < \omega_1$ ordinals such that for all $\pi < \rho$, $F(\pi, \boldsymbol{\mu})$ is defined. We aim to show that $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) < \omega_1$. Let

$$A = \{n : (\exists \pi < \rho)\, F(\pi, \boldsymbol{\mu}) = \|n\|\}.$$

Since $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) = \sup^+\{\|n\| : n \in A\}$, it would suffice, by the Boundedness Theorem IV.2.1(iii) to show that $A \in \Sigma^1_1$. This is not in general true, but we shall find a Σ^1_1 subset $B \subseteq A$ such that

$$(*) \qquad \sup^+\{\|n\| : n \in A\} = \sup^+\{\|n\| : n \in B\}.$$

Let a be an ω_1-index for F and let r and $\mathbf{m}$ be fixed notations in W for ρ and $\boldsymbol{\mu}$, respectively. Let R be the Π^1_1 relation defined by

$$R(n, p) \leftrightarrow \langle a, p, \mathbf{m}, n\rangle \in \Omega_W.$$

Let Sel_R be the selection function with Π^1_1 graph given by Lemma IV.2.5 and set

$$B = \{n : \exists p\, [p \prec_\Sigma r \wedge \mathrm{Sel}_R(p) = n]\}.$$

Since by the hypothesis, for each $p \prec_\Sigma r$ there exist n such that $R(n, p)$, also

$$B = \{n : \exists p\, [p \prec_\Sigma r \wedge \forall n'(n' \neq n \rightarrow \mathrm{Sel}_R(p) \neq n')]\}.$$

Hence $B \in \Sigma_1^1$. That (*) holds is clear; B contains a unique notation for each ordinal with a notation in A. $\square$

3.4 Theorem. *For all k and all $R \subseteq {}^k\omega$,*

(i) *R is ω_1-semi-recursive $\leftrightarrow R \in \Pi_1^1$;*

(ii) *R is ω_1-recursive $\leftrightarrow R \in \Delta_1^1$.*

Proof. Suppose first that R is ω_1-semi-recursive, say with ω_1-semi-index a. It is easy to define a recursive function F such that for all m, $F(m) \in W$ and $\|F(m)\| = m$ (cf. Exercise IV.1.20). Then

$$R(\mathbf{m}) \leftrightarrow \exists n. \langle a, F(m_0), \ldots, F(m_{k-1}), n \rangle \in \Omega_W.$$

Hence by Lemma 3.2, $R \in \Pi_1^1$.

For the implication ($\leftarrow$) of (i), suppose that R is a Π_1^1 relation. By Theorem III.3.2 there exists a Π_1^0 monotone operator Γ such that $R \leqslant \bar{\Gamma}$. It suffices to show that $\bar{\Gamma}$ is ω_1-semi-recursive.

Let S be a recursive relation such that for all m and A,

$$m \in \Gamma(A) \leftrightarrow \forall p\, S(m, \bar{K}_A(p)).$$

Consider the following pair of recursively defined functions:

$$G_{(\ )}(m, 0) = 1; \quad \text{for} \quad \sigma > 0,$$

$$G_{(\ )}(m, \sigma) = \begin{cases} 0, & \text{if} \quad (\exists \pi < \sigma)\, G(m, \pi) = 0; \\ 1, & \text{otherwise}; \end{cases}$$

$$G(m, \sigma) = \begin{cases} 0, & \text{if} \quad (\forall p < \omega)\, S(m, \bar{G}_{(\ )}(p, \sigma)); \\ 1, & \text{otherwise}. \end{cases}$$

It is straightforward to prove by induction on σ that for all m,

$$G_{(\ )}(m, \sigma) = 0 \leftrightarrow m \in \Gamma^{(\sigma)},$$

and

$$G(m, \sigma) = 0 \leftrightarrow m \in \Gamma^{\sigma}.$$

We shall prove below that $G_{(\ )}$ and G are ω_1-recursive. Then since by Theorem IV.2.15, $|\Gamma| \leqslant \omega_1$, we have

$$m \in \bar{\Gamma} \leftrightarrow (\exists \sigma < \omega_1)\, G(m, \sigma) = 0.$$

By the remarks following Corollary 2.8, $\bar{\Gamma}$ is ω_1-semi-recursive.

To see that $G_{(\ )}$ and G are ω_1-recursive, let H be defined by:

$$H(e, \rho, m, \sigma) \simeq \begin{cases} 1, & \text{if } \rho = \sigma = 0; \\ \inf_{\pi < \sigma} \{e\}_{\omega_1}(1, m, \pi), & \text{if } \rho = 0 < \sigma; \\ \sup_{\pi < \omega} K_S(m, \langle \{e\}_{\omega_1}(0, 0, \sigma), \ldots, \{e\}_{\omega_1}(0, \pi - 1, \sigma)\rangle), & \\ & \text{if } \rho > 0. \end{cases}$$

Clearly H is ω_1-partial recursive, so by the Recursion Theorem there exists an $\bar{e}$ such that

$$\{\bar{e}\}_{\omega_1}(\rho, m, \sigma) \simeq H(\bar{e}, \rho, m, \sigma).$$

We leave to the reader the easy verification that

$$G_{(\ )}(m, \sigma) \simeq \{\bar{e}\}_{\omega_1}(0, m, \sigma),$$

and

$$G(m, \sigma) \simeq \{\bar{e}\}_{\omega_1}(1, m, \sigma).$$

For (ii), we have by Corollary 2.13, Theorem 3.3, and the fact that $\{\pi : \pi < \omega\}$ is ω_1-recursive:

$$\begin{aligned} R \text{ is } \omega_1\text{-recursve} &\leftrightarrow R \text{ and } {}^k\omega \sim R \text{ are } \omega_1\text{-semi-recursive} \\ &\leftrightarrow R \text{ and } {}^k\omega \sim R \text{ are } \Pi_1^1 \\ &\leftrightarrow R \in \Delta_1^1. \quad \square \end{aligned}$$

It is interesting to compare the two recursion-theoretic characterizations of the classes of Π_1^1 and Δ_1^1 relations: (semi-) recursive in E and ω_1-(semi-) recursive. The image of any function $F : \omega \to \omega$ which is recursive in E is again recursive in E because Δ_1^1 is closed under existential number quantification ($\exists_\omega^0$). However, by Corollary 2.15 and the preceding theorem, every Π_1^1 set of numbers is the image of some ω_1-recursive function $G : \omega_1 \to \omega$. The difference, of course, lies in the domains of the functions; G has a "longer time" to enumerate its image. A Π_1^1 set is generable in ω_1 steps in any of several ways — via inductive definitions, reductions to recursive well-orderings, or ω_1-semi-recursion — and it is not surprising that it can be "effectively" enumerated in a sequence of length ω_1 but not, in general, in one of length ω. In fact, in any natural sense of effective enumeration, a set enumerable in fewer than ω_1 steps is necessarily Δ_1^1.

Another striking contrast arises in the theory of relative ω_1-recursion. As this topic is beyond the scope of this book, we shall merely sketch the facts and refer the reader to Shore [1977] or Simpson [1984?] for more details. Recall first the following two facts:

(1) (Friedberg–Mučnik Theorem (§ II.5)) there exist semi-recursive sets A and $B \subseteq \omega$ such that neither A is recursive in B nor B is recursive in A;

(2) (Theorem IV.2.12) for every two Π_1^1 sets A and $B \subseteq \omega$, either A is Δ_1^1 (hyperarithmetic) in B or B is Δ_1^1 in A.

Combining (2) with a relativized version of Corollary IV.1.9, we have also

(3) for every two sets A and $B \subseteq \omega$ which are semi-recursive in E, either A is recursive in B and E or B is recursive in A and E.

There are several possible generalizations of "recursive in" to "ω_1-recursive in", but we shall mention only two. The most natural in the present context arises by modifying the definition of $\Omega_{\omega_1\omega_1}$ to allow for the introduction of a parameter. For any $F: \omega_1 \to \omega_1$, let $\Omega_{\omega_1\omega_1}[F]$ be the smallest set satisfying clauses (0)–(4) as in the definition of $\Omega_{\omega_1\omega_1}$ and such that for all $\boldsymbol{\mu} \in {}^k\omega_1$,

$$(\langle 0, k, 3, i\rangle, \boldsymbol{\mu}, F(\mu_i)) \in \Omega_{\omega_1\omega_1}[F].$$

We then set $\{a\}_{\omega_1}^F(\boldsymbol{\mu}) \simeq \nu$ iff $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\omega_1\omega_1}[F]$ and define the other notions accordingly. We say that A is *weakly ω_1-computable* from B iff for some a, $K_A = \{a\}_{\omega_1}^{K_B}$. From the foregoing it is not hard to deduce that for any A and $B \subseteq \omega$,

$$A \text{ is weakly } \omega_1\text{-computable from } B \leftrightarrow A \text{ is } \Delta_1^1 \text{ in } B.$$

Hence by (3) there is no pair of ω_1-semi-recursive sets A and $B \subseteq \omega$ such that neither is weakly ω_1-computable from the other.

In general, the inductive definition of $\Omega_{\omega_1\omega_1}[F]$ will require more than ω_1 stages, as there is no reason to expect that ω_1 will be recursively-in-F regular — that is, closed under all functions (∞, ω_1)-partial recursive in F. One natural restriction on the reduction of A to B is to require that each computation of $K_A(\mu)$ from B take fewer than ω_1 steps. Accordingly, we set

$$[a]_{\omega_1}^F(\boldsymbol{\mu}) \simeq \nu \quad \text{iff} \quad (a, \boldsymbol{\mu}, \nu) \in \Omega_{\omega_1\omega_1}^{(\omega_1)}[F]$$

and say that A is *ω_1-computable* from B iff for some a, $K_A = [a]_{\omega_1}^F$. It is proved in Sacks [1971] that

(4) there exist ω_1-semi-recursive sets A and $B \subseteq \omega$ such that neither A is ω_1-computable from B nor B is ω_1-computable from A.

The intuitive explanation for the contrast between (3) and (4) is an extension of that concerning enumerability discussed above. If $A \subseteq \omega$ is Π_1^1 but not Δ_1^1, then by Theorems IV.2.11–12, $\omega_1[A] > \omega_1$. Just as any set of numbers which is "effectively" enumerable in fewer than ω_1 steps is Δ_1^1, so any set enumerable in fewer than $\omega_1[A]$ steps is Δ_1^1 in A. Every Π_1^1 set is enumerable in ω_1 steps and is thus Δ_1^1 in A. The restriction in (4) to computations of length less than ω_1 eliminates this possibility and permits the construction of incomparable sets.

If one relaxes the requirement that A and B be subsets of ω, then the results are completely analogous to (1):

(5) there exist ω_1-semi-recursive sets A and $B \subseteq \omega_1$ such that neither A is weakly ω_1-computable from B nor B is weakly ω_1-computable from A. Analogous results hold for every recursively regular κ in place of ω_1. A proof may be found in Sacks–Simpson [1972].

Although we have formally established only the existence of one recursively regular ordinal between ω and $\aleph_1$, we need almost no further work to show that there are in fact uncountably many such ordinals, namely the ordinals $\omega_1[\beta]$ for arbitrary $\beta \in {}^{\omega}\omega$ (cf. Exercise 3.12). However it is not in general true that

$$R \text{ is } \omega_1[\beta]\text{-recursive} \leftrightarrow R \in \Delta_1^1[\beta].$$

To see this, note that there are only countably many β which are even ∞-recursive. If β is not ∞-recursive, then $\beta \in \Delta_1^1[\beta]$ but β is not $\omega_1[\beta]$-recursive. The failure of the corresponding fact for semi-recursive relations will be shown in § 5.

We turn now to the second level of the analytical hierarchy. The relevant ordinal κ here is not, as might be expected, the next recursively regular ordinal greater than ω_1, but $\aleph_1$, the first *regular* ordinal greater than ω. In fact, however, we shall see in § 5 that for relations and functions on numbers, the same results are obtained ith $\kappa = \delta_2^1$, the least ordinal not the order-type of a Δ_2^1 well-ordering of ω.

For any $\gamma \in \mathrm{W}$, let

$$\Omega_\gamma = \{\langle a, \mathbf{m}, n\rangle : \{a\}_{\|\gamma\|}(|m_0|_\gamma, \ldots, |m_{k-1}|_\gamma) \simeq |n|_\gamma\},$$

and

$$O(u, \gamma) \leftrightarrow \gamma \in \mathrm{W} \wedge u \in \Omega_\gamma.$$

3.5 Lemma. *O is Π_1^1.*

Proof. For each $\gamma \in \mathrm{W}$, Ω_γ is the closure $\bar{\Gamma}_\gamma$ of a monotone Π_1^1 operator. Γ_γ is defined as in the proof of Lemma 3.2 with the relations $\preccurlyeq_\Sigma$ and $\preccurlyeq_\Pi$ both replaced by $\preccurlyeq_\gamma$. For $\gamma \notin \mathrm{W}$, set $\Gamma_\gamma(A) = \emptyset$ for all A. Then O is inductively defined by the family $\{\Gamma_\gamma : \gamma \in {}^{\omega}\omega\}$ and is thus Π_1^1 by Theorem III.3.15. $\square$

3.6 Lemma. *For any R and S, if S is $\aleph_1$-recursive and for all ρ and $\boldsymbol{\mu}$,*

$$R(\rho, \boldsymbol{\mu}) \leftrightarrow \exists g\, (\forall p < \omega)[g(p) < \rho \wedge S(\bar{g}(p), \boldsymbol{\mu})],$$

then also R is $\aleph_1$-recursive.

Proof. For each ρ, let Γ_ρ be the monotone operator defined by:

$$\langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho(B) \leftrightarrow \sigma < \bar{\rho} \wedge [\sim S(\sigma, \boldsymbol{\mu}) \vee (\forall \xi < \rho)\langle \boldsymbol{\mu}, \sigma * \langle \xi\rangle\rangle \in B],$$

where for each ρ, $\bar{\rho}$ is an ordinal larger than all $\langle \boldsymbol{\pi}\rangle$ with all $\pi_i < \rho$ ($\bar{\rho} = \overset{+}{\sup}_{k<\omega}\langle \rho_0, \ldots, \rho_k\rangle$ with all $\rho_j = \rho$). Exactly as in the proof of Theorem III.3.2 one may show that

$$\langle \boldsymbol{\mu}, \sigma\rangle \in \bar{\Gamma}_\rho \leftrightarrow \forall g\, (\exists p < \omega)[\rho \leqslant g(p) \vee \sim S(\sigma * \bar{g}(p), \boldsymbol{\mu})],$$

and hence that

$$R(\rho, \boldsymbol{\mu}) \leftrightarrow \langle \boldsymbol{\mu}, \langle\ \rangle\rangle \notin \bar{\Gamma}_\rho.$$

Let

$$P(\boldsymbol{\mu}, \sigma, \tau, \rho) \leftrightarrow \langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^\tau.$$

It is straightforward to verify as in the proof of Theorem 3.4 that P is $\aleph_1$-recursive, in fact, $(\infty, 0)$-recursive. Let

$$\begin{aligned} F(\rho, \boldsymbol{\mu}) &\simeq \aleph_1\text{-least } \tau (\forall \sigma < \bar{\rho})[\langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^\tau \leftrightarrow \langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^{(\tau)}] \\ &\simeq \aleph_1\text{-least } \tau (\forall \sigma < \bar{\rho})[P(\boldsymbol{\mu}, \sigma, \tau, \rho) \leftrightarrow (\exists \zeta < \tau) P(\boldsymbol{\mu}, \sigma, \zeta, \rho)]. \end{aligned}$$

Clearly F is $\aleph_1$-partial recursive. For a fixed $\boldsymbol{\mu}$ and ρ, Γ_ρ may be thought of as an operator over the countable set $\bar{\rho}$ and as such has a countable closure ordinal. In particular, for each $\sigma < \bar{\rho}$, there is a $\tau < \aleph_1$ such that

$$\langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^\tau \leftrightarrow \langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^{(\tau)}.$$

Since $\bar{\rho} < \aleph_1$ we have

$$F(\rho, \boldsymbol{\mu}) \simeq \sup_{\sigma < \bar{\rho}} (\aleph_1\text{-least } \tau)[\langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^\tau \leftrightarrow \langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^{(\tau)}]$$

and thus F is total, hence $\aleph_1$-recursive.

Finally,

$$\langle \boldsymbol{\mu}, \sigma\rangle \in \bar{\Gamma}_\rho \leftrightarrow \langle \boldsymbol{\mu}, \sigma\rangle \in \Gamma_\rho^{F(\rho, \boldsymbol{\mu})}$$

so that

$$R(\rho, \boldsymbol{\mu}) \leftrightarrow \sim P(\boldsymbol{\mu}, \langle \quad \rangle, F(\rho, \boldsymbol{\mu}), \rho).$$

Thus R is $\aleph_1$-recursive. $\square$

3.7 Theorem. *For all k and all $R \subseteq {}^k\omega$,*
(i) R is $\aleph_1$-semi-recursive $\leftrightarrow R \in \Sigma^1_2$;
(ii) R is $\aleph_1$-recursive $\leftrightarrow R \in \Delta^1_2$.

Proof. Part (ii) follows immediately from (i) and the fact that $\aleph_1$ is recursively regular. Suppose first that R is $\aleph_1$-semi-recursive, say with $\aleph_1$-semi-index a. It is easy to define an arithmetical functional F such that for $\gamma \in \mathsf{W}$, $\mathsf{F}(m, \gamma)$ is defined for all m and $|\mathsf{F}(m, \gamma)|_\gamma = m$. Then

$$\begin{aligned} R(\mathbf{m}) &\leftrightarrow \exists \nu . \{a\}_{\aleph_1}(\mathbf{m}) \simeq \nu \\ &\leftrightarrow \exists \nu\, (\exists \kappa < \aleph_1) . \{a\}_\kappa(\mathbf{m}) \simeq \nu \\ &\leftrightarrow \exists \nu\, (\exists \gamma \in \mathsf{W}) . \{a\}_{\|\gamma\|}(\mathbf{m}) \simeq \nu \\ &\leftrightarrow \exists \gamma\, [\gamma \in \mathsf{W} \wedge \exists n . \langle a, \mathsf{F}(m_0, \gamma), \ldots, \mathsf{F}(m_{k-1}, \gamma), n \rangle \in \Omega_\gamma]. \end{aligned}$$

The second equivalence follows from Lemma 2.5 and the recursive regularity of $\aleph_1$. R is Σ^1_2 by Lemma 3.5.

For the converse, suppose $R \in \Sigma^1_2$, say

$$R(\mathbf{m}) \leftrightarrow \exists \alpha\, \forall \beta\, \exists n\, P(\bar{\alpha}(n), \bar{\beta}(n), \mathbf{m})$$

for some recursive relation P. As in the proof of Theorem IV.1.1, for each $\mathbf{m}$ and α, there exists a linear ordering $\trianglelefteq^P_{\mathbf{m}, \alpha}$ such that

$$R(\mathbf{m}) \leftrightarrow \exists \boldsymbol{\alpha}\, [\trianglelefteq^P_{\mathbf{m}, \alpha} \text{ is a well-ordering}].$$

Of course, $\trianglelefteq^P_{\mathbf{m}, \alpha}$ is a countable ordering, hence is well-ordered just in case it is isomorphic to the usual $\leqslant$ on some ordinal $\pi < \aleph_1$. Hence

$$R(\mathbf{m}) \leftrightarrow \exists \alpha\, (\exists \pi < \aleph_1)\, \exists f\, \forall s\, \forall t\, [f(t) < \pi \wedge s \triangleleft^P_{\mathbf{m}, \alpha} t \leftrightarrow f(s) < f(t)].$$

For s and t less than any given p, the relation in brackets depends only on values of α and f for arguments less than p. Hence, there exist $(\infty, 0)$-recursive relations U and V such that

$$\begin{aligned} R(\mathbf{m}) &\leftrightarrow (\exists \pi < \aleph_1)\, \exists \alpha\, \exists f\, \forall p\, [f(p) < \pi \wedge \\ &\qquad\qquad (\forall s < p)(\forall t < p)\, U(\bar{\alpha}(p), \bar{f}(p), s, t, \mathbf{m})] \\ &\leftrightarrow (\exists \rho < \aleph_1)\, \exists g\, \forall p\, [g(p) < \rho \wedge V(\bar{g}(p), \mathbf{m})]. \end{aligned}$$

In the second implication ($\rightarrow$) we intend that $g(p) = \langle \alpha(p), f(p) \rangle$ and $\alpha = \langle \omega, \pi \rangle$. Then R is $\aleph_1$-semi-recursive by Lemma 3.6 and Corollary 2.11. $\square$

3.8–3.18 Exercises

3.8. Give another proof of Theorem 3.4 by showing directly that W is ω_1-semi-recursive.

3.9. Show that

(i) there exists an ω_1-partial recursive function F such that for all $c \in W$, $F(c) = \|c\|$;

(ii) there exists an ω_1-recursive function G such that for all $\rho < \omega_1$, $G(\rho) \in W$ and $\|G(\rho)\| = \rho$.

3.10. Let f be any ω_1-recursive function.

(i) Show that there exists a function g recursive in E such that for any $c \in W$, $g(c) \in W$ and $\|g(c)\| = f(\|c\|)$.

(ii) Show that there exists a primitive recursive function h such that for any $c \in W$, $h(c) \in W$ and $\|h(c)\| \geqslant f(\|c\|)$.

3.11. For any $R \subseteq {}^k\omega_1$, let

$$R_W(\mathbf{m}) \leftrightarrow m_0, \ldots, m_{k-1} \in W \quad \text{and} \quad R(\|\mathbf{m}\|).$$

Show that

(i) R is ω_1-semi-recursive $\leftrightarrow R_W \in \Pi_1^1$;

(ii) R is ω_1-finite $\leftrightarrow R_W \in \Delta_1^1$.

3.12. Show that for every $\beta \in {}^\omega\omega$, $\omega_1[\beta]$ is recursively regular, and if β is $\omega_1[\beta]$-recursive, then for any $R \subseteq {}^k\omega$,

$$R \text{ is } \omega_1[\beta]\text{-semi-recursive} \leftrightarrow R \in \Pi_1^1[\beta],$$

and

$$R \text{ is } \omega_1[\beta]\text{-recursive} \leftrightarrow R \in \Delta_1^1[\beta].$$

3.13. Characterize the least recursively regular ordinal $> \omega_1$.

3.14. Show that

(i) For all $\sigma < \omega_1$, the constant function with value σ is ω_1-recursive;

(ii) for all $\sigma < \delta_2^1$, the constant function with value σ is $\aleph_1$-recursive.

3.15. Show that if κ is recursively regular, α is κ-recursive and $R \subseteq {}^k\omega$ is $\Pi_1^1[\alpha]$, then R is κ-semi-recursive.

3.16. Suppose that κ is both recursively regular and a limit of recursively regular ordinals (recursively inaccessible — cf. § 6). Show

(i) if α is κ-recursive and $R \subseteq {}^k\omega$ is $\Pi_1^1[\alpha]$, then R is κ-recursive;

(ii) if $\alpha_0, \ldots, \alpha_{k-1}$ are κ-recursive and R is an arithmetical relation such that $\exists\beta\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)$, then $\exists\beta\, [\beta$ is κ-recursive $\wedge\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta)]$ (use a relativized version of Theorem III.4.7);

(iii) if $S \subseteq {}^k\omega$ is defined by

$$S(\mathbf{m}) \leftrightarrow (\exists \kappa\text{-recursive } \alpha)(\forall \kappa\text{-recursive } \beta)\, \mathsf{R}(\mathbf{m}, \alpha, \beta)$$

with R arithmetical, then S is κ-semi-recursive;

(iv) $\{\alpha : \alpha$ is κ-recursive$\}$ is a model of the Δ_2^1-Comprehension schema.

3.17. Show that for any $R \subseteq {}^k\omega$,

$$R \text{ is } \aleph_1\text{-(semi-) recursive} \leftrightarrow R \text{ is } \delta_2^1\text{-(semi-) recursive}$$

(use the Basis Theorem IV.7.9).

3.18. Show that for any $A \subseteq \omega$ and any $R \subseteq {}^k\omega$, if A is $\aleph_1$-recursive in parameters and $R \in \Delta_2^1[A]$, then also R is $\aleph_1$-recursive in parameters (relativize the proof of Theorem 3.7).

3.19 Notes. The results here are first stated explicitly in Kripke [1964 a], but some of the ideas are present in Kino–Takeuti [1962, – a].

The various possible notions of relative ω_1- (or κ-) semi-recursiveness are discussed in Kreisel–Sacks [1965]. Simpson [1974] is another good survey of the (then) current state of knowledge about relative ordinal recursion.

4. *Ordinal Recursion and Type-2 Functionals*

We shall see in § 5 that the results of the preceding section cannot be extended to higher levels of the analytical hierarchy. They do, however, have some natural extensions to classes of relations (semi-) recursive in a type-2 functional I. In view of Theorems VI.1.8–9 and the fact that $\omega_1[\mathsf{E}] = \omega_1$, Theorem 3.4 may be phrased as: for all $R \subseteq {}^k\omega$,

$$R \text{ is } \omega_1[\mathsf{E}]\text{-(semi-) recursive} \leftrightarrow R \text{ is (semi-) recursive in } \mathsf{E}.$$

The primary goal of this section is to establish conditions under which this equivalence holds with E replaced by other functionals. In the process we shall also develop some machinery which will be useful in § 6.

It is easy to see that the equivalence cannot hold for arbitrary I, even assuming that E is recursive in I. There are only countably many ∞-recursive relations on ω and if R is any non-∞-recursive relation, there exist I such that R and E are recursive in I (take I to code R and E), but clearly R is not $\omega_1[\mathsf{I}]$-recursive.

For this section I is a fixed functional such that E is recursive in I. The proofs below depend heavily on the results and techniques of §§ VI.2–4 and the constructions are in many cases similar to ones which appear there. We shall omit much of the tedious detail and hope that the reader who has followed the proofs of these earlier sections will feel confident that he could fill in these details.

We first recall from § VI.4 that

$$U^{\mathsf{I}} = \{\langle a, \mathbf{m}\rangle : \{a\}^{\mathsf{I}}(\mathbf{m}) \text{ is defined}\}$$

and that there exist relations $\leqslant_+$ and $<_+$ semi-recursive in I and $\leqslant_-$ and $<_-$ co-semi-recursive in I such that whenever at least one of u and v belongs to U^{I},

$$u \leqslant_+ v \leftrightarrow |u|^{\mathsf{I}} \leqslant |v|^{\mathsf{I}} \leftrightarrow u \leqslant_- v,$$

and

$$u <_+ v \leftrightarrow |u|^{\mathsf{I}} < |v|^{\mathsf{I}} \leftrightarrow u <_- v.$$

By Theorem VI.4.17, the ordinals $|u|^{\mathsf{I}}$ for $u \in U^{\mathsf{I}}$ are exactly the ordinals less than $\omega_1[\mathsf{I}]$. We shall write $|\mathbf{m}|^{\mathsf{I}}$ for $(|m_0|^{\mathsf{I}}, \ldots, |m_{k-1}|^{\mathsf{I}})$ and omit the superscript in most cases.

4.1 Lemma. *There exist functions F, G, and H partial recursive in I such that for all u, m, p, and q,*

(i) *$F(m) \in U^{\mathsf{I}}$ and $|F(m)| = m$;*

(ii) *if $m \in U^{\mathsf{I}}$, then $G(m) \in U^{\mathsf{I}}$ and $|G(m)| = |m| + 1$;*

(iii) *$H(p, q)$ is defined iff $|p|, |q| < \omega$, and if so,*

$$|H(p, q)| = \mathrm{Sb}_0(|p|, |q|).$$

Proof. Exercise 4.17. $\square$

4.2 Lemma. *There exists a primitive recursive function f such that for all a and all* $\mathbf{m} \in U^{\mathsf{I}}$,

(i) $\{f(a)\}^{\mathsf{I}}(\mathbf{m})$ *is defined iff* $\{a\}_{\omega_1[\mathsf{I}]}(|\mathbf{m}|)$ *is defined*;
(ii) $\{a\}_{\omega_1[\mathsf{I}]}(|\mathbf{m}|) \simeq |\{f(a)\}^{\mathsf{I}}(\mathbf{m})|$.

Proof. We define f via the Primitive Recursion Theorem to satisfy the following conditions:

(0) if $a = \langle 0, k, 0, n\rangle$, then $\{f(a)\}^{\mathsf{I}}(\mathbf{m}) \simeq F(n)$;
if $a = \langle 0, k, 1, i\rangle$, then $\{f(a)\}^{\mathsf{I}}(\mathbf{m}) \simeq m_i$;
if $a = \langle 0, k, 2, i\rangle$, then $\{f(a)\}^{\mathsf{I}}(\mathbf{m}) \simeq G(m_i)$;
if $a = \langle 0, k+4, 4\rangle$, then

$$\{f(a)\}^{\mathsf{I}}(p, q, r, s, \mathbf{m}) \simeq \begin{cases} p, & \text{if} \quad r \leqslant_+ s \quad \text{and} \quad s \leqslant_+ r; \\ q, & \text{if} \quad r <_+ s \quad \text{or} \quad s <_+ r; \end{cases}$$

if $a = \langle 0, k+2, 5\rangle$, then $\{f(a)\}^{\mathsf{I}}(p, q, m) \simeq H(p, q)$;

(1) if for some $k', b, c_0, \ldots, c_{k'-1}$, $a = \langle 1, k, b, c_0, \ldots, c_{k'-1}\rangle$, then $f(a) = \langle 1, k, 0, f(b), f(c_0), \ldots, f(c_{k'-1})\rangle$;

(2) if $a = \langle 2, k+1\rangle$, then $f(a)$ is defined as in case (2) of Lemma VI.2.2;

(3) if for some b, $a = \langle 3, k+1, b\rangle$, let R be the relation semi-recursive in I defined by:

$$\begin{aligned} R(n, r, \mathbf{m}) \leftrightarrow \forall p\, [p <_- r \rightarrow \exists q\, (q <_+ n \wedge \{f(b)\}^{\mathsf{I}}(p, \mathbf{m}) \simeq q)] \\ \wedge \neg \exists n' (n' <_- n \wedge \forall p\, [p <_+ r \rightarrow \exists q\, (q <_- n' \wedge \{f(b)\}^{\mathsf{I}}(p, \mathbf{m}) \simeq q)]), \end{aligned}$$

then choose $f(a)$ to be an index such that

(a) $\{f(a)\}^{\mathsf{I}}(r, \mathbf{m}) \simeq \mathrm{Sel}_R(r, \mathbf{m})$, and

(b) for any r, $\mathbf{m}$, and n such that $\{f(a)\}^{\mathsf{I}}(r, \mathbf{m}) \simeq n$, the set of subcomputations of $(f(a), r, \mathbf{m}, n)$ contains an element of the form $(f(b), p, \mathbf{m}, q)$ for each p such that $|p| < |r|$;

(4) if for some b, $a = \langle 4, k, b\rangle$, let S be the relation semi-recursive in I defined by:

$$\begin{aligned} S(n, \mathbf{m}) \leftrightarrow \forall p\, [p <_- n \rightarrow \exists q\, (|q| > 0 \wedge \{f(b)\}^{\mathsf{I}}(p, \mathbf{m}) \simeq q)] \\ \wedge \exists q\, (|q| = 0 \wedge \{f(b)\}^{\mathsf{I}}(n, \mathbf{m}) \simeq q), \end{aligned}$$

then choose $f(a)$ such that

(a) $\{f(a)\}^{\mathsf{I}}(\mathbf{m}) \simeq \mathrm{Sel}_S(\mathbf{m})$, and

(b) for any $\mathbf{m}$ and n such that $\{f(a)\}^{\mathsf{I}}(\mathbf{m}) \simeq n$, the set of subcomputations of $(f(a), \mathbf{m}, n)$ includes an element of the form $(f(b), p, \mathbf{m}, q)$ for each p such that $|p| \leqslant |n|$;

(5) if a is of none of these forms, $f(a) = 0$.

We leave to the reader the straightforward proof that f has the desired properties. The proof of the implication (←) of (i) does not use clause (b) of cases (3) and (4) but for the implication (→) these conditions are just what is needed to carry out an induction on subcomputations as in the proof of Theorem VI.2.8. □

4.3 Corollary. *For any k and any $R \subseteq {}^k\omega$,*

(i) *R is $\omega_1[\mathsf{I}]$-semi-recursive $\rightarrow R$ is semi-recursive in* I;

(ii) *R is $\omega_1[\mathsf{I}]$-recursive $\rightarrow R$ is recursive in* I.

Proof. For (i), let F and f be as in 4.1 and 4.2, respectively. Then if R is $\omega_1[\mathsf{I}]$-semi-recursive, say with $\omega_1[\mathsf{I}]$-semi-index a,

$$R(\mathbf{m}) \leftrightarrow \{f(a)\}^{\mathsf{I}}(F(m_0), \ldots, F(m_{k-1})) \text{ is defined}$$

so R is semi-recursive in I. (ii) follows immediately by Corollary VI.4.5. □

4.4 Theorem. $\omega_1[\mathsf{I}]$ *is recursively regular.*

Proof. The proof is similar to that of Theorem 3.3. Let F be any $\omega_1[\mathsf{I}]$-partial recursive function and ρ and $\boldsymbol{\mu}$ ordinals less than $\omega_1[\mathsf{I}]$ such that for all $\pi < \rho$, $F(\pi, \boldsymbol{\mu})$ is defined. We aim to show that $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) < \omega_1[\mathsf{I}]$. Let

$$A = \{n : (\exists \pi < \rho)\, F(\pi, \boldsymbol{\mu}) = |n|\}.$$

Choose r and $\mathbf{m}$, fixed elements of U^{I}, such that $|r| = \rho$ and each $|m_i| = \mu_i$, and set

$$R(n, p) \leftrightarrow F(|p|, |\mathbf{m}|) \simeq |n|.$$

It follows from Lemma 4.2 that R is semi-recursive in I. Let Sel_R be a selection function for R partial recursive in I (Corollary VI.4.2) and set

$$B = \{n : \exists p\, [p <_{-} r \wedge \forall n' (n' \neq n \rightarrow \mathrm{Sel}_R(p) \neq n')]\}.$$

Clearly $B \subseteq A$, B is co-semi-recursive in I, and

$$\sup{}^+\{|n| : n \in B\} = \sup{}^+\{|n| : n \in A\}.$$

Hence by the Boundedness Theorem VI.4.10 and Theorem VI.4.17,

$$\sup{}^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) = \sup{}^+\{|n| : n \in B\} < \kappa^{\mathsf{I}} = \omega_1[\mathsf{I}]. \quad \square$$

We consider next under what conditions the converse of Corollary 4.3 holds.

A natural requirement is suggested by Exercise 3.12. We want in some sense to require that I be $\omega_1[\mathsf{I}]$-recursive. Of course, this is not literally possible in our current formulation, but a weaker version of this condition, which asserts that I is $\omega_1[\mathsf{I}]$ computable *on* $\omega_1[\mathsf{I}]$-recursive functions, is both possible and sufficient.

We denote by $\{a\}_\kappa^\omega$ the restriction of $\{a\}_\kappa$ to arguments from ω.

4.5 Definition. For any κ, I is κ*-effective* iff there exists a κ-partial recursive function I such that for all a such that $\{a\}_\kappa^\omega \in {}^\omega\omega$,

$$\mathsf{I}(\{a\}_\kappa^\omega) = I(a).$$

A κ-index for I is also called a κ*-index* for I.

4.6 Example. For any $\kappa > \omega$, let

$$E(a) \simeq \inf_{\pi<\omega} \mathrm{sg}^+(\{a\}_\kappa(\pi)).$$

Then E is κ-partial recursive and if $\{a\}_\kappa^\omega \in {}^\omega\omega$, $E(a) \simeq \mathsf{E}(\{a\}_\kappa^\omega)$. Hence E is κ-effective for all $\kappa > \omega$.

4.7 Theorem. *There exists a primitve recursive function g such that for any d and any $\kappa > \omega$, if I is κ-effective with κ-index d, then for all a, $\mathbf{m} < \omega$,*

$$\{g(a, d)\}_\kappa(\mathbf{m}) \simeq \{a\}^{\mathsf{I}}(\mathbf{m}).$$

Proof. The proof is quite similar to that of Theorem VI.2.14. We define g via the Primitive Recursion Theorem to satisfy the following conditions:

(0) if $a = \langle 0, k, 0, \ldots\rangle$, then $g(a, d) = \langle 0, k, \ldots\rangle$;

(1) if for some $k', b, c_0, \ldots, c_{k'-1}$, $a = \langle 1, k, 0, b, c_0, \ldots, c_{k'-1}\rangle$, then $g(a, d) = \langle 1, k, g(b, d), g(c_0, d), \ldots, g(c_{k'-1}, d)\rangle$;

(2) if $a = \langle 2, k+1, 0\rangle$, then $g(a, d)$ is defined as in the corresponding case of the proof of Lemma VI.2.2;

(3) if for some b, $a = \langle 3, k, 0, b\rangle$, then $g(a, d)$ is the natural index such that

$$\{g(a, d)\}_\kappa(\mathbf{m}) \simeq 0 \cdot \sup_{\pi<\omega} \{g(b, d)\}_\kappa(\pi, \mathbf{m}) + \{d\}_\kappa(c),$$

where c is an index such that $\{c\}_\kappa(\pi) \simeq \{g(b, d)\}_\kappa(\pi, \mathbf{m})$; such a c may be computed from $\mathbf{m}$ and an index for g;

(4) if a is of none of these forms, $g(a, d) = 0$.

The proof that g has the required property also goes much as in Theorem VI.2.14 and we make only a few remarks on the differences. In case (3) of the proof that

$$\{a\}^{\mathsf{I}}(\mathbf{m}) \simeq n \rightarrow \{g(a, d)\}_\kappa(\mathbf{m}) \simeq n,$$

we have as induction hypothesis that for some β and all p,

$$\{b\}^{\mathsf{I}}(p,\mathbf{m}) \simeq \{g(b,d)\}_\kappa(p,\mathbf{m}) \simeq \beta(p).$$

Then with c as in case (3) of the definition, $\{c\}_\kappa^\omega = \beta$, so

$$\begin{aligned}\{g(a,d)\}_\kappa(\mathbf{m}) &\simeq 0 \cdot \sup_{\pi<\omega}\{g(b,d)\}_\kappa(\pi,\mathbf{m}) + \{d\}_\kappa(c)\\ &\simeq 0 + \mathsf{I}(\{c\}_\kappa^\omega)\\ &\simeq \mathsf{I}(\beta) = n.\end{aligned}$$

For the converse implication we need to define a notion of subcomputation for $\Omega_{\kappa\kappa}$ and prove an analogue of Theorem VI.2.5. This presents no problems, technical or otherwise, and cases (0)–(2) of the proof proceed as before. In case (3), if $a = \langle 3, k, 0, b\rangle$ and $\{g(a,d)\}_\kappa(\mathbf{m}) \simeq n$, we have in particular that

$$\sup_{\pi<\omega}\{g(b,d)\}_\kappa(\pi,\mathbf{m}) \text{ is defined}$$

so that for some $\beta \in {}^\omega\omega$ and all p, $\{g(b,d)\}_\kappa(p,\mathbf{m}) \simeq \beta(p)$ and $\{c\}_\kappa^\omega = \beta$. Then by the induction hypothesis, $\beta = \lambda p\,.\{b\}^{\mathsf{I}}(p,\mathbf{m})$, so

$$\begin{aligned}\{a\}^{\mathsf{I}}(\mathbf{m}) &\simeq \mathsf{I}(\lambda p\,.\{b\}^{\mathsf{I}}(p,\mathbf{m}))\\ &\simeq \mathsf{I}(\{c\}_\kappa^\omega)\\ &\simeq \{d\}_\kappa(c) \simeq n. \quad \square\end{aligned}$$

4.8 Corollary. *For any κ such that I is κ-effective and any $R \subseteq {}^k\omega$,*
(i) *R is semi-recursive in $\mathsf{I} \to R$ is κ-semi-recursive;*
(ii) *R is recursive in $\mathsf{I} \to R$ is κ-recursive.* $\square$

4.9 Corollary. *For any recursively regular κ, if I is κ-effective, then $\omega_1[\mathsf{I}] \leq \kappa$.*

Proof. Suppose that I is κ-effective and $\sigma < \omega_1[\mathsf{I}]$. Choose γ recursive in I such that $\|\gamma\| = \sigma$. By Corollary 4.8, γ is κ-recursive and thus by Lemma 3.1, $\sigma < \kappa$. $\square$

4.10 Corollary. *If I is $\omega_1[\mathsf{I}]$-effective, then for all $R \subseteq {}^k\omega$,*
(i) *R is $\omega_1[\mathsf{I}]$-semi-recursive $\leftrightarrow R$ is semi-recursive in I;*
(ii) *R is $\omega_1[\mathsf{I}]$-recursive $\leftrightarrow R$ is recursive in I.*

Proof. Immediate from Corollaries 4.3 and 4.8. $\square$

Of course, we already knew this for $\mathsf{I} = \mathsf{E}$ and to this point we have no other examples of functionals I which are $\omega_1[\mathsf{I}]$-effective. The rest of this section is devoted to finding some.

4.11 Definition. For any $d \in \omega$,

(i) $\mathrm{Om}[\mathsf{I}] = \{\omega_1[\mathsf{H}] : \mathsf{I}$ is recursive in $\mathsf{H}\}$;

(ii) $\mathrm{Ef}_d[\mathsf{I}] = \{\kappa : \kappa$ is recursively regular and I is κ-effective with index $d\}$;

(iii) I is *effective with index d* iff $\mathrm{Om}[\mathsf{I}] \subseteq \mathrm{Ef}_d[\mathsf{I}]$;

(iv) I is *effective* iff it is effective with some index.

Of course, E is effective and the conclusions of Corollary 4.10 hold for any effective I. Our main technique for finding other effective functionals is to show that if I is effective, so is its superjump I^{sJ}, which we recall is the functional such that

$$\mathsf{I}^{\mathsf{sJ}}((\langle a, \mathbf{m}\rangle) * \alpha) = \begin{cases} 0, & \text{if } \{a\}^{\mathsf{I}}(\mathbf{m}, \alpha) \text{ is defined;} \\ 1, & \text{otherwise.} \end{cases}$$

Recall also that for any set A of ordinals,

$$\mathrm{Lim}(A) = \{\kappa : \kappa \in A \wedge (\forall \sigma < \kappa)(\exists \lambda \in A) . \sigma < \lambda < \kappa\}.$$

4.12 Lemma. $\mathrm{Om}[\mathsf{I}^{\mathsf{sJ}}] \subseteq \mathrm{Lim}(\mathrm{Om}[\mathsf{I}])$.

Proof. Since I is recursive in I^{sJ}, $\mathrm{Om}[\mathsf{I}^{\mathsf{sJ}}] \subseteq \mathrm{Om}[\mathsf{I}]$. Suppose $\kappa \in \mathrm{Om}[\mathsf{I}^{\mathsf{sJ}}]$. It suffices to show that for any $\sigma < \kappa$ there exists a $\lambda \in \mathrm{Om}[\mathsf{I}]$ such that $\sigma < \lambda < \kappa$. By definition, $\kappa = \omega_1[\mathsf{H}]$ for some H such that I^{sJ} is recursive in H. Then for any $\sigma < \kappa$, there exists a γ recursive in H such that $\|\gamma\| = \sigma$. Then clearly,

$$\sigma < \omega_1[\gamma] \leqslant \omega_1[\mathsf{I}, \gamma].$$

Let G be a functional which codes I and γ, say $\mathsf{G}(\alpha) = \langle \gamma(\alpha(0)), \mathsf{I}(\alpha)\rangle$, and set

$$\lambda = \omega_1[\mathsf{I}, \gamma] = \omega_1[\mathsf{G}].$$

Then $\lambda \in \mathrm{Om}[\mathsf{I}]$ and $\sigma < \lambda$, so it remains to show that $\lambda < \kappa$.

By the techniques of Theorem VI.4.17 there exists a relation R semi-recursive in I and γ which well-orders a subset of U^{I}_{γ} in type λ. By Theorem VI.1.11, R is recursive in I^{sJ} and γ and thus recursive in H. Hence $\lambda < \omega_1[\mathsf{H}] = \kappa$. $\square$

We next need a slight generalization of Theorem 4.7:

4.13 Lemma. *There exists a primitive recursive function g such that for any d and* *any* $\kappa > \omega$, *if* I *is* κ*-effective with* κ*-index* d, *then for all* a, e, *and* $\mathbf{m} < \omega$ *such that* $\{e\}_\kappa^\omega \in {}^\omega\omega$,

$$\{g(a, d, e)\}_\kappa(\mathbf{m}) \simeq \{a\}^{\mathsf{I}}(\mathbf{m}, \{e\}_\kappa^\omega).$$

Proof. We need make only a few minor changes in the proof of Theorem 4.7. First, each case hypothesis should be changed from $a = \langle i, k, 0, \dots\rangle$ to $a = \langle i, k, 1, \dots\rangle$ to take account of the fact that $\{a\}^{\mathsf{I}}$ now has one function argument. The function g is defined exactly as before in cases (1)–(4) and in case (0) as long as $a \neq \langle 0, k, 1, 3, i, 0\rangle$ for some $i < k$. For such a we choose $g(a, d, e)$ to be an index such that

$$\{g(a, d, e)\}_\kappa(\mathbf{m}) \simeq \{e\}_\kappa(m_i) \simeq \{a\}^{\mathsf{I}}(\mathbf{m}, \{e\}_\kappa^\omega). \quad \square$$

4.14 Theorem. *For any* d, *there exists a* d' *such that*

$$\mathrm{Lim}(\mathrm{Ef}_d[\mathsf{I}]) \subseteq \mathrm{Ef}_{d'}[\mathsf{I}^{\mathrm{sJ}}].$$

Proof. Suppose $\kappa \in \mathrm{Lim}(\mathrm{Ef}_d[\mathsf{I}])$. We shall need the following two facts:

(a) $\mathrm{Ef}_d[\mathsf{I}] \cap \kappa$ is κ-recursive;

(b) for any e, *if* $\{e\}_\kappa^\omega \in {}^\omega\omega$, then there exists a $\lambda \in \mathrm{Ef}_d[\mathsf{I}] \cap \kappa$ such that $\{e\}_\lambda^\omega = \{e\}_\kappa^\omega$.

Suppose for now that (a) and (b) are true. Note that under the conditions of (b), for any λ, $\{e\}_\lambda^\omega = \{e\}_\kappa^\omega$ just in case $\mathrm{Dm}\{e\}_\lambda^\omega = \omega$. Set

$$h(e) \simeq \kappa\text{-least } \lambda[\lambda \in \mathrm{Ef}_d[\mathsf{I}] \wedge (\forall p < \omega)(\exists v < \lambda)T(e, \langle p\rangle, v)],$$

where T is as defined in § 2. By (a), h is κ-partial recursive and by (b), if $\{e\}_\kappa^\omega \in {}^\omega\omega$, then $h(e)$ is defined, $h(e) \in \mathrm{Ef}_d[\mathsf{I}]$, and $\{e\}_{h(e)}^\omega = \{e\}_\kappa^\omega$.

Now if g is the function of Lemma 4.13, we have for any e such that $\{e\}_\kappa^\omega \in {}^\omega\omega$ and all a and $\mathbf{m}$,

$$\begin{aligned}\mathsf{I}^{\mathrm{sJ}}(\langle a, \mathbf{m}\rangle) * \{e\}_\kappa^\omega) \simeq 0 &\leftrightarrow \{a\}^{\mathsf{I}}(\mathbf{m}, \{e\}_\kappa^\omega) \text{ is defined}\\ &\leftrightarrow \{a\}^{\mathsf{I}}(\mathbf{m}, \{e\}_{h(e)}^\omega) \text{ is defined}\\ &\leftrightarrow \{g(a, d, e)\}_{h(e)}(\mathbf{m}) \text{ is defined.}\end{aligned}$$

Hence if

$$F(\langle a, \mathbf{m}\rangle, e) \simeq \begin{cases} 0, & \text{if } \; h(e)\downarrow \wedge\, (\exists v < h(e))T(g(a, d, e), \langle\mathbf{m}\rangle, v);\\ 1, & \text{if } \; h(e)\downarrow \wedge \neg\,(\exists v < h(e))T(g(a, d, e), \langle\mathbf{m}\rangle, v);\\ \text{undefined}, & \text{otherwise;}\end{cases}$$

and $\{e^+\}_\kappa(m) \simeq \{e\}_\kappa(m+1)$, then F is κ-partial recursive and

$$I^{sJ}(\{e\}_\kappa^\omega) \simeq I^{sJ}((\{e\}_\kappa(0)) * \{e^+\}_\kappa^\omega) \simeq F(\{e\}_\kappa(0), e^+).$$

Thus it suffices to choose d' so that

$$\{d'\}_\kappa(e) \simeq F(\{e\}_\kappa(0), e^+),$$

as then $\kappa \in \mathrm{Ef}_{d'}[\mathsf{I}^{sJ}]$.

To establish (a), note that for $\lambda < \kappa$,

$$\lambda \in \mathrm{Ef}_d[\mathsf{I}] \leftrightarrow \lambda \text{ is recursively regular } \wedge$$
$$(\forall a < \omega)(\{a\}_\lambda^\omega \in {}^\omega\omega \rightarrow \{d\}_\lambda(a) \text{ is defined}).$$

The implication ($\rightarrow$) is immediate from the definition. Conversely, if the right-hand side holds and $\{a\}_\lambda^\omega \in {}^\omega\omega$, then $\{a\}_\lambda^\omega = \{a\}_\kappa^\omega$ and as $\{d\}_\lambda(a)$ is defined, we have

$$\{d\}_\lambda(a) \simeq \{d\}_\kappa(a) \simeq \mathsf{I}(\{a\}_\kappa^\omega) \simeq \mathsf{I}(\{a\}_\lambda^\omega).$$

Thus

$$\lambda \in \mathrm{Ef}_d[\mathsf{I}] \leftrightarrow \lambda \text{ is recursively regular } \wedge$$
$$(\forall a < \omega)((\forall p < \omega)(\exists \upsilon < \lambda)[T(a, \langle p\rangle, \upsilon) \wedge (\upsilon)_0 < \omega] \rightarrow (\exists \upsilon < \lambda)\, T(d, \langle a\rangle, \upsilon)),$$

which, with Corollary 2.19, implies (a).

For (b), suppose that $\{e\}_\kappa^\omega \in {}^\omega\omega$ and define a function G by

$$G(e, p) \simeq \kappa\text{-least } \upsilon\,.\, T(e, \langle p\rangle, \upsilon).$$

As κ is recursively regular and $\kappa > \omega$,

$$\sup_{p<\omega} G(e, p) < \kappa.$$

By the assumption that $\kappa \in \mathrm{Lim}(\mathrm{Ef}_d[\mathsf{I}])$, there exists a $\lambda \in \mathrm{Ef}_d[\mathsf{I}]$ such that $\sup_{p<\omega} G(e, p) < \lambda < \kappa$. Then $\{e\}_\lambda^\omega = \{e\}_\kappa^\omega$. □

4.15 Corollary. *If* I *is effective, then also* I^{sJ} *is effective.*

Proof. Immediate from 4.12 and 4.14. □

We recall the sequence of functionals defined following Theorem VI.6.11:

$$\mathsf{E}_0 = \mathsf{E} \quad \text{and} \quad \mathsf{E}_{r+1} = (\mathsf{E}_r)^{sJ}.$$

4.16 Corollary. *For all r and all $R \subseteq {}^k\omega$,*

(i) *R is $\omega_1[\mathsf{E}_r]$-semi-recursive $\leftrightarrow R$ is semi-recursive in E_r;*

(ii) *R is $\omega_1[\mathsf{E}_r]$-recursive $\leftrightarrow R$ is recursive in E_r.*

Proof. It is immediate by induction on r that each E_r is effective, hence $\omega_1[\mathsf{E}_r]$-effective. The result follows from Corollary 4.10. □

4.17–4.20 Exercises

4.17. Prove Lemma 4.1.

4.18. Show that if κ is recursively regular, α is κ-recursive, I is κ-effective, and $R \subseteq {}^k\omega$ is semi-recursive in I and α, then R is κ-semi-recursive. (This extends Exercise 3.15).

4.19 (Grilliot). Show that even if E is not recursive in I, $\omega_1[\mathsf{I}]$ is still recursively regular. (Use the result of Exercise VI.2.29.)

4.20. Show that if E is recursive in I and I is $\omega_1[\mathsf{I}]$-effective, then there exists an $\omega_1[\mathsf{I}]$-recursive function G such that for all $\rho < \omega_1[\mathsf{I}]$, $G(\rho) \in U^{\mathsf{I}}$ and $|G(\rho)|^{\mathsf{I}} = \rho$. (cf. Exercise 3.9)

4.21 Notes. The results of this section had the status of folklore already in the late 1960's and were probably known to Kripke and Platek even earlier, but we know of no published exposition prior to Aczel–Hinman [1974, § 2].

5. Stability

The fundamental definitions for this section are:

5.1 Definition. For any κ and λ,

(i) κ is *stable* iff κ is closed under all ∞-partial recursive functions;

(ii) κ is λ-*stable* iff κ is closed under all (∞, λ)-partial recursive functions;

(iii) κ is *weakly stable* iff κ is λ-stable for some $\lambda > \kappa$.

Note that κ is recursively regular iff κ is κ-stable. Then it is immediate from the definitions that

$$\kappa \text{ is stable} \leftrightarrow \kappa \text{ is } \lambda\text{-stable for all } \lambda \rightarrow \kappa \text{ is weakly stable} \rightarrow$$
$$\kappa \text{ is recursively regular.}$$

We shall see that weak stability is a much stronger property than recursive regularity and that stability is also much stronger than weak stability.

The main result we prove in this section is that for every uncountable cardinal ρ, there are ρ stable ordinals less than ρ, hence also ρ recursively regular ordinals less than ρ. For $\rho = \aleph_1$, the last assertion follows from Exercise 3.12. The technique of the proof is an adaptation of the downward Löwenheim–Skolem Theorem of first-order logc. Suppose λ is recursively regular, $\sigma < \lambda$, and

$$A = \{F(\boldsymbol{\mu}) : \boldsymbol{\mu} \leqslant \sigma \wedge F \text{ is } (\infty, \lambda)\text{-partial recursive}\}.$$

As λ is recursively regular, $A \subseteq \lambda$. Furthermore, as the (∞, λ)-partial recursive functions are closed under composition, A is closed under all (∞, λ)-partial recursive functions. Hence if it should happen that A is in fact an ordinal κ, κ would be λ-stable. Since the projection functions are (∞, λ)-partial recursive, $\kappa > \sigma$. Of course, if $\kappa = \lambda$, this would be of no interest, but we shall see that if λ is chosen sufficiently large (for example, if $\mathrm{Card}(\lambda) > \mathrm{Card}(\sigma)$), then A will indeed turn out to be a λ-stable ordinal $\kappa < \lambda$. If we use all ∞-partial recursive functions to form A, then κ will be stable.

To show that A is an ordinal, that is, that A is transitive, we show that it cannot be collapsed. The reader should review that part of I.1.5 which deals with the collapsing function φ_A and its properties (1)–(5), which will be used in the following proof. We write $\varphi_A(\boldsymbol{\mu})$ for $(\varphi_A(\mu_0), \ldots, \varphi_A(\mu_{k-1}))$.

5.2 Lemma. *For any λ and any $A \subseteq \lambda$ [any $A \subseteq \mathrm{Or}$], if A is closed under all (∞, λ)-partial recursive functions [all ∞-partial recursive functions], then for all $\boldsymbol{\mu}$ and ν in A and all (∞, λ)-partial recursive functions F [all ∞-partial recursive functions F],*

$$F(\boldsymbol{\mu}) \simeq \nu \rightarrow F(\varphi_A(\boldsymbol{\mu})) \simeq \varphi_A(\nu).$$

Proof. Let A be a fixed subset of λ which is closed under all (∞, λ)-partial recursive functions. To improve legibility we shall write $\bar{\boldsymbol{\mu}}$ instead of $\varphi_A(\boldsymbol{\mu})$. Then we need to prove that for all $\boldsymbol{\mu}$ and $\nu \in A$ and all a,

$$\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu \rightarrow \{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq \bar{\nu}.$$

Formally, this is a proof by induction over $\Omega_{\kappa\lambda}$, simultaneously for all $\kappa \geqslant \lambda$, that

$$(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda} \rightarrow (a, \bar{\boldsymbol{\mu}}, \bar{\nu}) \in \Omega_{\kappa\lambda},$$

but we shall state matters directly in terms of (∞, λ)-computations.

We consider cases (0)–(4). If $A = \varnothing$, the result is vacuous and we assume otherwise.

(0) If $a = \langle 0, k, 0, n\rangle$ and $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq n$, then $\{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq n = \bar{n}$ by (4) of I.1.5

($n \subseteq \omega \subseteq A$ because all constant functions with natural number values are (∞, λ)-recursive); if $a = \langle 0, k, 1, i\rangle$ and $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \mu_i$, then $\overline{\{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}})} \simeq \bar{\mu}_i$; if $a = \langle 0, k, 2, i\rangle$ and $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \mu_i + 1$, then $\{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq \bar{\mu}_i + 1 = \overline{\mu_i + 1}$ by (3) of I.1.5; if $a = \langle 0, k+4, 4\rangle$ and $\{a\}_{\infty\lambda}(\pi, \rho, \sigma, \tau, \boldsymbol{\mu}) \simeq \pi$, then $\sigma = \tau$ so $\bar{\sigma} = \bar{\tau}$ and $\{a\}_{\infty\lambda}(\bar{\pi}, \bar{\rho}, \bar{\sigma}, \bar{\tau}, \bar{\boldsymbol{\mu}}) \simeq \bar{\pi}$; if $a = \langle 0, k+4, 4\rangle$ and $\{a\}_{\infty\lambda}(\pi, \rho, \sigma, \tau, \boldsymbol{\mu}) \simeq \rho$, then $\sigma \neq \tau$, so by (1) of I.1.5, $\bar{\sigma} \neq \bar{\tau}$ and thus $\{a\}_{\infty\lambda}(\bar{\pi}, \bar{\rho}, \bar{\sigma}, \bar{\tau}, \bar{\boldsymbol{\mu}}) \simeq \bar{\rho}$; if $a = \langle 0, k+2, 5\rangle$ and $\{a\}_{\infty\lambda}(p, q, \boldsymbol{\mu}) \simeq \mathrm{Sb}_0(p, q)$, then $\{a\}_{\infty\lambda}(\bar{p}, \bar{q}, \bar{\boldsymbol{\mu}}) \simeq \{a\}_{\infty\lambda}(p, q, \bar{\boldsymbol{\mu}}) \simeq \mathrm{Sb}_0(p, q) \simeq \overline{\mathrm{Sb}_0(p, q)}$.

(1) If $a = \langle 1, k, b, c_0, \ldots, c_{k'-1}\rangle$ and $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \{b\}_{\infty\lambda}(\boldsymbol{\xi}) \simeq \nu$, where for $i < k'$, $\{c_i\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \xi_i$, then the closure property of A ensures that all $\xi_i \in A$, so by the induction hypothesis, $\{c_i\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq \bar{\xi}_i$ and $\{b\}_{\infty\lambda}(\bar{\boldsymbol{\xi}}) \simeq \bar{\nu}$. Hence

$$\{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq \{b\}_{\infty\lambda}(\bar{\boldsymbol{\xi}}) \simeq \bar{\nu}.$$

(2) If $a = \langle 2, k+1\rangle$ and $\{a\}_{\infty\lambda}(b, \boldsymbol{\mu}) \simeq \{b\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu$, then $\{a\}_{\infty\lambda}(\bar{b}, \bar{\boldsymbol{\mu}}) \simeq \{b\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}) \simeq \bar{\nu}$ by the induction hypothesis and the fact that $\bar{b} = b$.

(3) If $a = \langle 3, k+1, b\rangle$ and $\{a\}_{\infty\lambda}(\rho, \boldsymbol{\mu}) \simeq \sup^+_{\pi<\rho}\{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu}) \simeq \nu$, then

$$\begin{aligned}
\{a\}_{\infty\lambda}(\bar{\rho}, \bar{\boldsymbol{\mu}}) &\simeq \sup\nolimits^+_{\sigma<\bar{\rho}} \{b\}_{\infty\lambda}(\sigma, \bar{\boldsymbol{\mu}}) \\
&\simeq \sup\nolimits^+_{\pi<\rho \,\wedge\, \pi\in A}\{b\}_{\infty\lambda}(\bar{\pi}, \bar{\boldsymbol{\mu}}) \\
&\simeq \sup\nolimits^+_{\pi<\rho \,\wedge\, \pi\in A}\overline{\{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu})}.
\end{aligned}$$

This second equality follows from (2) of I.1.5 and the third from the induction hypothesis. Let this value be ζ; we aim to show that $\zeta = \bar{\nu}$.

Since for $\pi < \rho$ and $\pi \in A$, $\{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu}) \in A$ and $\{b\}_{\infty\lambda}(\rho, \boldsymbol{\mu}) < \nu$, also $\overline{\{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu})} < \bar{\nu}$. Hence $\zeta \leqslant \bar{\nu}$. Suppose for a contradiction that $\zeta < \bar{\nu}$. Then by (2) of I.1.5, $\zeta = \bar{\tau}_0$ for some τ_0 such that $\tau_0 < \nu$ and $\tau_0 \in A$. Since $\tau_0 < \nu$, there exists a $\pi < \rho$ such that $\tau_0 \leqslant \{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu})$. Let

$$h(\tau, \boldsymbol{\mu}) \simeq \lambda\text{-“least” } \pi[\tau \leqslant \{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu})].$$

h is (∞, λ)-partial recursive and as $\lambda > \rho$, $h(\tau_0, \boldsymbol{\mu})$ is defined, say with value π_0. Since τ_0 and $\boldsymbol{\mu}$ are elements of A, also $\pi_0 \in A$ by the closure property of A. But

$$\tau_0 \leqslant \{b\}_{\infty\lambda}(\pi_0, \boldsymbol{\mu})$$

so

$$\bar{\tau}_0 \leqslant \overline{\{b\}_{\infty\lambda}(\pi_0, \boldsymbol{\mu})} < \zeta,$$

a contradiction.

(4) Suppose $a = \langle 4, k, b\rangle$ and

$$\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq (\lambda\text{-“least”}\ \pi . \{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu}) \simeq 0) \simeq \nu.$$

For any $\sigma < \bar{\nu}$ there exists by (2) of I.1.5 a $\pi < \nu$ such that $\pi \in A$ and $\sigma = \bar{\pi}$. Then by the induction hypothesis,

$$\begin{aligned} \{b\}_{\infty\lambda}(\sigma, \bar{\boldsymbol{\mu}}) &\simeq \{b\}_{\infty\lambda}(\bar{\pi}, \bar{\boldsymbol{\mu}}) \\ &\simeq \overline{\{b\}_{\infty\lambda}(\pi, \boldsymbol{\mu})} > 0. \end{aligned}$$

On the other hand,

$$\{b\}_{\infty\lambda}(\bar{\nu}, \bar{\boldsymbol{\mu}}) \simeq \overline{\{b\}_{\infty\lambda}(\nu, \boldsymbol{\mu})} \simeq \bar{0} = 0.$$

Hence

$$\bar{\nu} \simeq \lambda\text{-“least”}\ \sigma[\{b\}_{\infty\lambda}(\sigma, \bar{\boldsymbol{\mu}}) \simeq 0] \simeq \{a\}_{\infty\lambda}(\bar{\boldsymbol{\mu}}).$$

The proof of the alternate version in brackets is exactly the same with all references to λ removed. □

5.3 Theorem. *For any* $\sigma \geqslant \omega$,

(i) *for any recursively regular* $\lambda > \sigma$, *there exists an ordinal* κ *such that* $\sigma < \kappa \leqslant \lambda$, $\mathrm{Card}(\kappa) = \mathrm{Card}(\sigma)$, *and* κ *is* λ*-stable*;

(ii) *there exists an ordinal* κ *such that* $\sigma < \kappa$, $\mathrm{Card}(\kappa) = \mathrm{Card}(\sigma)$, *and* κ *is stable.*

Proof. For (i), suppose λ is recursively regular and $\sigma < \lambda$, and set

$$A = \{F(\boldsymbol{\mu}) : \boldsymbol{\mu} \leqslant \sigma \wedge F \text{ is } (\infty, \lambda)\text{-partial recursive}\}.$$

As we noted above, A is closed under all (∞, λ)-partial recursive functions. Because λ is recursively regular, $A \subseteq \lambda$. Since the projection functions are (∞, λ)-recursive, $\sigma \subseteq A$ so by (4) of I.1.5, $\varphi_A(\boldsymbol{\mu}) = \boldsymbol{\mu}$ for all $\boldsymbol{\mu} \leqslant \sigma$.

For any $\nu \in A$ there exists $\boldsymbol{\mu} \leqslant \sigma$ and an (∞, λ)-partial recursive function F such that $F(\boldsymbol{\mu}) \simeq \nu$. By Lemma 5.2,

$$\varphi_A(\nu) \simeq F(\varphi_A(\boldsymbol{\mu})) \simeq F(\boldsymbol{\mu}) \simeq \nu.$$

It follows from (5) of I.1.5 that A is an ordinal $\kappa \leqslant \lambda$.

Clearly κ is λ-stable. Since there are only countably many (∞, λ)-partial recursive functions,

$$\mathrm{Card}(\kappa) = \mathrm{Card}(A) = \mathrm{Card}(\sigma) \cdot \aleph_0 = \mathrm{Card}(\sigma).$$

The proof of (ii) is obtained similarly from the bracketed portion of Lemma 5.2. □

5.4 Corollary. *For any cardinal $\rho > \omega$,*

$$\{\kappa : \kappa < \rho \wedge \kappa \text{ is stable}\} \quad \textit{and} \quad \{\kappa : \kappa < \rho \wedge \kappa \text{ is recursively regular}\}$$

both have cardinality ρ.

Proof. If ρ is an uncountable cardinal and $\sigma < \rho$, then by Theorem 5.3 there exists a stable κ such that $\mathrm{Card}(\kappa) = \mathrm{Card}(\sigma)$ and $\sigma < \kappa$, hence $\sigma < \kappa < \rho$. Thus the set of stable κ is cofinal in ρ. If ρ is regular, it follows that this set has cardinality ρ. The result for singular ρ is then immediate. Since a stable ordinal is recursively regular, there are also ρ recursively regular ordinals less than ρ. □

5.5 Lemma. *For any λ and ρ, if ρ is a limit of λ-stable ordinals [of stable ordinals], then ρ is λ-stable [stable].*

Proof. Suppose ρ is a limit of λ-stable ordinals. For any $\boldsymbol{\mu} < \rho$ there exists a κ such that $\boldsymbol{\mu} < \kappa < \rho$ and κ is λ-stable. Then for any (∞, λ)-partial recursive function F, $F(\boldsymbol{\mu}) < \kappa < \rho$. Hence ρ is λ-stable. The proof for stability is identical. □

5.6 Corollary. *Every uncountable cardinal is stable and therefore recursively regular.*

Proof. It follows from Corollary 5.4 (or directly from 5.3) that any uncountable cardinal is a limit of stable ordinals, hence is stable by Lemma 5.5. □

We conclude this section by showing that the least stable ordinal is δ_2^1, the least ordinal not the order-type of a Δ_2^1 wellordering of ω. First we prove a generalization of Theorem 2.2.

5.7 Theorem. *For all κ and all $\lambda \geqslant \kappa$, the following are equivalent:*

(i) *κ is λ-stable;*

(ii) *for all $a \in \omega$, all $\boldsymbol{\mu} < \kappa$, and all ν,*

$$\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu;$$

and if λ is recursively regular, also

(iii) *for all $a \in \omega$, all $\boldsymbol{\mu} < \kappa$ and all ν,*

$$\{a\}_\lambda(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu.$$

Proof. For recursively regular λ, the equivalence of (ii) and (iii) follows immediately from Theorem 2.2. From (ii) we may conclude that any (∞, λ)-partial recursive function applied to arguments from κ has the same value as some κ-partial recursive function, hence has value less than κ. Thus κ is λ-stable.

Suppose now that κ is λ-stable. Then an argument very similar to that used for the implication ((b)$\to$(a)) of Theorem 2.2 will show that for all τ, a, $\boldsymbol{\mu}$, and ν,

$$\boldsymbol{\mu} < \kappa \wedge (a, \boldsymbol{\mu}, \nu) \in \Omega_{\tau\lambda} \to (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}.$$

Cases (0)–(2) are done in exactly the same way. For case (3) we use the fact that κ is recursively regular. For case (4), if $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\tau\lambda}$, then $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu$ and because κ is closed under (∞, λ)-partial recursive functions, $\nu < \kappa$ and the proof proceeds as before. □

5.8 Theorem. *For all κ, the following are equivalent:*
(i) *κ is stable;*
(ii) *for all $a \in \omega$, all $\boldsymbol{\mu} < \kappa$, and all ν,*

$$\{a\}_\infty(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu.$$

Proof. As for Theorem 5.7. □

5.9 Theorem. *δ_2^1 is the least stable ordinal.*

Proof. We show first that no $\kappa < \delta_2^1$ is stable. For any such κ there exists a $\gamma \in \mathsf{W} \cap \Delta_2^1$ such that $\|\gamma\| = \kappa$. By Theorem 3.7, γ is $\aleph_1$-recursive. If κ were stable, hence $\aleph_1$-stable, then by the preceding theorem, γ would be κ-recursive. This contradicts Lemma 3.1.

To prove that δ_2^1 is stable, suppose $\boldsymbol{\mu} < \delta_2^1$ and $\{a\}_\infty(\boldsymbol{\mu}) \simeq (\nu)$. It suffices to show that $\nu < \delta_2^1$. Since $\aleph_1$ is stable by Corollary 5.6, also $\{a\}_{\aleph_1}(\boldsymbol{\mu}) \simeq \nu$. Let $\boldsymbol{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{k-1})$ be a sequence of elements of $\mathsf{W} \cap \Delta_2^1$ such that for $i < k$, $\|\varepsilon_i\| = \mu_i$. Then

$$\exists\gamma\, \exists\mathbf{m}\, \exists n\, [|\mathbf{m}|_\gamma = \|\boldsymbol{\varepsilon}\| \wedge O(\langle a, \mathbf{m}, n\rangle, \gamma)].$$

By Lemma 3.5, the expression in brackets defines a Δ_2^1 relation. Then by the Basis Theorem (IV.7.9).

$$(\exists\gamma \in \Delta_2^1)\, \exists\mathbf{m}\, \exists n\, [|\mathbf{m}|_\gamma = \|\boldsymbol{\varepsilon}\| \wedge O(\langle a, \mathbf{m}, n\rangle, \gamma)].$$

But then $\nu = |n|_\gamma$ for some $\gamma \in \mathsf{W} \cap \Delta_2^1$ so $\nu < \|\gamma\| < \delta_2^1$. □

5.10 Corollary. *For any $R \subseteq {}^k\omega$, the following are equivalent:*

(i) $R \in \Delta_2^1$ (Σ_2^1);

(ii) *R is δ_2^1-(semi-) recursive;*

(iii) *R is κ-(semi-) recursive for any $\kappa \geqslant \delta_2^1$;*

(iv) *R is ∞-(semi-) recursive.*

Proof. Immediate from Theorem 3.7 and the preceding two results. □

It follows, of course, that there is no analogue of Theorem 3.7 for higher levels of the analytical hierarchy. The equivalence of (i) and (ii) provides another hierarchy of length δ_2^1 for the Δ_2^1 relations on numbers (cf. V.6.8–9).

5.11 Corollary. *For any ∞-recursive set A of ordinals, and any stable ordinal κ,*

(i) *if $A \neq \varnothing$, then $A \cap \delta_2^1 \neq \varnothing$;*

(ii) *if $A \not\subseteq \kappa$, then $A \cap \kappa$ is cofinal in κ;*

(iii) *if κ is a cardinal and $A \not\subseteq \kappa$, then* $\mathrm{Card}(A \cap \kappa) = \kappa$.

Proof. Let A be ∞-recursive and set

$$F(\rho) \simeq \text{least } \mu . \mu \in A.$$

F is ∞-partial recursive and total if $A \neq \varnothing$. Since $0 < \delta_2^1$, $F(0) < \delta_2^1$ and $F(0) \in A$. Similarly, set

$$G(\rho, \sigma) \simeq \text{least } \mu[\mu \in A \wedge \mu > \sigma].$$

If $A \not\subseteq \kappa$, then for any $\sigma < \kappa$, $G(\rho, \sigma)$ is defined for all ρ. Hence $\sigma < G(0, \sigma) < \kappa$ and $G(0, \sigma) \in A$. (iii) is immediate from (ii). □

5.12–5.24 Exercises

5.12. Show that if $\kappa < \lambda$ and λ is stable, then κ is stable iff κ is λ-stable.

5.13. Show that ω_1 is not weakly stable.

5.14. Do there exist 0-stable ordinals κ such that $\omega < \kappa < \omega_1$?

5.15. Show that the least weakly stable ordinal is less than the least stable ordinal.

5.16. Suppose that both κ and $\lambda > \kappa$ are recursively regular. Show that κ is λ-stable iff for all λ-recursive relations R and all $\mu < \kappa$, $(\exists \pi < \lambda) R(\pi, \mu) \leftrightarrow (\exists \pi < \kappa) R(\pi, \mu)$.

5.17. Show that if κ is a limit of stable ordinals, then for any $R \subseteq {}^k\omega$, if R is κ-semi-recursive in parameters, then R is in fact κ-recursive in parameters.

5.18. Show that for any $A \subseteq \omega$, $\delta_2^1[A]$ is stable.

5.19. Show that for all $r \geqslant 3$, δ_r^1 is a limit of stable ordinals, hence is stable.

5.20. Show that for all $r \geqslant 1$ and all $R \subseteq {}^k\omega$, if R is δ_r^1-recursive in parameters, then $R \in \Delta_r^1$.

5.21. Prove the following extension of Corollary 5.10: for any $R \subseteq {}^k\omega$, R is ∞-(semi-) recursive in parameters iff R is $\delta_2^1[R]$-(semi-) recursive in parameters. (Suppose R is ∞-recursive in the parameters $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_{n-1})$. Apply Theorem 5.2 to the set $A = \{F(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \omega \cup \{\rho_0, \ldots, \rho_{n-1}\} \wedge F$ is ∞-partial recursive$\}$ to show that R is $\aleph_1$-recursive in parameters. Then apply a relativized version of the technique used in the proof of Theorem 5.9.)

5.22. Show that for any $A \subseteq \omega$, if A is ∞-recursive in parameters, then for all $R \subseteq {}^k\omega$,

(i) R is $\delta_2^1[A]$-semi-recursive in parameters $\leftrightarrow R \in \Sigma_2^1[A]$;

(ii) R is $\delta_2^1[A]$-recursive in parameters $\leftrightarrow R \in \Delta_2^1[A]$.

(Use the preceding exercise and Exercise 3.18.)

5.23. Suppose that every $R \subseteq {}^k\omega$ is ∞-recursive in parameters (this follows from the hypothesis V = L). Show that for all $r \geqslant 1$ and all $R \subseteq {}^k\omega$,

$$R \text{ is } \delta_r^1\text{-recursive in parameters} \leftrightarrow R \in \Delta_r^1.$$

5.24. Let Γ be an arithmetical inductive operator (not necessarily monotone) over ω. Show that $|\Gamma|$ is less than or equal to the least weakly stable ordinal.

5.25 Notes. The notion of stability is implicit already in Takeuti [1960] — he shows that uncountable cardinals are stable. Most of the other results are due to Kripke [1964] and Platek [1966].

6. *Recursively Large Ordinals*

The recursively regular ordinals are, of course, analogous to the regular cardinals of set theory. Furthermore, we know from Corollary 5.4 that every uncountable cardinal ρ is the ρ-th recursively regular ordinal. In particular, the recursively regular ordinals form a proper class. Let τ_ρ denote the ρ-th recursively regular ordinal — that is,

$$\tau_\rho = \text{least } \sigma[\sigma \text{ is recursively regular} \wedge (\forall \pi < \rho)\sigma > \tau_\pi].$$

Then $\tau_0 = \omega$, $\tau_1 = \omega_1$ and for all π and ρ, $\rho \leqslant \tau_\rho$ and $\pi < \rho \to \tau_\pi < \tau_\rho$. The fact mentioned above may now be stated: for any uncountable cardinal ρ, $\tau_\rho = \rho$. In set theory a fixed point in the enumeration of the regular cardinals is called (weakly) inaccessible. Accordingly, we say

6.1 Definition. For any κ, κ is *recursively inaccessible* iff $\kappa = \tau_\kappa$.

Thus every uncountable cardinal is recursively inaccessible, but these are by no means the only recursively inaccessible ordinals.

6.2 Lemma. *The function F such that $F(\rho) \simeq \tau_\rho$ is ∞-recursive. For any κ, the partial function F_κ such that $F_\kappa(\rho) \simeq \tau_\rho$ iff $\tau_\rho < \kappa$ is (∞, κ)-partial recursive and κ-recursive if κ is recursively regular.*

Proof. This follows from Theorem 1.9, Corollary 2.19, and the fact that the sequence τ_ρ is defined by the following primitive recursion:

$$\tau_0 = \omega;$$

$$\tau_{\rho+1} = \text{least } \sigma[\sigma \text{ is recursively regular} \wedge \sigma > \tau_\rho];$$

$$\tau_\rho = \text{least } \sigma[\sigma \text{ is recursively regular} \wedge \sigma \geqslant \text{sup}^+_{\pi<\rho}\tau_\pi],$$

$$\text{if } \rho \text{ is a limit ordinal.} \quad \square$$

6.3 Corollary. *For any κ, if κ is stable or even weakly stable, then κ is recursively inaccessible.*

Proof. Suppose first that κ is stable. Since the function F of the Lemma is ∞-recursive, κ is closed under F — that is, $\rho < \kappa \to \tau_\rho < \kappa$. Since κ is itself recursively regular, $\tau_\kappa \leqslant \kappa$. But always $\kappa \leqslant \tau_\kappa$ so in fact $\kappa = \tau_\kappa$.

If κ is λ-stable for some $\lambda > \kappa$, then κ is closed under F_λ. Hence, for all $\rho < \kappa$, if $F_\lambda(\rho)$ is defined (i.e., if $\tau_\rho < \lambda$), then $\tau_\rho < \kappa$. As in the previous case it suffices to show that $\rho < \kappa \to \tau_\rho < \kappa$. Suppose otherwise and let ρ_0 be the least $\rho < \kappa$ such that $\tau_{\rho_0} \geqslant \kappa$. Then as κ is itself recursively regular, $\tau_{\rho_0} = \kappa < \lambda$. But then $\tau_{\rho_0} < \kappa$, a contradiction. $\square$

6.4 Corollary. *The least recursively inaccessible ordinal is less than the least stable ordinal.*

Proof. This is immediate from Corollary 6.3 and Exercise 5.15, but we also give a direct proof. Let τ^1_ρ denote the ρ-th recursively inaccessible ordinal:

$$\tau^1_\rho = \text{least } \sigma[\tau_\sigma = \sigma \wedge (\forall \pi < \rho)\sigma > \tau^1_\pi].$$

By a proof similar to that of Lemma 6.2, the function F^1 such that $F^1(\rho) = \tau^1_\rho$ is ∞-recursive. Hence δ^1_2 is closed under F^1 so in particular, $F^1(0) < \delta^1_2$. □

The proof of this corollary suggests that these ideas may be carried much further. Again by analogy with set-theoretic terminology we call ordinals κ such that $\kappa = \tau^1_\kappa$ *recursively hyper-inaccessible*. Then every (weakly) stable κ is also recursively hyper-inaccessible and the least such ordinal is less than δ^1_2. If τ^2_ρ enumerates the recursively hyper-inaccessible ordinals, then an ordinal κ such that $\kappa = \tau^2_\kappa$ is recursively *hyper-hyper-inaccessible*, etc. (cf. Exercise 6.23).

Clearly the first recursively regular ordinals $\tau_0, \tau_1, \ldots, \tau_\omega, \ldots, \tau_{\tau_\omega}, \ldots$ are all less than the least recursively inaccessible. The next two results give characterizations of this ordinal.

6.5 Theorem. *For all κ, κ is recursively inaccessible iff κ is recursively regular and a limit of recursively regular ordinals.*

Proof. Suppose first that κ is recursively inaccessible. Thus $\kappa = \tau_\kappa$ so κ is recursively regular. Furthermore, for any $\sigma < \kappa$, $\sigma \leqslant \tau_\sigma < \tau_\kappa = \kappa$, so κ is also a limit of recursively regular ordinals.

Conversely, suppose κ is recursively regular and a limit of recursively regular ordinals. Suppose further that κ is not recursively inaccessible so that $\kappa < \tau_\kappa$. The sequence τ_ρ is increasing so $\kappa = \tau_\rho$ for some $\rho < \kappa$. Then because κ is a limit of the recursively regular ordinals less than κ, $\kappa = \sup^+_{\pi<\rho} \tau_\pi$. Since the sequence τ_π for $\pi < \rho$ is κ-partial recursive, this contradicts the recursive regularity of κ. □

6.6 Theorem. *The least recursively inaccessible ordinal is $\omega_1[\mathsf{E}_1]$.*

Proof. By Lemma 4.12, Exercise VI.1.20, and the preceding theorem,

$$\begin{aligned}\omega_1[\mathsf{E}_1] \in \mathrm{Om}[\mathsf{E}_1] &\subseteq \mathrm{Lim}(\mathrm{Om}[\mathsf{E}]) \subseteq \mathrm{Lim}\{\kappa : \kappa \text{ is recursively regular}\}\\ &\subseteq \{\kappa : \kappa \text{ is recursively inaccessible}\}.\end{aligned}$$

Hence $\omega_1[\mathsf{E}_1]$ is recursively inaccessible. To see that it is the least such, let κ be any recursively inaccessible ordinal. By Example 4.6 there is an index d such that for every $\lambda > \omega$, E is λ-effective with index d. Hence by Theorem 4:14 and the preceding theorem, there is an index d' such that

$$\kappa \in \mathrm{Lim}\{\lambda : \lambda \text{ is recursively regular}\} \subseteq \mathrm{Lim}(\mathrm{Ef}_d[\mathsf{E}]) \subseteq \mathrm{Ef}_{d'}[\mathsf{E}_1].$$

In particular, E_1 is κ-effective so that by Corollary 4.9, $\omega_1[\mathsf{E}_1] \leqslant \kappa$. □

Next in the galaxy of "large cardinals" of set theory are the Mahlo cardinals; these also have their "recursive analogue".

6.7 Definition. For any κ, κ is *recursively Mahlo* iff for any $F : \kappa \to \kappa$ which is κ-recursive in parameters, there exists a recursively regular ordinal $\lambda < \kappa$ which is closed under F — that is, $(\forall \pi < \lambda) F(\pi) < \lambda$.

Compare Definition 6.7 with Exercise 2.23. Analogously as in set theory, we have

6.8 Lemma. *For any κ, if κ is recursively Mahlo, then κ is recursively regular, recursively inaccessible, recursively hyper-inaccessible, etc.*

Proof. Suppose that κ is recursively Mahlo. Let F be κ-partial recursive and let $\rho, \boldsymbol{\mu} < \kappa$ be such that $F(\pi, \boldsymbol{\mu})$ is defined for all $\pi < \rho$. Let G be the function κ-partial recursive in parameters defined by:

$$G(0) \simeq \rho, \quad \text{and for} \quad \pi > 0,$$

$$G(\pi) \simeq \begin{cases} F(\pi, \boldsymbol{\mu}), & \text{if} \quad \pi < \rho; \\ 0, & \text{otherwise.} \end{cases}$$

There exists a recursively regular $\lambda < \kappa$ which is closed under G. In particular, $\rho = G(0) < \lambda$ so also

$$\sup^+_{\pi < \rho} F(\pi, \boldsymbol{\mu}) = \sup^+_{\pi < \rho} G(\pi) < \lambda < \kappa.$$

Thus κ is recursively regular.

To show that κ is recursively inaccessible, it suffices by Theorem 6.5 to show that κ is a limit of recursively regular ordinals. For any $\sigma < \kappa$, set $H(\pi) = \sigma$ for all $\pi < \kappa$. Then H is κ-recursive in parameters so there exists a recursively regular ordinal $\lambda < \kappa$ which is closed under H. In particular, $\sigma = H(0) < \lambda$.

We leave to the reader (Exercise 6.24) the proof that κ is recursively hyper-inaccessible, etc. $\square$

6.9 Theorem. *For any κ, if κ is stable or even weakly stable, then κ is recursively Mahlo.*

Proof. Suppose first that κ is stable and F is κ-recursive in parameters, say $F(\pi) \simeq \{a\}_\kappa(\pi, \boldsymbol{\mu})$ with $\boldsymbol{\mu} < \kappa$. Since κ is recursively regular and $F(\pi)$ is defined for all $\pi < \kappa$, we have by Theorem 2.6

$$(\forall \pi < \kappa)(\exists \upsilon < \kappa)\, T(a, \langle \pi, \boldsymbol{\mu} \rangle, \upsilon).$$

Let G be the function defined by, for any $\boldsymbol{\rho}$,

$$G(\boldsymbol{\rho}) \simeq \text{least } \lambda[\lambda \text{ is recursively regular } \wedge$$
$$(\forall \pi < \lambda)(\exists \upsilon < \lambda)\, T(a, \langle \pi, \boldsymbol{\rho} \rangle, \upsilon)].$$

Clearly G is ∞-recursive and $G(\boldsymbol{\mu})$ is defined. Since κ is stable and $\boldsymbol{\mu} < \kappa$, also $G(\boldsymbol{\mu}) < \kappa$. Then $G(\boldsymbol{\mu})$ is a recursively regular ordinal less than κ which is closed under F.

The argument in case κ is only weakly stable is nearly identical. $\square$

The least recursively Mahlo ordinal is $\omega_1[\mathsf{sJ}]$, but the proof of this fact is beyond the scope of this book (Harrington [1974]). It is not difficult to verify directly that this ordinal is less than δ_2^1; this also follows from Corollaries 6.12 and 6.19 below or from Corollary VII.1.9.

We next consider the notion of projectibility. Recursive regularity is an effective analogue of the set-theoretic property of being a regular cardinal. Nonprojectibility is in a sense an effective analogue of the property of being a cardinal, but the logical relationship between the notions is lost in the translation.

6.10 Definition. For any κ and λ,

(i) κ is *projectible to* λ iff there exists a one–one function F κ-recursive in parameters such that $\operatorname{Im} F \subseteq \lambda$;

(ii) κ^*, the *projectum* of κ, is the least λ such that κ is projectible to λ;

(iii) κ is *projectible* iff $\kappa^* < \kappa$; otherwise κ is *nonprojectible.*

We have already encountered numerous examples of ordinals which are projectible. In Exercise 3.9 is constructed an ω_1-recursive function G such that for all $\rho < \omega_1$,

$$G(\rho) \in W \quad \text{and} \quad \|G(\rho)\| = \rho.$$

Clearly G is one–one and thus it establishes that ω_1 is projectible to ω. Hence $\omega_1^* = \omega$. Similarly, the $\omega_1[\mathsf{I}]$-recursive function G of Exercise 4.20 such that for all $\rho < \omega_1[\mathsf{I}]$,

$$G(\rho) \in U^{\mathsf{I}} \quad \text{and} \quad |G(\rho)|^{\mathsf{I}} = \rho$$

shows that when E is recursive in I and I is $\omega_1[\mathsf{I}]$-effective, $\omega_1[\mathsf{I}]$ is projectible to ω. In particular, for all n, $\omega_1[\mathsf{E}_n]^* = \omega$. We could also show by similar means from the results at the end of § 3 that δ_2^1 is projectible to ω, but we shall have a simpler proof of this fact below (Corollary 6.15).

Of course, every infinite cardinal is nonprojectible, but there are many more nonprojectible ordinals.

6.11 Lemma. $\{\kappa : \kappa$ is recursively regular and nonprojectible$\}$ *is* $(\infty, 0)$-*recursive.*

Proof. By Corollary 2.19, $\{\kappa : \kappa$ is recursively regular$\}$ is $(\infty, 0)$-recursive. Let

$$R(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle) \leftrightarrow \boldsymbol{\mu} < \kappa \wedge (\forall \rho < \kappa)(\exists \upsilon < \kappa)[T(a, \langle \rho, \boldsymbol{\mu} \rangle, \upsilon) \wedge (\upsilon)_0 < \lambda]$$

and

$$S(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle) \leftrightarrow (\forall \rho \rho' \upsilon \upsilon' < \kappa)[T(a, \langle \rho, \boldsymbol{\mu} \rangle, \upsilon) \wedge \\ T(a, \langle \rho', \boldsymbol{\mu} \rangle, \upsilon') \wedge (\upsilon)_0 = (\upsilon')_0 \rightarrow \rho = \rho'].$$

$R(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle)$ holds whenever the index a defines a function F κ-recursive in parameters $\boldsymbol{\mu} < \kappa$ such that $\operatorname{Im} F \subseteq \lambda$. $S(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle)$ holds in addition just in case this function is 1–1. Hence if κ is recursively regular,

$$\kappa \text{ is projectible} \leftrightarrow (\exists a < \omega)(\exists \lambda < \kappa)(\exists \boldsymbol{\mu} < \kappa) \\ [R(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle) \wedge S(a, \kappa, \lambda, \langle \boldsymbol{\mu} \rangle)].$$

Since R and S are obviously $(\infty, 0)$-recursive, this proves the Lemma. □

6.12 Corollary. *The least recursively regular nonprojectible ordinal is less than* δ^1_2. *There are* κ *recursively regular nonprojectible ordinals less than every uncountable cardinal* κ.

Proof. By Corollary 5.11. □

Although it is by no means obvious from the definitions, the least recursively regular nonprojectible ordinal is a quite large ordinal. We show this by use of techniques from § 5. For any σ and λ, let

$$\operatorname{st}_\lambda(\sigma) = \{F(\boldsymbol{\mu}) : \boldsymbol{\mu} \leqslant \sigma \wedge F \text{ is } (\infty, \lambda)\text{-partial recursive}\}$$

and

$$\operatorname{st}(\sigma) = \{F(\boldsymbol{\mu}) : \boldsymbol{\mu} \leqslant \sigma \wedge F \text{ is } \infty\text{-partial recursive}\}.$$

In the proof of Theorem 5.3, we essentially showed:

6.13 Lemma. *For any* $\sigma \geqslant \omega$ *and any recursively regular* λ,

(i) $\mathrm{st}_\lambda(\sigma)$ *is the least* λ*-stable ordinal greater than* σ; $\mathrm{st}_\lambda(\sigma) \leq \lambda$ *and* $\mathrm{Card}(\sigma) = \mathrm{Card}(\mathrm{st}_\lambda(\sigma))$;

(ii) $\mathrm{st}(\sigma)$ *is the least stable ordinal greater than* σ; $\mathrm{Card}(\sigma) = \mathrm{Card}(\mathrm{st}(\sigma))$. $\square$

For example, $\mathrm{st}(\omega) = \delta_2^1$, the least stable ordinal. If κ is recursively regular, $\sigma < \kappa$, and there is no recursively regular λ such that $\sigma < \lambda < \kappa$, then $\mathrm{st}_\kappa(\sigma) = \kappa$. In particular, $\mathrm{st}_{\omega_1}(\omega) = \omega_1$.

We shall call an ordinal σ *sequence closed* iff for all $\boldsymbol{\mu} < \sigma$, also $\langle \boldsymbol{\mu} \rangle < \sigma$. For the most part we shall use only the obvious fact that if σ is either recursively regular or a limit of recursively regular ordinals, then σ is sequence closed.

6.14 Theorem. *For any recursively regular* λ *and any sequence closed ordinal* σ *such that* $\omega \leq \sigma < \lambda$, $\mathrm{st}_\lambda(\sigma)$ *and* $\mathrm{st}(\sigma)$ *are both projectible to* σ.

Proof. Let λ and σ be as stated and set $\kappa = \mathrm{st}_\lambda(\sigma)$. Since κ is λ-stable, for all $\boldsymbol{\mu} \leq \sigma$,

$$\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu \leftrightarrow \{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu.$$

Furthermore, if $\{a\}_{\infty\lambda}(\boldsymbol{\mu}) \simeq \nu$ with $\boldsymbol{\mu} \leq \sigma$, it is not hard to see that there exist a' and $\boldsymbol{\mu}'$ such that $\boldsymbol{\mu}' < \sigma$ and $\{a'\}_{\infty\lambda}(\boldsymbol{\mu}', \sigma) \simeq \nu$. Hence

$$\kappa = \{F(\boldsymbol{\mu}, \sigma) : \boldsymbol{\mu} < \sigma \wedge F \text{ is } \kappa\text{-partial recursive}\}.$$

By Theorem 2.6, for all ρ,

$$(*) \qquad \rho < \kappa \rightarrow (\exists a < \omega)(\exists \boldsymbol{\mu} < \sigma)(\exists \upsilon < \kappa)[T(a, \langle \boldsymbol{\mu}, \sigma \rangle, \upsilon) \wedge (\upsilon)_0 = \rho].$$

Let $G(\rho)$ be the least $((a, \boldsymbol{\mu}), \upsilon)$ for which the inside part holds — that is,

$$G(\rho) \simeq (\kappa\text{-least } \pi[(\pi)_0 < \sigma \wedge T((\pi)_{0,0}, (\pi)_{0,1} * \langle \sigma \rangle, (\pi)_1) \wedge (\pi)_{1,0} = \rho])_0.$$

Then G is κ-partial recursive in the parameter σ and by the definition of G, $\mathrm{Im}\, G \subseteq \sigma$. By $(*)$ and the assumption that σ is sequence closed, G is total. Furthermore, if $G(\rho) \simeq \nu$, then there exist π, a, $\boldsymbol{\mu}$, and ν such that $\pi = \langle \nu, \upsilon \rangle = \langle\langle a, \langle \boldsymbol{\mu} \rangle\rangle, \upsilon \rangle$ and $\{a\}_\kappa(\boldsymbol{\mu}, \sigma) \simeq \rho$. It follows that G is 1–1 and is thus a projection of κ to σ.

The proof for $\mathrm{st}(\sigma)$ is nearly identical. $\square$

6.15 Corollary. δ_2^1 *is projectible to* ω. *For any* κ, *the least stable or recursively regular ordinal greater than* κ *is projectible to* κ.

Proof. We noted above that $\delta_2^1 = \mathrm{st}(\omega)$ and ω is sequence closed. Since a limit of sequence closed ordinals is sequence closed, for any κ there exists a largest sequence closed ordinal $\sigma \leqslant \kappa$. Then the least stable ordinal greater than κ is $\mathrm{st}(\sigma)$ and if λ is the least recursively regular ordinal greater than κ, then $\lambda = \mathrm{st}_\lambda(\kappa)$. By Theorem 6.14, $\mathrm{st}(\sigma)$ and $\mathrm{st}_\lambda(\sigma)$ are projectible to σ, hence also to κ. □

This result yields yet another construction of a Σ_2^1 well–ordering R of order-type δ_2^1 — namely, if F is a δ_2^1-recursive function which projects δ_2^1 to ω, let

$$R(m, n) \leftrightarrow m, n \in \mathrm{Im}(F) \wedge F^{-1}(m) < F^{-1}(n)$$

(cf. Exercises V.1.30, 33).

The key to the "largeness" properties of nonprojectible ordinals is:

6.16 Theorem. *For any recursively regular κ, if $\kappa^* > \omega$, then κ^* is a limit of κ-stable ordinals.*

Proof. Suppose κ is recursively regular and $\kappa^* > \omega$. Let

$$\sigma = \sup\{\lambda : \lambda < \kappa^* \wedge (\lambda \text{ is } \kappa\text{-stable} \vee \lambda = \omega)\}.$$

Clearly $\sigma \leqslant \kappa^*$. Suppose, for a contradiction, that $\sigma < \kappa^*$. Then either $\sigma = \omega$ or σ is the largest κ-stable ordinal less than κ^* (because a limit of κ-stable ordinals is itself κ-stable by Lemma 5.5). In either case σ is sequence closed so by Theorem 6.14, $\mathrm{st}_\kappa(\sigma)$ is projectible to σ. But $\mathrm{st}_\kappa(\sigma)$ is a κ-stable ordinal greater than σ so $\mathrm{st}_\kappa(\sigma) \geqslant \kappa^*$. Thus κ is projectible to $\mathrm{st}_\kappa(\sigma)$ and composing these projections yields a projection of κ to σ, a contradiction. Hence $\sigma = \kappa^*$, and since $\kappa^* > \omega$, κ^* is a limit of the κ-stable ordinals $\lambda < \kappa^*$. □

6.17 Corollary. *For any recursively regular κ, κ^* is also recursively regular, and if $\kappa^* > \omega$, then κ^* is κ-stable.*

Proof. If $\kappa^* = \omega$, then it is certainly recursively regular. Otherwise, κ^* is a limit of κ-stable ordinals, hence is itself κ-stable and recursively regular. □

6.18 Corollary. *For any recursively regular $\kappa > \omega$, if κ is nonprojectible, then κ is a limit of smaller κ-stable ordinals.* □

6.19 Corollary. *For any recursively regular $\kappa > \omega$, if κ is nonprojectible, then κ is recursively Mahlo, recursively inaccessible, recursively hyper-inaccessible, etc.*

Proof. Suppose that κ is recursively regular and nonprojectible, and let F be any

(∞, κ)-partial recursive function and $\mu < \kappa$ such that $(\forall \pi < \kappa) F(\pi, \mu)$ is defined. By Corollary 6.18 there exists $\lambda < \kappa$ such that $\mu < \lambda$ and λ is κ-stable. Then $(\forall \pi < \lambda) F(\pi, \mu) < \lambda$. Hence κ is recursively Mahlo. The other conclusions follow from Lemma 6.8. □

A concept closely related to projectibility is the following:

6.20 Definition. For any recursively regular κ and any λ and $\rho < \kappa$,

(i) λ is *κ-projectible to ρ* iff there exists a 1–1 function F which is κ-partial recursive in parameters such that $\mathrm{Dm}\, F = \lambda$ and $\mathrm{Im}\, F \subseteq \rho$;

(ii) λ is a *κ-cardinal* iff λ is not κ-projectible to any $\rho < \lambda$.

We shall show in the next section that λ is a κ-cardinal iff "λ is a cardinal" is true in L_κ, the class of sets constructible before κ. The proof of the following is left as Exercise 6.41.

6.21 Theorem. *For any recursively regular $\kappa > \omega$,*

(i) *ω is a κ-cardinal;*

(ii) *for any κ-cardinal $\lambda > \omega$, λ is a limit of κ-stable ordinals and is thus κ-stable;*

(iii) *if $\kappa^* < \kappa$, then κ^* is the greatest κ-cardinal.* □

6.22–6.42 Exercises

6.22. Let σ_ρ denote the ρ-th ordinal which is either recursively regular or a limit of recursively regular ordinals. Show that there exist ρ such that $\rho = \sigma_\rho$ but ρ is not recursively inaccessible.

6.23. Let $\tau_\rho^0 = \tau_\rho$, $\tau_\rho^{\nu+1} = \rho$-th ordinal κ such that $\tau_\kappa^\nu = \kappa$, and for limit ν, $\tau_\rho^\nu = \rho$-th ordinal κ such that $\tau_\kappa^\pi = \kappa$ for all $\pi < \nu$. $\lambda\rho.\tau_\rho^\nu$ enumerates the ordinals which are *ν-recursively inaccessible.* Show that

(i) if κ is recursively Mahlo and $\rho, \nu < \kappa$, then also $\tau_\rho^\nu < \kappa$;

(ii) for all $n < \omega$, $\tau_0^n = \omega_1[\mathsf{E}_n]$;

(iii) the least ordinal κ such that $\tau_0^\kappa = \kappa$ is less than the least recursively Mahlo ordinal.

6.24. Show that if κ is recursively Mahlo, then κ is recursively hyper-inaccessible.

6.25. For any recursively regular κ and any $\lambda < \kappa$ such that κ is projectible to λ,

$$\kappa = \sup{}^+\{\sigma : \sigma \text{ is the order-type of a } \kappa\text{-recursive well-ordering of } \lambda\}.$$

6.26. Show that for any recursively regular κ and any $\lambda \leqslant \kappa$, the following are equivalent:

(i) κ is projectible to λ;

(ii) there exists a set $A \subseteq \lambda$ and a function G κ-partial recursive in parameters such that $A \subseteq \operatorname{Dm} G$ and $G''A = \kappa$;

(iii) there exists a set $A \subseteq \lambda$ and a function H λ-partial recursive in parameters such that $A \subseteq \operatorname{Dm} H$ and $\sup^+ H''A = \kappa$;

(iv) there exists a set $B \subseteq \lambda$ such that B is κ-semi-recursive in parameters but not κ-recursive in parameters.

6.27. Show that if κ is recursively regular and not projectible to ω, then $\{\alpha : \alpha$ is κ-recursive$\}$ is a model of the Σ_2^1-Comprehension schema (use Exercise 3.16).

6.28. Suppose that $\mathsf{A} \subseteq {}^\omega\omega$ *has the following property*: *for any arithmetical relation* R *and any* $\alpha_0, \ldots, \alpha_{l-1} \in \mathsf{A}$,

$$\exists \beta\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \rightarrow (\exists \beta \in \mathsf{A})\, \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}, \beta);$$

(A is said to be a *β-model*). Let

$$\omega_1(\mathsf{A}) = \sup{}^+\{\|\gamma\| : \gamma \in \mathsf{A} \cap \mathsf{W}\}.$$

Show that

(i) if A is a model of Δ_0^1-Comprehension, then $\omega_1(\mathsf{A})$ is either recursively regular or a limit of recursively regular ordinals;

(ii) if A is a model of Π_1^1-Comprehension, then $\omega_1(\mathsf{A})$ is a limit of recursively regular ordinals;

(iii) if A is a model of Δ_2^1-Comprehension, then $\omega_1(\mathsf{A})$ is recursively inaccessible;

(iv) if A is a model of Σ_2^1-Comprehension, then $\omega_1(\mathsf{A})$ is not projectible to ω.
For (iii) use Lemma 3.5 and the result of Exercise IV.7.18 that Δ_2^1-Comprehension implies Σ_2^1-Choice.

6.29. In the notation of the preceding exercise, let $\omega_1^\kappa = \omega_1(\{\alpha : \alpha$ is κ-recursive in parameters$\})$. Show that for $\kappa > \omega$,

(i) if κ is recursively regular, then $\omega_1^\kappa \leqslant \kappa$;

(ii) there exist κ such that $\omega_1^\kappa < \kappa$;

(iii) if κ is recursively regular and projectible to ω, then $\omega_1^\kappa = \kappa$;

6.30. Show that

(i) if κ is projectible to ω and $\{\alpha : \alpha$ is κ-recursive in parameters$\}$ is a β-model and satisfies the Δ_2^1-Comprehension schema, then κ is recursively inaccessible;

(ii) if $\{\alpha : \alpha$ is κ-recursive in parameters$\}$ is a β-model and satisfies the Σ_2^1-Comprehension schema, then κ is not projectible to ω.

6.31. Show that for any recursively regular κ and any $R \subseteq {}^k\omega$, if R is κ-(semi-) recursive in parameters, then R is also ω_1^κ-(semi-) recursive in parameters. (Cf. Exercise 5.21. Use the same technique as there but now show that A collapses to some ordinal $\lambda \leqslant \omega_1^\kappa$ and that R is λ-recursive in parameters.)

6.32. Show that if κ is recursively regular, then ω_1^κ is recursively regular (cf. Exercise 6.28 (i)).

6.33. Show that for any κ, if E_1 is κ-effective, then ω_1^κ is recursively inaccessible, hence for κ projectible to ω, E_1 is κ-effective iff κ is recursively inaccessible.

6.34. Is δ_3^1 projectible?

6.35. Show that the class of sequence-closed ordinals is $(\infty, 0)$-recursive and if $\mathrm{Sqc}(\sigma)$ denotes the σ-th sequence-closed ordinal, then the function Sqc is $(\infty, 0)$-recursive.

6.36. Show that every stable ordinal is a limit of recursively regular nonprojectible ordinals.

6.37. Prove the converse of Corollary 6.18: if κ is a limit of smaller κ-stable ordinals, then κ is nonprojectible.

6.38. Say that κ is *projectible onto* λ iff there exists a one–one function F κ-recursive in parameters such that $\operatorname{Im} F = \lambda$ — that is, there is a one–one correspondence between κ and λ that is κ-recursive in parameters. κ is *strongly projectible* iff κ is projectible onto some $\lambda < \kappa$. Show

(i) if κ is nonprojectible or recursively regular, then κ is not strongly projectible;

(ii) if κ is a limit of sequence-closed ordinals but is not recursively regular, then κ is strongly projectible;

(iii) if κ is nonprojectible and a limit of sequence-closed ordinals, then κ is recursively regular.

6.39. Let $\kappa = \sup^+\{\tau_n : n < \omega\}$. What is κ^*?

6.40. Show that for any recursively regular κ and any $\lambda < \kappa$, λ is a κ-cardinal iff there is no one–one function F partial κ-recursive in parameters such that $\operatorname{Dm} F = \lambda$ and $\operatorname{Im} F$ is an ordinal $\rho < \lambda$.

6.41. Prove Theorem 6.21.

6.42. A κ-cardinal λ is *κ-regular* iff there is no $\rho < \lambda$ and no F which is

κ-partial recursive in parameters such that $\lambda = \sup_{\pi<\rho} F(\pi)$. Show that if λ is the least κ-cardinal greater than ω, then λ is κ-regular.

7. Ordinal Recursion and Constructible Sets

The perspective of this book has been recursion-theoretic. When several treatments of a topic were possible, we have chosen the one which seemed most in the spirit of recursion theory. This is particularly true of the present chapter where we have developed the theory of recursion on ordinals as a natural generalization of ordinary recursion theory and completely neglected an alternative and quite distinct set-theoretic description of the theory. In this section we present this description and prove its equivalence with ordinal recursion theory as given above.

In a nutshell, the point is that the theory of recursion on ordinals may equally well be viewed as a theory of definability over the universe L of constructible sets (introduced in § V.2). The foundation of this correspondence lies in the results of § III.5 where we see that ordinary recursion theory may be described in terms of definability over $\mathfrak{N}$, the standard model of arithmetic. If we add to this the facts that (1) there is a "simple" one–one correspondence between ω and L_ω, the class of sets constructible at finite levels, and (2) ordinary recursion theory is equivalent with ω-recursion theory (Theorem 1.4), we arrive at the conclusion that ω-recursion theory may be described in terms of definability over L_ω. The results below show that the same is true for any recursively regular ordinal κ in place of ω.

To provide ourselves with ample symbols, we shall need to adopt some new notational conventions for this section. In particular we shall use letters

a, b, c, d, t, u, v, w, x, y, z, M, and N for arbitrary sets

and

f, g, and h for functions from k-tuples of sets to sets.

These variables are used both in the metatheory and the formal theory. To the language $\mathscr{L}_{\mathrm{ZF}}$ of set theory as described in § V.2 we add also $\vee$ (or) as a logical symbol.

7.1 Definition. The class of Δ_0 *formulas* of $\mathscr{L}_{\mathrm{ZF}}$ is the smallest class X such that

(i) all atomic formulas of $\mathscr{L}_{\mathrm{ZF}}$ and their negations belong to X;

(ii) for any $\mathfrak{A}$, $\mathfrak{B} \in X$, both $\mathfrak{A} \wedge \mathfrak{B}$ and $\mathfrak{A} \vee \mathfrak{B}$ belong to X;

(iii) for any $\mathfrak{A} \in X$ and any variables x and y, both $(\exists x \in y)\mathfrak{A}$ and $(\forall x \in y)\mathfrak{A}$ belong to X.

Of course, $(\exists x \in y)\mathfrak{A}$ is an abbreviation for $\exists x\,(x \in y \wedge \mathfrak{A})$, but we think of this *bounded quantifier* as a primitive symbol. The point of Δ_0 formulas is that no quantifiers occur without bounds. Note that the negation of a Δ_0 formula is logically equivalent to a Δ_0 formula.

Many of the elementary concepts of set theory are definable using only Δ_0 formulas. Among these are: $\varnothing$ (the empty set), the Boolean operations, ordered pair, cartesian product, relation and function, domain and range, ordinal, ordinal successor, finite ordinal, and supremum of a set of ordinals. As an example,

$$\begin{aligned} a = \mathrm{Dm}\, f \leftrightarrow (\forall x \in a)(\exists z \in f)(\exists w \in z)(\exists y \in w)\,.\, z = \langle x, y\rangle \\ \wedge (\forall z \in f)(\forall w \in z)(\forall x \in w)(\forall y \in w)\,.\, z = \langle x, y\rangle \rightarrow x \in a. \end{aligned}$$

For a more detailed exposition of these facts see Barwise [1975].

The other important feature of Δ_0 formulas is that they are *absolute* for transitive $\in$-models. That is, for any transitive sets $M \subseteq N$, any Δ_0 formula $\mathfrak{A}$ with free variables included among $x_0, \ldots, x_{k-1}$ and any $u_0, \ldots, u_{k-1} \in M$,

$$(M, \in) \models \mathfrak{A}[\mathbf{u}] \quad \text{iff} \quad (N, \in) \models \mathfrak{A}[\mathbf{u}].$$

This is easily proved by induction on the class of Δ_0 formulas. In particular, this is true when N is the "real world" so that, for example, if $\mathfrak{D}$ is the Δ_0 formula given above that defines "domain," then for $a, f \in M$,

$$(M, \in) \models \mathfrak{D}[a, f] \quad \text{iff} \quad a = \mathrm{Dm}\, f.$$

Similarly,

$$(M, \in) \models \text{"}a \text{ is an ordinal"} \quad \text{iff} \quad a \text{ is an ordinal.}$$

These and similar facts are used repeatedly below without special mention.

7.2 Definition. The class of Σ *formulas* of $\mathscr{L}_{\mathrm{ZF}}$ is the smallest class X such that (i)–(iii) as in Definition 7.1 and

$(\text{iv})_\Sigma$ for any $\mathfrak{A} \in X$ and any variable x, $\exists x\, \mathfrak{A}$ belongs to X.

The class of Π *formulas* of $\mathscr{L}_{\mathrm{ZF}}$ is the smallest class X such that (i)–(iii) as in Definition 7.1 and

$(\text{iv})_\Pi$ for any $\mathfrak{A} \in X$ and any variable x, $\forall x\, \mathfrak{A}$ belongs to X.

Clearly every Δ_0 formula is both a Σ formula and a Π formula. The negation of a Σ formula is logically equivalent to a Π formula and vice versa.

For any set M, a relation $R \subseteq {}^kM$ is called Σ (Π)-*definable* over M iff for some Σ (Π) formula $\mathfrak{A}$ and some $u_0, \ldots, u_{l-1} \in M$,

$$R(v_0, \ldots, v_{k-1}) \leftrightarrow (M, \in) \models \mathfrak{A}[\mathbf{v}, \mathbf{u}].$$

R is Δ-*definable* over M iff R is both Σ-definable and Π-definable over M. Note that any total function $f : {}^kM \to M$ is Δ-definable over M iff it is Σ-definable over M because

$$f(\mathbf{x}) = y \leftrightarrow \forall z\,[z \neq y \to f(\mathbf{x}) \neq z].$$

To state the main results of this section we need one more concept, admissibility. The theory of admissible sets and its generalizations have proved of enormous importance in set theory, generalized recursion theory, and even model theory. We shall not be able here even to touch upon any aspect of this theory except its relationship to ordinal recursion theory, but we recommend most strongly that any reader of this section take time to read at least the beginning of Barwise [1975].

7.3 Definition. For any set M, M is *admissible* iff M is transitive and $(M, \in)$ satisfies the universal closures of the following formulas:

(i) (*Pair*) $\exists a\,(x \in a \wedge y \in a)$;

(ii) (*Union*) $\exists b\,(\forall y \in a)(\forall x \in y) . x \in b$;

(iii) (Δ_0-*Separation*) $\exists b\,\forall x\,(x \in b \leftrightarrow x \in a \wedge \mathfrak{A})$ for all Δ_0 formulas $\mathfrak{A}$ in which b does not occur free;

(iv) (Δ_0-*Collection*) $(\forall x \in a)\exists y\,\mathfrak{A} \to \exists b\,(\forall x \in a)(\exists y \in b)\mathfrak{A}$ for all Δ_0 formulas $\mathfrak{A}$ in which b does not occur free.

We shall prove that for any ordinal κ,

(1) L_κ is admissible iff κ is recursively regular;

(2) if L_κ is admissible, then for any relation $R \subseteq {}^k\kappa$,

(i) R is κ-semi-recursive in parameters iff R is Σ-definable over L_κ;

(ii) R is κ-recursive in parameters iff R is Δ-definable over L_κ;

We proceed first towards the implications ($\to$). We shall need a number of facts about admissible sets. To give proofs for them all would take us too far from our topic, so we shall merely state the results and again refer the reader to Barwise [1975].

7.4 Theorem. *For any admissible set M,*

(i) (Δ-*Separation*) *for any Δ-definable set $a \subseteq M$ and any $b \in M$, also $a \cap b \in M$;*

(ii) (Σ-*Collection*) $(M, \in)$ *satisfies the universal closure of*

$$(\forall x \in a)\exists y\,\mathfrak{A} \to \exists b\,(\forall x \in a)(\exists y \in b)\mathfrak{A}$$

for all Σ *formulas* $\mathfrak{A}$ *in which* b *does not occur free*;

(iii) (*Transitive Closure*) *for any* $a \in M$, $\mathrm{TC}(a)$, *the smallest transitive set* b *such that* $a \in b$, *also belongs to* M;

(iv) (Σ-*Recursion*) *for any function* $g : {}^{k+2}M \to M$ *which is* Σ-*definable over* M, *the* (unique) *function* $f : {}^{k+1}M \to M$ *such that for all* $x_0, \dots, x_{k-1}, y \in M$

$$f(\mathbf{x}, y) = g(\mathbf{x}, y, f \restriction \mathrm{TC}(y))$$

where

$$f \restriction \mathrm{TC}(y) = \{(z, f(\mathbf{x}, z)) : z \in \mathrm{TC}(y)\},$$

is also Σ-*definable over* M. $\square$

Let $o(M)$ denote the least ordinal not in M. If M is transitive, then $o(M)$ is exactly the set of ordinals in M. Note that $o(\mathrm{L}_\kappa) = \kappa$.

7.5 Lemma. *For any* κ, *any admissible set* M *with* $\kappa = o(M)$, *any* ρ, $\boldsymbol{\mu} < \kappa$, *and any partial function* F *from* ${}^k\kappa$ *into* κ, *if* F *is* Σ-*definable over* M *and* $F(\pi, \boldsymbol{\mu})$ *is defined for all* $\pi < \rho$, *then* $\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) < \kappa$.

Proof. Let F be defined over M by the Σ formula $\mathfrak{A}$ and the parameters $\mathbf{u}$:

$$F(\pi, \boldsymbol{\mu}) \simeq \nu \leftrightarrow (M, \in) \models \mathfrak{A}[\pi, \boldsymbol{\mu}, \nu, \mathbf{u}].$$

Then by assumption we have

$$(M, \in) \models (\forall \pi \in \rho) \exists \nu\, \mathfrak{A}$$

and thus by Σ-Collection

$$(M, \in) \models \exists b\, (\forall \pi \in \rho)(\exists \nu \in b)\mathfrak{A}.$$

For such a, $b \in M$,

$$\sup^+_{\pi<\rho} F(\pi, \boldsymbol{\mu}) \leqslant \sup{}^+\{\nu : \nu \in b\} \leqslant \bigcup\{\nu : \nu \in b\} + 1 < \kappa. \quad \square$$

For any κ, let

$$T_\kappa(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma) \leftrightarrow \kappa', \lambda, \sigma < \kappa \wedge (a, \boldsymbol{\mu}, \nu) \in \Omega^\sigma_{\kappa'\lambda}.$$

7.6 Theorem. *For any* κ *and any admissible set* M *with* $\kappa = o(M)$, T_κ *is* Δ-*definable over* M.

Proof. The proof of Lemma 2.4 gives a recursive definition of the characteristic function F of T_κ in the form

$$F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma) = G(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma, F \upharpoonright \sigma).$$

If we establish that G is Σ-definable over M, then it follows by Σ-recursion that also F is Σ definable over M. Then since

$$\begin{aligned} T_\kappa(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma) &\leftrightarrow F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma) = 0 \\ &\leftrightarrow F(a, \langle \boldsymbol{\mu} \rangle, \nu, \kappa', \lambda, \sigma) \neq 1, \end{aligned}$$

it follows that T_κ is Δ-definable over M.

The class of Σ-definable functions is clearly closed under definition by Σ cases. By inspection we see that the functions of ordinal arithmetic are introduced by Σ-recursion, and hence that their restrictions to ordinals $< \kappa$ are Σ-definable over M. The coding functions for finite sequences of natural numbers and ordinals are defined recursively from ordinal exponentiation, hence they are Σ-definable. The decoding functions use in addition the bounded search operation. If $f(\sigma, \mathbf{x}) = \text{least } \pi < \sigma \,.\, g(\pi, \mathbf{x}) = 0$, then

$$\begin{aligned} f(\sigma, \mathbf{x}) = y \leftrightarrow & [y \in \sigma \wedge g(y, \mathbf{x}) = 0 \wedge (\forall u \in y)(\exists z \neq 0) g(u, \mathbf{x}) = z] \\ & \vee [y = \sigma \wedge (\forall y \in \sigma)(\exists z \neq 0) \,.\, g(y, \mathbf{x}) = z]. \end{aligned}$$

Hence if g is Σ-definable over M, so is f. Finally, the equivalence,

$$f(\mathbf{x}, g(\mathbf{x})) = y \leftrightarrow \exists z \, [g(\mathbf{x}) = z \wedge f(\mathbf{x}, z) = y]$$

shows that the class of Σ-definable functions is closed under composition.

With these facts in mind, it is straightforward to check that G is Σ-definable. The cases depend on Σ conditions concerning the indices. The five subclauses of clause (0) which concern the constant, projection, and successor functions, and the characteristic function of equality are Δ_0 by the remarks following Definition 7.2; the last subclause involving the sequence coding functions is Σ by the preceding remarks. In the remaining clauses, all quantifiers are bounded and the conditions under which F has value 0 are thus Δ. In clause (1) we need to observe that the function H is defined by recursion from the sequence coding functions and is thus Σ-definable. It follows that the (complementary) conditions under which F has value 1 are also Δ. $\square$

7.7 Theorem. *For any κ, any admissible set M with $\kappa = o(M)$, and all a, $\boldsymbol{\mu}$, and ν,*

$$(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa} \leftrightarrow (\exists \lambda < \kappa)(\exists \sigma < \kappa) \,.\, (a, \boldsymbol{\mu}, \nu) \in \Omega^{\sigma}_{\lambda\lambda}.$$

Proof. We proceed almost exactly as in the proof of Lemma 2.5. The admissibility of M easily implies that κ is a limit ordinal. In clause (3), the relation R is Δ-definable over M by Theorem 7.6. Hence the function F is Σ-definable and Lemma 7.5 gives the inequality $\tau < \kappa$. The condition of recursive regularity may similarly be replaced by admissibility in clause (4). □

7.8 Corollary. *For any κ and any admissible set M with $\kappa = o(M)$, the relation*

$$R(a, \langle\boldsymbol{\mu}\rangle, \nu) \leftrightarrow \{a\}_\kappa(\boldsymbol{\mu}) \simeq \nu$$

is Σ-definable over M. Every function κ-partial recursive in parameters has graph Σ-definable over M. Every relation κ-semi-recursive in parameters is Σ-definable over M. Every relation Σ-recursive in parameters is Δ-definable over M.

Proof. These are all immediate from Theorems 7.6 and 7.7. □

7.9 Corollary. *For any κ, if there exists an admissible set M such that $\kappa = o(M)$, then κ is recursively regular.*

Proof. Immediate from Corollary 7.8, Lemma 7.5 and Theorem 2.2. □

The implications $(\rightarrow)$ of (1) and (2) now follow by taking $M = \mathrm{L}_\kappa$ in Corollaries 7.8 and 7.9. To establish the converse implications we shall assign ordinals to the members of L_κ in such a way that Δ-definability over L_κ corresponds to recursiveness of the associated relations of ordinals. The assignment takes place in two steps: we first construct a language $\mathscr{L}$ in which every constructible set has a name and then "Gödel number" the language in such a way that whenever κ is recursively regular, names for elements of L_κ are assigned ordinals less than κ.

The symbols of $\mathscr{L}$ are $\dot{\in}$, $\dot{=}$, $\neg$, $\vee$, $\exists$, a variable v_i^σ for each ordinal σ and each $i < \omega$, and a constant symbol $\dot{\sigma}$ for each ordinal σ. The syntax of $\mathscr{L}$ is designed to imitate the process by which the levels of the hierarchy of constructible sets are defined. The variables v_i^σ intuitively range over L_σ. The terms of $\mathscr{L}$ include, in addition to the variables and constant symbols, *abstraction terms* $\hat{v}_i^\sigma\mathfrak{A}$. $\hat{v}_i^\sigma\mathfrak{A}$ is intended as a name for the element of $\mathrm{L}_{\sigma+1}$ defined over L_σ by the formula $\mathfrak{A}$. To make this work, $\mathfrak{A}$ should contain names only for elements of L_σ. This requirement leads us to define the classes of terms and formulas of $\mathscr{L}$ simultaneously by induction along with a notion of *degree*. The intention is that terms of degree $\leq \sigma$, other than variables, are names for elements of L_σ.

For all $i < \omega$ and all ordinals σ,

(i) v_i^σ is a term of degree σ;

(ii) $\dot{\sigma}$ is a term of degree $\sigma + 1$;

(iii) for any terms t and u, $t \dot{\in} u$ and $t \dot{=} u$ are formulas of degree $\max\{\deg(t), \deg(u)\}$;

(iv) for any formula $\mathfrak{A}$, $\neg\mathfrak{A}$ is a formula of degree $\deg(\mathfrak{A})$;

(v) for any formulas $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \vee \mathfrak{B}$ is a formula of degree $\max\{\deg(\mathfrak{A}), \deg(\mathfrak{B})\}$;

(vi) for any formula $\mathfrak{A}$ of degree $\leqslant \sigma$, $\exists v_i^\sigma \mathfrak{A}$ is a formula of degree σ;

(vii) for any formula $\mathfrak{A}$ of degree $\leqslant \sigma$ with no free variables other than v_i^σ, $\hat{v}_i^\sigma \mathfrak{A}$ is a term of degree $\sigma + 1$;

(viii) an expression is a formula or term only if it is by virtue of (i)–(vii).

A term of $\mathscr{L}$ is called *closed* iff it is not a variable; a formula of $\mathscr{L}$ is *closed* iff it has no free variables. The intended interpretation of $\mathscr{L}$ is given by means of a function Val defined on closed terms and a predicate $\models$ defined on closed formulas. $\mathrm{Val}(t)$ is the set named by t and $\models \mathfrak{A}$ holds just in case $\mathfrak{A}$ is true under the interpretation. The definition is by a simultaneous recursion on degree with a subsidiary recursion on formulas at each level.

For all $i < \omega$ and all ordinals σ,

(i) $\mathrm{Val}(\dot{\sigma}) = \sigma$;

(ii) for any closed terms t and u, $\models t \dot{\in} u$ iff $\mathrm{Val}(t) \in \mathrm{Val}(u)$ and $\models t \dot{=} u$ iff $\mathrm{Val}(t) = \mathrm{Val}(u)$;

(iii) for any closed formula $\mathfrak{A}$, $\models \neg\mathfrak{A}$ iff *not* $\models \mathfrak{A}$;

(iv) for any closed formulas $\mathfrak{A}$ and $\mathfrak{B}$, $\models \mathfrak{A} \vee \mathfrak{B}$ iff $\models \mathfrak{A}$ or $\models \mathfrak{B}$;

(v) for any closed formula $\exists v_i^\sigma \mathfrak{A}$, $\models \exists v_i^\sigma \mathfrak{A}$ iff for some closed term t of degree $\leqslant \sigma$, $\models \mathfrak{A}(t/v_i^\sigma)$;

(vi) for any term $\hat{v}_i^\sigma \mathfrak{A}$,

$$\mathrm{Val}(\hat{v}_i^\sigma \mathfrak{A}) = \{\mathrm{Val}(t) : t \text{ is a closed term of degree} \leqslant \sigma \text{ such that} \models \mathfrak{A}(t/v_i^\sigma)\}.$$

7.10 Theorem. *For all* κ,

$$\mathrm{L}_\kappa = \{\mathrm{Val}(t) : t \text{ is a closed term of degree } \leqslant \kappa\}.$$

Proof. We proceed by induction on κ. For $\kappa = 0$, there are no closed terms of degree 0 and both sides are empty. Assume as induction hypothesis that the result holds for any $\sigma < \kappa$ in place of κ. It κ is a limit ordinal, the result is immediate from the fact that the degree of a closed term is always a successor ordinal.

Suppose that $\kappa = \sigma + 1$. Clearly $\mathrm{Val}(\dot{\sigma}) = \sigma \in \mathrm{L}_\kappa$. We need to show that a subset of L_σ is definable over $(\mathrm{L}_\sigma, \in)$ (by a formula of $\mathscr{L}_{\mathrm{ZF}}$) just in case it is of the form $\mathrm{Val}(\hat{v}_i^\sigma \mathfrak{A})$ for some formula $\mathfrak{A}$ of $\mathscr{L}$ of degree $\leqslant \sigma$ with no free variables other than v_i^σ.

Let $\mathscr{L}_\sigma^*$ be a language obtained from $\mathscr{L}_{\mathrm{ZF}}$ by adding names $\dot{a}$ for each $a \in \mathrm{L}_\sigma$ and variables x_i^τ for each $i < \omega$ and $\tau < \sigma$. $(\mathrm{L}_\sigma, \in)^*$ denotes the corresponding expansion of $(\mathrm{L}_\sigma, \in)$ in which each name $\dot{a}$ is intepreted as a. Clearly the same subsets of L_σ are definable over $(\mathrm{L}_\sigma, \in)^*$ by formulas of $\mathscr{L}_\sigma^*$ as are definable over $(\mathrm{L}_\sigma, \in)$ by formulas of $\mathscr{L}_{\mathrm{ZF}}$. Note that over $(\mathrm{L}_\sigma, \in)^*$ we have no need of parameters in the definitions as these are already built into the language $\mathscr{L}_\sigma^*$. To each term t and formula $\mathfrak{A}$ of $\mathscr{L}$ of degree $\leqslant \sigma$ we assign a term t^* and a formula $\mathfrak{A}^*$ of $\mathscr{L}_\sigma^*$, respectively, as follows: for all $i < \omega$ and all ordinals $\tau < \sigma$,

(i) $(v_i^\tau)^*$ is x_i^τ; $(v_i^\sigma)^*$ is x_i;
(ii) if $\mathrm{Val}(t) = a$, then $t^* = \dot{a}$;
(iii) $(t \dot{\in} u)^*$ is $(t^* \dot{\in} u^*)$ and $(t \doteq u)^*$ is $(t^* \doteq u^*)$;
(iv) $(\neg\mathfrak{A})^*$ is $(\neg\mathfrak{A}^*)$ and $(\mathfrak{A} \vee \mathfrak{B})^*$ is $\mathfrak{A}^* \vee \mathfrak{B}^*$;
(v) $(\exists v_i^\tau \mathfrak{A})^*$ is $(\exists x_i^\tau \in \dot{\mathrm{L}}_\tau)\mathfrak{A}^*$;
(vi) $(\exists v_i^\sigma \mathfrak{A})^*$ is $\exists x_i\, \mathfrak{A}^*$.

We claim that for all closed formulas $\mathfrak{A}$ of $\mathscr{L}$ of degree $\leqslant \sigma$,

$$\models \mathfrak{A} \quad \text{iff} \quad (\mathrm{L}_\sigma, \in)^* \models \mathfrak{A}^*.$$

The proof is by induction on closed formulas. For atomic formulas, the claim is obvious and the steps corresponding to $\neg$ and $\vee$ are trivial. We have then for $\tau < \sigma$,

$$\begin{aligned} \models \exists v_i^\tau \mathfrak{A} \ &\text{iff for some closed term } t \text{ of degree} \leqslant \tau, \models \mathfrak{A}(t/v_i^\tau) \\ &\text{iff for some closed term } t \text{ of degree} \leqslant \tau, (\mathrm{L}_\sigma, \in)^* \models \mathfrak{A}^*(\dot{\mathrm{Val}}(t)/x_i^\tau) \\ &\text{iff for some } a \in \mathrm{L}_\tau, (\mathrm{L}_\sigma, \in)^* \models \mathfrak{A}^*(\dot{a}/x_i^\tau) \\ &\text{iff } (\mathrm{L}_\sigma, \in)^* \models (\exists x_i^\tau \in \dot{\mathrm{L}}_\tau)\mathfrak{A}^*. \end{aligned}$$

The second equivalence uses the easily proved fact that

$$\mathfrak{A}(t/v_i^\tau)^* = \mathfrak{A}^*(\mathrm{Val}(t)/x_i^\tau)$$

and the induction hypothesis on $\mathfrak{A}$. The third equivalence uses the induction hypothesis on τ. The step for $\exists v_i^\sigma \mathfrak{A}$ is similar.

Now we have for any closed formula $\mathfrak{A}$ of degree $\leqslant \sigma$ with at most v_i^σ free, and any $a \in \mathrm{L}_\sigma$,

$$\begin{aligned} a \in \mathrm{Val}(\hat{v}_i^\sigma \mathfrak{A}) \ &\text{iff for some closed term } t \text{ of degree} \leqslant \sigma, \\ &\qquad a = \mathrm{Val}(t) \quad \text{and} \quad \models \mathfrak{A}(t/v_i^\sigma) \\ &\text{iff } (\mathrm{L}_\sigma, \in)^* \models \mathfrak{A}^*(\dot{a}/x_i). \end{aligned}$$

It follows that $\mathrm{Val}(\hat{v}_i^\sigma \mathfrak{A})$ is definable over $(\mathrm{L}_\sigma, \in)^*$ and thus belongs to $\mathrm{L}_{\sigma+1} = \mathrm{L}_\kappa$.

For the converse inclusion suppose $b \in \mathrm{L}_\kappa$, so for some closed formula $\mathfrak{B}$ of $\mathscr{L}_\sigma^*$,

$$b = \{a : (\mathrm{L}_\sigma, \in)^* \models \mathfrak{B}(\dot{a}/x_i)\}.$$

We may clearly assume that neither the symbol $\wedge$ nor any of the variables x_i^τ occur in $\mathfrak{B}$. By the induction hypothesis, for every symbol $\dot{a}$ which occurs in $\mathfrak{B}$, there is a closed term t of $\mathscr{L}$ such that $t^* = \dot{a}$. It follows that there is a formula $\mathfrak{A}$ of $\mathscr{L}$ of degree $\leqslant \sigma$ with no free variables other than v_i^σ such that $\mathfrak{A}^* = \mathfrak{B}$. Thus $b = \mathrm{Val}(\hat{v}_i^\sigma \mathfrak{A})$. □

We now proceed to assign Gödel numbers to the terms and formulas of $\mathscr{L}$. The assignment is routine except for one trick in the choice of $\mathrm{gn}(v_i^\sigma)$. Let $\mathrm{Sqc}(\sigma)$ denote the σ-th sequence closed ordinal. Exercise 6.35 established that the function Sqc is $(\infty, 0)$-recursive. In particular, if κ is recursively regular and $\sigma < \kappa$, then also $\mathrm{Sqc}(\sigma) < \kappa$. We set

$$\mathrm{gn}(\dot{\sigma}) = \langle 0, \sigma \rangle;$$
$$\mathrm{gn}(v_i^\sigma) = \langle 1, i, \mathrm{Sqc}(\sigma) \rangle;$$
$$\mathrm{gn}(\hat{v}_i^\sigma \mathfrak{A}) = \langle 2, \mathrm{gn}(v_i^\sigma), \mathrm{gn}(\mathfrak{A}) \rangle;$$
$$\mathrm{gn}(t \doteq u) = \langle 3, \mathrm{gn}(t), \mathrm{gn}(u) \rangle;$$
$$\mathrm{gn}(t \dot{\in} u) = \langle 4, \mathrm{gn}(t), \mathrm{gn}(u) \rangle;$$
$$\mathrm{gn}(\neg \mathfrak{A}) = \langle 5, \mathrm{gn}(\mathfrak{A}) \rangle;$$
$$\mathrm{gn}(\mathfrak{A} \vee \mathfrak{B}) = \langle 6, \mathrm{gn}(\mathfrak{A}), \mathrm{gn}(\mathfrak{B}) \rangle;$$
$$\mathrm{gn}(\exists v_i^\sigma \mathfrak{A}) = \langle 7, \mathrm{gn}(v_i^\sigma), \mathrm{gn}(\mathfrak{A}) \rangle.$$

7.11 Lemma. *For all* σ

(i) *for any formula* $\mathfrak{A}$ *of* $\mathscr{L}$ *of degree* $< \sigma$, $\mathrm{gn}(\mathfrak{A}) < \mathrm{Sqc}(\sigma)$;

(ii) *for any closed term* t *of* $\mathscr{L}$ *of degree* $\leqslant \sigma$, $\mathrm{gn}(t) < \mathrm{Sqc}(\sigma)$.

Proof. We proceed by induction on σ and assume as induction hypothesis that (i) and (ii) hold with τ in place of σ for all $\tau < \sigma$. First, if $\mathfrak{A}$ is of degree $< \sigma$, then any term t occurring in $\mathfrak{A}$ is of degree some $\tau < \sigma$. If t is closed, then $\mathrm{gn}(t) < \mathrm{Sqc}(\tau) < \mathrm{Sqc}(\sigma)$ by (ii) of the induction hypothesis. If t is v_i^τ, then $\mathrm{gn}(t) = \langle 1, i, \mathrm{Sqc}(\tau) \rangle < \mathrm{Sqc}(\sigma)$ because $\mathrm{Sqc}(\sigma)$ is sequence-closed. It follows in turn by induction on the subformulas of $\mathfrak{A}$ that $\mathrm{gn}(\mathfrak{A}) < \sigma$.

Now if t is a closed term of degree σ, then necessarily σ is a successor ordinal, say $\tau + 1$, and either $t = \dot{\tau}$ or for some i and some formula $\mathfrak{A}$ of degree $\leqslant \tau$ with at most v_i^τ free, $t = \hat{v}_i^\tau \mathfrak{A}$. In the first case, $\mathrm{gn}(t) = \langle 0, \tau \rangle < \mathrm{Sqc}(\sigma)$ because $\tau \leqslant \mathrm{Sqc}(\tau) < \mathrm{Sqc}(\sigma)$. In the second case, $\mathrm{gn}(\mathfrak{A}) < \mathrm{Sqc}(\sigma)$ by (i), so $\mathrm{gn}(t) = \langle 2, \langle 1, i, \mathrm{Sqc}(\tau) \rangle, \mathrm{gn}(\mathfrak{A}) \rangle < \mathrm{Sqc}(\sigma)$. □

7.12 Corollary. *For any* σ, *any formula* $\mathfrak{A}$ *of* $\mathscr{L}$ *of degree* $\leqslant \sigma$, *and any closed term* t *of degree* $\leqslant \sigma$,

(i) $\mathrm{gn}\,\mathfrak{A}(t/v_i^\sigma) < \mathrm{gn}(t \dot{\in} \hat{v}_i^\sigma \mathfrak{A})$, $\mathrm{gn}(\exists v_i^\sigma \mathfrak{A})$;

(ii) *for any* $a \in \mathrm{Val}(t)$, *there exists a term* u *such that* $a = \mathrm{Val}(u)$ *and* $\mathrm{gn}(u) < \mathrm{gn}(t)$.

Proof. (i) follows easily from the fact that under the given conditions, $\mathrm{gn}(t) < \mathrm{Sqc}(\sigma) < \mathrm{gn}(v_i^\sigma)$, so that $\mathrm{gn}\,\mathfrak{A}(t/v_i^\sigma) \leqslant \mathrm{gn}(\mathfrak{A})$. For (ii), if t is $\dot{\tau}$ with $\tau < \sigma$, then a is an ordinal $< \tau$ and $\mathrm{gn}(\dot{a}) = \langle 0, a\rangle < \langle 0, \tau\rangle = \mathrm{gn}(t)$. Otherwise, t is of the form $\hat{v}_i^\tau \mathfrak{A}$ with $\tau < \sigma$ and $\mathfrak{A}$ a formula of degree $\leqslant \tau$ with at most v_i^τ free. If $a \in \mathrm{Val}(t)$, then $a = \mathrm{Val}(u)$ for some closed term u of degree $\leqslant \tau$, and $\mathrm{gn}(u) < \mathrm{Sqc}(\tau) < \mathrm{gn}(v_i^\tau) < \mathrm{gn}(t)$. □

7.13 Lemma. *There exist* $(\infty, 0)$*-recursive functions and relations which satisfy the following conditions for any terms t and u and formula* $\mathfrak{A}$ *of* $\mathscr{L}$:

(i) $\mathrm{Ord}(\mu) \leftrightarrow \exists\sigma.\ \mu = \mathrm{gn}(\dot{\sigma})$;

(ii) $\mathrm{Var}(\mu) \leftrightarrow \exists\sigma\,\exists i.\ \mu = \mathrm{gn}(v_i^\sigma)$;

(iii) $\mathrm{Dg}(\mathrm{gn}(t)) = \deg(t)$;
$\mathrm{Dg}(\mathrm{gn}(\mathfrak{A})) = \deg(\mathfrak{A})$;

(iv) $\mathrm{Fr}(\mathrm{gn}(v_i^\sigma), \mathrm{gn}(\mathfrak{A})) \leftrightarrow v_i^\sigma$ *occurs free in* $\mathfrak{A}$;

(v) $\mathrm{Te}(\mu) \leftrightarrow \mu$ *is the Gödel number of some term of* $\mathscr{L}$;

(vi) $\mathrm{Fm}(\mu) \leftrightarrow \mu$ *is the Gödel number of some formula of* $\mathscr{L}$;

(vii) $\mathrm{Sb}(\mathrm{gn}(t), \mathrm{gn}(v_i^\sigma), \mathrm{gn}(\mathfrak{A})) = \mathrm{gn}(\mathfrak{A}(t/v_i^\sigma))$.

Proof. The calculations are routine and we leave them as Exercise 7.23. □

7.14 Theorem. *The relation* Tr *defined by*

$\mathrm{Tr}(\mu) \leftrightarrow \mu$ *is the Gödel number of a closed formula* $\mathfrak{A}$ *of* $\mathscr{L}$ *such that* $\models \mathfrak{A}$

is $(\infty, 0)$*-recursive.*

Proof. We give a few of the conditions which Tr must satisfy and leave it to the reader to supply the remainder. The existence of Tr follows by application of the Recursion Theorem.

$$\mathrm{Tr}(\langle 4, \langle 0, \sigma\rangle, \langle 0, \tau\rangle\rangle) \leftrightarrow \sigma < \tau;$$

$$\mathrm{Tr}(\langle 4, \langle 0, \sigma\rangle, \mathrm{gn}(\hat{v}_j^\tau \mathfrak{B})\rangle) \leftrightarrow \sigma < \tau \wedge \mathrm{Tr}(\mathrm{gn}(\mathfrak{B}(\dot{\sigma}/v_j^\tau)));$$

$\mathrm{Tr}(\langle 4, \mathrm{gn}(\hat{v}_i^\sigma \mathfrak{A}), \mathrm{gn}(\hat{v}_j^\tau \mathfrak{B})\rangle) \leftrightarrow (\exists\mu < \mathrm{Sqc}(\tau))$

μ is the Gödel number of a closed term t of degree $\leqslant \tau$, $\mathrm{Tr}(\mathrm{gn}\,\mathfrak{B}(t/v_j^\tau))$, and $(\forall\nu < \mathrm{Sqc}(\sigma))$ if ν is the Gödel number of a closed term u of degree $\leqslant \sigma$, then $\mathrm{Tr}(\mathrm{gn}(\mathfrak{A}(u/v_i^\sigma)))$ iff $\mathrm{Tr}(\mathrm{gn}(u \dot{\in} t))$;

$$\mathrm{Tr}(\mathrm{gn}(\mathfrak{A} \vee \mathfrak{B})) \leftrightarrow \mathrm{Tr}(\mathrm{gn}(\mathfrak{A})) \text{ or } \mathrm{Tr}(\mathrm{gn}(\mathfrak{B}));$$

$$\mathrm{Tr}(\mathrm{gn}(\exists v_i^\sigma \mathfrak{A})) \leftrightarrow (\exists \mu < \mathrm{Sqc}(\sigma))\ \mu \text{ is the Gödel number of a closed term } t \text{ of degree } \leq \sigma \text{ and } \mathrm{Tr}(\mathfrak{A}(t/v_i^\sigma)).$$

In several cases we rely on Corollary 7.12 to ensure that the Gödel numbers on the right-hand side of each equivalence are less than those on the left. □

7.15 Corollary. *Every constructible set A of ordinals is $(\infty, 0)$-recursive in parameters. For every recursively regular κ, if $A \in \mathrm{L}_\kappa$, then A is κ-recursive in parameters.*

Proof. If $A \in \mathrm{L}_\kappa$, then by Theorem 7.10 there exists a closed term t of degree $\leq \kappa$ such that $A = \mathrm{Val}(t)$. Then for any σ,

$$\sigma \in A \leftrightarrow \models \dot{\sigma} \dot{\in} t \leftrightarrow \mathrm{Tr}(\langle 4, \langle 0, \sigma \rangle, \mathrm{gn}(t) \rangle).$$

Thus A is $(\infty, 0)$ recursive in the parameter $\mathrm{gn}(t)$. If κ is recursively regular, then $\mathrm{gn}(t) < \mathrm{Sqc}(\kappa) = \kappa$ and by Theorem 2.2, A is κ-recursive in the parameter $\mathrm{gn}(t)$. □

For any μ, let

$$^\circ\mu = \begin{cases} \mathrm{Val}(t), & \text{if } \mu \text{ is the Gödel number of a closed term } t; \\ \varnothing, & \text{otherwise.} \end{cases}$$

7.16 Lemma. *For any recursively regular κ,*

$$\mathrm{L}_\kappa = \{ {}^\circ\mu : \mu < \kappa \}.$$

Proof. As in the preceding proof, if $A \in \mathrm{L}_\kappa$, then $A = \mathrm{Val}(t)$ for some t with $\mathrm{gn}(t) < \kappa$. Thus $A = {}^\circ\mathrm{gn}(t)$. On the other hand, it is easy to prove that for all terms, $\mathrm{dg}(t) \leq \mathrm{gn}(t)$ so that for any $\mu < \kappa$, if $\mu = \mathrm{gn}(t)$, then $\mathrm{dg}(t) < \kappa$ and by Theorem 7.10, $^\circ\mu = \mathrm{Val}(t) \in \mathrm{L}_\kappa$. □

For any formula $\mathfrak{A}$ of $\mathscr{L}_{\mathrm{ZF}}$ with free variables among $x_0, \ldots, x_{k-1}$ and any $\mu_0, \ldots, \mu_{k-1} < \kappa$, let

$$\mathrm{Sat}_{\mathfrak{A}, \kappa}(\boldsymbol{\mu}) \leftrightarrow (\mathrm{L}_\kappa, \in) \models \mathfrak{A}[{}^\circ\boldsymbol{\mu}].$$

7.17 Theorem. *For any recursively regular κ and any Δ_0-formula $\mathfrak{A}$ of $\mathscr{L}_{\mathrm{ZF}}$, $\mathrm{Sat}_{\mathfrak{A}, \kappa}$ is κ-recursive.*

Proof. We proceed by induction on Δ_0 formulas. For atomic formulas, the result is immediate from Theorem 7.14. The $\neg$, $\wedge$, and $\vee$ steps are trivial and we consider only bounded quantification. Suppose that $\mathfrak{A}$ is a Δ_0 formula and that $\mathrm{Sat}_{\mathfrak{A},\kappa}$ is κ-recursive. Then for ρ, $\boldsymbol{\mu} < \kappa$

$$\begin{aligned}(\mathrm{L}_\kappa, \in) \models (\exists x \in y)\,\mathfrak{A}[{}^\circ\rho, {}^\circ\boldsymbol{\mu}] \\ &\leftrightarrow (\exists a \in {}^\circ\rho)\,.\,(\mathrm{L}_\kappa, \in) \models \mathfrak{A}[a, {}^\circ\boldsymbol{\mu}] \\ &\leftrightarrow (\exists \pi < \rho)\,.\,\mathrm{Tr}(\langle 4, \pi, \rho\rangle) \wedge \mathrm{Sat}_{\mathfrak{A},\kappa}(\pi, \boldsymbol{\mu}).\end{aligned}$$

The second equivalence depends on Corollary 7.12 (ii). $\square$

7.18 Corollary. *For any recursively regular κ and any $R \subseteq {}^k\kappa$,*

(i) *if R is Σ-definable over L_κ, then R is κ-semi-recursive in parameters;*

(ii) *if R is Δ-definable over L_κ, then R is κ-recursive in parameters.*

Proof. It clearly suffices to prove (i), so suppose $\mathfrak{A}$ is a Δ_0 formula and R is defined by

$$R(\boldsymbol{\mu}) \leftrightarrow (\mathrm{L}_\kappa, \in) \models \exists x\, \mathfrak{A}[\boldsymbol{\mu}, \boldsymbol{b}].$$

Let $\rho_0, \ldots, \rho_{l-1}$ be ordinals $< \kappa$ such that $(\forall j < l)$. ${}^\circ\rho_j = b_j$. Then

$$\begin{aligned}R(\boldsymbol{\mu}) &\leftrightarrow (\exists \nu < \kappa)\,.\,(\mathrm{L}_\kappa, \in) \models \mathfrak{A}[{}^\circ\nu, {}^\circ\langle 0, \mu_0\rangle, \ldots, {}^\circ\langle 0, \mu_{k-1}\rangle, {}^\circ\rho_0, \ldots, {}^\circ\rho_{l-1}] \\ &\leftrightarrow (\exists \nu < \kappa)\mathrm{Sat}_{\mathfrak{A},\kappa}(\nu, \langle 0, \mu_0\rangle, \ldots, \langle 0, \mu_{k-1}\rangle, \rho_0, \ldots, \rho_{l-1}).\end{aligned}$$

Thus R is κ-semi-recursive in the parameters $\rho_0, \ldots, \rho_{l-1}$. $\square$

7.19 Corollary. *For any κ, if κ is recursively regular, then L_κ is admissible.*

Proof. It is trivial to check that for any limit ordinal κ, $(\mathrm{L}_\kappa, \in)$ satisfies the pair, union, and Δ_0-Separation axioms. Suppose that κ is recursively regular, that $\mathfrak{A}$ is a Δ_0 formula in which b does not occur free, and that for some $\boldsymbol{\mu}$, $\rho < \kappa$,

$$(\mathrm{L}_\kappa, \in) \models (\forall x \in a)\exists y\, \mathfrak{A}[{}^\circ\rho, {}^\circ\boldsymbol{\mu}].$$

Then

$$(\forall x \in {}^\circ\rho)(\exists \sigma < \kappa)\,.\,(\mathrm{L}_\kappa, \in) \models \mathfrak{A}[{}^\circ\sigma, {}^\circ\boldsymbol{\mu}]$$

and

$$(\forall \pi < \rho)(\exists \sigma < \kappa)[\mathrm{Tr}(\langle 4, \pi, \rho\rangle) \rightarrow \mathrm{Sat}_{\mathfrak{A},\kappa}(\sigma, \boldsymbol{\mu})].$$

By the recursive regularity of κ and Theorem 7.14 and 7.17, if

$$\tau = \sup^{+}_{\pi<\rho}.\ \text{least}\ \sigma[\mathrm{Tr}(\langle 4, \pi, \rho\rangle) \to \mathrm{Sat}_{\mathfrak{A},\kappa}(\sigma, \boldsymbol{\mu})],$$

then $\tau < \kappa$, and if

$$\lambda = \sup^{+}_{\sigma<\tau}\mathrm{Dg}(\sigma),$$

then also $\lambda < \kappa$. Furthermore,

$$(\forall x \in {}^{\circ}\rho)\, \exists\sigma\, ({}^{\circ}\sigma \in \mathrm{L}_\lambda \text{ and } (\mathrm{L}_\kappa, \in) \models \mathfrak{A}[{}^{\circ}\sigma, {}^{\circ}\boldsymbol{\mu}]),$$

so

$$(\mathrm{L}_\kappa, \in) \models (\forall x \in a)(\exists y \in b)\mathfrak{A}[\mathrm{L}_\lambda, {}^{\circ}\rho, {}^{\circ}\boldsymbol{\mu}],$$

where b is interpreted as L_λ. Thus we have the conclusion of Δ_0-Collection. □

Finally we draw some interesting conclusions about relations on numbers.

7.20 Theorem. *For all $R \subseteq {}^{k}\omega$, the following are equivalent*:

(i) *R is ∞-recursive*;
(ii) *R is δ^1_2-recursive*;
(iii) *R is Δ^1_2*;
(iv) *$R \in \mathrm{L}_{\delta^1_2}$*.

Proof. The equivalence of (i), (ii), and (iii) is Corollary 5.10. If R is δ^1_2-recursive, then by Corollary 7.8, R is Δ-definable over $\mathrm{L}_{\delta^1_2}$. Since δ^1_2 is recursively regular, $\mathrm{L}_{\delta^1_2}$ is admissible and satisfies Δ-Separation. Hence $R \in \mathrm{L}_{\delta^1_2}$. On the other hand, if $R \in \mathrm{L}_{\delta^1_2}$, then by Corollary 7.15, R is δ^1_2-recursive in parameters. But by Exercise 3.14 (ii), R is then δ^1_2-recursive. □

7.21 Theorem. *For all $R \subseteq {}^{k}\omega$, the following are equivalent*:

(i) *R is ∞-recursive in parameters*;
(ii) *R is $\aleph_1$-recursive in parameters*;
(iii) *R is constructible*;
(iv) *$R \in \mathrm{L}_{\aleph_1}$*.

Proof. We prove that (ii)→(iv)→(iii)→(i)→(ii). If R is $\aleph_1$-recursive in parameters, then R is Δ-definable over $\mathrm{L}_{\aleph_1}$ and hence by Δ-Separation, $R \in \mathrm{L}_{\aleph_1}$. That (iv)→(iii) is trivial. That (iii)→(i) is immediate from Corollary 7.15. The implication (i)→(ii) is a weak version of Exercise 5.21. □

Note that using the full strength of Exercise 5.21 we could replace (ii) and (iv) by:

(ii)′ R is $\delta_2^1[R]$ recursive in parameters;
(iv)′ $R \in \mathrm{L}_{\delta_2^1[R]}$.

Note also that the equivalence of (ii) and (iii) implies that there are at most $\aleph_1$ constructible subsets of ω. Hence if $\mathrm{V} = \mathrm{L}$ then the Continuum Hypothesis holds.

7.22–7.29 Exercises

7.22. Prove Lemma 7.13.

7.23. Σ-*Separation* is the set of axioms

$$\exists b\, \forall x\, (x \in b \leftrightarrow x \in a \wedge \mathfrak{A})$$

for all Σ formulas $\mathfrak{A}$ in which b does not occur free. Show that for any admissible set M, $(M, \in)$ satisfies Σ-separation iff $(M, \in)$ satisfies the following schema of *strong* Δ_0-Collection:

$$\exists b\, (\forall x \in a)[\exists y\, \mathfrak{A} \rightarrow (\exists y \in b)\mathfrak{A}]$$

for all Δ_0 formulas $\mathfrak{A}$ in which b does not occur free.

7.24. Show that for any recursively regular κ, κ is non-projectible iff $(\mathrm{L}_\kappa, \in)$ satisfies Σ-Separation.

7.25. Show that for any recursively regular ordinal λ and any $\kappa \leqslant \lambda$, κ is λ-stable iff $(\mathrm{L}_\kappa, \in)$ is a Σ-*substructure* of $(\mathrm{L}_\lambda, \in)$ — that is, for any Σ formula $\mathfrak{A}$ with free variables among $x_0, \ldots, x_{k-1}$ and any $a_0, \ldots, a_{k-1} \in \mathrm{L}_\kappa$,

$$(\mathrm{L}_\kappa, \in) \models \mathfrak{A}[\boldsymbol{a}] \quad \text{iff} \quad (\mathrm{L}_\lambda, \in) \models \mathfrak{A}[\boldsymbol{a}]$$

(cf. Exercise 5.16).

7.26. Show that the special names $\dot{\sigma}$ in $\mathscr{L}$ are superfluous. Specifically, let $\mathscr{L}^-$ be the language obtained from $\mathscr{L}$ by omitting the names $\dot{\sigma}$. Show that for each ordinal σ there is a closed term t_σ of $\mathscr{L}^-$ of degree $\sigma + 1$ such that $\mathrm{Val}(t_\sigma) = \sigma$. Describe an assignment of Gödel numbers to $\mathscr{L}^-$ which makes $\lambda\sigma.\mathrm{gn}(t_\sigma)$ an $(\infty, 0)$-recursive function and check that all of the remaining results go through.

7.27. Define the classes of $\exists_r^0$ and $\forall_r^0$ formulas of $\mathscr{L}_{\mathrm{ZF}}$ as in Definition III.5.3, with $\exists_0^0 = \forall_0^0 = \Delta_0$. Note that if M is admissible, then every Σ formula is equivalent in M to an $\exists_1^0$ formula. Show that for any κ,

(i) κ is recursively regular iff L_κ is $\forall_2^0$-*reflecting* — that is, for any $\forall_2^0$ formula $\mathfrak{A}$ and any $\mathbf{u} \in \mathrm{L}_\kappa$, if $(\mathrm{L}_\kappa, \in) \models \mathfrak{A}[\mathbf{u}]$, then $(\exists \lambda < \kappa).(\mathrm{L}_\lambda, \in) \models \mathfrak{A}[\mathbf{u}]$;

(ii) κ is recursively Mahlo iff L_κ is $\forall_2^0$-*reflecting on the class of recursively regular ordinals* — that is, as above except that λ is required to be recursively regular.

7.28. Show that the Hypothesis of Constructibility is equivalent to: for all infinite cardinals κ, for any $\lambda < \kappa$, every subset of λ is κ-recursive in parameters.

7.29 (Shoenfield [1961]). Show that every Σ_2^1 relation $R \subseteq {}^k\omega$ is constructible.

7.30 Notes. The concepts and results of §§6 and 7 are again due to Kripke [1964] and Platek [1966]. Lévy [1965] first isolated the classes of Δ_0 and Σ formulas.