Chapter VII
Recursion in a Type-3 Functional

We first hasten to assure the reader that this chapter is not the second in an infinite sequence. Although there are several important differences between the theories of recursion relative to functionals of types 2 and 3, most of the theory of recursion relative to functionals of types greater than 3 can be obtained from type-3 theory with essentially only notational changes. This is discussed in § 4.

In § 1 we consider the basic definitions and facts about recursion in a type-3 functional and examples which illustrate the differences between types 2 and 3. For example, although E and OJ are each recursive in the other, the same is not true of their type-3 analogues $\mathbb{E}$ and $\mathbf{s}\mathbb{J}$. In § 2 we see that although the basic structure of the class of relations semi-recursive in a type-3 functional is superficially similar to the corresponding structure of type-2, the differences begin to be more important. Finally in § 3 we see that with respect to hierarchies the situation for recursion in a type-3 functional is very different from that for type-2.

1. Basic Properties

We shall consider in detail only the notion of recursion relative to a single fixed total type-3 functional $\mathbb{I}$: ${}^{({}^{\omega}\omega)}\omega \to \omega$. From this may easily be derived, by the usual sorts of coding, notions of recursion relative to several type-3 functionals and, as in § VI.7, the notion of a recursive type-4 functional. Although we are still primarily interested in the properties of relations and functionals over ${}^{k,l}\omega$, we shall also need the notion of relative recursiveness among type-3 functionals. For this reason we state the basic definitions in terms of functionals and relations over ${}^{k,l,l'}\omega$.

Consider first the intuitive notion of a functional $\mathbb{F}$ being calculable relative to $\mathbb{I}$. We must stretch our imagination one step further to conceive of an idealized computer prepared to receive inputs of the form $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$ and connected to a memory device M which contains the graph of $\mathbb{I}$. The inputs are considered to be stored before the beginning of the computation in infinite memory devices, which may be consulted at any time during the computation. Arguments β are presented to one of the $\mathsf{I}_{j'}$ just as before via a subsidiary infinite memory unit M' which is "loaded" with the graph of β. To apply $\mathbb{I}$, the computer requires another memory unit M'' which can be "loaded" with the graph of a functional F in order to obtain the value of $\mathbb{I}(\mathsf{F})$. As before, the "loading" of M' and M'' is considered to occur during the computation, which is in general therefore infinite in length. In fact, since it takes at least $2^{\aleph_0}$ "steps" to load M'' with the graph of a functional F, computations relative to $\mathbb{I}$ are generally uncountable. Clearly, the value of a computation depends on values $\mathbb{I}(\mathsf{F})$ only for F which are themselves computable from $\mathbb{I}$, $\mathbf{I}$, and $\boldsymbol{\alpha}$.

The precise definition is obtained from Definition VI.7.1 by adding a clause which insures that

$$\{\langle 4, k, l, l'\ b\rangle\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \mathbb{I}(\lambda\beta\,.\,\{b\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})).$$

1.1 Definition. For any total functional $\mathbb{I}: {}^{({}^{\omega}\omega)}\omega \to \omega$, $\Omega[\mathbb{I}]$ is the smallest set such that for all k, l, l', n, p, q, r, and s, *all* $i<k$, $j<k$, and $j'<l'$, and all $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \in {}^{k,l,l'}\omega$,

(0)
(1)
(2)
(3)
identical to the corresponding clauses of Definition VI. 7.1;

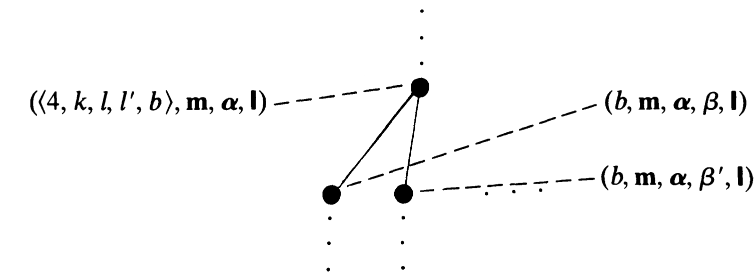
(4) for any b and any F, if for all β, $(b, \mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}, \mathsf{F}(\beta)) \in \Omega[\mathbb{I}]$, then $(\langle 4, k, l, l', b\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \mathbb{I}(\mathsf{F})) \in \Omega[\mathbb{I}]$.

The definition of $\Omega[\mathbb{I}]$ is easily interpreted as a closure under functions of rank $2^{\aleph_0}$ so that the closure ordinal is at most the least cardinal greater than $2^{\aleph_0}$. Without the Axiom of Choice we can conclude that the closure ordinal is at most $o({}^{\omega}\omega)$. The reader should be now find it obvious that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and $\mathbf{I}$, there is at most one n such that $(a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, n) \in \Omega[\mathbb{I}]$. We set

$$\{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq n \quad \text{iff} \quad (a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, n) \in \Omega[\mathbb{I}].$$

A functional $\mathbb{F}$ is *partial recursive in* $\mathbb{I}$ iff $\mathbb{F} = \{a\}^{\mathbb{I}}$ for some $a \in \omega$, etc. Since $\Omega^3 \subseteq \Omega[\mathbb{I}]$, every recursive functional $\mathbb{F}$ is recursive in $\mathbb{I}$. The $\mathbb{I}$-Recursion Theorem is proved exactly as in all other cases and it follows that the class of functionals partial recursive in $\mathbb{I}$ is closed under primitive and course-of-values recursion and unbounded search. It follows from Corollary II.3.3 that every partial recursive type-2 functional F is also partial recursive in $\mathbb{I}$ and by a slightly more complicated proof that the same is true for type-3 functionals $\mathbb{F}$. Definition by cases with relations recursive in $\mathbb{I}$ and functionals partial recursive in $\mathbb{I}$ is established as is Theorem II.2.12. The class of relations recursive in $\mathbb{I}$ is a Boolean algebra closed under composition with functionals recursive in $\mathbb{I}$.

Computations relative to $\mathbb{I}$ may be thought of as arranged in labeled trees. Nodes corresponding to clauses (0)–(3) are as before, while for clause·(4) we have nodes of the form:



where $\beta, \beta', \ldots$ includes all members of ${}^{\omega}\omega$. Of course, it is not necessary that the successor nodes be arranged in a well-ordered sequence.

Every relation recursive in $\mathbb{I}$ is also semi-recursive in $\mathbb{I}$ but not conversely. The class of relations semi-recursive in $\mathbb{I}$ is closed under finite intersection and bounded universal number quantification. Analogously to Lemma VI.1.2, the class of relations semi-recursive in $\mathbb{I}$ is seen to be closed under universal function quantification ($\forall^1$) and hence also under universal number quantification ($\forall^0$).

To illustrate the theory, we consider the type-3 analogues of the functionals E and oJ:

$$\mathbb{E}(\mathsf{I}) = \begin{cases} 0, & \text{if } \exists\alpha . \mathsf{I}(\alpha) = 0; \\ 1, & \text{otherwise;} \end{cases}$$

$$\mathsf{s}\mathbb{J}((a, \mathbf{m}, \alpha) * \mathsf{I}) = \begin{cases} 0, & \text{if } \{a\}^{\mathsf{I}}(\mathbf{m}, \alpha) \text{ is defined;} \\ 1, & \text{otherwise;} \end{cases}$$

where $(a, \mathbf{m}, \alpha) * \mathsf{I}$ is the functional F such that for any γ, $\mathsf{F}((0) * \gamma) = \mathsf{I}(\gamma)$, $\mathsf{F}((1) * \gamma) = \langle a, \mathbf{m}\rangle$, and for all p, $\mathsf{F}((p+2) * \gamma) = \alpha(p)$. Of course, $\mathsf{s}\mathbb{J}$ is just the natural encoding of the superjump:

$$\mathsf{s}\mathbb{J}((a, \mathbf{m}, \alpha) * \mathsf{I}) = \mathsf{I}^{\mathsf{sJ}}((\langle a, \mathbf{m}\rangle) * \alpha).$$

1.2 Lemma. *For all* $\mathbb{R}$, *if* $\mathbb{R}$ *is analytical, then* $\mathbb{R}$ *is recursive in* $\mathbb{E}$.

Proof. For any $\mathbb{R}$,

$$\mathbb{K}_{\exists^1\mathbb{R}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}) = \mathbb{E}(\lambda\beta . \mathbb{K}_{\mathbb{R}}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathsf{I}))$$

so that the class of relations recursive in $\mathbb{E}$ is closed under $\exists^1$. It follows by complementation that it is closed under $\forall^1$, hence also under $\exists^0$ and $\forall^0$. Since this class contans all recursive relations, it contains all arithmetical relations and hence all analytical relations. $\square$

1.3 Corollary. sJ *is recursive in* $\mathbb{E}$.

Proof. It suffices to show that (the graph of) sJ is analytical. For each I, let $\Gamma_{\alpha,\mathsf{I}}$ be a monotone operator defined as in the proof of Theorem VI.1.5 such that

$$\{a\}^{\mathsf{I}}(\mathbf{m},\alpha)\simeq n \leftrightarrow \langle a,\langle\mathbf{m}\rangle,n\rangle\in\bar{\Gamma}_{\alpha,\mathsf{I}}.$$

From the definition it is clear that the relation

$$\mathbb{R}(p,\alpha,A,\mathsf{I})\leftrightarrow p\in\Gamma_{\alpha,\mathsf{I}}(A)$$

is analytical. Since

$$p\in\bar{\Gamma}_{\alpha,\mathsf{I}}\leftrightarrow\forall A[\Gamma_{\alpha,\mathsf{I}}(A)\subseteq A\rightarrow p\in A],$$

also the relation

$$\mathbb{S}(p,\alpha,\mathsf{I})\leftrightarrow p\in\bar{\Gamma}_{\alpha,\mathsf{I}}$$

is analytical. Finally,

$$\begin{aligned}\mathsf{sJ}((a,\mathbf{m},\alpha)*\mathsf{I})=r\leftrightarrow&[r=0\wedge\exists n.\langle a,\langle\mathbf{m}\rangle,n\rangle\in\bar{\Gamma}_{\alpha,\mathsf{I}}]\vee\\&[r=1\wedge\forall n.\langle a,\langle\mathbf{m}\rangle,n\rangle\notin\bar{\Gamma}_{\alpha,\mathsf{I}}]\end{aligned}$$

and thus sJ is analytical. $\square$

The analogy with the results of § VI.1 suggests that we should expect also that $\mathbb{E}$ is recursive in sJ and that the class of relations recursive in either is Δ^2_1. This is, however, far from the case, and we show this next. For any $\mathbb{I}$, let

$$\mathbb{V}^{\mathbb{I}}(a,\langle\mathbf{m}\rangle,n,\langle\boldsymbol{\alpha}\rangle,\langle\mathbf{I}\rangle)\leftrightarrow(a,\mathbf{m},\boldsymbol{\alpha},\mathbf{I},n)\in\Omega[\mathbb{I}].$$

1.4 Theorem. *For any* $\mathbb{I}$ *and any* $r\geqslant 1$, *if* $\mathbb{I}\in\Delta^2_r$, *then* $\mathbb{V}^{\mathbb{I}}\in\Delta^2_r$.

Proof. For each I we define operators $\Gamma_{\mathsf{I},0},\ldots,\Gamma_{\mathsf{I},4}$ and $\Lambda_{\mathsf{I},4}$ much as in the proof of Theorem VI.1.5 but "one type up". For any $\mathsf{R}\subseteq\omega\times{}^{\omega}\omega$,

$$\Gamma_{\mathsf{I},0}(\mathsf{R}) = \{(\langle\langle 0,k,l,l',0,n\rangle,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) : (\mathbf{m},\boldsymbol{\alpha}) \in {}^{k,l}\omega\}$$
$$\cup \cdots \cup \{(\langle\langle 0,k+2,l,l',5\rangle,\langle p,q,\mathbf{m}\rangle,\mathrm{Sb}_0(p,q)\rangle,\langle\boldsymbol{\alpha}\rangle) : k,p,q \in \omega \wedge (\mathbf{m},\boldsymbol{\alpha}) \in {}^{k,l}\omega\}$$

$$\left.\begin{array}{l}\Gamma_{\mathsf{I},1}\\ \\ \Gamma_{\mathsf{I},2}\end{array}\right\} \begin{array}{l}\text{defined by similar modifications of the corresponding}\\ \\ \text{clauses of Theorem VI.1.5;}\end{array}$$

$$\Gamma_{\mathsf{I},3}(\mathsf{R}) = \{(\langle\langle 3,k,l,l',j',b\rangle,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) : j' < l' \wedge \exists\beta\,[\forall p\, \mathsf{R}(\langle b,\langle p,\mathbf{m}\rangle,\beta(p)\rangle,\langle\boldsymbol{\alpha}\rangle) \wedge \mathsf{I}_{j'}(\beta) = n]\};$$

$$\Gamma_{\mathsf{I},4}(\mathsf{R}) = \{(\langle\langle 4,k,l,l',b\rangle,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) : \exists\mathsf{F}\,[\forall\beta\, \mathsf{R}(\langle b,\langle\mathbf{m}\rangle,\mathsf{F}(\beta)\rangle,\langle\boldsymbol{\alpha},\beta\rangle) \wedge \mathbb{I}(\mathsf{F}) = n]\};$$

$$\Lambda_{\mathsf{I},4}(\mathsf{R}) = \{(\langle\langle 4,k,l,l',b\rangle,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) : \forall\beta\,\exists q\, \mathsf{R}(\langle b,\langle\mathbf{m}\rangle,q\rangle,\langle\boldsymbol{\alpha},\beta\rangle) \wedge \forall\mathsf{F}\,[\forall\beta\,\forall q\,(\mathsf{R}(\langle b,\langle\mathbf{m}\rangle,q\rangle,\langle\boldsymbol{\alpha},\beta\rangle) \rightarrow \mathsf{F}(\beta) = q) \rightarrow \mathbb{I}(\mathsf{F}) = n]\}.$$

Defining $\Gamma_{\langle\mathsf{I}\rangle}$ and $\Lambda_{\langle\mathsf{I}\rangle}$ as in VI.1.5, one can prove as before that

$$(a,\mathbf{m},\boldsymbol{\alpha},\mathsf{I},n) \in \Omega[\mathbb{I}] \leftrightarrow (\langle a,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) \in \bar{\Gamma}_{\langle\mathsf{I}\rangle}$$
$$\leftrightarrow (\langle a,\langle\mathbf{m}\rangle,n\rangle,\langle\boldsymbol{\alpha}\rangle) \in \bar{\Lambda}_{\langle\mathsf{I}\rangle}.$$

If $\mathbb{I} \in \Delta^2_r$, then clearly the decomposable operator Γ is Σ^2_r and Λ is Π^2_r. It then follows from VI.7.10 and VI.7.15 that $\mathbb{V}^{\mathbb{I}} \in \Delta^2_r$. □

This proof depends essentially on the fact that the subcomputations of a computation $\{a\}^{\mathbb{I}}(\mathbf{m},\boldsymbol{\alpha},\mathsf{I})$ all have the same list I of type-2 arguments, so that I may be treated as a parameter. The type-1 arguments may not be so treated, as subcomputations of $\{a\}^{\mathbb{I}}(\mathbf{m},\boldsymbol{\alpha},\mathsf{I})$ will in general involve longer lists $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of type-1 arguments. Of course, Theorem VI.1.5 works exactly because for computations relative to a type-2 functional I, the type-1 arguments do behave as parameters.

1.5 Corollary. *For any $r \geq 1$ and any $\mathbb{I} \in \Delta^2_r$,*
 (i) *if $\mathbb{R}$ is semi-recursive in $\mathbb{I}$, then $\mathbb{R} \in \Delta^2_r$;*
 (ii) $\{\mathbb{R} : \mathbb{R}$ is recursive in $\mathbb{I}\}$ *is a proper subset of* Δ^2_r. □

In particular, the class of relations recursive in $\mathbb{E}$ is a proper subset of Δ^2_1.

Before we consider further the relationship between $\mathbb{E}$ and $\mathbf{s}\mathbb{J}$, we discuss the substitution properties of recursion in a type-3 functional. The basic results are the following:

1.6 Theorem. *There exist primitive recursive functions f, g, and h such that for all $\mathbb{I}$, a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, and I,*

(i) $\{f(a, d)\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p . \{d\}^{\mathbb{I}}(p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}), \mathbf{I})$;
(ii) $\{g(a, d)\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \lambda \beta . \{d\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I}))$;
(iii) *if* $\lambda \mathsf{G} . \{d\}^{\mathbb{I}}(\mathsf{G})$ *is a total type*-3 *functional* $\mathbb{H}$, *then*

$$\{h(a, d)\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq \{a\}^{\mathbb{H}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}).$$

Proof. The proof of (i) is very similar to that of Theorem VI.2.10. (ii) should be seen as a uniform version of Theorem VI.2.14; we first define a function g_1 which works if $\lambda \beta . \{d\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})$ is total and then choose g such that

$$\{g(a, d)\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \simeq 0 \cdot \mathbb{I}(\lambda \beta . \{d\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{I})) + \{g_1(a, d)\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}).$$

(iii) is related to (ii) as is VI.2.14 to VI.2.10. □

1.7. Corollary. *For any* $\mathbb{F}$, $\mathbb{H}$, $\mathbb{I}$, β, *and* I, *if* $\mathbb{F}$ *is partial recursive in* β, I, *and* $\mathbb{H}$, *and* β, I, *and* $\mathbb{H}$ *are recursive in* $\mathbb{I}$, *then* $\mathbb{F}$ *is partial recursive in* $\mathbb{I}$. *In particular, the relation "recursive in" is transitive among objects of type*-3. □

Our next aim is to show that sJ is recursive in $\mathsf{E}_1^{\#}$. We have not in fact defined the notion of a type-3 functional being recursive in an extended type-2 functional. What is needed, of course, is a set $\Omega^3[\mathsf{E}_1^{\#}]$ defined exactly as in VI.7.1 but with an extra clause to handle application of $\mathsf{E}_1^{\#}$. We leave it to the reader to formulate explicitly these definitions and convince him- or herself that this notion has all of the usual properties. In particular, we shall need that if a functional $\mathbb{F}$ is recursive in $\mathbb{H}$ and $\mathbb{H}$ is recursive in $\mathsf{E}_1^{\#}$, then $\mathbb{F}$ is recursive in $\mathsf{E}_1^{\#}$.

1.8 Theorem. sJ *is recursive in* $\mathsf{E}_1^{\#}$.

Proof. The proof is essentially a uniform version of that of Theorem VI.6.11. First, the theory of recursion in $\mathsf{E}_1^{\#}$ must be extended to allow for type-2 arguments. In particular, there are sets $M_{\boldsymbol{\alpha},\mathsf{I}}$ and a primitive recursive function f similar to that of Corollary VI.6.8 such that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, I and n,

$$\{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}) \text{ is defined} \leftrightarrow f(a, \langle \mathbf{m} \rangle) \in M_{\boldsymbol{\alpha},\mathsf{I}}.$$

Corresponding to the ordinal comparison functional H of Theorem VI.6.9, there is a functional $\mathbb{H}$ partial recursive in $\mathsf{E}_1^{\#}$ such that for all u, v, $\boldsymbol{\alpha}$, and I,
(i) if $u \in M_{\boldsymbol{\alpha},\mathsf{I}}$ and $|u|^{\#}_{\boldsymbol{\alpha},\mathsf{I}} \leq |v|^{\#}_{\boldsymbol{\alpha},\mathsf{I}}$, then $\mathbb{H}(u, v, \langle \boldsymbol{\alpha} \rangle, \mathsf{I}) \simeq 0$;
(ii) if $v \in M_{\boldsymbol{\alpha},\mathsf{I}}$ and $|v|^{\#}_{\boldsymbol{\alpha},\mathsf{I}} < |u|^{\#}_{\boldsymbol{\alpha},\mathsf{I}}$, then $\mathbb{H}(u, v, \langle \boldsymbol{\alpha} \rangle, \mathsf{I}) \simeq 1$.

Now, by the methods of Theorem VI.6.11, construct primitive recursive functions g and h such that

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \text{ is defined} \leftrightarrow \{g(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}) \text{ is defined},$$

and

$$\{a\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \text{ is undefined} \leftrightarrow \{h(a)\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}) \text{ is defined.}$$

Then we claim that

$$\mathsf{sJ}((a, \mathbf{m}, \alpha) * \mathsf{I}) = \mathbb{H}(f(g(a), \langle \mathbf{m} \rangle), f(h(a), \langle \mathbf{m} \rangle), \langle \alpha \rangle, \mathsf{I}).$$

If $\{a\}^{\mathsf{I}}(\mathbf{m}, \alpha)$ is defined, then $f(g(a), \langle \mathbf{m} \rangle) \in M_{\alpha, \mathsf{I}}$ and $f(h(a), \langle \mathbf{m} \rangle) \notin M_{\alpha, \mathsf{I}}$, so

$$|f(g(a), \langle \mathbf{m} \rangle)|_{\alpha, \mathsf{I}}^{\#} < |f(h(a), \langle \mathbf{m} \rangle)|_{\alpha, \mathsf{I}}^{\#}$$

and thus $\mathbb{H}(f(g(a), \langle \mathbf{m} \rangle), f(h(a), \langle \mathbf{m} \rangle), \langle \alpha \rangle, \mathsf{I}) = 0$. If $\{a\}^{\mathsf{I}}(\mathbf{m}, \alpha)$ is undefined, then the inequality is reversed and the value is 1, as required. □

1.9 Corollary.
(i) *The class of relations recursive in* sJ *is a proper subset of* Δ_2^1;
(ii) $\mathbb{E}$ *is not recursive in* sJ.

Proof. Since the graph of $\mathsf{E}_1^{\#}$ is Δ_2^1, there is a Δ_2^1 decomposable operator Γ such that for all a, $\mathbf{m}$, $\boldsymbol{\alpha}$, and n

$$\langle a, \mathbf{m}, n \rangle \in \bar{\Gamma}_{\boldsymbol{\alpha}} \leftrightarrow \{a\}^{\mathsf{E}_1^{\#}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n$$

(cf. Exercise VI.6.20). In particular every relation semi-recursive in $\mathsf{E}_1^{\#}$ is Δ_2^1 and the class of relations recursive in $\mathsf{E}_1^{\#}$ is a proper subclass of Δ_2^1. (i) follows by Theorem 1.8 and the remarks preceding it. Then (ii) is immediate from Lemma 1.2 and Corollary 1.7. □

Note that it is *not* true that every relation semi-recursive in sJ is semi-recursive in $\mathsf{E}_1^{\#}$. E_1 is equivalent to E^{sJ} (Exercise VI.1.20), hence E_1 is recursive in sJ and thus every Σ_1^1 relation is recursive in sJ. Since the class of relations semi-recursive in sJ is closed under $\forall^1$, every Π_2^1 relation is semi-recursive in sJ. From this we can also conclude that there are relations R such that both R and $\sim R$ are semi-recursive in sJ, but R is not recursive in sJ. We shall see in the next section that a functional in which $\mathbb{E}$ is recursive does not exhibit this pathological behavior.

1.10–1.13 Exercises

1.10. Show that any partial recursive type-3 functional $\mathbb{F}$ is partial recursive in any type-3 $\mathbb{I}$.

1.11. Show in set theory without the Axiom of Choice that the closure ordinal of the monotone operator which defines $\Omega[\mathbb{I}]$ is at most $o({}^{\omega}\omega)$, the least ordinal not the type of a pre-wellordering of ${}^{\omega}\omega$.

1.12. Fill in some of the details in the proof of Theorem 1.6.

1.13. Define $\mathbf{b}\mathbb{E}$, the *bounded type-3 quantifier functional* by:

$$\mathbf{b}\mathbb{E}(\mathsf{I}) = \begin{cases} 0, & \text{if } \exists\alpha\,[\alpha \text{ recursive in } \mathsf{E}, \mathsf{I} \wedge \mathsf{I}(\alpha) = 0]; \\ 1, & \text{otherwise.} \end{cases}$$

Show that $\mathbf{b}\mathbb{E}$ and $\mathbf{s}\mathbb{J}$ are each recursive in the other. (That $\mathbf{b}\mathbb{E}$ is recursive in $\mathbf{s}\mathbb{J}$ is easy. For the converse let P be defined as in the proof of Lemma VI.6.1 and $<_{\boldsymbol{\alpha}}$ the transitive closure of the relation $<'_{\boldsymbol{\alpha}}$ defined by:

$$v <'_{\boldsymbol{\alpha}} u \leftrightarrow \mathsf{P}(u, v, \langle\boldsymbol{\alpha}\rangle).$$

(cf. Exercise VI.6.26). For any $\langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}}$, let $\delta_{a,\mathbf{m},\boldsymbol{\alpha}}$ be the characteristic function of $<_{\boldsymbol{\alpha}}$ restricted to $u \leqslant_{\boldsymbol{\alpha}} \langle a, \mathbf{m}\rangle$; otherwise, $\delta_{a,\mathbf{m},\boldsymbol{\alpha}} = \lambda p.0$. Show that $\mathsf{C}^{\mathsf{I}}_{\boldsymbol{\alpha}} = \{\delta\colon \text{for some } \langle a, \mathbf{m}\rangle \in U^{\mathsf{I}}_{\boldsymbol{\alpha}},\ \delta = \delta_{a,\mathbf{m},\boldsymbol{\alpha}}\}$ is recursive in E_1 and I, uniformly in $\boldsymbol{\alpha}$. Let

$$\mathsf{I}_{a,\mathbf{m},\boldsymbol{\alpha}}(\delta) = \begin{cases} 0, & \text{if} \quad \delta \in \mathsf{C}^{\mathsf{I}}_{\boldsymbol{\alpha}} \quad \text{and} \quad \delta = \delta_{a,\mathbf{m},\boldsymbol{\alpha}}; \\ 1, & \text{if} \quad \delta \in \mathsf{C}^{\mathsf{I}}_{\boldsymbol{\alpha}} \quad \text{and} \quad \delta \neq \delta_{a,\mathbf{m},\boldsymbol{\alpha}}; \\ 2, & \text{otherwise.} \end{cases}$$

Show that every function recursive in E and I is recursive in $\mathsf{I}_{a,\mathbf{m},\boldsymbol{\alpha}}$ and that

$$\mathbf{s}\mathbb{J}((a, \mathbf{m}, \boldsymbol{\alpha}) * \mathsf{I}) = \mathbf{b}\mathbb{E}(\mathsf{I}_{a,\mathbf{m},\boldsymbol{\alpha}}).\,)$$

1.14. Notes. The basic definition here is again due to Kleene [1959]. Theorem 1.4 for $\mathbb{E}$ was announced in Hinman [1964] and proved in Hinman [1966] but never published. The substitution theorems (1.6) are again due to Kleene [1963] with improvements from Hinman [1966]. Theorem 1.8 appears in Aczel [1970] where it is attributed to Gandy. Corollary 1.9 and Exercise 1.13 are due to Gandy [1967a]. Some variations on Exercise 1.13 appear in Aczel–Hinman [1974].

2. *Relations Semi-Recursive in a Type-3 Functional*

When $\mathbb{E}$ is recursive in $\mathbb{I}$, the class of functionals semi-recursive in $\mathbb{I}$ is similar in structure to the class of relations semi-recursive in a type-2 functional. In most instances, $\mathbb{E}$ plays a role analogous to that of E. This analogy fails at one point —

it is not in general true that the class of relations semi-recursive in $\mathbb{I}$ is closed under $\exists^1$. This fact is closely related to the considerations of § VI.6.

For any $\mathbb{I}$, we set

$$\mathbb{U}^{\mathbb{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{l} \rangle) \leftrightarrow \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{l}) \text{ is defined;}$$

$$\mathsf{U}^{\mathbb{I}}_{\mathbf{l}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) \leftrightarrow \mathbb{U}^{\mathbb{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{l} \rangle);$$

$$\mathsf{U}^{\mathbb{I}} = \mathsf{U}^{\mathbb{I}}_{\varnothing}.$$

Of course, $\mathbb{U}^{\mathbb{I}}$ is universal for the class of relations semi-recursive in $\mathbb{I}$. To each element of $\mathbb{U}^{\mathbb{I}}$ we assign an ordinal which measures the "length" of the corresponding computation:

$$|a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{l}|^{\mathbb{I}} = \text{least } \sigma\,.\,(a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{l}, \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{l})) \in \Omega[\mathbb{I}]^{\sigma}.$$

If $\sim\mathbb{U}^{\mathbb{I}}(a, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \mathbf{l} \rangle)$, we set $|a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{l}|^{\mathbb{I}} = o({}^{\omega}\omega)$. Then an appropriate modification of Lemma VI.3.2 holds with the additional clause:

(4) if for some b, $a = \langle 4, k, l, l', b \rangle$ and $\{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{l})$ is defined, then $|a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{l}|^{\mathbb{I}} = \sup^{+}\{|b, \mathbf{m}, \boldsymbol{\alpha}, \beta, \mathbf{l}|^{\mathbb{I}} : \beta \in {}^{\omega}\omega\}$.

In the following we shall write (as in several earlier sections) x^i for $(a^i, \langle \mathbf{m}^i \rangle, \langle \boldsymbol{\alpha}^i \rangle)$ and $\mathbb{H}(x^0, x^1, \langle \mathbf{l} \rangle)$ for $\mathbb{H}(a^0, a^1, \langle \mathbf{m}^0 \rangle, \langle \mathbf{m}^1 \rangle, \langle \boldsymbol{\alpha}^0 \rangle, \langle \boldsymbol{\alpha}^1 \rangle, \langle \mathbf{l} \rangle)$. We take $\mathbb{I}$ to be a fixed type-3 functional such that $\mathbb{E}$ is recursive in $\mathbb{I}$, and write $|x|^{\mathbb{I}}_{\mathbf{l}}$ for $|x, \mathbf{l}|^{\mathbb{I}}$.

2.1 Ordinal Comparison Theorem. *There exists a functional $\mathbb{H}$ partial recursive in $\mathbb{I}$ such that for all x^0, x^1, and $\mathbf{l}$,*

(i) *if $x^0 \in \mathsf{U}^{\mathbb{I}}_{\mathbf{l}}$ and $|x^0|^{\mathbb{I}}_{\mathbf{l}} \leq |x^1|^{\mathbb{I}}_{\mathbf{l}}$, then $\mathbb{H}(x^0, x^1, \langle \mathbf{l} \rangle) \simeq 0$;*
(ii) *if $x^1 \in \mathsf{U}^{\mathbb{I}}_{\mathbf{l}}$ and $|x^1|^{\mathbb{I}}_{\mathbf{l}} < |x^0|^{\mathbb{I}}_{\mathbf{l}}$, then $\mathbb{H}(x^0, x^1, \langle \mathbf{l} \rangle) \simeq 1$.*

Proof. We shall not give many details here as the reader may easily supply most of them by translating the proof of Theorem VI.3.3, the Ordinal Comparison Theorem for recursion relative to a type-2 functional. For simplicity let $\mathbf{l} = \varnothing$ and write $\mathbb{H}(x^0, x^1)$ for $\mathbb{H}(x^0, x^1, \langle\ \rangle)$. $\mathbb{H}$ is defined by effective transfinite recursion via the $\mathbb{I}$-Recursion Theorem and a functional $\mathbb{F}$ partial recursive in $\mathbb{I}$. $\mathbb{F}$ is defined in 25 cases labeled (r, s) for $0 \leq r, s \leq 4$ plus two "otherwise" cases. In all cases except those in which $r = 4$ or $s = 4$, the definition is almost exactly as before. We shall consider cases (2,4) and (3,4) and leave the remainder to the reader.

Case (2,4). Suppose $x^0 = (\langle 2, k^0 + 1, l^0, 0 \rangle, \langle b^0, \mathbf{m}^0 \rangle, \langle \boldsymbol{\alpha}^0 \rangle)$ and $x^1 = (\langle 4, k^1, l^1, 0, b \rangle, \langle \mathbf{m}^1 \rangle, \langle \boldsymbol{\alpha}^1 \rangle)$. We define $\mathbb{F}$ in such a way as to ensure that

$$\mathbb{H}(x^0, x^1) \simeq \mathbb{E}(\lambda\beta\,.\,\mathbb{H}((b^0, \langle \mathbf{m}^0 \rangle, \langle \boldsymbol{\alpha}^0 \rangle), (b^1, \langle \mathbf{m}^1 \rangle, \langle \boldsymbol{\alpha}^1, \beta \rangle))).$$

If $x^0 \in \mathsf{U}^{\mathbb{I}}$ and $|x^0|^{\mathbb{I}} \leqslant |x^1|^{\mathbb{I}}$, then for some β,

$$|b^0, \mathbf{m}^0, \boldsymbol{\alpha}^0|^{\mathbb{I}} \leqslant |b^1, \mathbf{m}^1, \boldsymbol{\alpha}^1, \beta|^{\mathbb{I}}$$

and thus $\mathbb{H}(x^0, x^1) \simeq 0$. If $x^1 \in \mathsf{U}^{\mathbb{I}}$ and $|x^1|^{\mathbb{I}} < |x^0|^{\mathbb{I}}$, then for all β,

$$|b^1, \mathbf{m}^1, \boldsymbol{\alpha}^1, \beta|^{\mathbb{I}} < |b^0, \mathbf{m}^0, \boldsymbol{\alpha}^0|^{\mathbb{I}}$$

and $\mathbb{H}(x^0, x^1) \simeq 1$.

Case (3,4). Suppose $x^0 = (\langle 3, k^0, l^0, 0, b^0\rangle, \langle \mathbf{m}^0\rangle, \langle \boldsymbol{\alpha}^0\rangle)$ and $x^1 = (\langle 4, k^1, l^1, 0, b^1\rangle, \langle \mathbf{m}^1\rangle, \langle \boldsymbol{\alpha}^1\rangle)$. We define $\mathbb{F}$ in such a way that

$$\mathbb{H}(x^0, x^1) \simeq \mathsf{E}^\circ(\lambda p\,.\,\mathbb{E}(\lambda\beta\,.\,\mathbb{H}((b^0, \langle p, \mathbf{m}^0\rangle, \langle \boldsymbol{\alpha}^0\rangle), (b^1, \langle \mathbf{m}^1\rangle, \langle \boldsymbol{\alpha}^1, \beta\rangle)))))$$

where E° is the "dual" of E, which is, of course, recursive in $\mathbb{E}$ and hence in $\mathbb{I}$. If $x^0 \in \mathsf{U}^{\mathbb{I}}$ and $|x^0|^{\mathbb{I}} \leqslant |x^1|^{\mathbb{I}}$, then for all p there exists a β such that

$$|b^0, p, \mathbf{m}^0, \boldsymbol{\alpha}^0|^{\mathbb{I}} \leqslant |b^1, \mathbf{m}^1, \boldsymbol{\alpha}^1, \beta|^{\mathbb{I}}$$

and thus $\mathbb{H}(x^0, x^1) \simeq 0$. If $x^1 \in \mathsf{U}^{\mathbb{I}}$ and $|x^1|^{\mathbb{I}} < |x^0|^{\mathbb{I}}$, then for some p and all β,

$$|b^1, \mathbf{m}^1, \boldsymbol{\alpha}^1, \beta|^{\mathbb{I}} < |b^0, p, \mathbf{m}^0, \boldsymbol{\alpha}^0|^{\mathbb{I}}$$

and $\mathbb{H}(x^0, x^1) \simeq 1$. $\square$

2.2 Selection Theorem. *There exists a function* $\mathrm{Sel}^{\mathbb{I}}$ *partial recursive in* $\mathbb{I}$ *such that for all* a, $\mathbf{m}$, $\boldsymbol{\alpha}$, *and* $\mathbf{I}$, *the following are equivalent*:

(i) $\exists p\,.\,\{a\}^{\mathbb{I}}(p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$ *is defined*;

(ii) $\{a\}^{\mathbb{I}}(\mathrm{Sel}^{\mathbb{I}}(a, \langle \mathbf{m}\rangle, \langle \boldsymbol{\alpha}\rangle, \langle \mathbf{I}\rangle), \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$ *is defined.*

Proof. Exactly as for Theorem VI.4.1. $\square$

2.3 Corollary. *For any relation* $\mathbb{R}$ *semi-recursive in* $\mathbb{I}$, *there exists a functional* $\mathrm{Sel}_{\mathbb{R}}$ *partial recursive in* $\mathbb{I}$ *such that for all* $\mathbf{m}$, $\boldsymbol{\alpha}$, *and* $\mathbf{I}$,

$$\exists p\, R(p, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \leftrightarrow \mathbb{R}(\mathrm{Sel}_{\mathbb{R}}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}), \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}). \quad \square$$

We may now derive corollaries corresponding to VI.4.3–6, 8–9 with very similar proofs, which we leave to the reader. The boundedness and hierarchy results take a little more care. Let

$$\boldsymbol{\kappa}^{\mathbb{I}} = \sup{}^{+}\{|x|^{\mathbb{I}} : x \in \mathsf{U}^{\mathbb{I}}\}.$$

Since $\mathsf{U}^{\mathbb{I}}$ is of power $2^{\aleph_0}$ there is no reason to expect that $\boldsymbol{\kappa}^{\mathbb{I}}$ is a countable ordinal (cf. Exercise 2.15). As in § V.1 it will be useful to consider the ordinals associated with the "countable part" of $\mathsf{U}^{\mathbb{I}}$. Let

$$U^{\mathbb{I}} = \{\langle a, \mathbf{m}\rangle : \mathsf{U}^{\mathbb{I}}(a, \langle \mathbf{m}\rangle, \langle\quad\rangle)\}.$$

The pre-wellordering on $\mathsf{U}^{\mathbb{I}}$ associated with the norm $|\quad|^{\mathbb{I}}$ induces a pre-wellordering on $U^{\mathbb{I}}$ in an obvious way. We denote by $|\quad|_0^{\mathbb{I}}$ the norm associated with this pre-wellordering and set

$$\kappa^{\mathbb{I}} = \sup{}^{+}\{|a, \mathbf{m}|_0^{\mathbb{I}} : U^{\mathbb{I}}(\langle a, \mathbf{m}\rangle)\}.$$

Note that $U^{\mathbb{I}}$ is *not* an initial segment of $\mathsf{U}^{\mathbb{I}}$ (Exercise 2.18) so $|\quad|_0^{\mathbb{I}}$ is not simply the restriction of $|\quad|^{\mathbb{I}}$ to $U^{\mathbb{I}}$. We use the notations $\mathsf{U}_\rho^{\mathbb{I}}$ and $U_\rho^{\mathbb{I}}$ as in § V.1.

2.4 Boundedness Theorem. (i) *for any A co-semi-recursive in $\mathbb{I}$, if $A \subseteq U^{\mathbb{I}}$, then* $\sup^{+}\{|u|_0^{\mathbb{I}} : u \in A\} < \kappa^{\mathbb{I}}$;

(ii) *for any β and any R co-semi-recursive in $\mathbb{I}$ and β, if $\mathsf{R} \subseteq \mathsf{U}^{\mathbb{I}}$, then* $\sup^{+}\{|x|^{\mathbb{I}} : x \in \mathsf{R}\} < \boldsymbol{\kappa}^{\mathbb{I}}$.

Proof. The proof of Theorem V.1.5 applies because the class of relations co-semi-recursive in $\mathbb{I}$ is closed under both $\exists^0$ and $\exists^1$. □

2.5 Hierarchy Theorem. (i) *for all relations R on numbers, R is recursive in $\mathbb{I}$ iff $R \leqslant U_\rho^{\mathbb{I}}$ for some $\rho < \kappa^{\mathbb{I}}$;*

(ii) *for all relations R, R is recursive in $\mathbb{I}$ and some β iff $\mathsf{R} \lll \mathsf{U}_\rho^{\mathbb{I}}$ for some $\rho < \boldsymbol{\kappa}^{\mathbb{I}}$.* □

Part (ii) of this theorem of course also provides a hierarchy for the relations R which are recursive in $\mathbb{I}$ alone, but it is somewhat unnatural in that although there are only countably many such R, the hierarchy is indexed by uncountably many ordinals, so that "most" of the levels are empty. The same phenomenon occurs at the odd levels beyond the first of the analytical hierarchy under the hypothesis PD (cf. Theorems V.1.6 and V.3.3 (iv)(b)). At the first level, of course, there is a natural hierarchy of length ω_1 for the class of Δ_1^1 relations of types 1 and 2 (Theorem IV.2.2); the situation is similar here. For each $\rho < \kappa^{\mathbb{I}}$, let $\bar{\rho}$ be the unique ordinal such that for some $u \in U^{\mathbb{I}}$, $|u|_0^{\mathbb{I}} = \rho$ and $|u|^{\mathbb{I}} = \bar{\rho}$.

2.6 Theorem. *For all relations R, R is recursive in $\mathbb{I}$ iff $\mathsf{R} \leqslant \mathsf{U}_{\bar{\rho}}^{\mathbb{I}}$ for some $\rho < \kappa^{\mathbb{I}}$.*

Proof. For any $\rho < \kappa^{\mathbb{I}}$ and any u such that $|u|_0^{\mathbb{I}} = \rho$,

$$x \in \mathsf{U}_{\bar{\rho}}^{\mathbb{I}} \leftrightarrow |x|^{\mathbb{I}} < \bar{\rho} = |u|^{\mathbb{I}} \leftrightarrow \mathbb{H}(x, (u, \langle\quad\rangle)) = 0,$$

where $\mathbb{H}$ is the ordinal comparison functional of Theorem 2.1. Hence $\mathsf{U}^{\mathbb{I}}_{\bar{\rho}}$ is recursive in $\mathbb{I}$ and thus so is any R reducible to it.

For the implication ($\rightarrow$), let R be recursive in $\mathbb{I}$, say with index a from $\mathbb{I}$, and let b and c be indices such that

$$\{b\}^{\mathbb{I}}(\mathbf{m}) \simeq \mathbb{I}(\lambda\alpha_0 \,.\, \mathbb{I}(\lambda\alpha_1 \cdots \mathbb{I}(\lambda\alpha_{l-1} \,.\, \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}))\cdots));$$

$$\{c\}^{\mathbb{I}}(\quad) \simeq \mathbb{I}(\lambda\,(\beta_0 \,.\, \mathbb{I}(\lambda\beta_1 \cdots \mathbb{I}(\lambda\beta_{k-1} \,.\, \{b\}^{\mathbb{I}}(\beta_0(0), \ldots, \beta_{k-1}(0)))\cdots)).$$

Clearly for each $\mathbf{m}$ and $\boldsymbol{\alpha}$,

$$|a, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} \leqslant |b, \mathbf{m}, \varnothing|^{\mathbb{I}} \leqslant |c, \varnothing, \varnothing|^{\mathbb{I}}.$$

Let d be an index such that

$$\{d\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \begin{cases} 0, & \text{if } \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}) = 0; \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

and there exists a number q such that if $\{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}) = 0$, then $|d, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} < |a, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} + q$ (i.e., d is a natural index). Then if $\rho = |c, \varnothing|^{\mathbb{I}}_0 + q$, $\rho < \kappa^{\mathbb{I}}$ and

$$\begin{aligned} \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) &\leftrightarrow (d, \mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{U}^{\mathbb{I}} \\ &\leftrightarrow |d, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} < |x, \varnothing, \varnothing|^{\mathbb{I}} + q \leqslant \bar{\rho} \\ &\leftrightarrow (d, \mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{U}^{\mathbb{I}}_{\bar{\rho}}. \quad \square \end{aligned}$$

Of course, it is not true in general that if $\rho < \sigma$ then strictly more relations are reducible to $\mathsf{U}^{\mathbb{I}}_{\bar{\tau}}$ for $\tau \leqslant \sigma$ than for $\tau \leqslant \rho$ (cf. Exercise 2.16).

2.7 Upper Classification Theorem. $\{\alpha : \alpha$ is recursive in $\mathbb{I}\}$ *is semi-recursive in* $\mathbb{I}$.

Proof. As for Theorem VI.4.12. $\square$

We leave to the reader the formulation and proof of choice principles parallel to Theorem VI.4.15 (cf. Exercise 2.17).

The ordinals $\kappa^{\mathbb{I}}$ and $\boldsymbol{\kappa}^{\mathbb{I}}$ may be evaluated much as in previous similar cases. Let $\omega_1[\mathbb{I}]$ be the least ordinal not recursive in $\mathbb{I}$ and $\boldsymbol{\omega}_1[\mathbb{I}]$ the least ordinal not the order-type of a pre-wellordering of ${}^{\omega}\omega$ recursive in $\mathbb{I}$ and some β.

2.8 Theorem. $\kappa^{\mathbb{I}} = \omega_1[\mathbb{I}]$ *and* $\boldsymbol{\kappa}^{\mathbb{I}} = \boldsymbol{\omega}_1[\mathbb{I}]$.

Proof. Similar to that of Theorem VI.4.17 (cf. Exercise 2.19). $\square$

To this point there has been a close analogy between the roles played by E and $\mathbb{E}$ in recursion relative to functionals of types 2 and 3, respectively. The remaining results of this section indicate some differences. One that we have already seen is that in contrast with the fact that the relations recursive in E are exactly the Δ_1^1 relations, those recursive in $\mathbb{E}$ form a proper subclass of Δ_1^2. The basis for this discrepancy is that the property of well-foundedness for type-2 relations is Δ_1^2 (in fact $\Delta_{(\omega)}^1$, Lemma VI.7.11), whereas well-foundedness for type-1 relations is Π_1^1 but not Δ_1^1. We now exploit the same fact in a different way to show that the class of relations semi-recursive in a type-3 functional $\mathbb{I}$ is not closed under existential function quantification ($\exists^1$) (cf. discussion at the beginning of § VI.6).

2.9 Lemma. *There exists a relation* $\mathsf{P}^{\mathbb{I}}$ *semi-recursive in* $\mathbb{I}$ *such that for all* u, v, $\boldsymbol{\alpha}$, *and* $\boldsymbol{\beta}$,

(i) $(u,\langle\boldsymbol{\alpha}\rangle)\in \mathsf{U}^{\mathbb{I}} \wedge \mathsf{P}^{\mathbb{I}}(u,v,\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle)\rightarrow |v,\boldsymbol{\beta}|^{\mathbb{I}}<|u,\boldsymbol{\alpha}|^{\mathbb{I}}$;

(ii) $(u,\langle\boldsymbol{\alpha}\rangle)\notin \mathsf{U}^{\mathbb{I}}\rightarrow \exists v\,\exists\boldsymbol{\beta}\,[(v,\langle\boldsymbol{\beta}\rangle)\notin \mathsf{U}^{\mathbb{I}} \wedge \mathsf{P}^{\mathbb{I}}(u,v,\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle)]$.

Proof. Much as in the proof of Lemma VI.6.1, we define $\mathsf{P}^{\mathbb{I}}(u,v,\langle\boldsymbol{\alpha}\rangle,\langle\boldsymbol{\beta}\rangle)$ to hold just in case one of the following holds for some k, $\mathbf{m}\in{}^k\omega$, and $l=\lg(\boldsymbol{\alpha})$:

(1) for some $b, c_0,\dots,c_{k'-1}$, $u=\langle\langle 1,k,l,0,b,c_0,\dots,c_{k'-1}\rangle,\mathbf{m}\rangle$, $\boldsymbol{\beta}=\boldsymbol{\alpha}$, and either $v=\langle c_i,\mathbf{m}\rangle$ for some $i<k'$ or there exist $q_0,\dots,q_{k'-1}$ such that for all $i<k'$, $\{c_i\}^{\mathbb{I}}(\mathbf{m},\boldsymbol{\alpha})\simeq q_i$ and $v=\langle b,\mathbf{q}\rangle$;

(2) for some b, $u=\langle\langle 2,k+1,l,0\rangle,b,\mathbf{m}\rangle$, $\boldsymbol{\beta}=\boldsymbol{\alpha}$, and $v=\langle b,\mathbf{m}\rangle$;

(4) for some b, $u=\langle\langle 4,k,l,0,b\rangle,\mathbf{m}\rangle$, $v=\langle b,\mathbf{m}\rangle$, and for some γ, $\boldsymbol{\beta}=\boldsymbol{\alpha}*(\gamma)$;

(5) u is of none of these forms, $|u,\boldsymbol{\alpha}|^{\mathbb{I}}\neq 0$, $\boldsymbol{\beta}=\boldsymbol{\alpha}$, and $v=0$.

Because we are (for simplicity) omitting type-2 parameters ($l'=0$), there is no need for a clause (3).

The proof that $\mathsf{P}^{\mathbb{I}}$ satisfies (i) and (ii) is essentially as before. □

2.10 Theorem *For all* a, $\mathbf{m}$, *and* $\boldsymbol{\alpha}$,

$$\{a\}^{\mathbb{I}}(\mathbf{m},\boldsymbol{\alpha}) \textit{ is defined} \leftrightarrow \neg\exists\delta\,\exists\varepsilon\,\forall p\,[\mathsf{P}^{\mathbb{I}}(\delta(p),\delta(p+1),(\varepsilon)^p,(\varepsilon)^{p+1})\wedge \delta(0)=\langle a,\mathbf{m}\rangle\wedge(\varepsilon)^0=\langle\boldsymbol{\alpha}\rangle].$$

Proof. This follows from Lemma 2.9 in very much the same way that Theorem VI.6.2 follows from Lemma VI.6.1. The difference is that the descending sequence of undefined computations can no longer be described by a Suslin quantifier ($\mathscr{A}$) as the type-1 function arguments may vary. □

2.11 Corollary. *The class of relations semi-recursive in* $\mathbb{I}$ *is not closed under existential function quantification* ($\exists^1$). □

As in § VI.6, this part of the analogy between E and $\mathbb{E}$ can be repaired by replacing $\mathbb{E}$ by an extended type-3 functional $\mathbb{E}^{\#}$ defined by: for any partial functional $\mathsf{F}: {}^{\omega}\omega \to \omega$,

$$\mathbb{E}^{\#}(\mathsf{F}) \simeq \begin{cases} 0, & \text{if} \quad \exists\alpha\,.\,\mathsf{F}(\alpha) \simeq 0; \\ 1, & \text{if} \quad \forall\alpha\,(\exists n > 0)\,.\,\mathsf{F}(\alpha) \simeq n; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

The theory of recursion relative to $\mathbb{E}^{\#}$ may be developed in a way closely parallel with the theory of recursion relative to $\mathsf{E}_1^{\#}$ in § VI.6. It is clear that the class of relations semi-recursive in $\mathbb{E}^{\#}$ is closed under $\exists^1$. That every relation semi-recursive in $\mathbb{E}$ is also semi-recursive in $\mathbb{E}^{\#}$ may be proved as in VI.6.6 and thus both $\mathsf{U}^{\mathbb{E}}$ and $\sim\mathsf{U}^{\mathbb{E}}$ are semi-recursive in $\mathbb{E}^{\#}$. To establish an ordinal comparison theorem for $\mathbb{E}^{\#}$ we define inductively a relation M as follows: M is the smallest relation such that for all $d \in \mathrm{Pri}$ and all $\boldsymbol{\alpha}$,

(i) $(0, \langle\boldsymbol{\alpha}\rangle) \in \mathsf{M}$;

(ii) if $\exists\beta\,([d](\boldsymbol{\alpha}, \beta), \langle\boldsymbol{\alpha}, \beta\rangle) \in \mathsf{M}$, then $(\langle 0, d\rangle, \langle\boldsymbol{\alpha}\rangle) \in \mathsf{M}$;

(iii) if $\forall\beta\,([d](\boldsymbol{\alpha}, \beta), \langle\boldsymbol{\alpha}, \beta\rangle) \in \mathsf{M}$, then $(\langle 1, d\rangle, \langle\boldsymbol{\alpha}\rangle) \in \mathsf{M}$.

The analogues of Theorems VI.6.7–9 all hold with very similar proofs, and we conclude in particular that $\mathsf{U}^{\mathbb{E}}$ is recursive in $\mathbb{E}^{\#}$.

If we include type-2 parameters everywhere, we can similarly prove that $\mathbb{U}^{\mathbb{E}}$ is recursive in $\mathbb{E}^{\#}$. Let $s\mathcal{J}$ be the type-4 jump operator defined by:

$$s\mathcal{J}(\mathbb{I})((a, \mathbf{m}, \alpha) * \mathsf{I}) = \begin{cases} 0, & \text{if} \quad \{a\}^{\mathbb{I}}(\mathbf{m}, \alpha, \mathsf{I}) \text{ is defined;} \\ 1, & \text{otherwise.} \end{cases}$$

Then the same argument shows that if $\mathbb{I}$ is recursive in $\mathbb{E}^{\#}$, so is $s\mathcal{J}(\mathbb{I})$. Thus if we set

$$\mathbb{E}_0 = \mathbb{E} \quad \text{and} \quad \mathbb{E}_{r+1} = s\mathcal{J}(\mathbb{E}_r),$$

then $\mathbb{E}_r$ is recursive in $\mathbb{E}_s$ iff $r \leq s$ and all $\mathbb{E}_r$ are recursive in $\mathbb{E}^{\#}$. The relations recursive in $\mathbb{E}$ are called *hyperanalytical* while those recursive in $\mathbb{E}^{\#}$ are called *hyperprojective*. Note that it follows from Corollary VI.7.13 that the relations semi-recursive in $\mathbb{E}^{\#}$ (*semi-hyperprojective*) still form a subset of Δ_1^2. The subset is proper because it is not closed under complementation.

We conclude this section by establishing an analogue of the Spector–Gandy Theorem (VI.4.18) for recursion in $\mathbb{I}$. The proof is a modification of that sketched in Exercise VI.6.26 and is substantially simpler than that of VI.4.18. We denote by $\Sigma_1^{2,\mathbb{I}}$ the class of relations $\mathbb{R}$ such that for some $\mathbb{S}$ recursive in $\mathbb{I}$,

$$\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}) \leftrightarrow (\exists\mathsf{F} \text{ recursive in } \boldsymbol{\alpha}, \mathbf{I}, \text{ and } \mathbb{I})\,\mathbb{S}(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I}, \mathsf{F}).$$

2.12 Theorem. *For all* $\mathbb{R}$, $\mathbb{R} \in \Sigma_1^{2,\mathbb{I}}$ *iff* $\mathbb{R}$ *is semi-recursive in* $\mathbb{I}$.

Proof. The implication ($\rightarrow$) is an immediate consequence of the Substitution Theorem (1.6): if $\mathbb{R}$ and $\mathbb{S}$ are as above, then

$$\mathbb{R}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}) \leftrightarrow \exists a\,.\,\mathbb{S}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \lambda\beta\,.\,\{a\}^{\mathsf{I}}(\boldsymbol{\alpha}, \beta, \mathsf{I})).$$

For simplicity, we shall prove the implication ($\leftarrow$) only for relations with arguments of types 0 and 1; the general case follows by an obvious modification. It will suffice to show that $\mathsf{U}^{\mathsf{I}} \in \Sigma_1^{2,\mathsf{I}}$. The method is analogous to that of Exercise VI.6.26. Let $\mathbb{P}$ be a relation such that $\mathbb{P}(u, v, \langle\boldsymbol{\alpha}\rangle, \langle\boldsymbol{\beta}\rangle, \mathsf{G})$ holds under exactly the same conditions as does $\mathsf{P}^{\mathsf{I}}(u, v, \langle\boldsymbol{\alpha}\rangle, \langle\boldsymbol{\beta}\rangle)$ in Lemma 2.9 except that in clause (1) we replace the condition

$$\{c_i\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i \quad \text{by} \quad \mathsf{G}(\langle c_i, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) = q_i.$$

Clearly $\mathbb{P}$ is recursive in I. Let $<_{\mathsf{G}}$ denote the transitive closure of the relation $<'_{\mathsf{G}}$ defined by

$$(v, \langle\boldsymbol{\beta}\rangle) <'_{\mathsf{G}} (u, \langle\boldsymbol{\alpha}\rangle) \leftrightarrow \mathbb{P}(u, v, \langle\boldsymbol{\alpha}\rangle, \langle\boldsymbol{\beta}\rangle, \mathsf{G}).$$

We say that a functional H is *closed for* G iff for all u, v, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$,

$$\mathsf{H}(u, \langle\boldsymbol{\alpha}\rangle) = 0 \wedge (v, \langle\boldsymbol{\beta}\rangle) <_{\mathsf{G}} (u, \langle\boldsymbol{\alpha}\rangle) \rightarrow \mathsf{H}(v, \langle\boldsymbol{\beta}\rangle) = 0.$$

H is *well-founded for* G iff

$$\neg\exists\beta\,\exists\gamma\,\forall p\,[\mathsf{H}(\beta(p), (\gamma)^p) = 0 \wedge (\beta(p+1), (\gamma)^{p+1}) <_{\mathsf{G}} (\beta(p), (\gamma)^p)].$$

Finally, we shall say that G is *locally correct for* H iff for all k, l, and $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$ and all u such that $\mathsf{H}(u, \langle\boldsymbol{\alpha}\rangle) = 0$:

(0) if $u = \langle\langle 0, k, l, 0, \ldots\rangle, \mathbf{m}\rangle$ and $\langle 0, k, l, 0, \ldots\rangle$ is an index of the proper form for $(\mathbf{m}, \boldsymbol{\alpha})$, then $\mathsf{G}(u, \langle\boldsymbol{\alpha}\rangle) = \{\langle 0, k, l, 0, \ldots\rangle\}^{\mathsf{I}}(\mathbf{m}, \boldsymbol{\alpha})$;

(1) if for some $b, c_0, \ldots, c_{k'-1}$, $u = \langle\langle 1, k, l, 0, b, c_0, \ldots, c_{k'-1}\rangle, \mathbf{m}\rangle$, then

$$\mathsf{G}(u, \langle\boldsymbol{\alpha}\rangle) = \mathsf{G}(\langle b, \mathsf{G}(\langle c_0, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle), \ldots, \mathsf{G}(\langle c_{k'-1}, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)\rangle, \langle\boldsymbol{\alpha}\rangle);$$

(2) if for some b, $u = \langle\langle 2, k+1, l, 0\rangle, b, \mathbf{m}\rangle$, then

$$\mathsf{G}(u, \langle\boldsymbol{\alpha}\rangle) = \mathsf{G}(\langle b, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle);$$

(4) if for some b, $u = \langle\langle 4, k, l, 0, b\rangle, \mathbf{m}\rangle$, then

$$\mathsf{G}(u, \langle\boldsymbol{\alpha}\rangle) = \mathsf{I}(\lambda\delta\,.\,\mathsf{G}(\langle b, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}, \delta\rangle)).$$

We now claim that for all d, $\mathbf{n}$, and $\boldsymbol{\beta}$,

$$\mathsf{U}^{\mathbb{I}}(d, \langle\mathbf{n}\rangle, \langle\boldsymbol{\beta}\rangle) \leftrightarrow (\exists \mathsf{G}, \mathsf{H} \text{ recursive in } \mathbb{I}, \boldsymbol{\beta})[\mathsf{G} \text{ is locally correct for } \mathsf{H} \wedge$$
$$\mathsf{H}(\langle d, \mathbf{n}\rangle, \langle\boldsymbol{\beta}\rangle) = 0 \wedge \mathsf{H} \text{ is closed for } \mathsf{G} \wedge \mathsf{H} \text{ is well-founded for } \mathsf{G}].$$

For the implication ($\rightarrow$), suppose that $\mathsf{U}^{\mathbb{I}}(d, \langle\mathbf{n}\rangle, \langle\boldsymbol{\beta}\rangle)$ holds and let

$$\mathsf{G}(\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) = \begin{cases} \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}), & \text{if } \quad |a, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} \leqslant |d, \mathbf{n}, \boldsymbol{\beta}|^{\mathbb{I}}; \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathsf{H}(\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) = \begin{cases} 0, & \text{if } \quad |a, \mathbf{m}, \boldsymbol{\alpha}|^{\mathbb{I}} \leqslant |d, \mathbf{n}, \boldsymbol{\beta}|^{\mathbb{I}}; \\ 1, & \text{otherwise.} \end{cases}$$

It follows from the Ordinal Comparison Theorem (2.1) that G and H are recursive in $\mathbb{I}$, $\boldsymbol{\beta}$, and it is straightforward to check that they satisfy the other conditions.

Suppose now that for some fixed d, $\mathbf{n}$, and $\boldsymbol{\beta}$, G and H are functionals recursive in $\mathbb{I}$, $\boldsymbol{\beta}$ which satisfy the condition in brackets. Let $\prec$ denote the restriction of $\prec_{\mathsf{G}}$ to $\{(u, \langle\boldsymbol{\alpha}\rangle) : \mathsf{H}(u, \langle\boldsymbol{\alpha}\rangle) = 0\}$. Then $\prec$ is a well-founded partial order whose field contains $(\langle d, \mathbf{n}\rangle, \langle\boldsymbol{\beta}\rangle)$ and it will suffice to prove by induction on $\prec$ that for all pairs $(\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)$ in its field,

$$(*) \qquad \mathsf{G}(\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \simeq \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}).$$

By the definition of $\mathbb{P}$, the proof breaks into four cases where $\mathbf{m} \in {}^{k}\omega$ and $l = \mathrm{lg}(\boldsymbol{\alpha})$.

(1) if for some $b, c_0, \ldots, c_{k'-1}$, $a = \langle 1, k, l, 0, b, c_0, \ldots, c_{k'-1}\rangle$, then because H is closed for G, for all $i < k'$

$$(\langle c_i, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \prec (\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) \quad \text{and} \quad (\langle b, \mathbf{q}\rangle, \langle\boldsymbol{\alpha}\rangle) \prec (\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)$$

where $q_i = \mathsf{G}(\langle c_i, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle)$. Then by the induction hypothesis, for all $i < k'$, $\{c_i\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}) \simeq q_i$ and thus by the induction hypothesis again and the local correctness of G,

$$\mathsf{G}(\langle a, \mathbf{m}\rangle, \langle\boldsymbol{\alpha}\rangle) = \mathsf{G}(\langle b, \mathbf{q}\rangle, \langle\boldsymbol{\alpha}\rangle) \simeq \{b\}^{\mathbb{I}}(\mathbf{q}, \boldsymbol{\alpha}) \simeq \{a\}^{\mathbb{I}}(\mathbf{m}, \boldsymbol{\alpha}).$$

$$\left.\begin{array}{l}(2)\\ \\ (4)\end{array}\right\} \text{ Similarly.}$$

(5) If $\langle a, \mathbf{m}\rangle$ is of none of the preceding forms, then necessarily

$|\langle a,\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle|^{\mathbb{I}}=0$ — that is, $a=\langle 0,k,l,0,\ldots\rangle$ and is an index. Otherwise, $(\langle a,\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle),(0,\langle\boldsymbol{\alpha}\rangle),(0,\langle\boldsymbol{\alpha}\rangle)\ldots$ would be a descending $<$-chain. Hence $(*)$ is satisfied by clause (0) of the definition of local correctness. □

2.13–2.21 Exercises

2.13. Show that if $\mathbb{H}$ and $\mathbb{I}$ are two type-3 functionals such that $\mathbb{H}(\mathsf{G})=\mathbb{I}(\mathsf{G})$ for all G recursive in $\mathbb{I}$ and some $\beta\in{}^{\omega}\omega$, then for all R, R is (semi-) recursive in $\mathbb{I}$ iff R is (semi-) recursive in $\mathbb{H}$. (cf. Exercise VI.1.13).

2.14. Show that $s\mathscr{J}(\mathsf{sJ})$ and $\mathbb{E}$ are each recursive in the other.

2.15. For any $\mathbb{I}$ such that $\mathbb{E}$ is recursive in $\mathbb{I}$, $\boldsymbol{\kappa}^{\mathbb{I}}>\aleph_1$.

2.16. Formulate and prove the analogue of Exercise VI.4.29 for the hierarchies of Theorems 2.5 and 2.6.

2.17. Formulate and prove choice principles for recursion in a type-3 functional $\mathbb{I}$ parallel to those of Theorem VI.4.15.

2.18. Show that $U^{\mathbb{I}}$ is not an initial segment of $\mathsf{U}^{\mathbb{I}}$ — that is, there exists $u\in U^{\mathbb{I}}$ and $\sigma<|u|^{\mathbb{I}}$ such that $\sigma\neq|v|^{\mathbb{I}}$ for any $v\in U^{\mathbb{I}}$.

2.19. Prove Theorem 2.8.

2.20. Show that there exists a type-3 functional $\mathbb{I}$ such that $\mathbb{E}$ is recursive in $\mathbb{I}$ and $\{\mathsf{I}:\mathsf{I}$ is recursive in $\mathbb{I}$ and some $\beta\in{}^{\omega}\omega\}$ is recursive in $\mathbb{I}$ (cf. Theorem VI.4.15). Is this true without the reference to β?

2.21. Show that for all R, the following are equivalent:
(i) R is semi-recursive in $\mathbb{E}^{\#}$;
(ii) R is reducible to the closure $\bar{\Gamma}$ of a Δ^1_2 monotone operator Γ over ${}^{\omega}\omega$;
(iii) for some recursive relation P,

$$\mathsf{R}(\mathbf{m},\boldsymbol{\alpha})\leftrightarrow\forall\beta_0\,\exists\beta_1\,\forall\beta_2\,\exists\beta_3\cdots\exists n\;\mathsf{P}(\mathbf{m},\boldsymbol{\alpha},\langle\beta_0,\ldots,\beta_{n-1}\rangle).$$

2.22 Notes. The basic theory here was worked out by Moschovakis [1967]. The functional $\mathbb{E}^{\#}$ and the (semi-) hyperprojective relations are studied in Hinman–Moschovakis [1971].

3. Hierarchies of Relations Recursive in a Type-3 Functional

In Theorem 2.6 of the preceding section we showed that when $\mathbb{E}$ is recursive in a type-3 functional $\mathbb{I}$, there is a natural hierarchy of (countable) length $\kappa^{\mathbb{I}}$ on the type-2 relations recursive in $\mathbb{I}$. It results from the previous construction of a hierarchy of uncountable length and the observation that there is a natural way of picking out a countable "cofinal" subhierarchy. Since there are only countably many relations recursive in $\mathbb{I}$, it seems somewhat unnatural that such a roundabout procedure should be necessary. In the case of recursion relative to a type-2 functional I there was an alternative method of generating inductively the type-1 relations recursive in I, that of iterating the jump operator J_{I} over a set of notations for ordinals less than $\omega_1[\mathsf{I}]$ (cf. Corollary VI.5.6). The aim of this section is to show that if $\mathbf{s}\mathbb{J}$ is recursive in $\mathbb{I}$, then a similar construction for a type-3 $\mathbb{I}$ breaks down before $\omega_1[\mathbb{I}]$ and classifies only a (small!) subclass of the relations recursive in $\mathbb{I}$.

For any type-3 functional $\mathbb{I}$, let $\mathbb{J}_{\mathbb{I}}: {}^{({}^{\omega}\omega)}\omega \to {}^{({}^{\omega}\omega)}\omega$ be defined by:

$$\mathbb{J}_{\mathbb{I}}(\mathsf{F})((\langle a, n\rangle) * \alpha) = \begin{cases} 0, & \text{if} \quad \mathbb{I}(\lambda\beta.\{a\}^{\mathsf{F}}(\alpha,\beta)) \simeq n; \\ 1, & \text{otherwise.} \end{cases}$$

3.1 Lemma. *There exists an index d and a primitive recursive function h such that for all c, F, G, and $\mathbb{I}$,*

(i) $\mathbf{s}\mathbb{J}(\mathsf{F})$ *is recursive in* $\mathbb{J}_{\mathbb{I}}(\mathsf{F})$ *with index* d;

(ii) *if* F *is recursive in* G *with index* c, *then* $\mathbb{J}_{\mathbb{I}}(\mathsf{F})$ *is recursive in* $\mathbb{J}_{\mathbb{I}}(\mathsf{G})$ *with index* $h(c)$.

(*cf. Definition* V.5.4).

Proof. Let $n_0 = \mathbb{I}(\lambda\beta.0)$ and let f be a primitive recursive function such that for all a, $\mathbf{m}$, α, and β,

$$\{f(a,\langle\mathbf{m}\rangle)\}^{\mathsf{F}}(\alpha,\beta) \simeq 0\cdot\{a\}^{\mathsf{F}}(\mathbf{m},\alpha).$$

Then

$$\begin{aligned} \mathbf{s}\mathbb{J}(\mathsf{F})((\langle a,\mathbf{m}\rangle)*\alpha) = 0 &\leftrightarrow \{a\}^{\mathsf{F}}(\mathbf{m},\alpha) \text{ is defined} \\ &\leftrightarrow \forall\beta.\{f(a,\langle\mathbf{m}\rangle)\}^{\mathsf{F}}(\alpha,\beta) \simeq 0 \\ &\leftrightarrow \mathbb{J}_{\mathbb{I}}(\mathsf{F})((\langle f(a,\langle\mathbf{m}\rangle), n_0\rangle)*\alpha) = 0 \\ &\leftrightarrow \{d\}((\langle a,\mathbf{m}\rangle)*\alpha, \mathbb{J}_{\mathbb{I}}(\mathsf{F})) = 0 \end{aligned}$$

for an appropriate d.

For (ii), suppose that for all γ, $\mathsf{F}(\gamma)=\{c\}^{\mathsf{G}}(\gamma)$ and let g be the primitive recursive function of Theorem VI.2.14 such that for any a, α, and β,

$$\{g(a,c)\}^{\mathsf{G}}(\alpha,\beta)\simeq\{a\}^{\mathsf{F}}(\alpha,\beta).$$

Then

$$\begin{aligned}\mathbb{J}_{\mathbb{I}}(\mathsf{F})((\langle a,n\rangle)*\alpha)=0 &\leftrightarrow \mathbb{I}(\lambda\beta.\{a\}^{\mathsf{F}}(\alpha,\beta))=n\\ &\leftrightarrow \mathbb{I}(\lambda\beta.\{g(a,c)\}^{\mathsf{G}}(\alpha,\beta))=n\\ &\leftrightarrow \mathbb{J}_{\mathbb{I}}(\mathsf{G})((\langle g(a,c),n\rangle)*\alpha)=0\end{aligned}$$

and from this it is easy to define an appropriate h. □

For the remainder of this section, let $\mathbb{I}$ be a fixed type-3 functional such that $\mathsf{s}\mathbb{J}$ is recursive in $\mathbb{I}$. We write simply $\mathbb{J}$ for $\mathbb{J}_{\mathbb{I}}$ and construct a hierarchy in a way entirely analgous to that of § VI.5. For a minor convenience we use functionals instead of sets as the "backbone" of the hierarchy.

3.2 Definition. $<^{\mathbb{J}}$ is the smallest subset of $\omega\times\omega$ such that for all w in the field of $<^{\mathbb{J}}$ there exist functionals $\mathsf{H}_w^{\mathbb{J}}$ which satisfy the following conditions:

(i) $1<^{\mathbb{J}}2$, $\mathsf{H}_1^{\mathbb{J}}=\lambda\beta.0$, and $\mathsf{H}_2^{\mathbb{J}}=\mathbb{J}(\mathsf{H}_1^{\mathbb{J}})$;

(ii) if $u<^{\mathbb{J}}v$, then $v<^{\mathbb{J}}2^v$ and $\mathsf{H}_{2^v}^{\mathbb{J}}=\mathbb{J}(\mathsf{H}_v^{\mathbb{J}})$;

(iii) if $u\in \mathrm{Fld}(<^{\mathbb{J}})$, for all p $\{a\}(p,\mathsf{H}_u^{\mathbb{J}})<^{\mathbb{J}}\{a\}(p+1,\mathsf{H}_u^{\mathbb{J}})$, and $\{a\}(0,\mathsf{H}_u^{\mathbb{J}})\simeq u$, then for all p,

$$\{a\}(p,\mathsf{H}_u^{\mathbb{J}})<^{\mathbb{J}}3^a5^u \quad\text{and}\quad \mathsf{H}_{3^a5^u}^{\mathbb{J}}((p)*\alpha)=\mathsf{H}^{\mathbb{J}}_{\{a\}(p,\mathsf{H}_u^{\mathbb{J}})}(\alpha);$$

(iv) if $u<^{\mathbb{J}}v$ and $v<^{\mathbb{J}}w$, then $u<^{\mathbb{J}}w$.

We write $O^{\mathbb{J}}$ for the field of $<^{\mathbb{J}}$, assign ordinals $|u|^{\mathbb{J}}$ to $u\in O^{\mathbb{J}}$ as in § IV.4, and set

$$\nabla(\mathbb{J})=\{\mathsf{R}:\mathsf{R}\text{ is recursive in }\mathsf{H}_u^{\mathbb{J}}\text{ for some }u\in O^{\mathbb{J}}\}.$$

We shall show that $\nabla(\mathbb{J})$ is a proper subset of the set of relations recursive in $\mathbb{I}$ by defining a functional K which is recursive in $\mathbb{I}$ but has the property that for all $u\in O^{\mathbb{J}}$, $\mathsf{H}_u^{\mathbb{J}}$ is recursive in K.

For $\gamma\in\mathsf{W}$, we define K^γ by recursion on $\|\gamma\|$ by:

$$\mathsf{K}^\gamma((p)*\alpha)=\begin{cases}\mathrm{o}\mathbb{J}(\alpha)(p), & \text{if }\ \|\gamma\|=0;\\ \mathbb{J}(\mathsf{K}^{(\gamma\restriction p)})(\alpha), & \text{otherwise.}\end{cases}$$

Roughly speaking, K^γ is the result of $\|\gamma\|$ applications of $\mathbb{J}$. Set

$$\mathsf{K}(\langle\alpha,\gamma\rangle) = \begin{cases} \mathsf{K}^{\gamma}(\alpha), & \text{if} \quad \gamma \in \mathsf{W}; \\ 0, & \text{otherwise.} \end{cases}$$

3.3 Lemma. K *is recursive in* $\mathbb{I}$.

Proof. Since $\mathsf{s}\mathbb{J}$ is recursive in $\mathbb{I}$, so is E_1 and hence so is W. Let

$$\mathsf{G}(e,(p)*\alpha,\gamma) \simeq \begin{cases} \mathsf{oJ}(\alpha)(p), & \text{if} \quad \gamma \in \mathsf{W} \quad \text{and} \quad \|\gamma\| = 0; \\ \mathbb{J}(\lambda\beta.\{e\}^{\mathbb{I}}(\beta,\gamma \restriction p))(\alpha), & \text{if} \quad \gamma \in \mathsf{W} \quad \text{and} \quad \|\gamma\| > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Because $\mathsf{s}\mathbb{J}$ is recursive in $\mathbb{I}$, $\mathbb{J}$ is calculable from $\mathbb{I}$ and thus G is partial recursive in $\mathbb{I}$. By the $\mathbb{I}$-Recursion Theorem there exists an index $\bar{e}$ such that

$$\{\bar{e}\}^{\mathbb{I}}((p)*\alpha,\gamma) \simeq \mathsf{G}(\bar{e},(p)*\alpha,\gamma).$$

A straightforward induction on $\|\gamma\|$ shows that for all γ,

$$\{\bar{e}\}^{\mathbb{I}}(\alpha,\gamma) = \mathsf{K}(\langle\alpha,\gamma\rangle)$$

and thus K is recursive in $\mathbb{I}$. $\square$

For each $u \in O^{\mathsf{J}}$, let δ_u be defined by:

$$\delta_u(\langle v,w\rangle) = \begin{cases} 0, & \text{if} \quad v \leqslant^{\mathsf{J}} w \leqslant^{\mathsf{J}} u; \\ 1, & \text{otherwise.} \end{cases}$$

An easy induction on $|u|^{\mathsf{J}}$ shows that for all $u \in O^{\mathsf{J}}$, $\delta_u \in \mathsf{W}$ and $\|\delta_u\| = |u|^{\mathsf{J}}$. Note that if $v <^{\mathsf{J}} u$, then $\delta_u \restriction v = \delta_v$.

3.4 Lemma. *There exist primitive recursive functions f and g such that for all $u \in O^{\mathsf{J}}$,*

(i) *δ_u is recursive in $\mathsf{H}_u^{\mathsf{J}}$ with index $f(u)$;*
(ii) *$\mathsf{H}_u^{\mathsf{J}}$ is recursive in K^{δ_u} with index $g(u)$.*

Proof. The existence of f may be established much as in Lemma VI.5.2 and we leave the details to Exercise 3.7. For (ii), we define g by effective transfinite recursion to satisfy the following conditions. If $u = 1$, $g(u)$ is any index of the recursive functional $\mathsf{H}_1^{\mathsf{J}}$ from K^{δ_1}. If $u = 2^v$, then $g(u)$ is an index for the following computation: from K^{δ_u} compute $\mathbb{J}(\mathsf{K}^{\delta_v}) = \mathbb{J}(\mathsf{K}^{\delta_u \restriction v})$ and from this (by 3.1(ii)) compute $\mathbb{J}(\mathsf{F})$, where F is the functional recursive in K^{δ_v} with index $g(v)$.

Since F is $\mathsf{H}_v^{\mathsf{J}}$, $g(u)$ is indeed an index for $\mathsf{H}_u^{\mathsf{J}}$ from K^{δ_u}. If $u = 3^a 5^v$, let $\varepsilon(p) = \{a\}(p, \mathsf{H}_v^{\mathsf{J}})$ and let $g(u)$ be an index from K^{δ_u} for the functional G described as follows. Let I be the functional recursive in $\mathsf{K}^{\delta_u \upharpoonright v}$ with index $g(v)$ and set $\varepsilon'(p) = \{a\}(p, \mathsf{I})$. For each p, let G_p be the functional recursive in $\mathsf{K}^{\delta_u \upharpoonright \varepsilon'(p)}$ with index $g(\varepsilon'(p))$. Finally, $\mathsf{G}((p) * \alpha) = \mathsf{G}_p(\alpha)$. It is now straightforward to prove by induction over O^{J} that (ii) is satisfied. □

3.5 Theorem. *There exists a primitive recursive function h such that for all $u \in O^{\mathsf{J}}$, $\mathsf{H}_u^{\mathsf{J}}$ is recursive in K with index $h(u)$.*

Proof. We again define h by effective transfinite recursion. $h(1)$ is any index for the recursive functional $\mathsf{H}_1^{\mathsf{J}}$ from K. If $u = 2^v$ and $\mathsf{H}_v^{\mathsf{J}}$ is recursive in K with index $h(v)$, then from K we can, by Lemma 3.4(i), compute an index for δ_v, and thence one for δ_u. By 3.4(ii), $\mathsf{H}_u^{\mathsf{J}}$ is recursive in K^{δ_u}, hence in K alone. We take $h(u)$ to be any index which describes this computation.

If $u = 3^a 5^v$ and $\varepsilon(p) = \{a\}(p, \mathsf{H}_v^{\mathsf{J}})$, then using $h(v)$ we can compute an index of ε from K. Then the indices $h(\varepsilon(p))$ uniformly compute $\mathsf{H}_{\varepsilon(p)}^{\mathsf{J}}$ from K and thus lead to an index of $\mathsf{H}_u^{\mathsf{J}}$ from K. □

3.6 Corollary. *There exists a set $A \subseteq \omega$ which is recursive in $\mathbb{I}$ but not recursive in any $\mathsf{H}_u^{\mathsf{J}}$ ($u \in O^{\mathsf{J}}$).*

Proof. Let $A = \{a : \{a\}^{\mathsf{K}}(a)$ is defined$\}$. A is recursive in $\mathsf{sJ}(\mathsf{K})$, hence in $\mathbb{I}$. A standard diagonal argument shows that A is not recursive in K and thus by Theorem 3.5 A is not recursive in any $\mathsf{H}_u^{\mathsf{J}}$ ($u \in O^{\mathsf{J}}$). □

It can also be shown that the type-1 functions and relations recursive in K are exactly those recursive in some $\mathsf{H}_u^{\mathsf{J}}$ ($u \in O^{\mathsf{J}}$) (Aczel–Hinman [1974, Theorem 1.10]). In the particular case $\mathbb{I} = \mathsf{sJ}$ there is a direct way to construct a countable hierarchy for the type-1 relations recursive in $\mathbb{I}$ (Aczel–Hinman [1974, Corollary 4.20]).

3.7 Exercise. Prove part (i) of Lemma 3.4.

3.8. Notes. The failure of hierarchies of the type discussed here to exhaust the class of objects recursive in $\mathbb{I}$ was shown in Moschovakis [1967] for $\mathbb{I}$ such that $\mathbb{E}$ is recursive in $\mathbb{I}$ and extended in Aczel–Hinman [1974] to all $\mathbb{I}$ such that sJ is recursive in $\mathbb{I}$.

4. Higher Types

For the most part, the definitions and results for recursion relative to functionals of types 4 and higher are straightforward generalizations of those for type 3. Rather than discuss these for arbitrary types, we sketch them for type 4 and leave it to the reader to formulate the general results. There is one new phenomenon which appears at type 4, a selection theorem which implies that when $\mathscr{E}$ (the type-4 quantifier functional) is recursive in a type-4 functional $\mathscr{I}$, then the class of relations semi-recursive in $\mathscr{I}$ is closed under existential function quantification ($\exists^1$).

We shall use script letters to denote functions and relations of type 4 — $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$, $\mathscr{I}$, and $\mathscr{J}$ for functions from ${}^{k,l,l',l''}\omega$ into ω and $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$. $\mathscr{U}$, and $\mathscr{V}$ for subsets of ${}^{k,l,l',l''}\omega$ (a typical element of ${}^{k,l,l',l''}\omega$ is of the form $(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}})$). As with types 2 and 3, we define recursion relative to a single type-4 functional $\mathscr{I}$ which takes a single argument $\mathbb{F}$. Examples of such functionals are:

$$\mathscr{E}(\mathbb{F}) = \begin{cases} 0, & \text{if } \exists \mathsf{G} \,.\, \mathbb{F}(\mathsf{G}) = 0; \\ 1, & \text{otherwise;} \end{cases}$$

$$s\mathscr{J}((a, \mathbf{m}, \alpha, \mathsf{I}) * \mathbb{F}) = \begin{cases} 0, & \text{if } \{a\}^{\mathsf{F}}(\mathbf{m}, \alpha, \mathsf{I}) \text{ is defined;} \\ 1, & \text{otherwise.} \end{cases}$$

Recursion relative to other sorts of type-4 functionals or relative to several type-4 functionals is then defined in the usual way by coding.

4.1 Definition. For any $\mathscr{I}$, $\Omega[\mathscr{I}]$ is the smallest set such that for all k, l, l', l'', n, p, q, r, and s, all $i < k$, $j < l$, $j' < l'$, and $j'' < l''$, and all $(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}}) \in {}^{k,l,l',l''}\omega$,

(0)–(4) identical to the corresponding clauses of Definition 1.1 with the addition of the parameter l'' and the provision in clause (4) for indices $\langle 4, k, l, l', l'', j'', b\rangle$ for the application of $\mathbb{I}_{j''}$;

(5) for any b and any $\mathbb{F}$,

if for all H, $(b, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \mathsf{H}, \vec{\mathbb{I}}, \mathbb{F}(\mathsf{H})) \in \Omega[\mathscr{I}]$,

then $(\langle 5, k, l, l', l'', b\rangle, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}}, \mathscr{I}(\mathbb{F})) \in \Omega[\mathscr{I}]$.

As always, we set

$$\{a\}^{\mathscr{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}}) \simeq n \quad \text{iff} \quad (a, \mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}}, n) \in \Omega[\mathscr{I}].$$

and say that $\mathscr{F}$ is *partial recursive in* $\mathscr{I}$ iff $\mathscr{F} = \{a\}^{\mathscr{I}}$ for some a, etc. The import of clause (5) is that

$$\{\langle 5, k, l, l', l'', b\rangle\}^{\mathscr{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \vec{\mathbb{I}}) \simeq \mathscr{I}(\lambda \mathsf{H} \,.\, \{b\}^{\mathscr{I}}(\mathbf{m}, \boldsymbol{\alpha}, \mathsf{I}, \mathsf{H}, \vec{\mathbb{I}})).$$

The simplest properties of recursion in a type-4 functional are established exactly as in previous cases and we leave their formulation and proof to the reader. Note that the class of relations semi-recursive in $\mathcal{I}$ is closed under universal (type-2) functional quantification ($\forall^2$). The substitution theorems are related to those for type 3 (Theorem 1.6) as those are related to the results for type 2 (Theorems VI.2.10, 14). The new cases here involve primitive recursive functions h and h' such that for all $\mathcal{I}$, a, d, $\mathbf{m}$, $\boldsymbol{\alpha}$, $\mathbf{I}$ and $\vec{\mathbb{I}}$,

(i) $\{h(a,d)\}^{\mathcal{I}}(\mathbf{m},\boldsymbol{\alpha},\mathbf{I},\vec{\mathbb{I}}) \simeq \{a\}^{\mathcal{I}}(\mathbf{m},\boldsymbol{\alpha},\mathbf{I},\vec{\mathbb{I}},\lambda \mathsf{H}\,.\{d\}^{\mathcal{I}}(\mathbf{m},\boldsymbol{\alpha},\mathbf{I},\mathsf{H},\vec{\mathbb{I}}))$;

(ii) if $\lambda \mathbb{G}\,.\{d\}^{\mathcal{I}}(\mathbb{G})$ is a total type-4 functional $\mathcal{H}$, then

$$\{h'(a,d)\}^{\mathcal{I}}(\mathbf{m},\boldsymbol{\alpha},\mathbf{I},\vec{\mathbb{I}}) \simeq \{a\}^{\mathcal{H}}(\mathbf{m},\boldsymbol{\alpha},\mathbf{I},\vec{\mathbb{I}}).$$

The ideas involved in the discussion of $s\mathcal{I}$ and $\mathbb{E}^{\#}$ following Corollary 2.11 can be extended to show that $s\mathcal{I}$ is recursive in $\mathbb{E}^{\#}$ and thus that the class of relations recursive in $s\mathcal{I}$ is a proper subset of Δ^2_1. On the other hand, all Π^2_1 relations are semi-recursive in $s\mathcal{I}$ and all $\Delta^2_{(\omega)}$ relations are recursive in $\mathscr{E}$. It follows from the proof of Theorem 1.4 that $s\mathcal{I}$ is recursive in $\mathscr{E}$, but the preceding remarks imply that $\mathscr{E}$ is not recursive in $s\mathcal{I}$. The classes Σ^3_r, Π^3_r, and Δ^3_r may be defined in the natural way and one may prove similarly as for Corollary VI.7.12 that for any inductive operator Γ over ${}^{({}^{\omega}\omega)}\omega$ and any $r \geqslant 1$, if $\Gamma \in \Delta^3_r$, then also $\bar{\Gamma} \in \Delta^3_r$. In particular we obtain by the method of Theorem 1.4 that for any $r \geqslant 1$, if $\mathcal{I} \in \Delta^3_r$, then the class of relations semi-recursive in $\mathcal{I}$ forms a proper subset of Δ^3_r. Of course, the example $\mathscr{E}$ shows that no Δ^2_r has this closure property.

For the rest of this section we shall assume that $\mathcal{I}$ is a fixed type-4 functional such that $\mathscr{E}$ is recursive in $\mathcal{I}$. For simplicity, we shall omit mention of type-3 parameters; type-2 arguments can no longer be regarded as parameters. We use x and y to denote typical elements $(\mathbf{m},\boldsymbol{\alpha},\mathbf{I})$ of ${}^{k,l,l'}\omega$, $\langle \mathsf{x}\rangle$ to denote the corresponding triple $(\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha}\rangle,\langle\mathbf{I}\rangle)$, $\langle \mathsf{x},\beta\rangle$ for $(\langle\mathbf{m}\rangle,\langle\boldsymbol{\alpha},\beta\rangle,\langle\mathbf{I}\rangle)$, etc.

Set

$$\mathbb{U}^{\mathcal{I}}(a,\langle \mathsf{x}\rangle) \leftrightarrow \{a\}^{\mathcal{I}}(\mathsf{x}) \text{ is defined}$$

and

$$|a,\mathsf{x}|^{\mathcal{I}} = \text{least } \sigma\,.\,(a,\mathsf{x},\{a\}^{\mathcal{I}}(\mathsf{x})) \in \Omega[\mathcal{I}]^{\sigma}$$

if such a σ exists; otherwise, $o({}^{({}^{\omega}\omega)}\omega)$ (the least ordinal not the type of a pre-wellordering of ${}^{({}^{\omega}\omega)}\omega$). The ordinals of computations introduced by clause (5) are given by: if $a = \langle 5,k,l,l',0,b\rangle$ and $\{a\}^{\mathcal{I}}(\mathsf{x})$ is defined, then

$$|a,\mathsf{x}|^{\mathcal{I}} = \sup{}^{+}\{|b,\mathsf{x},\mathsf{H}|^{\mathcal{I}} : \mathsf{H} \in {}^{({}^{\omega}\omega)}\omega\}.$$

4.2 Ordinal Comparison Theorem. *There exists a functional* $\mathbb{H}$ *partial recursive in* $\mathcal{I}$ *such that for all* a^0, a^1, x^0, *and* x^1,

(i) *if* $(a^0,\langle \mathsf{x}^0\rangle) \in \mathbb{U}^{\mathcal{I}}$ *and* $|a^0,\mathsf{x}^0|^{\mathcal{I}} \leqslant |a^1,\mathsf{x}^1|^{\mathcal{I}}$, *then*

$$\mathbb{H}(a^0, a^1, \langle \mathsf{x}^0 \rangle, \langle \mathsf{x}^1 \rangle) \simeq 0;$$

(ii) *if* $(a^1, \langle \mathsf{x}^1 \rangle) \in \mathbb{U}^{\mathscr{I}}$ *and* $|a^1, \mathsf{x}^1|^{\mathscr{I}} < |a^0, \mathsf{x}^0|^{\mathscr{I}}$, *then*

$$\mathbb{H}(a^0, a^1, \langle \mathsf{x}^0 \rangle, \langle \mathsf{x}^1 \rangle) \simeq 1.$$

Proof. The definition of $\mathbb{H}$ is entirely analogous to that in Theorem 2.1 and thus (in more detail) to that in Theorem VI.3.3. □

From this we obtain exactly as in § VI.4 and § 2 of this chapter the Selection Theorem and all the structural consequences analogous to 2.3–2.8. The proof of Lemma 2.9 can similarly be generalized and we obtain a relation $\mathbb{P}^{\mathscr{I}}$ semi-recursive in $\mathscr{I}$ such that for all a and $\mathsf{x} = (\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$,

$$\{a\}^{\mathscr{I}}(\mathsf{x}) \text{ is defined} \leftrightarrow \neg \exists \mathsf{F} \exists \delta \exists \varepsilon \forall p \, [\mathbb{P}^{\mathscr{I}}(\delta(p), \delta(p+1), (\varepsilon)^p, (\varepsilon)^{p+1}, (\mathsf{F})^p, (\mathsf{F})^{p+1}) \wedge \delta(0) = \langle a, \mathbf{m} \rangle \wedge (\varepsilon)^0 = \langle \boldsymbol{\alpha} \rangle \wedge (\mathsf{F})^0 = \langle \mathbf{I} \rangle].$$

Intuitively, $\mathbb{P}^{\mathscr{I}}(\langle a, \mathbf{m} \rangle, \langle b, \mathbf{n} \rangle, \langle \boldsymbol{\alpha} \rangle, \langle \boldsymbol{\beta} \rangle, \langle \mathbf{I} \rangle, \langle \mathbf{H} \rangle)$ means that the computation $(b, \mathbf{n}, \boldsymbol{\beta}, \mathbf{H})$ is an immediate predecessor of $(a, \mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$. In particular, the class of relations semi-recursive in $\mathscr{I}$ is not closed under existential functional quantification ($\exists^2$). We leave to the reader the definition and study of the functional $\mathscr{E}^{\#}$ and the formulation and proof of the analogue for $\mathscr{I}$ of Theorem 2.12 (Spector–Gandy Theorem).

We come now to showing that the class of relations semi-recursive in $\mathscr{I}$ *is* closed under existential function quantification ($\exists^1$).

4.3 Theorem. *There exists a functional* $\mathbb{F}$ *partial recursive in* $\mathscr{I}$ *such that for any* a *and* x, *the following are equivalent*:

(i) $\exists \beta . \{a\}^{\mathscr{I}}(\mathsf{x}, \beta)$ *is defined*;

(ii) $\forall \beta . \mathbb{F}(a, \langle \mathsf{x}, \beta \rangle)$ *is defined and* $\exists \beta . \mathbb{F}(a, \langle \mathsf{x}, \beta \rangle) \simeq 0$.

From this follows immediately that if $\mathbb{R}$ is semi-recursive in $\mathscr{I}$, say with semi-index a from $\mathscr{I}$, then

$$\exists \beta \, \mathbb{R}(\mathsf{x}, \beta) \leftrightarrow \mathbb{E}(\lambda \beta . \mathbb{F}(a, \langle \mathsf{x}, \beta \rangle)) \simeq 0$$

so that $\exists^1 \mathbb{R}$ is also semi-recursive in $\mathscr{I}$.

The proof of Theorem 4.3 is rather involved if all details are included, but the basic idea is relatively simple and we shall confine our discussion to this.

Proof of Theorem 4.3. The point of the functional $\mathbb{F}$ is to select from $\mathsf{B}_{a,\mathsf{x}} = \{\beta : \{a\}^{\mathscr{I}}(\mathsf{x}, \beta) \text{ is defined}\}$, whenever it is non-empty, a non-empty subset $\mathsf{A}_{a,\mathsf{x}} = \{\beta : \mathbb{F}(a, \langle \mathsf{x}, \beta \rangle) \simeq 0\}$ which is recursive in x and $\mathscr{I}$, uniformly in a and x — that is, with index from x and $\mathscr{I}$ computable from a independently of x. $\mathsf{A}_{a,\mathsf{x}}$ will be recursive (not merely semi-recursive) in x and $\mathcal{J}$ because when $\mathsf{B}_{a,\mathsf{x}} \neq \varnothing$, $\lambda\beta\, \mathbb{F}(a, \langle \mathsf{x}, \beta\rangle)$ is total.

It turns out to be more convenient to prove a slightly more general result. Let φ, ψ, and θ denote functions from ${}^{\omega}\omega$ into ${}^{2,1,1}\omega$, so $\varphi(\beta) = (a, m, \alpha, \mathsf{I})$. We may think of φ as a quadruple $(\mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I})$ of type-2 functionals, where

$$\varphi(\beta) = (\mathsf{F}(\beta), \mathsf{G}(\beta), \lambda p . \mathsf{H}(p, \beta), \lambda\gamma . \mathsf{I}(\beta, \gamma))$$

and thus as a single type-2 functional. We shall define a function $\mathbb{G}$ partial recursive in $\mathcal{J}$ which selects a set $\mathsf{A}_{\varphi} = \{\beta : \mathbb{G}(\beta, \varphi) \simeq 0\}$ such that whenever $\mathsf{B}_{\varphi} = \{\beta : \varphi(\beta) \in \mathbb{U}^{\mathcal{J}}\}$ is non-empty, then $\mathsf{A}_{\varphi} \subseteq \mathsf{B}_{\varphi}$ and A_{φ} is recursive in φ and $\mathcal{J}$, uniformly in φ. Then $\mathbb{F}$ is obtained by applying this result to the function $\varphi_{a,\mathsf{x}}$ defined by: $\varphi_{a,\mathsf{x}}(\beta) = (a, \langle \mathsf{x}, \beta\rangle)$.

For any φ, let

$$\min \varphi = \inf\{|\varphi(\beta)|^{\mathcal{J}} : \beta \in {}^{\omega}\omega\}$$

and

$$\boldsymbol{\kappa}^{\mathcal{J}} = \sup{}^{+}\{|a, \mathsf{x}|^{\mathcal{J}} : \mathbb{U}^{\mathcal{J}}(a, \langle \mathsf{x}\rangle)\}.$$

Clearly, $\exists\beta . \varphi(\beta) \in \mathbb{U}^{\mathcal{J}} \leftrightarrow \min \varphi < \boldsymbol{\kappa}^{\mathcal{J}}$.

We shall determine $\mathbb{G}$ so that for all β and φ,

(a) if $\min \varphi < \boldsymbol{\kappa}^{\mathcal{J}}$, then

$$\mathbb{G}(\beta, \varphi) = \begin{cases} 0, & \text{if } \quad |\varphi(\beta)|^{\mathcal{J}} = \min \varphi; \\ 1, & \text{otherwise;} \end{cases}$$

(b) $\mathbb{G}(\beta, \varphi) \simeq 0 \rightarrow \varphi(\beta) \in \mathbb{U}^{\mathcal{J}}$.

In fact, we need only be concerned about condition (a), for if $\mathbb{G}$ satisfies (a), and $\varphi(\beta) = (a, \langle \mathsf{x}\rangle)$, then the functional computed according to the following flow diagram satisfies both (a) and (b):

$$\begin{tikzcd} \boxed{\mathbb{G}(\beta, \varphi)} \arrow[r, "0"] \arrow[d, "1"] & \boxed{0 \cdot \{a\}^{\mathcal{J}}(\mathsf{x})} \arrow[r] & {} \\ {} & & \end{tikzcd}$$

We may thus define $\mathbb{G}$ by effective transfinite recursion on $\min \varphi$ for $\min \varphi < \boldsymbol{\kappa}^{\mathcal{J}}$. Suppose that for all ψ such that $\min \psi < \min \varphi$ and all β, $\mathbb{G}(\beta, \psi)$ has been defined to satisfy (a); we shall describe how to compute $\mathbb{G}(\beta, \varphi)$. If $\min \varphi = 0$, this is clear and we assume $\min \varphi > 0$.

Let $\circleddash$ denote the "augmented" immediate predecessor relation for computations relative to $\mathscr{I}$:

$$(a', \mathsf{x}') \circleddash (a, \mathsf{x}) \leftrightarrow \mathbb{P}^{\mathscr{I}}(a, a', \mathsf{x}, \mathsf{x}') \quad \text{or} \quad a = \langle 0, k, l, l', 0, \ldots \rangle$$
$$\text{is an index appropriate for } \mathsf{x} \text{ and}$$
$$(a', \mathsf{x}') = (a, \mathsf{x}).$$

The relation $\circleddash_0$ is the restriction of $\circleddash$ to those cases where the fact that (a', x') is an immediate predecessor of (a, x) can be determined without performing any computations. Precisely, let $\mathbb{P}_0^{\mathscr{I}}$ be the relation defined by replacing clause (1) of the definition of $\mathbb{P}^{\mathscr{I}}(u, v, \langle \boldsymbol{\alpha} \rangle, \langle \boldsymbol{\beta} \rangle, \langle \mathbf{H} \rangle, \langle \mathbf{I} \rangle)$ by

(1′) for some $b, c_0, \ldots, c_{k'-1}$, $u = \langle \langle 1, k, l, l', 0, b, c_0, \ldots, c_{k'-1} \rangle, \mathbf{m} \rangle$, $\boldsymbol{\beta} = \boldsymbol{\alpha}$, $\mathbf{H} = \mathbf{I}$, and $v = \langle c_i, \mathbf{m} \rangle$ for some $i < k'$;

Then the definition of $\circleddash_0$ is obtained from that of $\circleddash$ by replacing $\mathbb{P}^{\mathscr{I}}$ by $\mathbb{P}_0^{\mathscr{I}}$. Thus $\circleddash_0$ is a subrelation of $\circleddash$ which is recursive in $\mathscr{I}$. Although in general $\circleddash$ is only semi-recursive in $\mathscr{I}$, it is clear that for any (a, x),

(c) if for all $(a', \mathsf{x}') \circleddash_0 (a, \mathsf{x})$, $\{a'\}^{\mathscr{I}}(\mathsf{x}')$ is defined, then $\{(a', \mathsf{x}') : (a', \mathsf{x}') \circleddash (a, \mathsf{x})\}$ is recursive in x and $\mathscr{I}$, uniformly in x.

Let

$$\mathsf{B} = \{\beta : \text{for all } (a, \mathsf{x}) \circleddash_0 \varphi(\beta),\ |a, \mathsf{x}|^{\mathscr{I}} < \min \varphi\};$$
$$\mathsf{C} = \{\beta : \text{for all } (a, \mathsf{x}) \circleddash \varphi(\beta),\ |a, \mathsf{x}|^{\mathscr{I}} < \min \varphi\}.$$

Clearly $\mathsf{C} \subseteq \mathsf{B}$ and $|\varphi(\beta)|^{\mathscr{I}} = \min \varphi$ iff $\beta \in \mathsf{C}$, so it will suffice to show that C is recursive in $\mathscr{I}$ and φ, uniformly in φ, as we may then set $\mathbb{G}(\beta, \varphi) = \mathsf{K}_\mathsf{C}(\beta)$. Our approach to this is to show that B is recursive in $\mathscr{I}$ and φ and to define a set $\mathbb{A}$ also recursive in $\mathscr{I}$ and φ such that

(d) for all $\theta \in \mathbb{A}$, $\min \theta < \min \varphi$, but

(e) $\min \varphi = \sup^+\{\min \theta : \theta \in \mathbb{A}\}$.

Given such $\mathbb{A}$ and B, we have for any β,

$$\beta \in \mathsf{C} \leftrightarrow \beta \in \mathsf{B} \text{ and for all } (a, \mathsf{x}) \circleddash \varphi(\beta),\ (\exists \theta \in \mathbb{A}).\, |a, \mathsf{x}|^{\mathscr{I}} \leqslant \min \theta.$$

For $\beta \in \mathsf{B}$, the condition $(a, \mathsf{x}) \circleddash \varphi(\beta)$ is recursive in $\mathscr{I}$ and φ by (c). The quantifier $(\exists \theta \in \mathbb{A})$ is computable from $\mathscr{E}$ and $\mathbb{A}$, hence from $\mathscr{I}$ and φ. Finally, for $\theta \in \mathbb{A}$, it follows from (d) and the induction hypothesis that

(f) $$|a, \mathsf{x}|^{\mathscr{I}} \leqslant \min \theta \leftrightarrow \exists \gamma\, (\mathbb{G}(\gamma, \theta) = 0 \wedge |a, \mathsf{x}|^{\mathscr{I}} \leqslant |\theta(\gamma)|^{\mathscr{I}}).$$

That the condition on the right side is recursive in $\mathscr{I}$ and φ is immediate from the Ordinal Comparison Theorem and the fact that $\mathbb{E}$ is recursive in $\mathscr{I}$ and we have thus verified that C is recursive in $\mathscr{I}$ and φ.

Let λ denote the least cardinal greater than $2^{\aleph_0}$. For $\sigma < \lambda$ we define $\mathbb{A}_\sigma$ and B_σ by recursion as follows. Let $\mathsf{B}_{(\sigma)} = \bigcup\{\mathsf{B}_\tau : \tau < \sigma\}$. Then

$$\mathbb{A}_\sigma = \{\theta\colon \forall\beta\,[\beta \in \mathsf{B}_{(\sigma)} \wedge \theta(\beta) \circledcirc \varphi(\beta)] \vee [\beta \notin \mathsf{B}_{(\sigma)} \wedge \theta(\beta) \circledcirc_0 \varphi(\beta)]\};$$
$$\mathsf{B}_\sigma = \{\beta\colon \text{for all } (a, \mathsf{x}) \circledcirc_0 \varphi(\beta)\ (\exists\theta \in \mathbb{A}_\sigma)\,.\,|a, \mathsf{x}|^{\mathscr{I}} \leqslant \min\theta\}.$$

It is straightforward to prove by induction that for $\sigma \leqslant \rho$, $\mathbb{A}_\sigma \subseteq \mathbb{A}_\rho$ and $\mathsf{B}_\sigma \subseteq \mathsf{B}_\rho \subseteq \mathsf{B}$. Suppose that for all $\sigma < \lambda$, $\mathsf{B}_{(\sigma)} \subsetneq \mathsf{B}_\sigma$. Then by associating with each $\beta \in \mathsf{B}_{(\lambda)}$ the least σ such that $\beta \in \mathsf{B}_\sigma$, we obtain a pre-wellordering of ${}^\omega\omega$ of length λ, a contradiction. Hence, for some $\bar\sigma < \lambda$, $\mathsf{B}_{(\bar\sigma)} = \mathsf{B}_{\bar\sigma}$. It follows that for $\rho \geqslant \bar\sigma$, $\mathbb{A}_{\bar\sigma} = \mathbb{A}_\rho$ and $\mathsf{B}_{\bar\sigma} = \mathsf{B}_\rho$. We set $\mathbb{A} = \mathbb{A}_{\bar\sigma}$ and show below that $\mathsf{B} = \mathsf{B}_{\bar\sigma}$.

It is immediate that (d) and the inequality ($\geqslant$) of (e) are satisfied. Suppose that $\min\varphi > \sup^+\{\min\theta : \theta \in \mathbb{A}\}$. We shall construct a function ψ with the following two mutually contradictory properties: $\psi \in \mathbb{A}$ and $(\forall\theta \in \mathbb{A})\,\forall\beta\,(|\psi(\beta)|^{\mathscr{I}} > \min\theta)$. For $\beta \in \mathsf{B}_{\bar\sigma}$,

$$\sup^+\{\min\theta : \theta \in \mathbb{A}\} < \min\varphi \leqslant \varphi(\beta) = \sup^+\{|a, \mathsf{x}|^{\mathscr{I}} : (a, \mathsf{x}) \circledcirc \varphi(\beta)\}.$$

Hence there exists a pair $(a, \mathsf{x}) \circledcirc \varphi(\beta)$ such that for all $\theta \in \mathbb{A}$, $|a, \mathsf{x}|^{\mathscr{I}} > \min\theta$ and we set $\psi(\beta)$ equal to some such pair. For $\beta \notin \mathsf{B}_{\bar\sigma}$, there exists by the definition of $\mathsf{B}_{\bar\sigma}$ a pair $(a, \mathsf{x}) \circledcirc_0 \varphi(\beta)$ such that for all $\theta \in \mathbb{A}_{\bar\sigma} = \mathbb{A}$, $|a, \mathsf{x}|^{\mathscr{I}} > \min\theta$ and we set $\psi(\beta)$ equal to some such pair. That ψ has the announced properties follows from the fact that $\mathsf{B}_{(\bar\sigma)} = \mathsf{B}_{\bar\sigma}$.

It remains to show that $\mathbb{A}$ and B are recursive in $\mathscr{I}$ and φ. Note first that $\mathsf{B} = \mathsf{B}_{\bar\sigma}$, since if $\beta \in \mathsf{B}$, then by (e), for all $(a, \mathsf{x}) \circledcirc_0 \varphi(\beta)$, $|a, \mathsf{x}|^{\mathscr{I}} < \sup^+\{\min\theta : \theta \in \mathbb{A}\}$, so $\beta \in \mathsf{B}_\sigma$. Recall that $\mathbb{W}$ denotes the set of type-2 functionals I which code well-orderings of ${}^\omega\omega$. $\mathbb{W}$ is recursive in $\mathscr{I}$ and contains codes for all ordinals less than λ. We claim that there exist functionals $\mathbb{H}$ and $\mathbb{I}$ recursive in $\mathscr{I}$ such that for all $\mathsf{I} \in \mathbb{W}$, β, and φ,

$$\mathbb{H}(\theta, \varphi, \mathsf{I}) = 0 \leftrightarrow \theta \in \mathbb{A}_{\|\mathsf{I}\|} \quad \text{and} \quad \mathbb{I}(\beta, \varphi, \mathsf{I}) = 0 \leftrightarrow \beta \in \mathsf{B}_{\|\mathsf{I}\|}.$$

From this it follows that

$$\theta \in \mathbb{A} \leftrightarrow (\exists\mathsf{I} \in \mathbb{W})\,.\,\mathbb{H}(\theta, \varphi, \mathsf{I}) = 0$$

and

$$\beta \in \mathsf{B} \leftrightarrow (\exists\mathsf{I} \in \mathbb{W})\,.\,\mathbb{I}(\beta, \varphi, \mathsf{I}) = 0$$

so both are recursive in $\mathscr{J}$. $\mathbb{H}$ and $\mathbb{I}$ are defined by effective transfinite recursion parallel to the recursive definitions of A_σ and B_σ. The equivalence (f) is used again to justify the recursiveness of B_σ and in the definition of $\mathbb{H}$ we use the equivalence, for $\mathsf{I} \in \mathbb{W}$,

$$\begin{aligned}\beta \in \mathsf{B}_{(\|\mathsf{I}\|)} &\leftrightarrow \exists \mathsf{J}\,(\|\mathsf{J}\| < \|\mathsf{I}\| \wedge \beta \in \mathsf{B}_{\|\mathsf{J}\|}) \\ &\leftrightarrow \exists \mathsf{J}\,(\mathsf{J} < \mathsf{I} \wedge \beta \in \mathsf{B}_{\|\mathsf{J}\|})\end{aligned}$$

where

$$\mathsf{J} < \mathsf{I} \leftrightarrow \exists \mathsf{F}\,(\mathsf{F} \text{ is an isomorphism of } \leqslant_{\mathsf{J}} \text{ into a proper initial segment of } \leqslant_{\mathsf{I}}). \quad \square$$

4.4–4.5 Exercises

4.4. Prove that $s\mathscr{J}$ is recursive in $\mathbb{E}^{\#}$.

4.5. Sketch a proof that for all $r \geqslant 1$ and any inductive operator Γ over ${}^{({}^{\omega}\omega)}\omega$, if $\Gamma \in \Delta^3_r$, then also $\bar{\Gamma} \in \Delta^3_r$.

4.6 Notes. Theorem 4.3 was announced in Grilliot [1969] and is known as the Grilliot Selection Theorem. The proof there was incorrect and our proof is from Harrington–MacQueen [1976].